

**EXISTENCE AND MULTIPLICITY RESULTS
FOR SEMILINEAR ELLIPTIC EQUATIONS
WITH MEASURE DATA AND JUMPING NONLINEARITIES**

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ABSTRACT. We study existence and multiplicity results for semilinear elliptic equations of the type $-\Delta u = g(x, u) - te_1 + \mu$ with homogeneous Dirichlet boundary conditions. Here $g(x, u)$ is a jumping nonlinearity, μ is a Radon measure, t is a positive constant and $e_1 > 0$ is the first eigenfunction of $-\Delta$. Existence results strictly depend on the asymptotic behavior of $g(x, u)$ as $u \rightarrow \pm\infty$. Depending on this asymptotic behavior, we prove existence of two and three solutions for $t > 0$ large enough. In order to find solutions of the equation, we introduce a suitable action functional I_t by mean of an appropriate iterative scheme. Then we apply to I_t standard results from the critical point theory and we prove existence of critical points for this functional.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a connected open bounded domain with smooth boundary and let $n \geq 2$. We denote by $\mathcal{M}(\Omega)$ the space of Radon measures, i.e. the dual space of the Banach space $C_0(\overline{\Omega})$ of continuous functions which vanish on the boundary, endowed with the uniform norm. We study existence and multiplicity

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results for the Dirichlet problem

$$(1.1) \quad \begin{cases} -\Delta u = g(x, u) - te_1 + \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $g: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Caratheodory function, $t > 0$, $e_1 > 0$ is the first eigenfunction of $-\Delta$ with Dirichlet boundary conditions and $\mu \in \mathcal{M}(\Omega)$. In general, we will denote by $\lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots$ the eigenvalues of $-\Delta$ with Dirichlet boundary conditions and by e_i the corresponding eigenfunctions normalized with respect to the L^2 -norm.

According with [8], [11], [19], by solution of (1.1), we mean a function $u \in L^1(\Omega)$ such that $g(x, u) \in L^1(\Omega)$ and

$$(1.2) \quad \int_{\Omega} -u\Delta\varphi \, dx = \int_{\Omega} g(x, u)\varphi \, dx - t \int_{\Omega} e_1\varphi \, dx + \int_{\Omega} \varphi \, d\mu \quad \text{for all } \varphi \in C_0^2(\overline{\Omega})$$

where $C_0^2(\overline{\Omega})$ denotes the space of C^2 functions which vanish on the boundary.

Existence and nonexistence of solutions for second order semilinear and quasilinear elliptic equations with measure data is a widely studied problem, see for example [4]–[6], [8]. These papers deal with Dirichlet problems of the form

$$(1.3) \quad \begin{cases} -\Delta u = g(x, u) + \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with a nonlinearity $g(x, s)$ essentially satisfying the assumption $g(x, s)s \leq 0$ for large s . In these papers, existence of solutions of (1.3) is obtained by solving the “regularized” problems

$$(1.4) \quad \begin{cases} -\Delta u_m = g_m(x, u_m) + \mu_m & \text{in } \Omega, \\ u_m = 0 & \text{on } \partial\Omega, \end{cases}$$

where $g_m(x, s)$ is a suitable sequence of truncated functions of $g(x, s)$ and $\{\mu_m\}$ is a sequence of regular functions which converges to μ weakly in the sense of measures. Assuming suitable growth restrictions at infinity on the nonlinearity $g(x, s)$, the existence of a solution u of (1.3) follows passing to the limit in (1.4).

Thanks to the sign assumption $g(x, s)s \leq 0$, existence of solutions of (1.4) is obtained via global minimization for the corresponding action functional. Therefore the results in [4]–[6], [8] deal with the existence of at most one solution of (1.3) and uniqueness is obtained in [8] assuming that $s \mapsto g(x, s)$ is a concave function.

In [11] some existence and multiplicity results are obtained for problem (1.3) essentially with a nonlinearity satisfying $g(x, s)s \geq 0$ for large s . In [11] both asymptotically linear and superlinear nonlinearities are considered. In the asymptotically linear case, suitable nonresonance assumptions are needed in order to obtain existence of at least one solution.

Other existence results for (1.3) are obtained in [16] when $g(x, s) = \lambda s$ and $\mu \in L^1(\Omega)$.

In the present paper we study problem (1.1) when $g(x, s)$ is asymptotically linear at infinity but with different asymptotic behavior at $+\infty$ and $-\infty$. In other words we assume that $g(x, s)$ satisfies the so called “jumping” condition

$$(1.5) \quad \begin{cases} \lim_{s \rightarrow +\infty} \frac{g(x, s)}{s} = \alpha & \text{for a.e. } x \in \Omega, \\ \lim_{s \rightarrow -\infty} \frac{g(x, s)}{s} = \beta & \text{for a.e. } x \in \Omega, \end{cases}$$

with $\alpha, \beta \in \mathbb{R}$ and $\alpha \neq \beta$. More precisely we assume in our results that the nonlinearity $g(x, s)$ is a perturbation of the function $s \mapsto \alpha s^+ - \beta s^-$ in the sense given in (2.1)–(2.2) with $s^+ = \max\{s, 0\}$ and $s^- = -\min\{s, 0\}$.

A large number of results about the jumping problem (1.1) were obtained in the classical case with a function $h \in L^2(\Omega)$ in place of the measure μ , see for example [2], [3], [9], [10], [13]–[15], [18]. See also [16] for some results concerning problem (1.1) with $\mu \in L^1(\Omega)$. The existence of solutions strictly depends on the pair (α, β) with α, β as in (1.5) and in particular on their position with respect to the eigenvalues of $-\Delta$.

Our main purpose is to prove multiplicity results for problem (1.1) for any $\mu \in \mathcal{M}(\Omega)$. We study both the cases $\alpha < \beta$, $\alpha > \beta$ and we prove for t large existence of two and three solutions depending on the pair (α, β) . Critical point theory is the main tool used in our proofs and precisely some extensions [14] of the classical linking theorem [17].

The main difficulty here is that problem (1.1) does not admit an action functional defined in the whole Sobolev space $H_0^1(\Omega)$ due to the measure term μ . The functional associated to the Euler–Lagrange equation (1.1) would be

$$(1.6) \quad J_t(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \int_{\Omega} G(x, u) dx + t \int_{\Omega} e_1 u dx - \int_{\Omega} u d\mu$$

for all $u \in H_0^1(\Omega) \cap C_0(\overline{\Omega})$ with $G(x, s) = \int_0^s g(x, t) dt$. It is clear that the integral with respect to the measure μ which appears in (1.6) is well defined for continuous functions which vanish on the boundary but not for all functions in $H_0^1(\Omega)$. In order to overcome this problem, we define in the spirit of [11] a new functional I_t obtained from J_t formally by $I_t(u) = J_t(\gamma + u) - J_t(\gamma)$ for a suitable γ . It is worth noting that γ will not belong to $H_0^1(\Omega)$, so $J_t(\gamma + u)$ and $J_t(\gamma)$ do not make sense separately but their difference does. With this definition the functional I_t is defined in the whole $H_0^1(\Omega)$ and its critical points w are such that $u = w + \gamma$ is a solution of (1.1).

In order to obtain multiplicity results for (1.1), the first attempt is to apply the well known classical variational results for jumping type problems to the functional I_t . A main difficulty in this family of problems is the verification

of the Palais–Smale condition. If the pair (α, β) does not belong to the Fucik spectrum

$$\Sigma_\Omega = \{(\alpha, \beta) \in \mathbb{R}^2 : \text{there exists } u \in H_0^1(\Omega) \setminus \{0\} \text{ such that } -\Delta u = \alpha u^+ - \beta u^-\}$$

then it is standard to see that the Palais–Smale condition holds for I_t for any $t \in \mathbb{R}$. Unfortunately the Fucik spectrum is still widely unknown (see [10], [12] for some results concerning Σ_Ω) so that this condition is very difficult to check. In the case $\mu \in L^2(\Omega)$ [14] proved that the Palais–Smale condition holds true for t large without assuming that $(\alpha, \beta) \notin \Sigma_\Omega$. This result can be extended to our case, provided μ is a nonnegative Radon measure and $g(x, s) = \alpha s^+ - \beta s^-$. Hence, in this particular case one can prove directly the existence of two or three critical points for I_t applying for example the results in [14] (see Section 8).

When $\mu \in \mathcal{M}(\Omega)$ is a sign changing measure, we do not apply critical point theory directly to the functional I_t but we introduce a sequence of regular functions $\{\mu_m\}$ such that $\mu_m \rightharpoonup \mu$ weakly in the sense of measures and we define the corresponding functionals $J_t^{(m)}$ and $I_t^{(m)}(u) = J_t^{(m)}(\gamma^{(m)} + u) - J_t^{(m)}(\gamma^{(m)})$ for a suitable function $\gamma^{(m)}$ depending on the measure μ_m . Here $J_t^{(m)}$ is defined by (1.6) with μ_m in place of μ . We show that the functionals $I_t^{(m)}$ satisfy the Palais–Smale condition for any $t > t_0$ where t_0 is a positive number independent of m . Then we prove existence of two or three critical points for $I_t^{(m)}$ at different levels and we show that these critical points converge as $m \rightarrow \infty$ to distinct critical points for the limit functional I_t . The existence of solutions of (1.1) then follows immediately.

This paper is organized as follows. In Section 2 we give the statements of the main results the first two of which are devoted to the case $\alpha < \beta$ and the other two to the case $\alpha > \beta$. Then in Section 3 we define the main tools which will be fundamental in the proofs. Sections 4–5 and Sections 6–7 are devoted to the proofs of existence of two and three solutions for (1.1) respectively in the cases $\alpha < \beta$ and $\alpha > \beta$. Finally in Section 8 we give an alternative proof to our results when μ is a nonnegative Radon measure.

2. Main results

Suppose that there exist $\alpha, \beta \in \mathbb{R}$ such that

$$(2.1) \quad g(x, s) = \alpha s^+ - \beta s^- + \delta(x, s)$$

$$(2.2) \quad |\delta(x, s)| \leq a(x) \in L^p(\Omega) \quad \text{for a.e. } x \in \Omega, \text{ for all } s \in \mathbb{R}$$

with $p > (n/2)$. Note that under the assumptions (2.1)–(2.2) the function $g(x, s)$ also satisfies (1.5).

We recall that by a solution of (1.1) we mean a function $u \in L^1(\Omega)$ such that $g(x, u) \in L^1(\Omega)$ and

$$(2.3) \quad \int_{\Omega} -u\Delta\varphi \, dx = \int_{\Omega} g(x, u)\varphi \, dx - t \int_{\Omega} e_1\varphi \, dx + \int_{\Omega} \varphi \, d\mu \quad \text{for all } \varphi \in C_0^2(\overline{\Omega}).$$

Note that if $u \in L^1(\Omega)$ then the fact that $g(x, u) \in L^1(\Omega)$ follows immediately by (2.1) and (2.2).

Moreover by Theorem 8.1 in [19], it follows that any solution u of (2.3) belongs to the Sobolev space $W_0^{1,q}(\Omega)$ for any $q < n/(n-1)$ so that all solutions found in our results have this regularity.

