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A DEFORMATION LEMMA WITH AN APPLICATION TO A MEAN FIELD EQUATION

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Abstract. Given a Hilbert space $(\mathcal{H}, \langle\,\cdot\,,\,\cdot\,\rangle)$, Λ an interval of $\mathbb{R}$ and $K \in C^{1,1}(\mathcal{H}, \mathbb{R})$ whose gradient is a compact mapping, we consider a family of functionals of the type:

$$I(\lambda, u) = \frac{1}{2}\langle u, u\rangle - \lambda K(u), \quad (\lambda, u) \in \Lambda \times \mathcal{H}.$$

Though the Palais–Smale condition may fail under just these assumptions, we present a deformation lemma to detect critical points. As a corollary, if $I(\overline{\lambda}, \,\cdot\,)$ has a "mountain pass geometry" for some $\overline{\lambda} \in \Lambda$, we deduce the existence of a sequence $\lambda_n \to \overline{\lambda}$ for which each $I(\lambda_n, \,\cdot\,)$ has a critical point. To illustrate such results, we consider the problem:

$$-\Delta u = \lambda\bigg(\frac{e^u}{\int_\Omega e^u} - \frac{T}{|\Omega|}\bigg), \quad u \in H_0^1(\Omega),$$

where $\Omega \subset\subset \mathbb{R}^2$ and T belongs to the dual H^{-1} of $H_0^1(\Omega)$. It is known that the associated energy functional does not satisfy the Palais–Smale condition. Nevertheless, we can prove existence of multiple solutions under some smallness condition on $\|T - 1\|_{H^{-1}}$, where 1 denotes the constant function identically equal to 1 in the domain.

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1. Introduction

Consider a Hilbert space $(\mathcal{H}, \langle\,\cdot\,,\,\cdot\,\rangle)$ whose associated norm will be denoted by $|\cdot|$. It is well-known that if $I \in C^{1,1}(\mathcal{H}, \mathbb{R})$ does *not* have any Palais–Smale sequence in the set $\{u \in \mathcal{H} : a \leq I(u) \leq b\}$ $(a < b)$, namely a sequence $u_n \in \mathcal{H}$ satisfying:

$$\|DI_{(u_n)}\|_{H^{-1}} \to 0, \quad I(u_n) \in [a, b], \tag{1.1}$$

then $\{u \in \mathcal{H} : I(u) \leq a\}$ is a deformation retract of $\{u \in \mathcal{H} : I(u) \leq b\}$. This classical "Deformation Lemma" can be obtained by considering the gradient vector flow of the functional I (see [30]). Such a result can be used to derive existence of critical point if one knows that the functional satisfies the so-called Palais–Smale condition ((PS)-condition for short), i.e. that any sequence (1.1) admits a convergent subsequence.

Based on this classical deformation lemma, the arguments of Ambrosetti and Rabinowitz in [1] show that if I exhibits a "mountain pass geometry", one may find two level sets which cannot be deformed one to the other by deformation retract. Hence in such a case we deduce that the functional I admits a sequence satisfying (1.1) for some values a, b. If furthermore the (PS)-condition is satisfied, then one obtains a critical point $\overline{u}$ for the functional I with $I(\overline{u}) \in [a, b]$. This is their famous "Mountain Pass Theorem" which gives a simple as well as useful criteria to prove existence of critical points. But this compactness assumption of Palais–Smale could be a serious restriction to apply this theorem and since then a lot of works has been undertaken to handle such a difficulty.

The aim of the present paper is to investigate a way of overcoming the possible failure of the (PS)-condition for functionals which are of the form:

$$I(\lambda, u) = \frac{1}{2}\langle u, u\rangle - \lambda K(u), \quad (\lambda, u) \in \Lambda \times \mathcal{H}, \tag{1.2}$$

where Λ is an interval of $(0, \infty)$ and K is such that

$$K \in C^{1,1}(\mathcal{H}, \mathbb{R}) \quad \text{with } \nabla K \colon \mathcal{H} \to \mathcal{H} \text{ compact.} \tag{1.3}$$

Here the gradient ∇ is defined with respect to the inner product $\langle\,\cdot\,,\,\cdot\,\rangle$ and by "compact" we mean that $\nabla K(u_n)$ admits a subsequence converging in the strong topology of $\mathcal{H}$ for any bounded sequence u_n. Note that the assumptions (1.2) and (1.3) are not enough to ensure the (PS)-condition. Hence the classical flow defined by the vector-field $-\nabla_u I(\lambda, u)$ is not completely appropriate to derive a deformation lemma. To overcome this problem, we shall modify this usual flow by following an idea of Bahri [3]. We will then be able to prove the following "Deformation Lemma":

Proposition 1.1. *Consider a family of functional satisfying* (1.2) *and* (1.3). *Fix* $\overline{I} := I(\overline{\lambda}, \cdot)$ *for some* $\overline{\lambda} \in \Lambda$ *and consider* $a, b \in \mathbb{R}$ $(a < b)$. *Assume that there are no sequences* $(\lambda_n, u_n) \in \Lambda \times \mathcal{H}$ *satisfying:*

$$D_u I(\lambda_n, u_n) = 0, \quad a \leq \overline{I}(\lambda_n, u_n) \leq b, \quad \lambda_n \to \overline{\lambda}, \quad \lambda_n \leq \overline{\lambda}. \tag{1.4}$$

Then $\{u \in \mathcal{H} : \overline{I}(u) \leq a\}$ *is a deformation retract of* $\{u \in \mathcal{H} : \overline{I}(u) \leq b\}$.

The main idea in the proof of Proposition 1.1 is to find a vector flow along which $\overline{I}$ decreases but by keeping also the function K bounded from above. The seminal idea of such a flow can be found in the works [3] and [4, p. 18–21], where Bahri used it to handle some specific problems in the contact forms. In [22], we showed how such a Deformation Lemma can be applied to handle the lack of compactness of the functional

$$H_0^1(\Omega) \to \mathbb{R}, \quad u \mapsto \frac{1}{2}\int_\Omega |\nabla u|^2 - \lambda \log\left(\frac{1}{|\Omega|}\int_\Omega e^u\right). \tag{1.5}$$

In [23] we announced that this approach works in a general Hilbert framework. In the present paper we shall give a complete and more detailed proof of Proposition 1.1 and for simplicity we restrict the study to a Hilbert space. But actually a similar deformation lemma is available for a more general family of functionals defined on a Banach space. This extension will be presented in a coming work.

Now if more information is available on the geometry of the functional $I(\overline{\lambda}, \cdot)$, one can eventually exhibit two sublevel sets I which are not topologically equivalent. In this case, Proposition 1.1 yields a sequence of critical points satisfying (1.4) for some values $a < b$. This occurs if for example $I(\overline{\lambda}, \cdot)$ exhibits a "mountain pass geometry". In such a case, by applying above Deformation Lemma, we derive a version of the "Mountain Pass Theorem" which in its simpler form reads as follows:

Proposition 1.2. *Let* $I(\lambda, \cdot)$ *be a family of functionals satisfying* (1.2), (1.3). *Assume that for some* $\overline{\lambda} \in \Lambda$, $\overline{I} := I(\overline{\lambda}, \cdot)$ *has a strict local minimizer* h_0 *and that there exists* $h_1 \in \mathcal{H}$ *with* $|h_1| > |h_0|$ *such that* $\overline{I}(h_1) < \overline{I}(h_0)$. *Consider the set of paths* $\Gamma := \{\gamma \in C([0,1], \mathcal{H}) : \gamma(0) = h_0,\ \gamma(1) = h_1\}$, *and define the min-max value* $\overline{c} := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \{\overline{I}(\gamma(t))\}$. *Then, for each* $\varepsilon \in (0, \overline{c} - \overline{I}(h_0))$, *there exists a sequence* $(\lambda_n, u_n) \in \Lambda \times \mathcal{H}$ *satisfying*

$$\begin{cases} D_u I(\lambda_n, u_n) = 0, \quad \lambda_n \in (0, \overline{\lambda}), \quad \lambda_n \to \overline{\lambda}, \\ \overline{c} - \varepsilon < I(\lambda_n, u_n) < \overline{c} + \varepsilon. \end{cases} \tag{1.6}$$

Conclusion similar to (1.6) has been obtained previously by Struwe. For example, consider a family of functionals $I(\lambda, \cdot)$ exhibiting a mountain-pass geometry (for each $\lambda \in \Lambda$) whose associated min-max value $c(\lambda)$ is monotone in λ. Then, Struwe pointed out that this monotonicity can be exploited to construct bounded Palais–Smale sequences for a subset $D \subset \Lambda$ of full measure. This has been applied successfully among others in [31] and [12] to prove existence of critical points for functionals of the type (1.5). A more general setting to apply Struwe's trick has been given by Jeanjean–Toland in [14], where it has been noted among other that the monotonicity is not necessary.

Our approach and results differ in several points from these works. Firstly, in Proposition 1.2 we assume the mountain pass geometry to hold only at a single value $\overline{\lambda}$. Furthermore, while the strategy of Struwe or its extension done in [14] rely on some fine properties of functions, here we will combine the Deformation Lemma stated in Proposition 1.1 with the arguments of [1] used in their classical Mountain Pass Theorem. In particular, the conclusion (1.6) includes also an estimate of the energy of the critical points. Note moreover that in the application it is enough to derive existence of a critical points for a dense set of values of the parameter instead of a set of full measure.