We introduce some notations. We endow the Hilbert space $H_0^1(\Omega)$ with the scalar product defined by

$$(u, v) = \int_{\Omega} \nabla u \nabla v \, dx \quad \text{for all } u, v \in H_0^1(\Omega).$$

For any index $i \geq 1$ we set $H_i = \text{span}\{e_1, \dots, e_i\}$ and $H_i^\perp = \{u \in H_0^1(\Omega) : (u, v) = 0 \text{ for all } v \in H_i\}$. We recall that $\lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots$ denote the eigenvalues of $-\Delta$ and that e_i denote the corresponding eigenfunctions normalized with respect to the L^2 -norm. In the rest of the paper we introduce the following notations for balls and spheres in H_i and H_i^\perp :

$$\begin{aligned} B_i^-(\rho) &= \{v \in H_i : \|v\|_{H_0^1} \leq \rho\}, & S_i^-(\rho) &= \{v \in H_i : \|v\|_{H_0^1} = \rho\}, \\ B_i^+(\rho) &= \{v \in H_i^\perp : \|v\|_{H_0^1} \leq \rho\}, & S_i^+(\rho) &= \{v \in H_i^\perp : \|v\|_{H_0^1} = \rho\}. \end{aligned}$$

For any $\alpha, \beta \in \mathbb{R}$ let us set, for any $v \in H_0^1(\Omega)$,

$$\begin{aligned} Q_\alpha(v) &= \frac{1}{2} \int_{\Omega} (|\nabla v|^2 - \alpha v^2) \, dx \\ \text{and } Q_{\alpha,\beta}(v) &= \frac{1}{2} \int_{\Omega} (|\nabla v|^2 - \alpha(v^+)^2 - \beta(v^-)^2) \, dx. \end{aligned}$$

We start with the case $\beta > \alpha$. According with [14], for any index $i \geq 1$ we introduce the numbers

$$(2.4) \quad M_i(\alpha, \beta) = \sup_{v \in H_i} \left\{ Q_\alpha(v) + \frac{1}{2}(\alpha - \beta) \int_{\Omega} ((e_1 + v)^-)^2 \, dx \right\},$$

$$(2.5) \quad m_i(\rho, \alpha, \beta) = \inf_{w \in S_i^+(\rho)} \left\{ Q_\alpha(w) + \frac{1}{2}(\alpha - \beta) \int_{\Omega} ((e_1 + w)^-)^2 \, dx \right\}$$

and the set

$$(2.6) \quad E_i = \{(\alpha, \beta) \in \mathbb{R}^2 : \alpha < \beta, \text{ there exists } \rho_i > 0 \text{ such that } M_i(\alpha, \beta) < m_i(\rho_i, \alpha, \beta)\}.$$

Finally, for $i \geq 1$ and $e \in H_i^\perp \setminus \{0\}$ we define

$$\begin{aligned} \Sigma_i^-(e) &= \{z = \sigma e + v : \|z\|_{H_0^1} = 1, v \in H_i, \sigma \geq 0\}, \\ \alpha_{i+1} &= \sup \left\{ \int_{\Omega} |\nabla v|^2 dx : v \in H_i, \int_{\Omega} v^2 dx = 1, v \geq 0 \right\}, \\ \mu_{i+1}(\alpha) &= \inf \left\{ \beta \in \mathbb{R} : \inf_{e \in H_i^\perp \setminus \{0\}} \max_{z \in \Sigma_i^-(e)} Q_{\alpha, \beta}(z) < 0 \right\}. \end{aligned}$$

One can easily check that $\alpha_2 = \lambda_1$, $\alpha_{i+1} \in [\lambda_1, \lambda_i)$ and that the function $\alpha \mapsto \mu_{i+1}(\alpha)$ is finite if and only if $\alpha > \alpha_{i+1}$, see Lemma 4.12 in [14] for more details. In our first result we prove existence of two solutions for (1.1).

THEOREM 2.1. *Let $n \geq 2$ and $\mu \in \mathcal{M}(\Omega)$. Let $i \geq 1$ be such that $\lambda_i < \lambda_{i+1}$. Assume that $g(x, s)$ satisfies (2.1)–(2.2). If $\alpha \in [\lambda_i, \lambda_{i+1})$ and $\beta > \mu_{i+1}(\alpha)$ ($\Rightarrow \beta > \alpha$) then there exists $t_0 > 0$ such that (1.1) admits at least two solutions for any $t > t_0$.*

Before the statement of the next result we point out the following fact taken from [14]:

REMARK 2.2. Let $k \geq j \geq 2$ be such that $\lambda_{j-1} < \lambda_j = \dots = \lambda_k < \lambda_{k+1}$. Then the set of the pairs (α, β) such that $(\alpha, \beta) \in E_k \cap E_{j-1}$ and $\beta > \mu_{k+1}(\alpha)$ is an open nonempty set.

Thanks to Remark 2.2, the statement of the next result becomes meaningful.

THEOREM 2.3. *Let $n \geq 2$ and let $\mu \in \mathcal{M}(\Omega)$. Let $k \geq j \geq 2$ be such that $\lambda_{j-1} < \lambda_j = \dots = \lambda_k < \lambda_{k+1}$. Assume that $g(x, s)$ satisfies (2.1)–(2.2). If $(\alpha, \beta) \in E_k \cap E_{j-1}$ and $\beta > \mu_{k+1}(\alpha)$ ($\Rightarrow \beta > \alpha$) then there exists $t_0 > 0$ such that (1.1) admits at least three solutions for any $t > t_0$.*

Now we consider the case $\alpha > \beta$. We define

$$(2.7) \quad N_i(\rho, \alpha, \beta) = \sup_{v \in S_i^-(\rho)} \left\{ Q_\alpha(v) + \frac{1}{2}(\alpha - \beta) \int_{\Omega} ((e_1 + v)^-)^2 dx \right\},$$

$$(2.8) \quad n_i(\alpha, \beta) = \inf_{w \in H_i^\perp} \left\{ Q_\alpha(w) + \frac{1}{2}(\alpha - \beta) \int_{\Omega} ((e_1 + w)^-)^2 dx \right\}$$

and the set

$$(2.9) \quad F_i = \{(\alpha, \beta) \in \mathbb{R}^2 : \alpha > \beta, \text{ there exists } \rho_i > 0 \text{ such that } N_i(\rho_i, \alpha, \beta) < n_i(\alpha, \beta)\}.$$

Finally, for $i \geq 1$ and $e \in H_i \setminus \{0\}$ we define

$$\begin{aligned} \Sigma_i^+(e) &= \{z = \sigma e + w : \|z\|_{H_0^1} = 1, w \in H_i^\perp, \sigma \geq 0\}, \\ \nu_i(\alpha) &= \sup \left\{ \beta \in \mathbb{R} : \sup_{e \in H_i \setminus \{0\}} \inf_{z \in \Sigma_i^+(e)} Q_{\alpha, \beta}(z) > 0 \right\}. \end{aligned}$$

The map $\alpha \mapsto \nu_i(\alpha)$ is defined for any $\alpha \in \mathbb{R}$ as one can see by Lemma 6.5 in [14]. We prove

THEOREM 2.4. *Let $n \geq 2$ and $\mu \in \mathcal{M}(\Omega)$. Let $i \geq 1$ be such that $\lambda_i < \lambda_{i+1}$. Assume that $g(x, s)$ satisfies (2.1)–(2.2). If $\alpha \in (\lambda_i, \lambda_{i+1}]$ and $\beta < \nu_i(\alpha)$ ($\Rightarrow \beta < \alpha$) then there exists $t_0 > 0$ such that (1.1) admits at least two solutions for any $t > t_0$.*

As in the case $\beta > \alpha$, before the statement of the next result concerning the existence of three solutions, we point out the following fact taken from [14]:

REMARK 2.5. Let $k \geq j \geq 2$ be such that $\lambda_{j-1} < \lambda_j = \dots = \lambda_k < \lambda_{k+1}$. Then the set of the pairs (α, β) such that $(\alpha, \beta) \in F_k \cap F_{j-1}$ and $\beta < \nu_{j-1}(\alpha)$ is an open nonempty set.

Then we establish

THEOREM 2.6. *Let $n \geq 2$ and let $\mu \in \mathcal{M}(\Omega)$. Let $k \geq j \geq 2$ be such that $\lambda_{j-1} < \lambda_j = \dots = \lambda_k < \lambda_{k+1}$. Assume that $g(x, s)$ satisfies (2.1)–(2.2). Let $\alpha > \lambda_1$. If $(\alpha, \beta) \in F_k \cap F_{j-1}$ and $\beta < \nu_{j-1}(\alpha)$ ($\Rightarrow \beta < \alpha$) then there exists $t_0 > 0$ such that (1.1) admits at least three solutions for any $t > t_0$.*

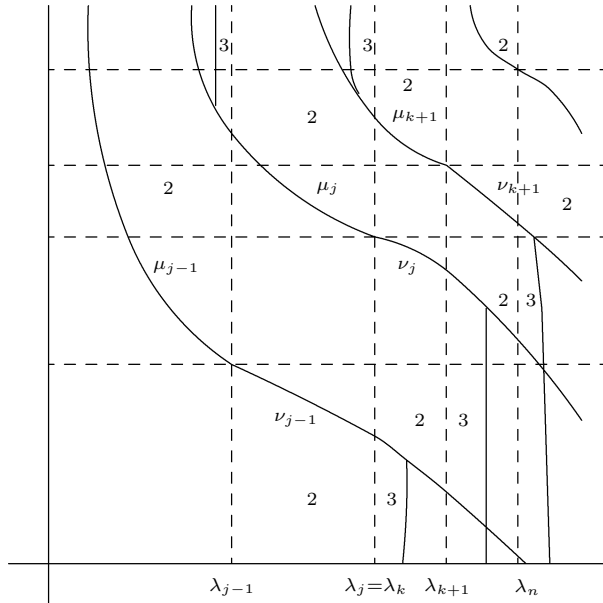


FIGURE 1. Multiplicity map for the solutions

3. Preliminary results

Consider the equation

$$(3.1) \quad \begin{cases} -\Delta u = g(x, u) - te_1 + \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $g(x, s)$ satisfies (2.1)–(2.2) and $\mu \in \mathcal{M}(\Omega)$.