To illustrate how above results can be applied, we shall investigate the set of critical points of the following functional:

$$\begin{gathered} J(\lambda,\,\cdot\,)\colon H_0^1(\Omega) \to \mathbb{R}, \\ u \mapsto \frac{1}{2}\int_\Omega |\nabla u|^2 - \lambda\bigg\{\log\bigg(\frac{1}{|\Omega|}\int_\Omega e^u\bigg) - \frac{T(u)}{|\Omega|}\bigg\}, \end{gathered} \tag{1.7}$$

where $\Omega \subset\subset \mathbb{R}^2$, λ a real number and $T \in H^{-1}$ (the topological dual of $H_0^1(\Omega)$). Based on the Moser–Trudinger inequality (see [27]), the functional (1.7) is of class C^∞ and its critical points are weak solutions to the non-local semilinear problem:

$$-\Delta u = \lambda\bigg(\frac{e^u}{\int_\Omega e^u} - \frac{T}{|\Omega|}\bigg), \quad u \in H_0^1(\Omega). \tag{1.8}$$

The study of equations involving exponential nonlinearity goes back to Liouville ([21]), who gave a representation of the solutions of $-\Delta u = e^u$ on simply-connected domain of $\mathbb{R}^2$. This type of PDE is geometrically meaningful since it is related to the problem of prescribing the Gauss curvature which has given rise to the Nirenberg Problem on the sphere [27] or Kazdan–Warner problem for compact Riemannian surface [16]. Similar problems arise also in statistical mechanics and we refer to [18] for a more detailed discussion. For example, in thermal equilibrium at a given temperature β^{-1} and chemical potential μ_{ch}, the spatial density u of a perfect gas contained in $\Omega \subset \mathbb{R}^2$ satisfies the *Poisson–Boltzmann* equation: $-\Delta u = 2\pi e^{\beta[\mu_{\mathrm{ch}}-u]}$. More recently, by considering the mean-field thermodynamic limit of a two-dimensional system with logarithmic singular pair interactions, Caglioti et al. [8] and independently Kiessling [17] have been led to Problem (1.8) with $T \equiv 0$:

$$-\Delta u = \lambda \frac{e^u}{\int_\Omega e^u}, \quad u \in H_0^1(\Omega). \tag{1.9}$$

Problem (1.8) with a constant but non-zero source term appears when considering the thermal equilibrium of a two-dimensional plasma confined by a magnetic field in a cylinder (see for example [29]). But considering more general distribution in the right-hand side of (1.8) is of relevance in connection with some Chern-Simons-Higgs model, where Dirac measures are coming into play (see [33]).

When $\lambda \leq 0$, existence and uniqueness of critical points for (1.7) follows easily. Hence we shall focus on the more interesting case $\lambda > 0$. In one dimension, namely if Ω is an interval (a,b), the continuous injection $H_0^1((a,b)) \hookrightarrow L^\infty((a,b))$ shows readily that the functional $J(\lambda,\,\cdot\,)$ admits a global minimizer for any $\lambda \in \mathbb{R}$. In dimension 2, the situation is more complex. For $\lambda < 8\pi$, as in [8] and [17] where the case $T \equiv 0$ was considered, the Moser–Trudinger inequality shows that $J(\lambda,\,\cdot\,)$ has a global minimizer. Several works have been devoted to understand the structure of the critical points beyond 8π. When the domain is a ball, the Pohozaev identity shows that problem (1.9) has no solutions for $\lambda \geq 8\pi$. For non-simply connected smooth domain, existence for (1.9) in the range $(8\pi, 16\pi)$ has been established by Ding et al. [12] via variational method. When the domain and T are of class C^2, topological arguments are also available. Indeed by applying results obtained by Y.-Y. Li [20] and later completed by Chen–Lin [10], [11], the total Leray–Schauder degree of the set of solutions for problem (1.8) can be calculated in terms of the Euler characteristic of the domain Ω for each $\lambda \neq 8\pi N$ ($N \in \mathbb{Z}^+$). Though these regularity assumptions allow to derive striking existence results for a wide class of T and of domains, they can be sometimes too restrictive. Furthermore, in some situations, the only knowledge of the degree does not give any information. This occurs for example if in problem (1.8), $u \equiv 0$ is a trivial solution. In such a case, a further analysis is needed to capture the eventual non-trivial solutions. In [24] we did this by showing that on any domain the associated functional has a mountain pass geometry when the parameter belongs to a certain interval $A = (8\pi, \alpha)$ (always non-empty). This structure was exploited to derive existence of a non-trivial critical point for each $\lambda \in A$.

In the present paper, we shall extend the existence results obtained in [24] by considering a general $T \in H^{-1}$. With this aim, we need to introduce the space:

$$\mathcal{U}(\Omega) = \left\{ \varphi \in H^1(\Omega) : \int_\Omega \varphi = 0 \text{ and } \varphi - c \in H_0^1(\Omega) \text{ for some } c \in \mathbb{R} \right\}, \tag{1.10}$$

and define

$$\Lambda_1 := \inf\left\{\frac{\int_\Omega |\nabla\varphi|^2}{\int_\Omega \varphi^2} : \varphi \in \mathcal{U}(\Omega)\setminus\{0\}\right\}. \tag{1.11}$$

Let us emphasize that $\Lambda_1|\Omega| > 8\pi$ (see [24], [13]). We also introduce the following function

$$-\Delta T_0 = T, \quad T_0 \in H^1_0(\Omega). \tag{1.12}$$

The result of existence that we will prove is the following:

PROPOSITION 1.3. *Let $\Omega \subset\subset \mathbb{R}^2$, $\lambda_0 < \Lambda_1|\Omega|$ and consider the family of functionals $J(\lambda,\,\cdot\,)$ defined by* (1.7). *Then the following hold.*

(a) *There exist $\varepsilon_0, \delta_0 > 0$ (depending on λ_0 and the geometry of Ω) such that $J(\lambda,\,\cdot\,)$ has a local minimizer m_λ with $J(\lambda, m_\lambda) \le 0$ whenever*

$$|\Omega|^{-1}\|T-1\|_{H^{-1}} < \varepsilon_0 \quad \text{and} \quad \lambda \in (\lambda_0-\delta_0, \lambda_0], \tag{1.13}$$

where 1 *denotes the function identically equal to* 1 *on the domain Ω;*

(b) *Assume $\lambda_0 \in (8\pi, \Lambda_1|\Omega|)$ and the linear form T satisfies* (1.13). *Then we can find a dense subset $D \subset (\lambda_0-\delta_0, \lambda_0] \cap (8\pi, \lambda_0]$ such that $J(\lambda,\,\cdot\,)$ has two critical points for each $\lambda \in D$.*

(c) *If $\lambda_0 = 8\pi$, assume T satisfies* (1.13) *and furthermore $(T_0)^- \notin L^\infty_{\mathrm{loc}}(\Omega)$. Then, there exists a dense subset $D \subset (8\pi-\delta_0, 8\pi]$ such that $J(\lambda,\,\cdot\,)$ has two critical points for each $\lambda \in D$.*

The last statement is of particular interest. Indeed, when Ω is a simply-connected domain of class $C^{2,\alpha}$ and $T \in C^{0,\alpha}(\Omega)$ with $T \ge 0$, a careful inspection of the arguments of Suzuki [31] and Chang et al. [9] shows that Problem (1.8) has a unique solution whenever $\lambda \le 8\pi$. The third statement of Proposition 1.3 points out that a similar uniqueness result cannot hold in full generality when T has a negative part. For simplicity we have stated our multiplicity result by assuming T_0 to be unbounded from below, but this condition can be refined. The main point is that under this hypothesis, we can easily prove that the functional has a mountain pass geometry at $\lambda = 8\pi$ without making further assumptions on the domain. Let us also emphasize that in this result we shall not need any information on $J(\lambda,\,\cdot\,)$ for $\lambda < 8\pi$.

The paper is organized as follows. In Section 2, we prove the Deformation Lemma given in Proposition 1.1. This result is applied in Section 3 to prove a version of the "Mountain Pass Theorem" slightly more general than the one stated in Proposition 1.2. In Section 4, we introduce a family of test functions that are useful to understand the geometry of the functional (1.7). The study of the local and global minimizers of the functional (1.7) will be undertaken in

Section 5. All these results will be applied in Section 6 to complete the proof of Proposition 1.3.

2. A Deformation Lemma

Throughout this section, we shall work in a Hilbert space $(\mathcal{H},\langle\,\cdot\,,\,\cdot\,\rangle)$ and consider a family of functional satisfying (1.2) and (1.3). We first introduce a definition:

Definition 2.1. Given two sets $A\subset B\subset\mathcal{H}$, we say that A is a deformation retract of B if there exists a map $\eta\colon[0,1]\times\mathcal{H}\to\mathcal{H}$ satisfying

(a) η is continuous;
(b) $\eta(t,u_0)=u_0$ for all $(t,u_0)\in[0,1]\times A$;
(c) for $t=1$, $\eta(1,\,\cdot\,)$ maps B on A.

The two following Lemma will be useful:

Lemma 2.2. *Let* $(X_n,Y_n)\in\mathcal{H}\times\mathcal{H}$ *and set* $Z_n:=-\{|Y_n|X_n+|X_n|Y_n\}$. *We have:*

(a) $\langle X_n,Z_n\rangle\le 0$ *and* $\langle Y_n,Z_n\rangle\le 0$*;*
(b) *If* $\inf_{n\in\mathbb{N}}|Y_n|>0$ *and* $\langle X_n,Z_n\rangle\to 0$, *then* $(Z_n/|Y_n|)\to 0$.

Proof. A straightforward calculation gives

$$\langle X_n,Z_n\rangle=-|X_n|\{|X_n||Y_n|+\langle X_n,Y_n\rangle\},\tag{2.1}$$

and the Cauchy–Schwarz inequality shows that the right handside of (2.1) is non-positive.

For the second statement of the lemma, we note that

$$|Z_n|^2=2|X_n||Y_n|\{(|X_n||Y_n|+\langle X_n,Y_n\rangle\}=-2|Y_n|\langle X_n,Z_n\rangle,$$

where we have used (2.1) in the last equality. Therefore, we deduce that

$$\left|\frac{Z_n}{|Y_n|}\right|^2=-\frac{2\langle X_n,Z_n\rangle}{|Y_n|}\to 0. \qquad\square$$

Lemma 2.3. *Assume* (1.2) *and* (1.3) *hold. Let* $(\lambda_n,u_n)\in\Lambda\times\mathcal{H}$ *be a sequence satisfying*

(a) $D_uI(\lambda_n,u_n)\to 0$ *strongly in* $\mathcal{H}^{-1}$*;*
(b) $\sup_{n\in\mathbb{N}}\{\lambda_n,I(\lambda_n,u_n),K(u_n)\}<\infty$.

Then, up to a subsequence, $(\lambda_n,u_n)\to(\widetilde{\lambda},\widetilde{u})$ *strongly in* $\mathbb{R}\times\mathcal{H}$ *and* $D_uI(\widetilde{\lambda},\widetilde{u})=0$.

Proof. Since λ_n is bounded and

$$\frac{1}{2}\langle u_n,u_n\rangle=I(\lambda_n,u_n)+\lambda_nK(u_n)\le C,$$

we have up to a subsequence that

$$\lambda_n \to \widetilde{\lambda} \quad \text{and} \quad u_n \to \widetilde{u} \text{ weakly in } \mathcal{H}. \tag{2.2}$$

On the other hand, since $D_u I(\lambda, u_n) \to 0$ (strongly), we get

$$\langle \nabla_u \overline{I}(\lambda_n, u_n), \varphi \rangle \to 0, \quad \text{for all } \varphi \in \mathcal{H}. \tag{2.3}$$

In particular, by choosing $u_n - \widetilde{u}$ as a test function in (2.3), we get

$$|u_n - \widetilde{u}|^2 + \langle \widetilde{u}, u_n - \widetilde{u} \rangle + \lambda_n \langle \nabla K(u_n), u_n - \widetilde{u} \rangle \to 0. \tag{2.4}$$

The assumption that ∇K is compact together with (2.2) and (2.4) imply that $|u_n - \widetilde{u}| \to 0$. □

We can now prove the Deformation Lemma as stated in the introduction.

Proof of Proposition 1.1. Let us introduce the following subsets of $\mathcal{H}$:

$$\overline{I}^a := \{u \in \mathcal{H} : \overline{I}(u) \le a\}, \quad \overline{I}^b := \{u \in \mathcal{H} : \overline{I}(u) \le b\},$$
$$\overline{I}_a^b := \{u \in \mathcal{H} : a \le \overline{I}(u) \le b\}.$$

By assumption there exists $\varepsilon > 0$ such that

$$D_u I(\lambda, u) \ne 0, \quad \text{for all } (\lambda, u) \in [\overline{\lambda} - \varepsilon, \overline{\lambda}] \times \overline{I}_a^b. \tag{2.5}$$

Under this hypothesis, we are going to construct a flow which deforms $\overline{I}^b$ on $\overline{I}^a$ by keeping K bounded along the flow-line. To do this let $Z \in C^{0,1}(\mathcal{H}, \mathcal{H})$ be defined by:

$$Z(u) := -\{|\nabla K(u)|\, \nabla \overline{I}(u) + |\nabla \overline{I}(u)|\, \nabla K(u)\}, \tag{2.6}$$

and choose $\omega_\varepsilon \in \mathcal{C}^\infty(\mathbb{R})$ such that

$$0 \le \omega_\varepsilon \le 1, \quad \omega_\varepsilon(\zeta) = 0 \quad \text{for all } \zeta \le \varepsilon, \qquad \omega_\varepsilon(\zeta) = 1 \quad \text{for all } \zeta \ge 2\varepsilon.$$

Consider then the local flow $\eta = \eta(t, u_0)$ defined by the Cauchy problem:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -\omega_\varepsilon\left(\frac{|\nabla \overline{I}(u)|}{|\nabla K(u)|}\right) \nabla \overline{I}(u) + Z(u), \quad u(0) = u_0, \tag{2.7}$$

where $\omega_\varepsilon(|\nabla \overline{I}(u)|/|\nabla K(u)|)$ is understood to be equal to 1 when $\nabla K(u) = 0$.

Roughly speaking, if K is constant in a subset of the Hilbert space, the flow (2.7) is just the classical flow of "steepest descent." While if K varies, then the vector-field $Z(u)$ comes into play and modify the trajectory given by the classical gradient flow. The main properties that we shall prove are:

(a) Along a flow-line, $\overline{I}$ is strictly decreasing and K may increase but not "too much";
(b) Given $u_0 \in \overline{I}^b$, we have $\eta(t, u_0) \in \overline{I}^a$ at some $t = t(u_0)$;
(c) We may associate to the flow (2.7) a deformation retract of $\overline{I}^b$ on $\overline{I}^a$.

Let us first make a simple observation. By setting

$$u := \eta(t, u_0), \qquad \widetilde{\omega}_\varepsilon(t) := \omega_\varepsilon\left(\frac{|\nabla \overline{I}(u)|}{|\nabla K(u)|}\right),$$

a straightforward calculation shows that

$$\frac{d}{dt}[\overline{I} \circ \eta(\,\cdot\,, u_0)](t) = -\,\widetilde{\omega}_\varepsilon(t)|\nabla \overline{I}(u)|^2 + \langle \nabla \overline{I}(u), Z(u)\rangle, \tag{2.8}$$

$$\frac{d}{dt}[K \circ \eta(\,\cdot\,, u_0)](t) = -\,\widetilde{\omega}_\varepsilon(t)\langle \nabla K(u), \nabla \overline{I}(u)\rangle + \langle \nabla K(u), Z(u)\rangle. \tag{2.9}$$

Using (2.8) and Lemma 2.2, we see that

$$\frac{d}{dt}[\overline{I} \circ \eta(\,\cdot\,, u_0)](t) \le -\widetilde{\omega}_\varepsilon(t)|\nabla \overline{I}(u)|^2 \le 0, \tag{2.10}$$

namely $\overline{I}$ decreases along the flow-line.

To go further, we need to study $K(\eta(t, u_0))$ and $|\eta(t, u_0)|$ when $\eta(t, u_0) \in \overline{I}_a^b$.

Claim 1. The variation of K along a flow-line can be estimated by the variation of $\overline{I}$ as follows:

$$\frac{d}{dt}[K \circ \eta(\,\cdot\,, u_0)](t) \le -\frac{1}{\varepsilon}\frac{d}{dt}[\overline{I} \circ \eta(\,\cdot\,, u_0)](t). \tag{2.11}$$

Indeed, if $u := \eta(t, u_0)$ is such that

$$\frac{|\nabla \overline{I}(u)|}{|\nabla K(u)|} < \varepsilon,$$

then (2.9) and Lemma 2.2 show that

$$\frac{d}{dt}[K \circ \eta(\,\cdot\,, u_0)](t) = \langle \nabla K(u), Z(u)\rangle \le 0. \tag{2.12}$$

Assume now that $u := \eta(t, u_0)$ is such that

$$\frac{|\nabla \overline{I}(u)|}{|\nabla K(u)|} \ge \varepsilon. \tag{2.13}$$

Then (2.9), Lemma 2.2, Cauchy–Schwarz inequality and (2.13) imply that

$$\begin{aligned} \frac{d}{dt}[K \circ \eta(\,\cdot\,, u_0)](t) &\le \widetilde{\omega}_\varepsilon(t)|\nabla K(u)||\nabla \overline{I}(u)| \\ &= \widetilde{\omega}_\varepsilon(t)\frac{|\nabla K(u)|}{|\nabla \overline{I}(u)|}|\nabla \overline{I}(u)|^2 \le \frac{1}{\varepsilon}\,\widetilde{\omega}_\varepsilon(t)|\nabla \overline{I}(u)|^2. \end{aligned} \tag{2.14}$$

From (2.14) together with (2.10), we get

$$\frac{d}{dt}[K \circ \eta(\,\cdot\,, u_0)](t) \le -\frac{1}{\varepsilon}\frac{d}{dt}[\overline{I} \circ \eta(\,\cdot\,, u_0)](t). \tag{2.15}$$

Note that the right hand-side of (2.15) is non-negative (by (2.10)) while the relation (2.12) is non-positive. Hence, we deduce that (2.11) holds at any t (for which $\eta(\,\cdot\,, u_0)$ is defined).

Claim 2. Given $u_0 \in \mathcal{H}$, there exists a constant $C := C(a, b, u_0) > 0$ such that

$$K(\eta(u_0, t)), |\eta(u_0, t)| \leq C(a, b, u_0), \quad \text{whenever } \eta(u_0, t) \in \overline{I}_a^b.$$

By integrating (2.11), we obtain:

$$K(\eta(t, u_0)) < K(u_0) + \frac{b - a}{\varepsilon} \quad \text{whenever } \eta(t, u_0) \in \overline{I}_a^b. \tag{2.16}$$

Hence, whenever $\eta(t, u_0)$ belongs to $\overline{I}_a^b$, we see by using (2.16) that

$$\frac{1}{2}|\eta(t, u_0)|^2 \leq \overline{\lambda} K(\eta(t, u_0)) + b \leq C(a, b, u_0, \varepsilon). \tag{2.17}$$

So the second claim follows. In particular we see also that $t \mapsto \eta(t, u_0)$ is globally defined.

Claim 3. There exists a constant $c := c(a, b, u_0, \varepsilon) > 0$ such that

$$\frac{d}{dt}[\overline{I} \circ \eta(\,\cdot\,, u_0)] \leq -c^2 < 0 \quad \text{whenever } \eta(t, u_0) \in \overline{I}_a^b. \tag{2.18}$$

Let us fixed $u_0 \in \overline{I}^b$ in (2.7). Assume the existence of a sequence $t_n \geq 0$ such that

$$\frac{d}{dt}[\overline{I} \circ \eta(\,\cdot\,, u_0)](t_n) \to 0, \quad \eta(t_n, u_0) \in \overline{I}_a^b. \tag{2.19}$$

Set $u_n := \eta(t_n, u_0)$ and note that $K(u_n) \leq C$ (by Claim 2). By recalling (2.8), $\langle \nabla \overline{I}(u_n), Z(u_n) \rangle \leq 0$ (Lemma 2.2), we get

$$\omega_\varepsilon \left(\frac{|\nabla \overline{I}(u_n)|}{|\nabla K(u_n)|} \right) |\nabla I(u_n)|^2 \to 0 \quad \text{and} \quad \langle \nabla \overline{I}(u_n), Z(u_n) \rangle \to 0.$$

Hence, we have two possibilities:

$$|\nabla \overline{I}(u_n)| \to 0 \tag{2.20}$$

or

$$\frac{|\nabla \overline{I}(u_n)|}{|\nabla K(u_n)|} \to \gamma \leq 2\varepsilon \quad \text{and} \quad \langle \nabla \overline{I}(u_n), Z(u_n) \rangle \to 0. \tag{2.21}$$

In the first case, Lemma 2.3 (applied with $\lambda_n = \overline{\lambda}$) implies that $u_n \to \widetilde{u}$ strongly. In particular $\nabla \overline{I}(\widetilde{u}) = 0$ and $\widetilde{u} \in \overline{I}_a^b$. This contradicts our initial assumption (2.5).

Let us consider the second case (2.21). Since $\langle \nabla \overline{I}(u_n), Z(u_n) \rangle \to 0$, Lemma 2.2 implies

$$\nabla \overline{I}(u_n) + \frac{|\nabla \overline{I}(u_n)|}{|\nabla K(u_n)|} \nabla K(u_n) \to 0.$$

By setting $\gamma_n := (|\nabla \overline{I}(u_n)|)/(|\nabla K(u_n)|)$ (which converges to some $\gamma \leq 2\varepsilon$), we obtain

$$D_u I(\overline{\lambda} - \gamma_n, u_n) \to 0 \text{ (strongly)}, \quad I(\overline{\lambda} - \gamma_n, u_n), K(u_n) \leq C.$$

Lemma 2.3 (applied with $\lambda_n = \overline{\lambda} - \gamma_n$) shows again that u_n converges strongly to some $\widetilde{u} \in \overline{I}^b_a$ with $D_u\overline{I}(\overline{\lambda} - \gamma, \widetilde{u}) = 0$. This contradicts our initial assumption (2.5). Therefore (2.19) is impossible, which concludes the proof of (2.18).