Let v_1 be the unique solution of

$$(3.2) \quad \begin{cases} -\Delta v_1 = \mu & \text{in } \Omega, \\ v_1 = 0 & \text{on } \partial\Omega, \end{cases}$$

and by iteration define for $k = 1, 2, \dots$

$$(3.3) \quad \begin{cases} -\Delta v_{k+1} = g\left(x, \sum_{i=1}^k v_i\right) - g\left(x, \sum_{i=1}^{k-1} v_i\right) & \text{in } \Omega, \\ v_{k+1} = 0 & \text{on } \partial\Omega. \end{cases}$$

The functions v_k are well defined in view of (2.1)–(2.2) and Theorem 8.1 in [19]. Moreover, for any $k \geq 1$ they satisfy

$$(3.4) \quad v_k \in L^q(\Omega) \quad \text{for all } q \geq 1 \text{ if } n = 2 \text{ and all } q \in \left[1, \frac{n}{n-2}\right) \text{ if } n > 2.$$

Suppose that u is a solution of (3.1). We introduce the functions $u_{k+1} = u_k - v_{k+1}$ for $k = 0, 1, 2, \dots$ where $u_0 = u$. Then u_{k+1} solves

$$(3.5) \quad \begin{cases} -\Delta u_{k+1} = g\left(x, u_{k+1} + \sum_{i=1}^{k+1} v_i\right) - g\left(x, \sum_{i=1}^k v_i\right) - te_1 & \text{in } \Omega, \\ u_{k+1} = 0 & \text{on } \partial\Omega. \end{cases}$$

Let $\gamma_k = \sum_{i=1}^k v_i$ for $k \geq 1$ and $\gamma_0 = 0$. For $k \geq 1$, introduce the function $h_k(x, s)$ defined by

$$(3.6) \quad h_k(x, s) = g(x, s + \gamma_k) - g(x, \gamma_k) \quad \text{for all } x \in \Omega \text{ and all } s \in \mathbb{R}$$

and the function f_k defined by

$$(3.7) \quad f_k = g(x, \gamma_k) - g(x, \gamma_{k-1}).$$

Then, by adding and subtracting $g(x, \gamma_{k+1})$ in (3.5), we see that $w = u_{k+1}$ solves

$$(3.8) \quad \begin{cases} -\Delta w = h_{k+1}(x, w) - te_1 + f_{k+1} & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

We recall that by a solution of (3.8) we mean a function $w \in L^1(\Omega)$ such that $h_{k+1}(x, w) \in L^1(\Omega)$ and, for all $\varphi \in C_0^2(\overline{\Omega})$,

$$\int_{\Omega} -w \Delta \varphi \, dx = \int_{\Omega} h_{k+1}(x, w) \varphi \, dx - t \int_{\Omega} e_1 \varphi \, dx + \int_{\Omega} f_{k+1} \varphi \, dx.$$

Then we prove

LEMMA 3.1. *Let $n \geq 2$ and assume that $g(x, s)$ satisfies (2.1)–(2.2). Then there exists $N \in \mathbb{N}$ such that $v_N \in L^\infty(\Omega)$ and $f_N \in L^p(\Omega)$ with p as in (2.2).*

PROOF. By (2.1)–(2.2) we have

$$(3.9) \quad |g(x, \gamma_k) - g(x, \gamma_{k-1})| \leq \max\{|\alpha|, |\beta|\}|v_k| + 2a(x)$$

for almost every $x \in \Omega$ and for all $k = 1, 2, \dots$ with $a(x) \in L^p(\Omega)$ and $p > n/2$. By (3.4) we may suppose that $v_k \in L^{q_k}(\Omega)$ for some $q_k > 1$ and some $k \geq 1$.

Case 1. If $q_k > n/2$ then we are done. Indeed by (3.3), (3.9), elliptic regularity [1] and Sobolev embedding we obtain $v_{k+1} \in W^{2, q_k}(\Omega) \subset L^\infty(\Omega)$.

Case 2. If $q_k = n/2$ with the same procedure we arrive to $v_{k+1} \in W^{2, q_k}(\Omega) \subset L^q(\Omega)$ for any $q \geq 1$ and with another iteration we obtain $v_{k+2} \in L^\infty(\Omega)$.

Case 3. If $q_k < n/2$, as in Cases 1–2, we obtain $v_{k+1} \in W^{2, q_k}(\Omega) \subset L^{q_{k+1}}(\Omega)$ with $q_{k+1} = (nq_k)/(n - 2q_k)$. After a finite number of iterations we find $\bar{k} \in \mathbb{N}$ such that $q_{\bar{k}} \geq n/2$ and applying Case 1 or Case 2 we obtain $v_{\bar{k}+2} \in L^\infty(\Omega)$. Choosing $N = \bar{k} + 2$ we obtain $v_N \in L^\infty(\Omega)$.

The fact that $f_N \in L^p(\Omega)$ follows immediately by (3.7), (3.9) and the fact that $v_N \in L^\infty(\Omega)$. \square

From now on we fix $N \in \mathbb{N}$ as given by Lemma 3.1. By (2.1)–(2.2) we have

$$(3.10) \quad |h_N(x, s)| = |g(x, \gamma_N + s) - g(x, \gamma_N)| \leq \max\{|\alpha|, |\beta|\}|s| + 2a(x)$$

for all $s \in \mathbb{R}$ and for almost every $x \in \Omega$ and hence in view of Lemma 3.1, we can look for solutions of the problem

$$(3.11) \quad \begin{cases} -\Delta w = h_N(x, w) - te_1 + f_N & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases}$$

in the space $H_0^1(\Omega)$. Therefore by a solution of (3.11) we mean a function $w \in H_0^1(\Omega)$ such that

$$\int_{\Omega} \nabla w \nabla z \, dx = \int_{\Omega} h_N(x, w)z \, dx - t \int_{\Omega} e_1 z \, dx + \int_{\Omega} f_N z \, dx \quad \text{for all } z \in H_0^1(\Omega).$$

Conversely if we find a solution $w \in H_0^1(\Omega)$ of (3.11) for a suitable t then it is easy to see that the function $u = w + \gamma_N$ solves (3.1) in the sense given in (1.2).

By (2.1), (2.2) and (3.6) we see that the function $h_N(x, s)$ satisfies

$$(3.12) \quad \begin{cases} \lim_{s \rightarrow \infty} \frac{h_N(x, s)}{s} = \alpha & \text{for a.e. } x \in \Omega, \\ \lim_{s \rightarrow -\infty} \frac{h_N(x, s)}{s} = \beta & \text{for a.e. } x \in \Omega. \end{cases}$$

Introduce the functional

$$(3.13) \quad I_{N,t}(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx - \int_{\Omega} H_N(x, u) \, dx + t \int_{\Omega} e_1 u \, dx - \int_{\Omega} f_N u \, dx$$

for all $u \in H_0^1(\Omega)$ where $H_N(x, s) = \int_0^s h_N(x, t) dt$. Then by Lemma 3.1 and (3.10) we deduce that $I_{N,t} \in C^1(H_0^1(\Omega))$ for any $t \in \mathbb{R}$ and its critical points solve problem (3.11).

4. Proof of Theorem 2.1

In the rest of this section we assume that $g(x, s)$ satisfies (2.1)–(2.2). Given $\mu \in \mathcal{M}(\Omega)$, we introduce a sequence $\{\mu_m\} \subset L^2(\Omega)$ such that $\mu_m \rightharpoonup \mu$ weakly in the sense of measures as $m \rightarrow \infty$. Applying the same iterative scheme introduced in (3.2), (3.3), (3.6) and (3.7) with μ_m in place of μ , we define the functions $v_k^{(m)}, \gamma_k^{(m)}, h_k^{(m)}(x, s), H_k^{(m)}(x, s)$ and $f_k^{(m)}$.

Next we define the sequence of functionals given by

$$I_{N,t}^{(m)}(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \int_{\Omega} H_N^{(m)}(x, u) dx + t \int_{\Omega} e_1 u dx - \int_{\Omega} f_N^{(m)} u dx$$

for all $u \in H_0^1(\Omega)$. Our purpose is to prove the existence of a sequence of critical points w_m of the functional $I_{N,t}^{(m)}$. First we show that the functionals $I_{N,t}^{(m)}$ satisfy the Palais–Smale condition for any large t uniformly with respect to m .

We recall that a functional I satisfies the Palais–Smale condition if any sequence $\{u_k\}$ such that

$$I'(u_k) \rightarrow 0 \quad \text{in } H^{-1}(\Omega), |I(u_k)| \leq C \quad \text{for all } k$$

admits a strongly convergent subsequence in $H_0^1(\Omega)$.

We start with the following preliminary lemma

LEMMA 4.1. *Let $\alpha \neq \beta$ or $(\alpha, \beta) \neq (\lambda_i, \lambda_i)$ for any $i > 1$. Then for any $M > 0$ there exists $\bar{t} > 0$ such that for any $t > \bar{t}$ any solution $z \in H_0^1(\Omega)$ of*

$$(4.1) \quad -\Delta z = \alpha z^+ - \beta z^-$$

which satisfies

$$(4.2) \quad \left| t \int_{\Omega} e_1 z dx \right| \leq M \|z\|_{L^\infty}$$

is identically equal to zero. We recall that by elliptic regularity estimates, $\|z\|_{L^\infty} < \infty$ for any solution $z \in H_0^1(\Omega)$ of (4.1).

PROOF. Suppose by contradiction that there exist $t_k \rightarrow \infty, z_k \in H_0^1(\Omega) \setminus \{0\}$ such that (4.1)–(4.2) hold respectively with t_k, z_k in place of t, z . Then (4.2) becomes

$$(4.3) \quad \left| t_k \int_{\Omega} e_1 z_k dx \right| \leq M \|z_k\|_{L^\infty}.$$

Replacing z_k with $z_k / \|z_k\|_{H_0^1}$ if necessary, we may suppose that $\|z_k\|_{H_0^1} = 1$ and, up to a subsequence, that $z_k \rightharpoonup z$ in $H_0^1(\Omega)$. Since the functions z_k solve the

equation (4.1), by compact embedding $H_0^1(\Omega) \subset L^q(\Omega)$ for any $q \in [1, \infty)$ if $n = 2$ and for any $q \in [1, 2n/(n-2))$ if $n > 2$, by elliptic regularity estimates [1] we deduce that $z_k \rightarrow z$ in $W^{2,q}(\Omega)$ with q in the given range. If $n > 2$ after a finite number of iterations which involve Sobolev embeddings and elliptic regularity estimates we also obtain $z_k \rightarrow z$ in $W^{2,q}(\Omega)$ for any $q \in [1, \infty)$. Therefore for any $n \geq 2$ by Sobolev embedding we obtain $z_k \rightarrow z$ in $L^\infty(\Omega)$ and hence the right hand side of (4.3) is bounded. Since $t_k \rightarrow \infty$, this implies

$$(4.4) \quad \int_{\Omega} e_1 z \, dx = \lim_{k \rightarrow \infty} \int_{\Omega} e_1 z_k \, dx = 0.$$

If $\alpha = \beta$ and α, β do not belong to the spectrum of $-\Delta$, since z solves (4.1) it follows that $z \equiv 0$. If $\alpha = \beta = \lambda_1$ by (4.1) and (4.4) we obtain $z \equiv 0$. If $\alpha \neq \beta$ we proceed as follows. By (4.4) we obtain

$$(4.5) \quad \int_{\Omega} z^+ e_1 \, dx = \int_{\Omega} z^- e_1 \, dx$$

and since $z \in H_0^1(\Omega)$ solves (4.1) we also have

$$(4.6) \quad \int_{\Omega} (\alpha z^+ - \beta z^-) e_1 \, dx = \int_{\Omega} \nabla z \nabla e_1 \, dx = \lambda_1 \int_{\Omega} e_1 z \, dx = 0.$$

Since $\alpha \neq \beta$, combining (4.5) and (4.6) we obtain $z \equiv 0$.