Claim 4. $\overline{I}^a$ is a deformation retract of $\overline{I}^b$.

Given $u_0 \in \overline{I}^b$, we deduce from Claim 2 that

$$\overline{I}(\eta(t, u_0)) \leq -c^2 t + \overline{I}(u_0).$$

Hence, there is a t such that $\overline{I}(\eta(t, u_0)) \leq a$. It is then meaningful to define

$$t_a(u_0) := \begin{cases} \inf\{t \geq 0 : \overline{I}(\eta(t, u_0)) \in \overline{I}^a\} & \text{if } \overline{I}(u_0) > a, \\ 0 & \text{if } \overline{I}(u_0) \leq a. \end{cases}$$

Consider now the mapping

$$\widetilde{\eta}\colon [0,1] \times \mathcal{H} \to \mathcal{H}, \quad (s, u_0) \mapsto \eta(s t_a(u_0), u_0).$$

Then, classical result on ODE shows that $\widetilde{\eta}$ is continuous. On the other hand the conditions (b) and (c) of Definition 2.1 can be easily verified. Therefore, $\widetilde{\eta}$ is a deformation retract of $\overline{I}^b$ on $\overline{I}^a$. $\square$

Let us emphasize that the assumptions (1.2) and (1.3) can be relaxed. In particular an extension to Banach spaces is also possible and will be discussed in another work. But for the model example we have in mind, namely the functional (1.7), the framework that we have chosen is quite sufficient.

3. A Mountain Pass Theorem

Based on the Deformation Lemma we have proved in previous section, we can derive the following version of the mountain pass Theorem:

Theorem 3.1. *Let $I(\lambda, \cdot)$ satisfy* (1.2), (1.3). *Assume that for some $\overline{I} := I(\overline{\lambda}, \cdot)$, there exist $h_0, h_1 \in \mathcal{H}$ and $\rho_0 > 0$ with the properties:*

$$\begin{cases} |h_1 - h_0| > \rho_0, \\ \overline{\alpha} := \max\{\overline{I}(h_0), \overline{I}(h_1)\} < \overline{\beta} := \inf_{|u-h_0|=\rho_0}\{\overline{I}(u)\}. \end{cases} \tag{3.1}$$

By setting $\Gamma := \{\gamma \in C([0,1], \mathcal{H}) : \gamma(0) = h_0,\ \gamma(1) = h_1\}$, let us define:

$$\overline{c} := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \{\overline{I}(\gamma(t))\} \quad (\geq \overline{\beta}). \tag{3.2}$$

Then, for each $\varepsilon \in (0, \overline{c} - \overline{\alpha})$, there exists a sequence $(\lambda_n, u_n) \in \Lambda \times \mathcal{H}$ satisfying

$$\begin{cases} D_u I(\lambda_n, u_n) = 0, \quad \lambda_n \in [0, \overline{\lambda}), \quad \lambda_n \to \overline{\lambda}, \\ \overline{c} - \varepsilon < I(\lambda_n, u_n) < \overline{c} + \varepsilon. \end{cases} \tag{3.3}$$

Proof. If the Palais–Smale condition would hold, the arguments of [1] show that the minmax value $\overline{c}$ defined by (3.2) is a critical value for the functional

$\overline{I}$. Without this compactness condition, these same arguments combined with the deformation lemma stated in Theorem 1.1 give a slightly different conclusion. More precisely, we first note that the assumption $|h_1 - h_0| \geq \rho$ implies $\overline{c} \geq \alpha$. Assume now there is no sequence (λ_n, u_n) satisfying (3.3). Then, by Proposition 1.1, $\overline{I}^{\overline{c}-\varepsilon}$ is a deformation retract of $\overline{I}^{\overline{c}+\varepsilon}$ through a continuous map $\eta\colon [0,1] \times \mathcal{H} \to \mathcal{H}$ where

$$\overline{I}^{\overline{c}+\varepsilon} := \{u : \overline{I}(u) < \overline{c} + \varepsilon\} \quad \text{and} \quad \overline{I}^{\overline{c}-\varepsilon} := \{u : \overline{I}(u) < \overline{c} - \varepsilon\}.$$

Notice that $h_0, h_1 \in \overline{I}^{\overline{c}-\varepsilon}$. Hence for each $\gamma \in \Gamma$ and $t \in [0,1]$, the deformed curve

$$\gamma_t\colon [0,1] \to \mathcal{H}, \quad s \mapsto \eta(t, \gamma(s)),$$

is still in the set Γ. Consider now any $\gamma_0 \in \Gamma$ having the property

$$\begin{cases} \gamma_0(s) \in \overline{I}^{\overline{c}+\varepsilon} & \text{for all } s \in [0,1], \\ \overline{I}(\gamma_0(s)) \in (\overline{c} - \varepsilon, \overline{c} + \varepsilon) & \text{for some } s \in [0,1], \end{cases} \tag{3.4}$$

(such curve exists by the definition of the minmax value $\overline{c}$). Then, we have

$$\inf_{\gamma \in \Gamma} \max_{s \in [0,1]} \overline{I}(\gamma(s)) \leq \max_{s \in [0,1]} \overline{I}(\eta(t, \gamma_0(s))) \quad \text{for all } t \in [0,1].$$

But at $t = 1$, the right hand-side is less than $\overline{c} - \varepsilon$ due to (3.4) and the fact that η is a deformation retract. A contradiction with the definition of $\overline{c}$. Hence, the property (3.3) must hold. □

The aim of the remaining paper will be to apply these general results to the family of functionals (1.7). By using the Moser–Trudinger inequality (see [27]), one check easily that the functionals $J(\lambda, \cdot\,)$ fulfills the assumptions (1.2) and (1.3) by setting

$$K(u) = \log\left(\frac{1}{|\Omega|} \int_\Omega e^u\right) - \frac{T(u)}{|\Omega|}.$$

In order to apply the above mountain pass Theorem, we need first to understand the geometry of each $J(\lambda, \cdot\,)$. We shall proceed as follows:

(a) introduce a family of test functions,
(b) study the existence of global and local minimizers.

4. Liouville equation

For each $a \in \Omega$, denoting by δ_a the Dirac measure at the point a, the *Green function* is defined as the function $G(a, \cdot\,)\colon x \mapsto G(a,x)$ solving:

$$-\Delta_x G(a, \cdot\,) = \delta_a \quad \text{in } \Omega, \qquad G(a, \cdot\,) = 0 \quad \text{on } \partial\Omega.$$

Its *singular part* is given by $\Gamma(a,x) = (1/(2\pi)) \log(1/|x-a|)$ and its *regular part* is the unique function $H(a, \cdot\,)\colon x \mapsto H(a,x)$ satisfying

$$\Delta_x H(a, \cdot\,) = 0 \quad \text{in } \Omega, \qquad H(a,x) = \Gamma(a,x) \quad \text{on } \partial\Omega.$$

Therefore, with above definitions, we have $G(a,x) = \Gamma(a,x) - H(a,x)$.

Inspired by the approach of Bahri–Coron to handle semilinear equation involving Sobolev critical exponent (see [5]), we consider for each $(a,\mu) \in \Omega\times[1,\infty)$ the family of functions:

$$\delta_{a,\mu}(x) = \log \frac{8\mu^2}{(1+\mu^2|x-a|^2)^2} \quad x \in \mathbb{R}^2, \tag{4.1}$$

which are solutions of the "Liouville equation":

$$-\triangle u = e^u \quad \text{in } \mathbb{R}^2, \qquad \int_{\mathbb{R}^2} e^u < \infty. \tag{4.2}$$

Consider the "projection on $H_0^1(\Omega)$" $\overline{\delta}_{a,\mu}$ of $\delta_{a,\mu}$ defined by

$$\Delta(\overline{\delta}_{a,\mu}) = \Delta(\delta_{a,\mu}), \quad \overline{\delta}_{a,\mu} \in H_0^1(\Omega). \tag{4.3}$$

These functions $\overline{\delta}_{a,\mu}$ will be used to study the geometry of the functional (1.7). For this purpose, we need several estimates on $\delta_{a,\mu}$ and $\overline{\delta}_{a,\mu}$ as $\mu\to\infty$. In [22], in order to study the Palais–Smale property, such estimates have been derived for $a\in\Omega$ fixed. But in the next section, we shall allow the point a to move inside Ω. Hence in the present paper we will be more precise and take into consideration both variables (a,μ) in our remainders. We first emphasize the following property of the functions (4.1):

$$\delta_{a,\mu}(x) + \log\left(\frac{\mu^2}{8}\right) = 8\pi\Gamma(a,x) - 2\log\left(1+\frac{1}{\mu^2|x-a|^2}\right), \tag{4.4}$$

$$\begin{aligned}(\nabla_x\delta_{a,\mu})(x) &= -4\frac{\mu^2(x-a)}{1+\mu^2|x-a|^2} = -4\frac{x-a}{|x-a|^2} + O\left(\frac{1}{\mu^2|x-a|^3}\right)\\ &= 8\pi\nabla_x\Gamma(a,x) + O\left(\frac{1}{\mu^2|x-a|^3}\right).\end{aligned} \tag{4.5}$$

The estimates on $\delta_{a,\mu}$ we will use are collected in the following proposition:

Proposition 4.1. *Let $\Omega \subset\subset \mathbb{R}^2$ be a domain of class C^1, $(a,\mu)\in\Omega\times[2,\infty)$ and set $\delta := \delta_{a,\mu}$. Given $f\in C^2(\overline{\Omega})$, we have*

$$\int_\Omega fe^\delta = 8\pi f(a) + \Delta f(a)\, O\left(\frac{\log\mu}{\mu^2}\right) + O\left(\frac{1}{\mu^2}\int_{\partial\Omega}\frac{d\xi}{|\xi-a|^3}\right). \tag{4.6}$$

Furthermore,

$$\int_\Omega e^{p\delta} = \frac{8^p\pi}{2p-1}\mu^{2p-2} + O\left(\frac{1}{\mu^{2p}}\int_{\mathbb{R}^2\setminus\Omega}\frac{dx}{|x-a|^{4p}}\right) \quad \text{for all } p\ge 1, \tag{4.7}$$

and by setting $\Psi(s) = s\log s$ $(s>0)$, we have

$$\int_\Omega e^\delta\left\{\delta+\log\left(\frac{\mu^2}{8}\right)\right\} = 16\pi\log\mu^2 - 16\pi + O\left(\frac{1}{\mu^2}\int_{\mathbb{R}^2\setminus\Omega}\Psi^+\left(\frac{1}{|x-a|^4}\right)dx\right). \tag{4.8}$$

PROOF. The definition of δ and the Green's identity imply

$$\begin{aligned}\int_\Omega fe^\delta &= -\int_\Omega f\Delta\Big(\delta+\log\Big(\frac{\mu^2}{8}\Big)\Big)\\ &= -\int_\Omega \Big(\delta+\log\Big(\frac{\mu^2}{8}\Big)\Big)\Delta f+\int_{\partial\Omega}\Big\{\Big(\delta+\log\Big(\frac{\mu^2}{8}\Big)\Big)\frac{\partial f}{\partial\nu}-f\frac{\partial\delta}{\partial\nu}\Big\}.\end{aligned}$$

Hence, by plugging in this identity the relations (4.4) and (4.5), we obtain

$$\begin{aligned}\int_\Omega fe^\delta = &-8\pi\int_\Omega \Gamma(a,\,\cdot\,)\Delta f+8\pi\int_{\partial\Omega}\Big\{\Gamma(a,\,\cdot\,)\frac{\partial f}{\partial\nu}-f\frac{\partial\Gamma(a,\,\cdot\,)}{\partial\nu}\Big\}\\ &+2\int_\Omega \log\Big(1+\frac{1}{\mu^2|x-a|^2}\Big)\Delta f\\ &-2\int_{\partial\Omega}\log\Big(1+\frac{1}{\mu^2|x-a|^2}\Big)\frac{\partial f}{\partial\nu}+O\Big(\frac{1}{\mu^2}\int_{\partial\Omega}\frac{1}{|x-a|^3}\Big).\end{aligned}\tag{4.9}$$

By the Green's representation formula, we have

$$f(a)=-\int_\Omega \Gamma(a,\,\cdot\,)\Delta f+\int_{\partial\Omega}\Big\{\Gamma(a,\,\cdot\,)\frac{\partial f}{\partial\nu}-f\frac{\partial\Gamma(a,\,\cdot\,)}{\partial\nu}\Big\},\tag{4.10}$$

and by considering a ball centered at a of radius $2\operatorname{diam}(\Omega)$, we check easily that:

$$\int_\Omega \log\Big(1+\frac{1}{\mu^2|x-a|^2}\Big)\Delta f=\Delta f(a)O\Big(\frac{\log\mu}{\mu^2}\Big).\tag{4.11}$$

So using (4.10), (4.11) in (4.9), we finally get

$$\int_\Omega fe^\delta = 8\pi f(a)+\Delta f(a)\,O\Big(\frac{\log\mu}{\mu^2}\Big)+O\Big(\frac{1}{\mu^2}\int_{\partial\Omega}\frac{1}{|\xi-a|^3}\Big),$$

which is the relation (4.6).

To prove (4.7), a straightforward calculation shows that

$$\begin{aligned}\int_{\mathbb{R}^2} e^{p\delta} &= 8^p\pi\int_0^\infty \frac{dt}{(1+t)^{2p}}=\frac{8^p\pi}{2p-1}\mu^{2p-2},\\ \int_{\mathbb{R}^2\setminus\Omega} e^{p\delta} &= \int_{\mathbb{R}^2\setminus\Omega}\Big|\frac{8\mu^2}{(1+\mu^2|x-a|^2)^2}\Big|^p\le\frac{8^p}{\mu^{2p}}\int_{\mathbb{R}^2\setminus\Omega}\frac{1}{|x-a|^{4p}},\end{aligned}$$

and so (4.7) follows.

To prove (4.8), we first note that

$$\int_{\mathbb{R}^2} e^\delta\Big\{\delta+\log\Big(\frac{\mu^2}{8}\Big)\Big\}=16\pi\log\mu^2-16\pi.\tag{4.12}$$

To estimate the remainder, we shall prove (by setting $\Psi(s)=s\log s$)

$$\int_{\mathbb{R}^2\setminus\Omega}\Psi\Big(\Big|\frac{\mu^2}{1+\mu^2|x-a|^2}\Big|^2\Big)=O\Big(\int_{\mathbb{R}^2\setminus\Omega}\Psi^+\Big(\frac{1}{|x-a|^2}\Big)\Big).\tag{4.13}$$

Let

$$\Omega_1:=\Big\{x\in\mathbb{R}^2\setminus\Omega:\frac{\mu^2}{1+\mu^2|x-a|^2}\ge 1\Big\}.$$

Since the function Ψ is negative on the interval $(0,1)$ and increasing on $(1,\infty)$, we get

$$(4.14)\quad \int_{\mathbb{R}^2\setminus\Omega} \Psi\left(\left|\frac{\mu^2}{1+\mu^2|x-a|^2}\right|^2\right) \le \int_{\Omega_1} \Psi\left(\frac{1}{|x-a|^4}\right) \le \int_{\mathbb{R}^2\setminus\Omega} \Psi^+\left(\frac{1}{|x-a|^4}\right).$$

In order to estimate from below the integral in (4.13), we set $M := 2/(1+2e^{1/2})$ and define:

$$(4.15)\quad \begin{aligned} \Omega_2 &:= \left\{x \in \mathbb{R}^2 \setminus \Omega : \frac{\mu^2}{1+\mu^2|x-a|^2} \in (0,M)\right\}, \\ \Omega_3 &:= \left\{x \in \mathbb{R}^2 \setminus \Omega : \frac{\mu^2}{1+\mu^2|x-a|^2} \in [M,1]\right\}. \end{aligned}$$

For $\mu^2 \ge 2$, we check easily that

$$e^{1/2} \le \frac{1}{M} - \frac{1}{2} \le |x-a|^2, \quad x \in \Omega_2, \qquad \frac{1}{2} \le |x-a|^2 \le \frac{1}{M} - 2, \quad x \in \Omega_3.$$

Hence, since Ψ is decreasing on $(0,e^{-1})$, we get

$$(4.16)\quad \int_{\Omega_2} \Psi\left(\left|\frac{\mu^2}{1+\mu^2|x-a|^2}\right|^2\right) \ge \int_{\Omega_2} \Psi\left(\frac{1}{|x-a|^4}\right) \ge -\int_{\mathbb{R}^2\setminus\Omega} \Psi^+\left(\frac{1}{|x-a|^4}\right).$$

Finally, since $|\Omega_3| = O(1)$, we obtain

$$(4.17)\quad \begin{aligned} &\int_{\Omega_3} \Psi\left(\left|\frac{\mu^2}{1+\mu^2|x-a|^2}\right|^2\right) \\ &\qquad \ge |\Omega_3| \min_{s\in(0,\infty)}\{\Psi(s)\} \ge -C\int_{\mathbb{R}^2\setminus\Omega} \Psi^+\left(\frac{1}{|x-a|^4}\right). \end{aligned}$$

From (4.14), (4.16) and (4.17), we obtain (4.13), which completes the proof of the estimate (4.8). □

By using the Green's representation formula, we can estimate the projection $\overline{\delta}_{a,\mu}$ as follows:

Proposition 4.2. *Let $\Omega \subset\subset \mathbb{R}^2$ be a domain of class C^1. Given $a \in \Omega$, consider $\overline{\delta} := \overline{\delta}_{a,\mu}$ defined by* (4.3). *Then,*

$$(4.18)\quad \overline{\delta}(x) = \delta(x) + \log\left(\frac{\mu^2}{8}\right) - 8\pi H(a,x) + R(a,\mu,x),$$

where the remainder $R(a,\mu,\cdot\,)$ is given by:

$$(4.19)\quad R(a,\mu,x) := -\int_{\partial\Omega} \log\left(1+\frac{1}{\mu^2|\xi-a|^2}\right)^2 \frac{\partial G}{\partial\nu}(x,\xi)\,d\xi.$$

Proof. The function $h := \delta - \overline{\delta} + \log(\mu^2/8)$ satisfies

$$\Delta h = 0 \quad \text{in } \Omega, \qquad h = \log\frac{\mu^4}{(1+\mu^2|x-a|^2)^2} \quad \text{on } \partial\Omega.$$

Therefore, by using the Green's function G of Ω, we have:

$$\begin{aligned} h(x) &= -\int_{\partial\Omega} \log \frac{\mu^4}{(1+\mu^2|\xi-a|^2)^2} \frac{\partial G}{\partial \nu}(x,\xi)\, d\xi \\ &= -\int_{\partial\Omega} \left\{ \log \frac{1}{|\xi-a|^4} + \log \left(\frac{\mu^2|\xi-a|^2}{1+\mu^2|\xi-a|^2} \right)^2 \right\} \frac{\partial G}{\partial \nu}(x,\xi)\, d\xi \\ &= 8\pi H(a,x) + \int_{\partial\Omega} \log \left(1 + \frac{1}{\mu^2|\xi-a|^2} \right)^2 \frac{\partial G}{\partial \nu}(x,\xi)\, d\xi, \end{aligned}$$

and so (4.18) follows. $\square$

When the domain is of class $C^{2,\alpha}$, the function

$$u(x) = \int_{\partial\Omega} \frac{\partial G}{\partial \nu}(x,\xi)\, d\xi$$

is in $L^\infty(\Omega)$. So for such domains, Proposition 4.2 yields:

$$\overline{\delta} = \delta + \log\left(\frac{\mu^2}{8}\right) - 8\pi H(a,\,\cdot\,) + O\biggl(\frac{1}{\mu^2}\int_{\partial\Omega} \frac{d\xi}{|\xi-a|^2}\biggr). \tag{4.20}$$

In particular, since $\int_\Omega |\nabla\overline{\delta}|^2 = \int_\Omega \overline{\delta} e^\delta$, we deduce from (4.20) together with Proposition 4.1 the following estimate:

$$\begin{aligned} \int_\Omega |\nabla\overline{\delta}|^2 = {}& 16\pi \log\mu^2 - 16\pi - (8\pi)^2 H(a,a) \\ & + O\biggl(\frac{1}{\mu^2}\int_{\mathbb{R}^2\setminus\Omega} \Psi^+(\frac{1}{|x-a|^4})dx + \frac{1}{\mu^2}\int_{\partial\Omega}\frac{dx}{|x-a|^2}\biggr). \end{aligned} \tag{4.21}$$