On the other hand we just proved before that $z_k \rightarrow z$ in $W^{2,q}(\Omega)$ for any $q \in [1, \infty)$ and in particular $z_k \rightarrow z$ in $H_0^1(\Omega)$. Since $\|z_k\|_{H_0^1} = 1$ for any k we obtain $\|z\|_{H_0^1} = 1$, a contradiction with $z \equiv 0$. \square

We are ready to prove the following

LEMMA 4.2. *There exists $\bar{t} > 0$ such that for any $t > \bar{t}$ and any $m \in \mathbb{N}$ then the functional $I_{N,t}^{(m)}$ satisfies the Palais–Smale condition.*

PROOF. Suppose that $\{u_k\}$ is a Palais–Smale sequence for the functional $I_{N,t}^{(m)}$. Suppose by contradiction that $\{u_k\}$ is not bounded in $H_0^1(\Omega)$. Up to a subsequence we may assume that $\|u_k\|_{H_0^1} \rightarrow \infty$ as $k \rightarrow \infty$. Define $\hat{u}_k = u_k / \|u_k\|_{H_0^1}$ so that $\{\hat{u}_k\}$ is bounded in $H_0^1(\Omega)$ and assume up to a subsequence that $\hat{u}_k \rightharpoonup \hat{u}$ in $H_0^1(\Omega)$. Since $\{u_k\}$ is a Palais–Smale sequence then $(I_{N,t}^{(m)})'(u_k) \rightarrow 0$ in $H^{-1}(\Omega)$ as $k \rightarrow \infty$ and in particular for any $w \in H_0^1(\Omega)$ we have

$$(4.7) \quad 0 = \lim_{k \rightarrow \infty} \frac{\langle (I_{N,t}^{(m)})'(u_k), w \rangle}{\|u_k\|_{H_0^1}} = \lim_{k \rightarrow \infty} \left(\int_{\Omega} \nabla \hat{u}_k \nabla w \, dx - \int_{\Omega} \frac{h_N^{(m)}(x, u_k)}{\|u_k\|_{H_0^1}} w \, dx + t \int_{\Omega} \frac{e_1}{\|u_k\|_{H_0^1}} w \, dx - \int_{\Omega} \frac{f_N^{(m)}}{\|u_k\|_{H_0^1}} w \, dx \right).$$

By (3.10), (3.12) and dominated convergence we deduce that

$$\frac{h_N^{(m)}(x, u_k)}{\|u_k\|_{H_0^1}} w \rightarrow (\alpha \widehat{u}^+ - \beta \widehat{u}^-) w \quad \text{in } L^1(\Omega) \text{ as } k \rightarrow \infty \text{ for all } w \in H_0^1(\Omega).$$

And this with (4.7) implies that $\widehat{u} \in H_0^1(\Omega)$ is a solution of (4.1) and hence $\widehat{u} \in L^\infty(\Omega)$.

Define

$$\varphi^{(m)}(x, s) = \delta(x, s + \gamma_N^{(m)}) - \delta(x, \gamma_N^{(m)})$$

and

$$\Phi^{(m)}(x, s) = \int_0^s \varphi^{(m)}(x, t) dt.$$

Since $\{u_k\}$ is a Palais–Smale sequence with $\|u_k\|_{H_0^1} \rightarrow \infty$ and $\widehat{u}_k \rightharpoonup \widehat{u}$ in $H_0^1(\Omega)$, by (2.1)–(2.2) we have

$$\begin{aligned} (4.8) \quad 0 &= \lim_{k \rightarrow \infty} \frac{2I_{N,t}^{(m)}(u_k) - \langle (I_{N,t}^{(m)})'(u_k), u_k \rangle}{\|u_k\|_{H_0^1}} \\ &= \lim_{k \rightarrow \infty} -\alpha \int_{\Omega} \left[\left(\widehat{u}_k + \frac{\gamma_N^{(m)}}{\|u_k\|_{H_0^1}} \right)^+ \gamma_N^{(m)} - (\gamma_N^{(m)})^+ \widehat{u}_k - \frac{((\gamma_N^{(m)})^+)^2}{\|u_k\|_{H_0^1}} \right] dx \\ &\quad + \lim_{k \rightarrow \infty} -\beta \int_{\Omega} \left[- \left(\widehat{u}_k + \frac{\gamma_N^{(m)}}{\|u_k\|_{H_0^1}} \right)^- \gamma_N^{(m)} + (\gamma_N^{(m)})^- \widehat{u}_k - \frac{((\gamma_N^{(m)})^-)^2}{\|u_k\|_{H_0^1}} \right] dx \\ &\quad - \lim_{k \rightarrow \infty} \frac{1}{\|u_k\|_{H_0^1}} \int_{\Omega} [2\Phi^{(m)}(x, u_k) - \varphi^{(m)}(x, u_k) u_k] dx \\ &\quad + \lim_{k \rightarrow \infty} \left(- \int_{\Omega} f_N^{(m)} \widehat{u}_k dx + t \int_{\Omega} e_1 \widehat{u}_k dx \right) \\ &= -\alpha \int_{\Omega} [\widehat{u}^+ \gamma_N^{(m)} - (\gamma_N^{(m)})^+ \widehat{u}] dx - \beta \int_{\Omega} [-\widehat{u}^- \gamma_N^{(m)} + (\gamma_N^{(m)})^- \widehat{u}] dx \\ &\quad - \int_{\Omega} f_N^{(m)} \widehat{u} dx + t \int_{\Omega} e_1 \widehat{u} dx \\ &\quad - \lim_{k \rightarrow \infty} \frac{1}{\|u_k\|_{H_0^1}} \int_{\Omega} [2\Phi^{(m)}(x, u_k) - \varphi^{(m)}(x, u_k) u_k] dx. \end{aligned}$$

The last identity is an immediate consequence of the fact that $\gamma_N^{(m)} \in L^2(\Omega)$ and $f_N^{(m)} \in L^p(\Omega)$ with p as in (2.2).

Since $\widehat{u} \in L^\infty(\Omega)$ and since by Theorem 8.1 in [19] and Lemma 3.1, we have that the sequences $\{\gamma_N^{(m)}\}$ and $\{f_N^{(m)}\}$ are bounded in $L^1(\Omega)$, then we obtain

$$\begin{aligned} (4.9) \quad &\left| \alpha \int_{\Omega} [\widehat{u}^+ \gamma_N^{(m)} - (\gamma_N^{(m)})^+ \widehat{u}] dx \right. \\ &\quad \left. + \beta \int_{\Omega} \left[-\widehat{u}^- \gamma_N^{(m)} + (\gamma_N^{(m)})^- \widehat{u} \right] dx + \int_{\Omega} f_N^{(m)} \widehat{u} dx \right| \\ &\leq 2(\alpha + \beta) \|\gamma_N^{(m)}\|_{L^1} \|\widehat{u}\|_{L^\infty} + \|f_N^{(m)}\|_{L^1} \|\widehat{u}\|_{L^\infty} \leq M \|\widehat{u}\|_{L^\infty} \end{aligned}$$

for a suitable $M > 0$ independent of m .

On the other hand by (2.1)–(2.2) we also have

$$(4.10) \quad \left| \lim_{k \rightarrow \infty} \frac{1}{\|u_k\|_{H_0^1}} \int_{\Omega} [2\Phi^{(m)}(x, u_k) - \varphi^{(m)}(x, u_k)u_k] dx \right| \\ \leq 6 \int_{\Omega} a(x) |\widehat{u}| dx \leq 6 \|a\|_{L^1} \|\widehat{u}\|_{L^\infty}.$$

Inserting (4.9) and (4.10) into (4.8) we obtain

$$\left| t \int_{\Omega} e_1 \widehat{u} dx \right| \leq (M + 6\|a\|_{L^1}) \|\widehat{u}\|_{L^\infty}.$$

Therefore Lemma 4.1 applies and hence there exists $\bar{t} > 0$ independent of m such that if $t > \bar{t}$ then $\widehat{u} \equiv 0$. Then by compact embedding $H_0^1(\Omega) \subset L^q(\Omega)$ we deduce that $\widehat{u}_k \rightarrow 0$ in $L^q(\Omega)$ for any $q \geq 1$ if $n = 2$ and for any $q \in [1, 2n/(n-2))$ if $n > 2$. Therefore, since $\{u_k\}$ is Palais–Smale sequence we obtain by (3.1) and Lemma 3.1

$$0 = \lim_{k \rightarrow \infty} \frac{I_{N,t}^{(m)}(u_k)}{\|u_k\|_{H_0^1}^2} = \frac{1}{2} - \lim_{k \rightarrow \infty} \frac{1}{\|u_k\|_{H_0^1}^2} \int_{\Omega} H_N^{(m)}(x, u_k) dx \\ + \lim_{k \rightarrow \infty} \left(\frac{t}{\|u_k\|_{H_0^1}} \int_{\Omega} e_1 \widehat{u}_k dx - \frac{1}{\|u_k\|_{H_0^1}} \int_{\Omega} f_N^{(m)} \widehat{u}_k dx \right) = \frac{1}{2}$$

a contradiction. This proves that $\{u_k\}$ is bounded and converges weakly in $H_0^1(\Omega)$ up to a subsequence.

Then by (3.10) it follows by standard arguments that $\{u_k\}$ converges strongly in $H_0^1(\Omega)$ up to a subsequence. This completes the proof of the lemma. \square

Next we prove that the functionals $I_{N,t}^{(m)}$ have a linking structure for large t uniformly with respect to m . First we prove the following technical lemma.

LEMMA 4.3. *Let $\Gamma^{(m)}(x, s) = H_N^{(m)}(x, s) + f_N^{(m)}s - (\alpha/2)(s^+)^2 - (\beta/2)(s^-)^2$. Then for any $\varepsilon > 0$ there exists $M > 0$ independent of m such that*

$$\int_{\Omega} |\Gamma^{(m)}(x, u)| dx \leq \varepsilon \|u\|_{H_0^1}^2 \quad \text{for all } \|u\|_{H_0^1} > M \text{ and all } m \in \mathbb{N}.$$

PROOF. Suppose by contradiction that there exist $\varepsilon > 0$, a sequence $\{u_k\}$ with $\|u_k\|_{H_0^1} \rightarrow \infty$ and a sequence $m_k \rightarrow \infty$ such that

$$(4.11) \quad \int_{\Omega} |\Gamma^{(m_k)}(x, u_k)| dx > \varepsilon \|u_k\|_{H_0^1}^2.$$

Define $\widehat{u}_k = u_k/\|u_k\|_{H_0^1}$ and assume up to a subsequence that $\widehat{u}_k \rightharpoonup u$ in $H_0^1(\Omega)$. Since $\gamma_N^{(m_k)}$ converges almost everywhere in Ω as $k \rightarrow \infty$ then by (2.1)–(2.2) one sees that

$$\frac{\Gamma^{(m_k)}(x, u_k)}{\|u_k\|_{H_0^1}^2} \rightarrow 0 \quad \text{for a.e. } x \in \Omega \text{ as } k \rightarrow \infty.$$

On the other hand, by compact embedding $H_0^1(\Omega) \subset L^2(\Omega)$ we have $\widehat{u}_k \rightarrow u$ in $L^2(\Omega)$ and by the proof of Lemma 3.1 we obtain $f_N^{(m_k)} \rightarrow f_N$ in $L^p(\Omega)$ with p as in (2.2). And this with (3.10) yields

$$\begin{aligned} \left| \frac{\Gamma^{(m_k)}(x, u_k)}{\|u_k\|_{H_0^1}^2} \right| &\leq \max\{|\alpha|, |\beta|\}(\widehat{u}_k(x))^2 \\ &+ \frac{2a(x)|\widehat{u}_k(x)|}{\|u_k\|_{H_0^1}} + \frac{|f_N^{(m_k)}(x)||\widehat{u}_k(x)|}{\|u_k\|_{H_0^1}} \leq \eta(x) \in L^1(\Omega) \end{aligned}$$

for almost every $x \in \Omega$ and for all $k \in \mathbb{N}$. By dominated convergence we deduce that

$$\lim_{k \rightarrow \infty} \int_{\Omega} \left| \frac{\Gamma^{(m_k)}(x, u_k)}{\|u_k\|_{H_0^1}^2} \right| dx = 0$$

which contradicts (4.11). \square

Arguing as in [14] we establish

LEMMA 4.4. *Let $i \geq 1$ be such that $\lambda_i < \lambda_{i+1}$. Let α, β be such that $\beta > \alpha > \lambda_1$. For any $t > 0$ define $s_t = t/(\alpha - \lambda_1)$. For any $\varepsilon > 0$ small enough there exists $d > 0$ such that:*