Furthermore, we also get

$$\begin{aligned} \log\left(\int_\Omega e^{\overline{\delta}}\right) &= \log\left(\frac{\mu^2}{8}\right) + \log\left(\int_\Omega e^{-8\pi H(a,\,\cdot\,)} e^\delta\right) \\ &\quad + O\biggl(\frac{1}{\mu^2}\int_{\partial\Omega}\frac{d\xi}{|\xi-a|^2}\biggr) \\ &= \log(\mu^2) + \log(\pi) - 8\pi H(a,a) \\ &\quad + e^{8\pi H(a,a)}\biggl\{ |\nabla_x H(a,a)|^2 O\biggl(\frac{\log\mu}{\mu^2}\biggr) \\ &\quad + O\biggl(\frac{1}{\mu^2}\int_{\partial\Omega}\frac{d\xi}{|\xi-a|^3}\biggr)\biggr\} + O\biggl(\frac{1}{\mu^2}\int_{\partial\Omega}\frac{d\xi}{|\xi-a|^2}\biggr). \end{aligned} \tag{4.22}$$

Hence consider a $C^{2,\alpha}$ domain Ω and $K \subset\subset \Omega$. Then, whenever $a \in K$, estimates (4.21) together with (4.22) yield

$$\begin{aligned} J(\lambda,\overline{\delta}) = {}& (8\pi-\lambda)\log\mu^2 + \frac{(8\pi)^2}{2} H(a,a) \\ & - 8\pi - 8\pi\log\left(\frac{\pi}{|\Omega|}\right) + T(\overline{\delta}) + O\biggl(\frac{\log\mu}{\mu^2}\biggr). \end{aligned} \tag{2.23}$$

In order to use these functions $\overline{\delta}$ without making any assumption on the boundary, we shall several times proceed as follows. Take a $C^{2,\alpha}$ domain $\widetilde{\Omega} \subset \Omega$ and consider in this domain the projection $\overline{\delta}_\mu := \overline{\delta}_{a_\mu,\mu}$ of the function $\delta_{a_\mu,\mu}$ defined by (4.1):

$$-\Delta\overline{\delta}_\mu = e^{\delta_{a_\mu,\mu}}, \quad \overline{\delta}_\mu \in H_0^1(\widetilde{\Omega}).$$

Extend then $\overline{\delta}_\mu$ in the $H_0^1(\Omega)$-function:

$$\widetilde{\delta}_{a,\mu}(x) := \begin{cases} \overline{\delta}_{a,\mu}(x) & \text{if } x \in \widetilde{\Omega}, \\ 0 & \text{if } x \in \Omega \setminus \widetilde{\Omega}. \end{cases} \tag{4.24}$$

But note that if the boundary $\partial\Omega$ is of class $C^{2,\alpha}$, the consideration of the open set $\widetilde{\Omega}$ and of $\widetilde{\delta}_{a,\mu}$ is useless. In such a case, it is sufficient to consider the projection of $\delta_{a,\mu}$ in Ω.

5. Existence of minimizers

5.1. Global minimizers. When $T \equiv 0$, it is well-known that as a consequence of the Moser–Trudinger inequality (see [27]), the functional $J(\lambda, \cdot)$ defined by (1.7) is bounded from below if and only if $\lambda \le 8\pi$. In the presence of the linear form T, we have the following:

Proposition 5.1. *Let $\Omega \subset\subset \mathbb{R}^2$. Consider the functionals $J(\lambda, \cdot)$ given by* (1.7) *and T_0 defined by* (1.12).

(a) *The functional $J(\lambda, \cdot)$ admits a minimizer for each $\lambda < 8\pi$;*
(b) *If $(T_0)^- \in L^\infty(\Omega)$, then $J(8\pi, \cdot)$ is bounded from below;*
(c) *If $(T_0)^- \notin L^\infty_{\mathrm{loc}}(\Omega)$, then $J(8\pi, \cdot)$ is unbounded from below. More precisely, there exists a sequence of functions $\widetilde{\delta}_{a_\mu,\mu}$ defined by* (4.24) *such that*

$$\lim_{\mu\to\infty} \|\nabla\widetilde{\delta}_{a_\mu,\mu}\| = \infty \quad \textit{and} \quad \lim_{\mu\to\infty} J(8\pi, \widetilde{\delta}_{a_\mu,\mu}) = -\infty; \tag{5.1}$$

(d) *If $\lambda > 8\pi$, then the family of functions $\widetilde{\delta}_{a,\mu}$ (for some fixed $a \in \widetilde{\Omega} \subset \Omega$) defined by* (4.24) *satisfies:*

$$\lim_{\mu\to\infty} \|\nabla\widetilde{\delta}_{a,\mu}\| = \infty \quad \textit{and} \quad \lim_{\mu\to\infty} J(\lambda, \widetilde{\delta}_{a,\mu}) = -\infty. \tag{5.2}$$

Proof. (a) As in [8], [17], the Moser–Trudinger inequality implies that the functional (1.7) is sequentially lower-semicontinuous for any λ and coercive when $\lambda < 8\pi$ (with respect to the weak topology). So the existence of a minimizer for each $\lambda < 8\pi$ follows.

(b) For each $\lambda < 8\pi$, consider a minimizer u_λ given by Proposition 5.1. Note that by using T_0 as a test function in the equation (1.8) satisfied by u_λ, we get

$$\begin{aligned}\int_\Omega \nabla u_\lambda \nabla T_0 =& \lambda \int_\Omega \frac{e^{u_\lambda}}{\int_\Omega e^{u_\lambda}} T_0 - \frac{\lambda}{|\Omega|}\int_\Omega |\nabla T_0|^2 \\ &\geq -\lambda \|(T_0)^-\|_\infty - \frac{\lambda}{|\Omega|}\|\nabla T_0\|_2^2.\end{aligned} \tag{5.3}$$

Therefore, the infimum of $J(\lambda,\,\cdot\,)$ can be estimated as follows

$$\begin{aligned}J(\lambda,u_\lambda) &= \frac{1}{2}\int_\Omega |\nabla u_\lambda|^2 - \lambda \log\left(\frac{1}{|\Omega|}\int_\Omega e^{u_\lambda}\right) + \frac{\lambda}{|\Omega|}\int_\Omega \nabla T_0 \nabla u_\lambda \\ &\geq \frac{1}{2}\int_\Omega |\nabla u_\lambda|^2 - \lambda \log\left(\frac{1}{|\Omega|}\int_\Omega e^{u_\lambda}\right) \\ &\quad - \frac{\lambda^2}{|\Omega|}\left(\|(T_0)^-\|_\infty + \frac{\|\nabla T_0\|_2^2}{|\Omega|}\right).\end{aligned} \tag{5.4}$$

The Moser–Trudinger inequality shows that the right hand-side of (5.4) is uniformly bounded from below as $\lambda \to 8\pi$. The conclusion of the second statement follows then easily.

(c) By assumption, there exists a ball $B \subset\subset \Omega$ such that $(T_0)^- \notin L^\infty(B)$. Since $C_0^\infty(\Omega)$ is dense in $H_0^1(\Omega)$, for each $\mu \geq 1$ we can choose $\varphi_\mu \in C_0^\infty(\Omega)$ such that

$$\|\nabla(\varphi_\mu - T_0)\|_2 < \frac{1}{\mu^2}. \tag{5.5}$$

Note that given $R > 0$, the set $\mathcal{B}_R^\infty := \{\varphi \in L^\infty(\Omega) : \|\varphi\|_\infty \leq R\}$ is closed in $L^2(\Omega)$ (since strong convergence in $L^2(\Omega)$ implies a.e. pointwise convergence, see Theorem IV.9 in [6]). In particular, the set $\{\varphi_\mu^- : \mu \geq 1\}$ cannot be bounded in $L^\infty(B)$. Hence along a sequence $a_\mu \in B$, we must have

$$\varphi_\mu(a_\mu) \to -\infty, \quad a_\mu \to p \in B. \tag{5.6}$$

Take a $C^{2,\alpha}$ domain $\widetilde{\Omega}$ such that $B \subset\subset \widetilde{\Omega} \subset \Omega$, and consider the functions $\widetilde{\delta}_\mu := \widetilde{\delta}_{a_\mu,\mu}$ defined by (4.24). In order to apply the estimates of Section 4, we note that:

$$\frac{|x-p|}{|x-b|} \leq 1 + \frac{\operatorname{diam}(B)}{\operatorname{dist}(\partial B, \partial\widetilde{\Omega})} \leq C_B, \quad \text{for all } (b,x) \in B \times (\mathbb{R}^2 \setminus \widetilde{\Omega}), \tag{5.7}$$

where p is defined in (5.6) and C_B is a finite constant since $\operatorname{dist}(\partial B, \partial\widetilde{\Omega}) > 0$. Consider now the Green function G of the domain $\widetilde{\Omega}$ and its regular part H.

Using (5.7) and applying (4.23) in the domain $\widetilde{\Omega}$ with $\lambda = 8\pi$, we get

$$
\begin{aligned}
(5.8)\quad J(8\pi, \widetilde{\delta}_\mu) &= \frac{1}{2}\int_{\widetilde{\Omega}} |\nabla \overline{\delta}_\mu| - \lambda \log\left(\frac{1}{|\widetilde{\Omega}|}\int_{\widetilde{\Omega}} e^{\overline{\delta}_\mu}\right) \\
&\quad - \lambda \log\left(\frac{|\widetilde{\Omega}|}{|\Omega|}\right) - \lambda \log\left(1 + \frac{|\widetilde{\Omega}| - |\Omega|}{\int_{\widetilde{\Omega}} e^{\overline{\delta}_\mu}}\right) - \frac{\lambda}{|\Omega|} T(\widetilde{\delta}) \\
&= \frac{(8\pi)^2}{2} H(a_\mu, a_\mu) - 8\pi - 8\pi \log\left(\frac{\pi}{|\Omega|}\right) + O\left(\frac{\log \mu}{\mu^2}\right) - \frac{\lambda}{|\Omega|} T(\widetilde{\delta}).
\end{aligned}
$$

By considering the trace of T_0 on $\widetilde{\Omega}$ and applying Green's Theorem, we have:

$$
\begin{aligned}
T(\widetilde{\delta}_\mu) &= \int_{\widetilde{\Omega}} \nabla T_0 \nabla \overline{\delta}_\mu = \int_{\partial\widetilde{\Omega}} T_0 \frac{\partial \overline{\delta}_\mu}{\partial \nu} + \int_{\widetilde{\Omega}} T_0 e^{\delta_\mu} \\
&= \int_{\partial\widetilde{\Omega}} T_0 \frac{\partial \overline{\delta}_\mu}{\partial \nu} + \int_{\widetilde{\Omega}} (T_0 - \varphi_\mu) e^{\delta_\mu} + \int_{\widetilde{\Omega}} \varphi_\mu e^{\delta_\mu}.
\end{aligned} \tag{5.9}
$$

By differentiating (4.18) with respect to the variable x, using (4.5) and the fact that $\partial\widetilde{\Omega}$ is of class $C^{2,\alpha}$, we obtain:

$$
\frac{\partial \overline{\delta}}{\partial \nu}(x) = 8\pi \frac{\partial G}{\partial \nu}(a, x) + O\left(\frac{1}{\mu^2}\int_{\partial\Omega} \frac{d\xi}{|\xi - a|^3}\right), \quad \text{for all } x \in \partial\widetilde{\Omega}. \tag{5.10}
$$

Therefore, by taking into account (5.7), the boundary integral in (5.9) can be written as

$$
\int_{\partial\widetilde{\Omega}} T_0 \frac{\partial \overline{\delta}_\mu}{\partial \nu} = 8\pi \int_{\partial\widetilde{\Omega}} T_0(\xi) \frac{\partial G(a_\mu, \xi)}{\partial \nu}\, d\xi + O\left(\frac{1}{\mu^2}\right). \tag{5.11}
$$

By applying (5.5), (4.7) and (5.7), we get

$$
\int_{\widetilde{\Omega}} |T_0 - \varphi_\mu| e^{\delta_\mu} \leq \|\nabla (T_0 - \varphi_\mu)\|_2 \, \|e^{\delta_\mu}\|_2 = O\left(\frac{1}{\mu}\right). \tag{5.12}
$$

The last integral in (5.9) can be estimated using (4.6) with (5.7) leading to:

$$
\int_{\widetilde{\Omega}} \varphi_\mu e^{\delta_\mu} = 8\pi \varphi_\mu(a_\mu) + O\left(\frac{\log \mu}{\mu^2}\right). \tag{5.13}
$$

By plugging in (5.8) the relation (5.9) together with the estimates (5.11) to (5.14), and reminding that a_μ converges to an interior point of $\widetilde{\Omega}$, we get

$$
J(8\pi, \widetilde{\delta}_\mu) = 8\pi \varphi_\mu(a_\mu) + O(1). \tag{5.14}
$$

Hence letting $\mu \to \infty$ in (5.14) and using (5.6), we deduce that $J(8\pi, \widetilde{\delta}_\mu)$ tends to $-\infty$, and furthermore $\widetilde{\delta}_{a,\mu}$ is unbounded in $H_0^1(\Omega)$ by (4.21).

(d) Choose $\widetilde{\Omega}$ of class $C^{2,\alpha}$ in Ω, $a \in \widetilde{\Omega}$ and the functions $\widetilde{\delta}_{a,\mu}$ defined in (4.24). On the one hand, (4.21) shows that $\widetilde{\delta}_{a,\mu}$ is unbounded in $H_0^1(\Omega)$. Furthermore, by applying (4.23) and using $|T(u)| \leq \|T\| \|\nabla u\|_2$, we deduce that

$$
J(\lambda, \widetilde{\delta}_{a,\mu}) \leq (8\pi - \lambda) \log \mu^2 + O(1) + O(\sqrt{\log \mu^2}). \tag{5.15}
$$

Hence, for each $\lambda > 8\pi$, we have $J(\lambda, \widetilde{\delta}_{a,\mu}) \to -\infty$ as $\mu \to \infty$. □

Under the assumption of the statement (b) of Proposition 5.1, the test functions (4.24) together with the estimates (4.21) and (4.23) show that the functional $J(8\pi, \cdot\,)$ is *never* coercive for the weak topology of $H_0^1(\Omega)$. So existence of global minimizer for this critical value becomes more subtle. When $T \equiv 0$, and Ω simply-connected, a condition on the domain ensuring the existence of a minimizer has been obtained in the work of Chang and al. [9].

5.2. Existence of local minimizers. Let us prove first that under a smallness condition on T, the functional $J(\lambda, \cdot\,)$ admits a local minimizer. To this end, besides the eigenvalue $\Lambda_1 := \Lambda_1(\Omega)$ defined by (1.11), we shall introduce the following constant:

$$S := \inf\left\{\frac{\|\nabla\varphi\|_2}{\|\varphi\|_6}, \varphi \in \mathcal{U}(\Omega)\setminus\{0\}\right\}, \tag{5.16}$$

where the space $\mathcal{U}(\Omega)$ is defined by (1.10). Before stating our first existence result, we need:

Lemma 5.2.

(a) *For any $\Omega \subset\subset \mathbb{R}^2$, we have $8\pi < \Lambda_1|\Omega|$.*

(b) *For each $\lambda < \Lambda_1|\Omega|$, consider the function $\omega := \omega(\lambda, \Omega, \cdot\,)$ defined by:*

$$\omega(t) := \left(1 - \frac{\lambda}{\Lambda_1|\Omega|}\right)\frac{t}{2} - \frac{\lambda}{S^3|\Omega|^{1/2}}\, t^2\, e^{t^2/(8\pi)}, \quad t > 0. \tag{5.17}$$

Then, $\sup_{t>0}\{\omega(t)\} > 0$ and there exists $R_0 > 0$ (unique) such that

$$\omega(R_0) = \sup_{t>0}\{\omega(t)\}. \tag{5.18}$$

Proof. The fact that $8\pi < \Lambda_1|\Omega|$ has been proved in [24] and a more detailed discussion can be found in [13]. To prove the existence of R_0, we just note that

$$\omega'(0) = \left(1 - \frac{\lambda}{\Lambda_1|\Omega|}\right)\frac{1}{2} > 0, \quad \omega'' < 0, \quad \lim_{t\to\infty}\omega(t) = -\infty.$$

Moreover, this maximum is strictly positive and depends only on the parameter λ and the geometry of the domain (actually the constants Λ_1 and S). □

The motivation of introducing the function (5.17) will become clear from the proof of the following proposition:

Proposition 5.3 (A local minimizer). *Let $\lambda < \Lambda_1|\Omega|$ and consider the constants R_0, $\omega(R_0)$ (depending only on λ, Ω) given by* (5.18). *Assume*

$$|\Omega|^{-1}\, \|T - 1\|_{H^{-1}} < \lambda^{-1}\omega(R_0). \tag{5.19}$$

Then, in the ball $B(0,R_0) \subset H_0^1(\Omega)$ we have

$$J(\lambda,0)=0 \quad \text{and} \quad J(\lambda,u)>0 \quad \text{for all } \|u\|=R_0. \tag{5.20}$$

In particular, $J(\lambda,\,\cdot\,)$ has a local minimizer $m_\lambda \in B(0,R_0)$ such that

$$J(\lambda,m_\lambda) \le 0.$$

Proof. Denoting by $\overline{u} := (1/|\Omega|)\int_\Omega u$, we have

$$\begin{aligned} J(\lambda,u) &= \frac{1}{2}\int_\Omega |\nabla(u-\overline{u})|^2 - \lambda \log\left(\frac{1}{|\Omega|}\int_\Omega e^{[u-\overline{u}]}\right) + \frac{\lambda}{|\Omega|}(T-1)(u) \\ &= Q(u) - \lambda R(u) \,+\, \frac{\lambda}{|\Omega|}(T-1)(u), \end{aligned} \tag{5.21}$$

where we have set

$$\begin{aligned} Q(u) &= \frac{1}{2}\left\{\int_\Omega |\nabla(u-\overline{u})|^2 - \frac{\lambda}{|\Omega|}\int_\Omega [u-\overline{u}]^2\right\}, \\ R(u) &= \log\left(\frac{1}{|\Omega|}\int_\Omega e^{[u-\overline{u}]}\right) - \frac{1}{2|\Omega|}\int_\Omega [u-\overline{u}]^2. \end{aligned}$$

On the on hand, by the definition of Λ_1 (see (1.11)), we have

$$Q(u) \ge \frac{1}{2}\left(1-\frac{\lambda}{\Lambda_1|\Omega|}\right)\|\nabla u\|_2^2. \tag{5.22}$$

Let us estimate $R(u)$. Setting $w := u-\overline{u}$ and using the inequality $\log(1+x) \le x$, we have:

$$\begin{aligned} R(u) &\le \frac{1}{|\Omega|}\int_\Omega \left\{\sum_{n=2}^{\infty}\frac{w^n}{n!}\right\} - \int_\Omega \frac{w^2}{2} = \int_\Omega \left\{w^3 \sum_{n=3}^{\infty}\frac{w^{n-3}}{n!}\right\} \\ &\le \int_\Omega |w|^3 e^{|w|} \;\le\; \left\{\int_\Omega |w|^6\right\}^{1/2}\left\{\int_\Omega e^{2w}\right\}^{1/2}. \end{aligned}$$

By using the Moser–Trudinger inequality, we know that

$$\frac{1}{|\Omega|}\int_\Omega e^w \le C_\Omega e^{\|\nabla w\|_2^2/(16\pi)}, \quad \text{for all } w \in \mathcal{U}(\Omega), \tag{5.23}$$

and by using a result of [25], we note that the best constant C_Ω in above inequality is given by $C_\Omega = 1$. Furthermore, from the definition of the constant S (see (5.16)), we have:

$$R(u) \le \frac{1}{S^3|\Omega|^{1/2}}\|\nabla u\|_2^3\, e^{\|\nabla w\|_2^2/(8\pi)}. \tag{5.24}$$

Hence from (5.21), (5.22) and (5.24), we obtain:

$$
\begin{aligned}
(5.25)\quad J(\lambda,u) &\geq \|\nabla u\|_2\bigg\{\bigg(1-\frac{\lambda}{\Lambda_1|\Omega|}\bigg)\frac{\|\nabla u\|_2}{2}-\frac{\lambda}{S^3|\Omega|^{1/2}}\|\nabla u\|_2^2\, e^{\|\nabla u\|_2^2/(8\pi)}\bigg\}\\
&\quad -\frac{\lambda}{|\Omega|}\|T-1\|_{H^{-1}}\|\nabla u\|_2\\
&= \|\nabla u\|_2\bigg\{\omega(\|\nabla u\|_2)-\frac{\lambda}{|\Omega|}\|T-1\|_{H^{-1}}\bigg\},
\end{aligned}
$$

where ω is the function defined by (5.17). Hence, the assumption (5.19) together with (5.25) imply the property (5.20).