(a) *For any $\rho > 0$, $t > 0$ and $m \in \mathbb{N}$ we have*

$$(4.12) \quad \inf_{w \in S_i^+(\rho s_t)} \{I_{N,t}^{(m)}(s_t e_1 + w) - I_{N,t}^{(m)}(s_t e_1)\} \geq s_t^2(m_i(\rho, \alpha, \beta) - 3\varepsilon\lambda_1 - 2\varepsilon\rho^2) - d.$$

(b) *If $M_i(\alpha, \beta) < \infty$ then there exists $C > 0$ such that for any $t > 0$ and $m \in \mathbb{N}$ we have*

$$(4.13) \quad \sup_{v \in H_i} \{I_{N,t}^{(m)}(s_t e_1 + v) - I_{N,t}^{(m)}(s_t e_1)\} \leq s_t^2(M_i(\alpha, \beta) + 3\varepsilon\lambda_1 + \varepsilon C) + d.$$

(c) *If we assume that $\alpha \in (\alpha_{i+1}, \lambda_{i+1})$ and $\beta > \mu_{i+1}(\alpha)$ then there exist $\bar{\sigma}_i > 0$, $\bar{t} > 0$, $e \in H_i^\perp \setminus \{0\}$ all independent of m such that for any $\sigma_i > \bar{\sigma}_i$, $t > \bar{t}$ and $m \in \mathbb{N}$, we have*

$$(4.14) \quad \sup_{v \in \sigma_i t \Sigma_i^-(e)} \{I_{N,t}^{(m)}(s_t e_1 + v) - I_{N,t}^{(m)}(s_t e_1)\} \leq 0.$$

PROOF. We follow closely the proof of Lemma 4.5 in [14]. For any $z \in H_0^1(\Omega)$ let $u = z/s_t$. We write

$$(4.15) \quad \begin{aligned} &I_{N,t}^{(m)}(s_t e_1 + z) - I_{N,t}^{(m)}(s_t e_1) \\ &= s_t^2 \{Q_\alpha(u) + \frac{\alpha - \beta}{2} \int_{\Omega} ((e_1 + u)^-)^2 dx\} + R^{(m)}(t, z) \\ &= s_t^2 \{Q_{\alpha, \beta}(u) + \frac{\alpha - \beta}{2} \int_{\Omega} [((e_1 + u)^-)^2 - (u^-)^2] dx\} + R^{(m)}(t, z) \end{aligned}$$

where $R^{(m)}(t, z) = \int_{\Omega} [\Gamma^{(m)}(x, s_t e_1) - \Gamma^{(m)}(x, s_t e_1 + z)] dx$ with $\Gamma^{(m)}(x, s)$ as in Lemma 4.3.

Then by Lemma 4.3 we infer that for any $\varepsilon > 0$ there exists $d > 0$ independent of m such that

$$(4.16) \quad |R^{(m)}(t, z)| \leq \varepsilon (\|s_t e_1 + z\|_{H_0^1}^2 + \|s_t e_1\|_{H_0^1}^2) + d \quad \text{for all } z \in H_0^1(\Omega).$$

(a) Choosing $z = w \in H_i^\perp$ with $\|w\|_{H_0^1} = \rho s_t$, by (2.5), (4.15) and (4.16) we immediately obtain (4.12).

(b) Since $M_i(\alpha, \beta) < \infty$ then by Lemma 5.1(b₂) in [14] we have

$$c = - \max_{v \in S_i^-(1)} Q_{\alpha, \beta}(v) > 0$$

so that by (4.15) and (4.16), it follows that for any $v \in H_i$

$$(4.17) \quad \begin{aligned} I_{N,t}^{(m)}(s_t e_1 + v) - I_{N,t}^{(m)}(s_t e_1) \\ \leq -c \|v\|_{H_0^1}^2 + (\beta - \alpha) \int_{\Omega} s_t e_1 v^- dx + 3\varepsilon \lambda_1 s_t^2 + 2\varepsilon \|v\|_{H_0^1}^2 + d \end{aligned}$$

and, by (2.4), we also have

$$(4.18) \quad I_{N,t}^{(m)}(s_t e_1 + v) - I_{N,t}^{(m)}(s_t e_1) \leq s_t^2 M_i(\alpha, \beta) + 3\varepsilon \lambda_1 s_t^2 + 2\varepsilon \|v\|_{H_0^1}^2 + d.$$

Using (4.17) with $\|v\|_{H_0^1} \geq s_t((\beta - \alpha)/(\sqrt{\lambda_1}(c - 2\varepsilon)))$ we obtain

$$(4.19) \quad I_{N,t}^{(m)}(s_t e_1 + v) - I_{N,t}^{(m)}(s_t e_1) \leq 3\varepsilon \lambda_1 s_t^2 + d$$

and using (4.18) with $\|v\|_{H_0^1} \leq s_t((\beta - \alpha)/(\sqrt{\lambda_1}(c - 2\varepsilon)))$ we also have

$$(4.20) \quad I_{N,t}^{(m)}(s_t e_1 + v) - I_{N,t}^{(m)}(s_t e_1) \leq s_t^2 \left(M_i(\alpha, \beta) + 3\varepsilon \lambda_1 + 2\varepsilon \frac{(\beta - \alpha)^2}{\lambda_1 (c - 2\varepsilon)^2} \right) + d.$$

Since $M_i(\alpha, \beta) \geq 0$ then by (4.19) we deduce that (4.20) holds for any $v \in H_i$.

(c) Since $\beta > \mu_{i+1}(\alpha)$ then there exists $e \in H_i^\perp \setminus \{0\}$ such that $-c = \max_{\Sigma_i^-(e)} Q_{\alpha, \beta} < 0$. Then by (4.15) and (4.16) we obtain for $v \in \sigma_i t \Sigma_i^-(e)$

$$\begin{aligned} I_{N,t}^{(m)}(s_t e_1 + v) - I_{N,t}^{(m)}(s_t e_1) \\ \leq s_t^2 \left(-c \left\| \frac{v}{s_t} \right\|_{H_0^1}^2 + (\beta - \alpha) \int_{\Omega} e_1 \frac{v^-}{s_t} dx + 3\varepsilon \lambda_1 + 2\varepsilon \left\| \frac{v}{s_t} \right\|_{H_0^1}^2 \right) + d. \end{aligned}$$

Therefore, if ε is small enough and $\|v/s_t\|_{H_0^1} = \sigma_i(\alpha - \lambda_1)$ and s_t are large enough, then (4.14) follows. \square

In the next lemma we prove that the functionals $I_{N,t}^{(m)}$ have the geometrical structure ‘‘links and bounds’’ described in [14] for large t uniformly with respect to m and we prove that they admit at least two critical points under this restriction on t .

LEMMA 4.5. *Let $i \geq 1$ be such that $\lambda_i < \lambda_{i+1}$. Let $(\alpha, \beta) \in E_i$ ($\Rightarrow \beta > \alpha$) be such that $\alpha \in (\alpha_{i+1}, \lambda_{i+1})$ and $\beta > \mu_{i+1}(\alpha)$. For any $t > 0$ let $s_t = t/(\alpha - \lambda_1)$. Then there exist $\rho_i > 0$, $t_0 > 0$, $e \in H_i^\perp \setminus \{0\}$, $\bar{\sigma}_i > \rho_i/(\alpha - \lambda_1)$ all independent of m such that if $\sigma_i > \bar{\sigma}_i$, $t > t_0$ and $m \in \mathbb{N}$ then $I_{N,t}^{(m)}$ admits two critical points $w_{1,m}$, $w_{2,m}$, respectively at levels $c_{1,m}$, $c_{2,m}$ with*

$$(4.21) \quad \inf_{z \in B_i^+(\rho_i s_t)} I_{N,t}^{(m)}(s_t e_1 + z) \leq c_{2,m} \leq \sup_{\Sigma_i^-} I_{N,t}^{(m)} \\ < \inf_{z \in S_i^+(\rho_i s_t)} I_{N,t}^{(m)}(s_t e_1 + z) \leq c_{1,m} \leq \sup_{\Delta_i^-} I_{N,t}^{(m)}$$

where

$$\Sigma_i^- = \{s_t e_1 + v : v \in B_i^-(\sigma_i t)\} \cup \{s_t e_1 + v : v \in \sigma_i t \Sigma_i^-(e)\}, \\ \Delta_i^- = \{z = v + \sigma e : v \in H_i, \sigma \geq 0, \|z\|_{H_0^1} \leq \sigma_i t\}.$$

PROOF. Since $(\alpha, \beta) \in E_i$ by (2.6) (see also Lemma 5.1(d) in [14]) it follows that there exists $\rho_i > 0$ small enough such that $m_i(\rho_i, \alpha, \beta) > M_i(\alpha, \beta) \geq 0$ so that if we choose $\varepsilon > 0$ small enough and $t > 0$ large enough in (4.12)–(4.13), we obtain

$$(4.22) \quad \inf_{w \in S_i^+(\rho_i s_t)} \{I_{N,t}^{(m)}(s_t e_1 + w) - I_{N,t}^{(m)}(s_t e_1)\} > 0,$$

$$(4.23) \quad \sup_{v \in H_i} I_{N,t}^{(m)}(s_t e_1 + v) < \inf_{w \in S_i^+(\rho_i s_t)} I_{N,t}^{(m)}(s_t e_1 + w).$$

By (4.14), (4.22) and (4.23) we also obtain

$$(4.24) \quad \sup_{\Sigma_i^-} I_{N,t}^{(m)} < \inf_{z \in S_i^+(\rho_i s_t)} I_{N,t}^{(m)}(s_t e_1 + z) \quad \text{for all } m \in \mathbb{N}.$$

By Lemma 4.2 we infer that there exists $\bar{t} > 0$ such that for any $t > \bar{t}$ and any $m \in \mathbb{N}$ the functional $I_{N,t}^{(m)}$ satisfies the Palais–Smale condition. We may choose $t_0 \geq \bar{t}$ large enough so that also (4.24) holds true. Now the existence of two critical points and the estimates in (4.21) follows immediately from Theorem 8.2 in [14]. \square

The next step is to prove that any sequence of critical points $\{w_m\}$ of $I_{N,t}^{(m)}$ such that $I_{N,t}^{(m)}(w_m)$ is uniformly bounded with respect to m , admits a subsequence strongly convergent in $H_0^1(\Omega)$.

LEMMA 4.6. *Let $\{w_m\}$ be a sequences of critical points for $I_{N,t}^{(m)}$ such that $I_{N,t}^{(m)}(w_m)$ is bounded. Then there exists $w \in H_0^1(\Omega)$ such that $w_m \rightarrow w$ strongly in $H_0^1(\Omega)$ up to a subsequence. Moreover, w is a critical point for $I_{N,t}$.*

PROOF. First we prove that $\{w_m\}$ is bounded in $H_0^1(\Omega)$. Suppose by contradiction that $\{w_m\}$ is not bounded in $H_0^1(\Omega)$ and assume up to a subsequence

that $\|w_m\|_{H_0^1} \rightarrow \infty$. Let $\widehat{w}_m = w_m/(\|w_m\|_{H_0^1})$ so that we may assume up to a subsequence that $\widehat{w}_m \rightharpoonup \widehat{w}$ in $H_0^1(\Omega)$. Since w_m is a critical point for $I_{N,t}^{(m)}$ then $\widehat{w}_m \in H_0^1(\Omega)$ solves the equation

$$-\Delta \widehat{w}_m = \frac{h_N^{(m)}(x, w_m)}{\|w_m\|_{H_0^1}} - t \frac{e_1}{\|w_m\|_{H_0^1}} + \frac{f_N^{(m)}}{\|w_m\|_{H_0^1}} \quad \text{in } H^{-1}(\Omega).$$

By (3.4) and the proof of Lemma 3.1 we infer that

$$(4.25) \quad \begin{aligned} \gamma_i^{(m)} &\rightarrow \gamma_i \quad \text{in } L^q(\Omega) \\ &\text{for all } q \in [1, \infty) \quad \text{if } n = 2, \\ &\text{for all } q \in [1, n/(n-2)) \quad \text{if } n > 2, \\ &\text{for all } i = 1, \dots, N, \end{aligned}$$

$$(4.26) \quad f_N^{(m)} \rightarrow f_N \quad \text{in } L^p(\Omega)$$

with p as in (2.2).

If $n = 2$ then by (3.10), (4.25), (4.26), compact embedding $H_0^1(\Omega) \subset L^q(\Omega)$ for any $q \in [1, \infty)$ and elliptic regularity estimates [1], we have $\widehat{w}_m \rightarrow \widehat{w}$ in $W^{2,q}(\Omega)$ for any $q \in [1, p]$.

If $n > 2$ at the first step we have $\widehat{w}_m \rightarrow \widehat{w}$ in $L^q(\Omega)$ and in turn $\widehat{w}_m \rightarrow \widehat{w}$ in $W^{2,q}(\Omega)$ for any $q \in [1, \min\{p, 2n/(n-2)\})$. Then by iteration, using Sobolev embeddings and elliptic regularity estimates we obtain the strong convergence $\widehat{w}_m \rightarrow \widehat{w}$ in $W^{2,q}(\Omega)$ for any $q \in [1, p]$ also for $n > 2$. Since $p > n/2$ by Sobolev embedding we infer that

$$(4.27) \quad \widehat{w}_m \rightarrow \widehat{w} \quad \text{in } L^\infty(\Omega) \text{ as } m \rightarrow \infty,$$

$$(4.28) \quad \widehat{w}_m \rightarrow \widehat{w} \quad \text{in } H_0^1(\Omega) \text{ as } m \rightarrow \infty.$$

By (3.10), (3.12) and dominated convergence we also obtain

$$\frac{h_N^{(m)}(x, w_m)}{\|w_m\|_{H_0^1}} z \rightarrow (\alpha \widehat{w}^+ - \beta \widehat{w}^-) z \quad \text{in } L^1(\Omega) \text{ as } m \rightarrow \infty \text{ for all } z \in H_0^1(\Omega)$$

so that, by (4.26), the function $\widehat{w} \in H_0^1(\Omega)$ solves the equation

$$(4.29) \quad -\Delta \widehat{w} = \alpha \widehat{w}^+ - \beta \widehat{w}^-.$$

Put

$$\chi(y) = \begin{cases} 1 & \text{if } y \geq 0, \\ 0 & \text{if } y < 0. \end{cases}$$

Since $(I_{N,t}^{(m)})'(w_m) = 0$ and the sequence $I_{N,t}^{(m)}(w_m)$ is uniformly bounded with respect to m , we obtain

$$\begin{aligned}
(4.30) \quad 0 &= \lim_{m \rightarrow \infty} \frac{2I_{N,t}^{(m)}(w_m) - \langle (I_{N,t}^{(m)})'(w_m), w_m \rangle}{\|w_m\|_{H_0^1}} \\
&= \lim_{m \rightarrow \infty} -\frac{\alpha}{\|w_m\|_{H_0^1}} \int_{\Omega} \gamma_N^{(m)} [(w_m + \gamma_N^{(m)})^+ - \chi(\gamma_N^{(m)})w_m - (\gamma_N^{(m)})^+] dx \\
&\quad + \lim_{m \rightarrow \infty} -\frac{\beta}{\|w_m\|_{H_0^1}} \\
&\quad \cdot \int_{\Omega} \gamma_N^{(m)} [-(w_m + \gamma_N^{(m)})^- - \chi(-\gamma_N^{(m)})w_m + (\gamma_N^{(m)})^-] dx \\
&\quad - \lim_{m \rightarrow \infty} \frac{1}{\|w_m\|_{H_0^1}} \int_{\Omega} [2\Phi^{(m)}(x, w_m) - \varphi^{(m)}(x, w_m)w_m] dx \\
&\quad + \lim_{m \rightarrow \infty} \left(- \int_{\Omega} f_N^{(m)} \widehat{w}_m dx + t \int_{\Omega} e_1 \widehat{w}_m dx \right)
\end{aligned}$$

with $\varphi^{(m)}(x, s)$ and $\Phi^{(m)}(x, s)$ as in the proof of Lemma 4.2.

By direct computation one sees that

$$(4.31) \quad \|w_m\|_{H_0^1}^{-1} |\gamma_N^{(m)}| |(w_m + \gamma_N^{(m)})^+ - \chi(\gamma_N^{(m)})w_m - (\gamma_N^{(m)})^+| \leq |\gamma_N^{(m)}| |\widehat{w}_m|,$$

$$\begin{aligned}
(4.32) \quad \|w_m\|_{H_0^1}^{-1} |\gamma_N^{(m)}| |-(w_m + \gamma_N^{(m)})^- - \chi(-\gamma_N^{(m)})w_m + (\gamma_N^{(m)})^-| \\
\leq |\gamma_N^{(m)}| |\widehat{w}_m|.
\end{aligned}$$

By (2.1)–(2.2), (4.25)–(4.27), (4.30)–(4.32) and dominated convergence, we obtain

$$\begin{aligned}
(4.33) \quad t \int_{\Omega} e_1 \widehat{w} dx &= \alpha \int_{\Omega} (\widehat{w}^+ \gamma_N - \gamma_N^+ \widehat{w}) dx \\
&\quad + \beta \int_{\Omega} (-\widehat{w}^- \gamma_N + \gamma_N^- \widehat{w}) dx + \int_{\Omega} f_N \widehat{w} dx \\
&\quad + \lim_{m \rightarrow \infty} \frac{1}{\|w_m\|_{H_0^1}} \int_{\Omega} [2\Phi^{(m)}(x, w_m) - \varphi^{(m)}(x, w_m)w_m] dx \\
&\leq (2(\alpha + \beta) \|\gamma_N\|_{L^1} + \|f_N\|_{L^1} + 6\|a\|_{L^1}) \|\widehat{w}\|_{L^\infty}.
\end{aligned}$$

By Lemma 4.1, we infer that for $t > \bar{t}$ any function in $H_0^1(\Omega)$ which satisfies (4.29) and (4.33) is identically equal to zero in Ω . In particular we deduce that $\widehat{w} \equiv 0$ which contradicts (4.28), i.e. $\|\widehat{w}\|_{H_0^1} = \lim_{m \rightarrow \infty} \|\widehat{w}_m\|_{H_0^1} = 1$. This proves that $\{w_m\}$ is bounded in $H_0^1(\Omega)$.

Up to a subsequence we may assume that $w_m \rightharpoonup w$ weakly in $H_0^1(\Omega)$ and since w_m satisfies

$$(4.34) \quad \int_{\Omega} \nabla w_m \nabla z dx = \int_{\Omega} [h_N^{(m)}(x, w_m) - te_1 + f_N^{(m)}] z dx,$$

for all $z \in H_0^1(\Omega)$, passing to the limit as $m \rightarrow \infty$, by (3.10), (4.25) and (4.26) we infer that w is a critical point for $I_{N,t}$. By (3.10), (4.25) and Sobolev embedding we also have

$$(4.35) \quad \int_{\Omega} h_N^{(m)}(x, w_m) w_m dx \rightarrow \int_{\Omega} h_N(x, w) w dx \quad \text{as } m \rightarrow \infty.$$

Once we use (4.26), (4.34) and (4.35), the proof of the strong convergence $w_m \rightarrow w$ in $H_0^1(\Omega)$ becomes standard and hence we omit it. \square

4.1. End of the proof of Theorem 2.1. If we suppose that $\beta > \mu_{i+1}(\alpha)$ ($\Rightarrow \beta > \alpha$) and $\alpha \in [\lambda_i, \lambda_{i+1})$ then by Lemma 4.4 in [14] we have that $(\alpha, \beta) \in E_i$ and hence Lemma 4.5 applies so that we may consider the sequences of critical points $\{w_{1,m}\}, \{w_{2,m}\}$ found in Lemma 4.5. We prove that the corresponding critical levels $c_{1,m} = I_{N,t}^{(m)}(w_{1,m})$, $c_{2,m} = I_{N,t}^{(m)}(w_{2,m})$ are uniformly bounded with respect to m . By (4.25)–(4.26) it follows that $I_{N,t}^{(m)} \rightarrow I_{N,t}$ as $m \rightarrow \infty$ uniformly on bounded sets with respect to the H_0^1 -norm (see the proof of Lemma 20 in [11] for more details), i.e. for any bounded set $B \subset H_0^1(\Omega)$ we have

$$(4.36) \quad \sup_{z \in B} |I_{N,t}^{(m)}(z) - I_{N,t}(z)| \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

Therefore we have

$$(4.37) \quad \begin{aligned} \inf_{z \in B_i^+(\rho_i s_t)} I_{N,t}^{(m)}(s_t e_1 + z) &\rightarrow \inf_{z \in B_i^+(\rho_i s_t)} I_{N,t}(s_t e_1 + z), \\ \sup_{\Sigma_i^-} I_{N,t}^{(m)} &\rightarrow \sup_{\Sigma_i^-} I_{N,t}, \end{aligned}$$

as $m \rightarrow \infty$ and

$$(4.38) \quad \begin{aligned} \inf_{z \in S_i^+(\rho_i s_t)} I_{N,t}^{(m)}(s_t e_1 + z) &\rightarrow \inf_{z \in S_i^+(\rho_i s_t)} I_{N,t}(s_t e_1 + z), \\ \sup_{\Delta_i^-} I_{N,t}^{(m)} &\rightarrow \sup_{\Delta_i^-} I_{N,t}, \end{aligned}$$

as $m \rightarrow \infty$. By (4.21) we deduce immediately that $\{c_{1,m}\}, \{c_{2,m}\}$ are bounded. By Lemma 4.6 we deduce that there exist $w_1, w_2 \in H_0^1(\Omega)$ such that up a subsequence

$$(4.39) \quad w_{1,m} \rightarrow w_1, \quad w_{2,m} \rightarrow w_2 \quad \text{in } H_0^1(\Omega) \text{ as } m \rightarrow \infty.$$

In particular by (4.25)–(4.26) and (4.39) we obtain

$$(4.40) \quad c_{1,m} = I_{N,t}^{(m)}(w_{1,m}) \rightarrow I_{N,t}(w_1), \quad c_{2,m} = I_{N,t}^{(m)}(w_{2,m}) \rightarrow I_{N,t}(w_2)$$

as $m \rightarrow \infty$. On the other hand since the estimates (4.12)–(4.14) in Lemma 4.4 are uniform with respect to m , by the pointwise convergence

$$I_{N,t}^{(m)}(z) \rightarrow I_{N,t}(z) \quad \text{as } m \rightarrow \infty \text{ for all } z \in H_0^1(\Omega),$$

arguing as in the proof of Lemma 4.5 we obtain

$$(4.41) \quad \sup_{\Sigma_i^-} I_{N,t} < \inf_{z \in S_i^+(\rho_i s_t)} I_{N,t}(s_t e_1 + z)$$

with ρ_i , Σ_i^- and $t > t_0$ as in Lemma 4.5.