Now, classical arguments show the existence of a local minimizer $m_\lambda \in B(0,R_0)$ for $J(\lambda,\,\cdot\,)$. More precisely, let u_n be a minimizing sequence:

$$
J(\lambda,u_n)\to\inf\{J(\lambda,u)\colon u\in B(0,R_0)\},\quad u_n\in B(0,R_0).
$$

Since u_n is bounded, it converges to some $m_\lambda\in\overline{B(0,R)}$ in the week topology of $H_0^1(\Omega)$. Recalling that the mapping $H_0^1(\Omega)\to\mathbb{R}$, $u\mapsto\int_\Omega e^u$ is weekly continuous (see [2]), and the same holds for T, we get $J(\lambda,u_n)\geq J(\lambda,m_\lambda)$. Therefore

$$
J(\lambda,m_\lambda)=\inf\{J(\lambda,u) : u\in\overline{B(0,R)}\}.
$$

Since $J(\lambda,m_\lambda)\leq J(\lambda,0)=0$ and $J(\lambda,\,\cdot\,)$ is strictly positive on $\partial B(0,R_0)$ by (5.20), we get $m_\lambda\in B(0,R_0)$ (namely m_λ cannot be on the boundary). So m_λ is a local minimizer of $J(\lambda,\,\cdot\,)$. ☐

Since $\Lambda_1|\Omega|>8\pi$ (see Lemma 5.2), above Proposition can also be applied when the parameter is equal to 8π. For example, let Ω be a disk of radius R. If $T\equiv 0$, the Pohozaev identity shows that the problem (1.8) has no solutions for $\lambda\geq 8\pi$. But let us choose for example T to be defined by the L^1-function:

$$
(5.26)\qquad T(x)=\begin{cases}1 & \text{if } \varepsilon<|x|<R,\\ -1 & \text{if } |x|<\varepsilon.\end{cases}
$$

Then, for ε small enough, Proposition 5.3 applies and shows that problem (1.8) has a solution when $\lambda=8\pi$.

6. Multiplicity for a mean field equation

It is known that the Palais–Smale condition for the functional (1.7) fails at each value $\lambda=8\pi\mathbb{N}$ (for each $N\in\mathbb{N}$), while this compactness condition is not well understood for the other values of the parameter. Based on our previous results, we will nevertheless be able to prove existence of critical points that are not minimizers.

Proof of Proposition 1.3. (a) By applying Proposition 5.3, we derive easily for each $\lambda\in(\lambda_0-\delta_0,\lambda_0]$ the existence of a local minimizer m_λ which has the property $J(\lambda,m_\lambda)\leq 0$.

(b) Consider $\overline{\lambda} \in (8\pi, \Lambda_1|\Omega|)$. By using (5.20) and (5.2), we see that the assumptions (3.1) are satisfied in the Hilbert space $H_0^1(\Omega)$ with

$$\overline{I} := J(\overline{\lambda}, \cdot\,), \quad h_0 \equiv 0, \quad \text{and} \quad h_1 := \widetilde{\delta}_{a,\mu} \quad (\text{with } \mu \text{ large enough}),$$

namely $J(\overline{\lambda}, \cdot\,)$ has a mountain pass geometry. By defining $\overline{c}$ as in (3.2), we note that $\overline{c} > 0$. Hence, Theorem 3.1 gives a sequence $(\lambda_n, u_n) \in (0, \overline{\lambda}] \times H_0^1(\Omega)$ such that

$$D_u J(\lambda_n, u_n) = 0, \quad \lambda_n \to \overline{\lambda}, \quad J(\lambda_n, u_n) > 0.$$

Hence for n large enough (to ensure $\lambda_n \in (\lambda_0 - \delta_0, \lambda_0])$ we have

$$J(\lambda_n, m_{\lambda_n}) \le 0 < J(\lambda_n, u_n),$$

and therefore $m_{\lambda_n} \neq u_n$.

(c) By using (5.201) and (5.1), the same proof of statement (b) allows to conclude. □

Remark 6.1. (a) Note that in dimension two any point has H^1-capacity zero. Therefore, given $a \in \Omega$ and $\varepsilon, M > 0$, it is always possible to find a function $f \in C_0^\infty(\Omega)$ such that:

$$\|\Delta f\|_{H^{-1}} < \varepsilon \quad \text{and} \quad f(a) < -M. \tag{6.1}$$

So one check easily the existence of a linear form T satisfying the assumptions of Proposition 1.3.

(b) Instead of assuming T_0 to be unbounded from below, it is enough to have an assumption on T_0 which ensures the infimum of $J(8\pi, \cdot\,)$ to be strictly less then the energy of the local minimizer. This can be investigate by considering again the projections $\overline{\delta} := \overline{\delta}_{a,\mu}$ as defined in (4.3). After some calculations one sees that the minima of the following function come into play:

$$\Omega \to \mathbb{R}, \quad a \mapsto 4\pi H(a,a) + T_0(a).$$

Let us now give a result about existence of radially symmetric critical points when the domain is a ball. Consider the following spaces of radial functions:

$$L^{\mathrm{ra}}(\Omega) := \{u \in L^2(\Omega) : u \text{ radial}\}, \quad H_0^{\mathrm{ra}}(\Omega) := H_0^1(\Omega) \cap L^{\mathrm{ra}}(\Omega). \tag{6.2}$$

The orthogonal of $H_0^{\mathrm{ra}}(\Omega)$ in $H_0^1(\Omega)$, with respect to the inner product defined by $(u,v) \mapsto \int_\Omega \nabla u \cdot \nabla v$, will be denoted by $H_0^{\text{n-ra}}(\Omega)$. We say that $T \in H^{-1}$ is radial if $T(\xi) = 0$ whenever $\xi \in H_0^{\text{n-ra}}(\Omega)$.

Proposition 6.2. *Let $\Omega = B(0,R)$ be a ball. Assume $\lambda_0 \in (8\pi, \Lambda_1|\Omega|)$ and T be a radial linear form satisfying* (1.13). *Then we can find a dense subset*

$D \subset (\lambda_0 - \delta_0, \lambda_0] \cap (8\pi, \lambda_0]$ *such that* $J(\lambda, \cdot)$ *has two radially symmetric critical points for each* $\lambda \in D$.

Proof. Let $J^{\mathrm{ra}}(\lambda, \cdot)$ be the restriction of $J(\lambda, \cdot)$ to $H_0^{\mathrm{ra}}(\Omega)$. Then we check easily that the conclusion of Proposition 5.3 holds for $J^{\mathrm{ra}}(\lambda, \cdot)$ in an interval $(\lambda_0 - \delta_0, \lambda_0]$. Then we can follow the proof of Proposition 1.3, but by working now in the Hilbert space $H_0^{\mathrm{ra}}(\Omega)$. Therefore given $\overline{\lambda} \in (\lambda_0 - \delta_0, \lambda_0] \cap (8\pi, \lambda_0]$, we deduce the existence of $u_n, m_n \in H_0^{\mathrm{ra}}(\Omega)$ such that

$$D_u J^{\mathrm{ra}}(\lambda_n, m_n) = D_u J^{\mathrm{ra}}(\lambda_n, u_n) = 0, \quad \lambda_n \to \overline{\lambda}, \quad \lambda_n \leq \overline{\lambda},$$
$$J^{\mathrm{ra}}(\lambda_n, u_n) > 0 \geq J^{\mathrm{ra}}(\lambda_n, m_n).$$

At this step (u_n, m_n) are two different critical points of $J^{\mathrm{ra}}(\lambda_n, \cdot)$. But well-known arguments show that if u_0 is a critical point of $J^{\mathrm{ra}}(\lambda, \cdot)$, than u_0 is also a critical point of $J(\lambda, \cdot)$. Indeed we have

$$\int_\Omega \nabla u_0 \nabla \xi^{\mathrm{ra}} = \lambda \int_\Omega \frac{e^{u_0}}{\int_\Omega e^{u_0}} \xi^{\mathrm{ra}} - \frac{\lambda}{|\Omega|} T(\xi^{\mathrm{ra}}), \quad \text{for all } \xi^{\mathrm{ra}} \in H_0^{\mathrm{ra}}(\Omega). \tag{6.3}$$

Decompose now $\xi \in H_0^1(\Omega)$ uniquely as $\xi = \xi^{\mathrm{ra}} + \xi^{\text{n-ra}}$. On one hand $T(\xi) = T(\xi^{\text{n-ra}})$ (by assumption). On the other hand, since $e^{u_0} \in L^{\mathrm{ra}}(\Omega)$ one can prove that $\int_\Omega e^{u_0} \xi^{\text{n-ra}} = 0$. Hence, we deduce readily that the identity (6.3) keeps holding for any test function $\xi \in H_0^1(\Omega)$. □

Remark 6.3. When the domain is an annulus A and T is radial, existence of radial critical points for the functional $J(\lambda, \dot{)}$ are easier to derive. Indeed, the continuous injection $H_0^{\mathrm{ra}}(A) \hookrightarrow L^\infty(A)$ implies readily that $J(\lambda, \cdot)$ restricted to $H_0^{\mathrm{ra}}(A)$ has a minimizer for any $\lambda \in \mathbb{R}$.

Our results give existence of two solutions for $D \subset \Lambda$. If furthermore some a priori estimates are known on the set of solutions we have constructed, then standard arguments will show that this multiplicity result keeps holding for any value $\lambda \in D$. If for example $\partial\Omega$ and $T_0 := \Delta_D^{-1}(T)$ are of class $C^{2,\alpha}$, the arguments of Ma-Wei [26], Brezis–Merle [7], Li–Shafrir [19] imply that the set of solutions to Problem (1.8) for λ belonging to some compact interval of $(0, \infty) \setminus \{8\pi N : N \in \mathbb{N}\}$ is bounded in norm $L^\infty(\Omega)$. In this case our existence results proved in this section hold on a full interval E removed from the eventual value $8\pi N$ contained in E. If furthermore $T > 0$, the refined blow-up analysis of Chen–Lin ([10, Theorem 6.2]) shows that actually the result of multiplicity hold on the full interval E (even at the possible value $8\pi N$ contained in E).

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