Moreover, w_1 and w_2 are critical points for $I_{N,t}$ and by (4.21), (4.36)–(4.41) we obtain

$$\begin{aligned} \inf_{z \in B_i^+(\rho_i s_t)} I_{N,t}(s_t e_1 + z) &\leq I_{N,t}(w_2) \leq \sup_{\Sigma_i^-} I_{N,t} \\ &< \inf_{z \in S_i^+(\rho_i s_t)} I_{N,t}(s_t e_1 + z) \leq I_{N,t}(w_1) \leq \sup_{\Delta_i^-} I_{N,t} \end{aligned}$$

and hence the functional $I_{N,t}$ admits two critical points at distinct levels.

As we explained in Section 3, to any solution w of (3.11) corresponds a solution u of (1.1) given by $u = w + \gamma_N$. Therefore the functions $u_1 = w_1 + \gamma_N$ and $u_2 = w_2 + \gamma_N$ are distinct solutions of (1.1). This completes the proof of the theorem. \square

5. Proof of Theorem 2.3

In this section we assume that $g(x, s)$ satisfies (2.1) and (2.2). As in the proof of Theorem 2.1 we introduce a sequence $\{\mu_m\} \subset L^2(\Omega)$ such that $\mu_m \rightharpoonup \mu$ weakly in the sense of measures as $m \rightarrow \infty$ and the corresponding functionals

$$I_{N,t}^{(m)}(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \int_{\Omega} H_N^{(m)}(x, u) dx + t \int_{\Omega} e_1 u dx - \int_{\Omega} f_N^{(m)} u dx$$

for all $u \in H_0^1(\Omega)$. By Lemma 4.2 we know that $I_{N,t}^{(m)}$ satisfies the Palais–Smale condition for $t > \bar{t}$. We want to prove that under the assumptions of Theorem 2.3, the functional $I_{N,t}^{(m)}$ admits at least three critical points at different levels for $t > \bar{t}$.

Taking into account Remark 2.2, we prove the $I_{N,t}^{(m)}$ has the geometrical structure “links in scale and bounds” described in [14] for $t > \bar{t}$ and we prove that it admits at least three critical points under this restriction on t .

LEMMA 5.1. *Let $k \geq j \geq 2$ be such that $\lambda_{j-1} < \lambda_j = \dots = \lambda_k < \lambda_{k+1}$. For any $t > 0$ let $s_t = t/(\alpha - \lambda_1)$. If $(\alpha, \beta) \in E_k \cap E_{j-1}$ and $\beta > \mu_{k+1}(\alpha)$ ($\Rightarrow \beta > \alpha$) then there exist $\rho_k > 0$, $\rho_{j-1} > 0$, $t_0 > 0$, $e \in H_k^\perp \setminus \{0\}$, $\bar{\sigma}_k > \rho_k/(\alpha - \lambda_1)$, $\bar{\sigma}_{j-1} > \rho_{j-1}/(\alpha - \lambda_1)$ all independent of m such that if $\sigma_k > \bar{\sigma}_k$, $\sigma_{j-1} > \bar{\sigma}_{j-1}$, $\sigma_k \geq \sigma_{j-1}$, $t > t_0$ and $m \in \mathbb{N}$ then $I_{N,t}^{(m)}$ admits three critical points $w_{1,m}$, $w_{2,m}$,*

$w_{3,m}$ at levels $c_{1,m}$, $c_{2,m}$, $c_{3,m}$ with

$$(5.1) \quad \inf_{z \in B_{j-1}^+(\rho_{j-1}s_t)} I_{N,t}^{(m)}(s_t e_1 + z) \leq c_{3,m} \leq \sup_{\Sigma_{j-1}^-} I_{N,t}^{(m)} \\ < \inf_{z \in S_{j-1}^+(\rho_{j-1}s_t)} I_{N,t}^{(m)}(s_t e_1 + z) \leq c_{2,m} \leq \sup_{\Delta_{j-1}} I_{N,t}^{(m)} \\ \leq \sup_{\Sigma_k^-} I_{N,t}^{(m)} < \inf_{z \in S_k^+(\rho_k s_t)} I_{N,t}^{(m)}(s_t e_1 + z) \leq c_{1,m} \leq \sup_{\Delta_k^-} I_{N,t}^{(m)}$$

where

$$\Sigma_{j-1}^- = \{s_t e_1 + v : v \in B_{j-1}^-(\sigma_{j-1}t)\} \cup \{s_t e_1 + v : v \in \sigma_{j-1}t \Sigma_{j-1}^-(e_j)\}, \\ \Sigma_k^- = \{s_t e_1 + v : v \in B_k^-(\sigma_k t)\} \cup \{s_t e_1 + v : v \in \sigma_k t \Sigma_k^-(e)\}, \\ \Delta_{j-1}^- = \{z = v + \sigma e_j : v \in H_{j-1}, \sigma \geq 0, \|z\|_{H_0^1} \leq \sigma_{j-1}t\}, \\ \Delta_k^- = \{z = v + \sigma e : v \in H_k, \sigma \geq 0, \|z\|_{H_0^1} \leq \sigma_k t\}.$$

PROOF. Since $(\alpha, \beta) \in E_{j-1}$, using the inequalities (4.9), (4.10) with $j-1$ in place of i and arguing as in the proof of Lemma 4.5, we infer that there exist $\rho_{j-1} > 0$ and $t_1 > 0$ such that

$$(5.2) \quad \inf_{w \in S_{j-1}^+(\rho_{j-1}s_t)} \{I_{N,t}^{(m)}(s_t e_1 + w) - I_{N,t}^{(m)}(s_t e_1)\} > 0$$

for all $t > t_1$ and all $m \in \mathbb{N}$ and

$$(5.3) \quad \sup_{v \in H_{j-1}} I_{N,t}^{(m)}(s_t e_1 + v) < \inf_{w \in S_{j-1}^+(\rho_{j-1}s_t)} I_{N,t}^{(m)}(s_t e_1 + w)$$

for all $t > t_1$ and all $m \in \mathbb{N}$. Since $(\alpha, \beta) \in E_k$ then by Lemma 5.1 (b₂) in [14] we have that $\max_{v \in S_k^-(1)} Q_{\alpha, \beta}(v) < 0$ and in particular $\max_{v \in \Sigma_{j-1}^-(e_j)} Q_{\alpha, \beta}(v) < 0$. Therefore arguing as in the proof of Lemma 4.4 (c), we infer that there exist $\bar{\sigma}_{j-1} > 0$ and $t_2 > 0$ such that for any $\sigma_{j-1} > \bar{\sigma}_{j-1}$ and $t > t_2$ we have

$$(5.4) \quad \sup_{v \in \sigma_{j-1}t \Sigma_{j-1}^-(e_j)} \{I_{N,t}^{(m)}(s_t e_1 + v) - I_{N,t}^{(m)}(s_t e_1)\} \leq 0 \quad \text{for all } m \in \mathbb{N}.$$

Combining (5.2)–(5.4) and choosing $t_3 > \max\{t_1, t_2\}$, we obtain

$$(5.5) \quad \sup_{\Sigma_{j-1}^-} I_{N,t}^{(m)} < \inf_{z \in S_{j-1}^+(\rho_{j-1}s_t)} I_{N,t}^{(m)}(s_t e_1 + z) \quad \text{for all } t > t_3 \text{ and all } m \in \mathbb{N}.$$

Since $(\alpha, \beta) \in E_k$ and $\beta > \mu_{k+1}(\alpha)$ then arguing as in the proof of Lemma 4.5 with k in place of i we also obtain

$$(5.6) \quad \sup_{\Sigma_k^-} I_{N,t}^{(m)} < \inf_{z \in S_k^+(\rho_k s_t)} I_{N,t}^{(m)}(s_t e_1 + z) \quad \text{for all } t > t_4, \text{ and all } m \in \mathbb{N}$$

for a suitable $\rho_k > 0$ and for $t_4 > 0$ large enough.

By Lemma 4.2 we infer that there exists $\bar{t} > 0$ such that for any $t > \bar{t}$ and any $m \in \mathbb{N}$ the functional $I_{N,t}^{(m)}$ satisfies the Palais–Smale condition. We may choose $t_0 > \max\{t_3, t_4, \bar{t}\}$ so that also (5.5) and (5.6) hold true.

Consider now the inequality (5.5). Then Theorem 8.2 in [14] applies and hence for any $t > t_0$ and any $m \in \mathbb{N}$, the functional $I_{N,t}^{(m)}$ admits two critical points $w_{2,m}, w_{3,m}$ respectively at levels $c_{2,m}, c_{3,m}$ with

$$(5.7) \quad \inf_{z \in B_{j-1}^+(\rho_{j-1} s_t)} I_{N,t}^{(m)}(s_t e_1 + z) \leq c_{3,m} \leq \sup_{\Sigma_{j-1}^-} I_{N,t}^{(m)} \\ < \inf_{z \in S_{j-1}^+(\rho_{j-1} s_t)} I_{N,t}^{(m)}(s_t e_1 + z) \leq c_{2,m} \leq \sup_{\Delta_{j-1}^-} I_{N,t}^{(m)}.$$

If we consider now (5.6), applying the classical linking theorem [17], we infer that for any $t > t_0$ and any $m \in \mathbb{N}$, the functional $I_{N,t}^{(m)}$ admits a critical point $w_{1,m}$ at level $c_{1,m}$ with

$$(5.8) \quad \sup_{\Sigma_k^-} I_{N,t}^{(m)} < \inf_{z \in S_k^+(\rho_k s_t)} I_{N,t}^{(m)}(s_t e_1 + z) \leq c_{1,m} \leq \sup_{\Delta_k^-} I_{N,t}^{(m)}.$$

By (5.7) and (5.8) we deduce that (5.1) holds true since $\sigma_{j-1} \leq \sigma_k$ and in turn $\Delta_{j-1}^- \subset \Sigma_k^-$. \square

5.1. End of the proof of Theorem 2.3. Since $(\alpha, \beta) \in E_k \cap E_{j-1}$ and $\beta > \mu_{k+1}(\alpha)$ then Lemma 5.1 applies so that we may consider the sequences of critical points $\{w_{1,m}\}, \{w_{2,m}\}, \{w_{3,m}\}$ found in Lemma 5.1. With the procedure introduced in the proof of Theorem 2.1 one can see that the critical levels $c_{1,m} = I_{N,t}^{(m)}(w_{1,m})$, $c_{2,m} = I_{N,t}^{(m)}(w_{2,m})$, $c_{3,m} = I_{N,t}^{(m)}(w_{3,m})$ are uniformly bounded with respect to m . By Lemma 4.6 we infer that there exist $w_1, w_2, w_3 \in H_0^1(\Omega)$ such that up to a subsequence

$$(5.9) \quad w_{1,m} \rightarrow w_1, \quad w_{2,m} \rightarrow w_2, \quad w_{3,m} \rightarrow w_3 \quad \text{in } H_0^1(\Omega) \text{ as } m \rightarrow \infty.$$

In particular by (4.22)–(4.23) and (5.9) we obtain as $m \rightarrow \infty$

$$(5.10) \quad \begin{aligned} c_{1,m} &= I_{N,t}^{(m)}(w_{1,m}) \rightarrow I_{N,t}(w_1), \\ c_{2,m} &= I_{N,t}^{(m)}(w_{2,m}) \rightarrow I_{N,t}(w_2), \\ c_{3,m} &= I_{N,t}^{(m)}(w_{3,m}) \rightarrow I_{N,t}(w_3). \end{aligned}$$

Arguing as in the proof of Theorem 2.1 we also obtain

$$(5.11) \quad \sup_{\Sigma_{j-1}^-} I_{N,t} < \inf_{z \in S_{j-1}^+(\rho_{j-1} s_t)} I_{N,t}(s_t e_1 + z)$$

$$(5.12) \quad \sup_{\Sigma_k^-} I_{N,t} < \inf_{z \in S_k^+(\rho_k s_t)} I_{N,t}(s_t e_1 + z)$$

with $\rho_{j-1}, \rho_k, \Sigma_{j-1}^-, \Sigma_k^-$ and $t > t_0$ as in Lemma 5.1.

Moreover, w_1, w_2, w_3 are critical points for $I_{N,t}$ and by (4.33), (5.1), (5.9)–(5.12) we infer that

$$\begin{aligned} \inf_{z \in B_{j-1}^+(\rho_{j-1}s_t)} I_{N,t}(s_t e_1 + z) &\leq I_{N,t}(w_3) \leq \sup_{\Sigma_{j-1}^-} I_{N,t} \\ &< \inf_{z \in S_{j-1}^+(\rho_{j-1}s_t)} I_{N,t}(s_t e_1 + z) \leq I_{N,t}(w_2) \leq \sup_{\Delta_{j-1}^-} I_{N,t} \\ &\leq \sup_{\Sigma_k^-} I_{N,t} < \inf_{z \in S_k^+(\rho_k s_t)} I_{N,t}(s_t e_1 + z) \leq I_{N,t}(w_1) \leq \sup_{\Delta_k^-} I_{N,t} \end{aligned}$$

which proves that $I_{N,t}$ admits three critical points at distinct levels. Finally the functions $u_1 = w_1 + \gamma_N$, $u_2 = w_2 + \gamma_N$, $u_3 = w_3 + \gamma_N$ are distinct solutions of (1.1). This completes the proof of the theorem. \square

6. Proof of Theorem 2.4

As in the proof of Theorems 2.1 and 2.3 we introduce again the sequence $\{\mu_m\} \subset L^2(\Omega)$ such that $\mu_m \rightharpoonup \mu$ weakly in the sense of measures as $m \rightarrow \infty$ and we define the functionals

$$I_{N,t}^{(m)}(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \int_{\Omega} H_N^{(m)}(x, u) dx + t \int_{\Omega} e_1 u dx - \int_{\Omega} f_N^{(m)} u dx$$

for all $u \in H_0^1(\Omega)$. By Lemma 4.2 we know that the $I_{N,t}^{(m)}$ satisfies the Palais–Smale condition for $t > \bar{t}$.

The proof of this result is very close to the proof of Theorem 2.1 in the case $\beta > \alpha$ and hence we will omit some details.

We start with the following

LEMMA 6.1. *Let $i \geq 1$ be such that $\lambda_i < \lambda_{i+1}$. Let α, β be such that $\alpha > \beta$ and $\alpha > \lambda_1$. For any $t > 0$ define $s_t = t/(\alpha - \lambda_1)$. For any $\varepsilon > 0$ small enough there exists $d > 0$ such that:*

(a) *If we assume that $n_i(\alpha, \beta) > -\infty$ then there exists $C > 0$ such that for any $t > 0$ and $m \in \mathbb{N}$ we have*

$$(6.1) \quad \inf_{w \in H_t^\perp} \{I_{N,t}^{(m)}(s_t e_1 + w) - I_{N,t}^{(m)}(s_t e_1)\} \geq s_t^2(n_i(\alpha, \beta) - 3\varepsilon\lambda_1 - \varepsilon C) - d.$$

(b) *For any $\rho > 0$, $t > 0$ and $m \in \mathbb{N}$ we have*

$$(6.2) \quad \sup_{v \in S_i^-(\rho s_t)} \{I_{N,t}^{(m)}(s_t e_1 + v) - I_{N,t}^{(m)}(s_t e_1)\} \leq s_t^2(N_i(\rho, \alpha, \beta) + 3\varepsilon\lambda_1 + 2\varepsilon\rho^2) + d.$$

(c) *If we assume that $\beta < \nu_i(\alpha)$ then there exist $\bar{\sigma}_i > 0$, $\bar{t} > 0$, $e \in H_i \setminus \{0\}$ all independent of m such that for any $\sigma_i > \bar{\sigma}_i$, $t > \bar{t}$ and $m \in \mathbb{N}$, we*

have

$$(6.3) \quad \sup_{w \in \sigma_t \Sigma_i^+(e)} \{I_{N,t}^{(m)}(s_t e_1 + w) - I_{N,t}^{(m)}(s_t e_1)\} \geq 0.$$

PROOF. For any $z \in H_0^1(\Omega)$ let $u = z/s_t$. We write

$$(6.4) \quad \begin{aligned} & I_{N,t}^{(m)}(s_t e_1 + z) - I_{N,t}^{(m)}(s_t e_1) \\ &= s_t^2 \left\{ Q_\alpha(u) + \frac{\alpha - \beta}{2} \int_\Omega ((e_1 + u)^-)^2 dx \right\} + R^{(m)}(t, z) \\ &= s_t^2 \left\{ Q_{\alpha,\beta}(u) + \frac{\alpha - \beta}{2} \int_\Omega [((e_1 + u)^-)^2 - (u^-)^2] dx \right\} + R^{(m)}(t, z) \end{aligned}$$

where $R^{(m)}(t, z) = \int_\Omega [\Gamma^{(m)}(x, s_t e_1) - \Gamma^{(m)}(x, s_t e_1 + z)] dx$ with $\Gamma^{(m)}(x, s)$ as in Lemma 4.3.

Then by Lemma 4.3 we infer that for any $\varepsilon > 0$ there exists $d > 0$ independent of m such that

$$(6.5) \quad |R^{(m)}(t, z)| \leq \varepsilon (\|s_t e_1 + z\|_{H_0^1}^2 + \|s_t e_1\|_{H_0^1}^2) + d \quad \text{for all } z \in H_0^1(\Omega).$$

(a) Since $n_i(\alpha, \beta) > -\infty$ then by Lemma 7.1 (b₁) in [14] we have that $c = \inf_{w \in S_i^+(1)} Q_{\alpha,\beta}(w) > 0$ so that by (6.4) and (6.5), it follows that for any $w \in H_i^\perp$ we have

$$(6.6) \quad \begin{aligned} & I_{N,t}^{(m)}(s_t e_1 + w) - I_{N,t}^{(m)}(s_t e_1) \\ & \geq c \|w\|_{H_0^1}^2 - (\alpha - \beta) \int_\Omega s_t e_1 w^- dx - 3\varepsilon \lambda_1 s_t^2 - 2\varepsilon \|w\|_{H_0^1}^2 - d \end{aligned}$$

and, by (2.8), we also have

$$(6.7) \quad I_{N,t}^{(m)}(s_t e_1 + w) - I_{N,t}^{(m)}(s_t e_1) \geq s_t^2 n_i(\alpha, \beta) - 3\varepsilon \lambda_1 s_t^2 - 2\varepsilon \|w\|_{H_0^1}^2 - d.$$

Using (6.6) with $\|w\|_{H_0^1} \geq s_t((\alpha - \beta)/(\sqrt{\lambda_1}(c - 2\varepsilon)))$ we obtain

$$(6.8) \quad I_{N,t}^{(m)}(s_t e_1 + w) - I_{N,t}^{(m)}(s_t e_1) \geq -3\varepsilon \lambda_1 s_t^2 - d$$

and using (6.7) with $\|w\|_{H_0^1} \leq s_t((\alpha - \beta)/(\sqrt{\lambda_1}(c - 2\varepsilon)))$ we also obtain

$$(6.9) \quad I_{N,t}^{(m)}(s_t e_1 + w) - I_{N,t}^{(m)}(s_t e_1) \geq s_t^2 \left(n_i(\alpha, \beta) - 3\varepsilon \lambda_1 - 2\varepsilon \frac{(\alpha - \beta)^2}{\lambda_1 (c - 2\varepsilon)^2} \right) - d.$$

Since $n_i(\alpha, \beta) \leq 0$ then by (6.8) we deduce that (6.9) holds for any $w \in H_i^\perp$.

(b) Choosing $z = v \in H_i$ with $\|v\|_{H_0^1} = \rho s_t$, by (2.7), (6.4) and (6.5) we immediately obtain (6.2).

(c) Since $\beta < \nu_i(\alpha)$ then there exists $e \in H_i \setminus \{0\}$ such that

$$c = \inf_{\Sigma_i^+(e)} Q_{\alpha,\beta} > 0.$$

Then by (6.4) and (6.5) we obtain for $w \in \sigma_i t \Sigma_i^+(e)$

$$\begin{aligned} & I_{N,t}^{(m)}(s_t e_1 + w) - I_{N,t}^{(m)}(s_t e_1) \\ & \geq s_t^2 \left(c \left\| \frac{w}{s_t} \right\|_{H_0^1}^2 - (\alpha - \beta) \int_{\Omega} e_1 \frac{w^-}{s_t} dx - 3\varepsilon \lambda_1 - 2\varepsilon \left\| \frac{w}{s_t} \right\|_{H_0^1}^2 \right) - d. \end{aligned}$$

Therefore, if ε is small enough and $\|w/s_t\|_{H_0^1} = \sigma_i(\alpha - \lambda_1)$ and s_t are large enough, then (6.3) follows. \square

Then we prove

LEMMA 6.2. *Let $i \geq 1$ be such that $\lambda_i < \lambda_{i+1}$. Let $(\alpha, \beta) \in F_i$ ($\Rightarrow \alpha > \beta$) and $\beta < \nu_i(\alpha)$. For any $t > 0$ let $s_t = t/(\alpha - \lambda_1)$. Then there exist $\rho_i > 0$, $t_0 > 0$, $e \in H_i \setminus \{0\}$, $\bar{\sigma}_i > \rho_i/(\alpha - \lambda_1)$ all independent of m such that if $\sigma_i > \bar{\sigma}_i$, $t > t_0$ and $m \in \mathbb{N}$ then $I_{N,t}^{(m)}$ admits two critical points $w_{1,m}$, $w_{2,m}$ respectively at levels $c_{1,m}$, $c_{2,m}$ with*

$$(6.10) \quad \inf_{\Delta_i^+} I_{N,t}^{(m)} \leq c_{2,m} \leq \sup_{z \in S_i^-(\rho_i s_t)} I_{N,t}^{(m)}(s_t e_1 + z) < \inf_{\Sigma_i^+} I_{N,t}^{(m)} \leq c_{1,m} \leq \sup_{z \in B_i^-(\rho_i s_t)} I_{N,t}^{(m)}(s_t e_1 + z)$$

where

$$\begin{aligned} \Sigma_i^+ &= \{s_t e_1 + w : w \in B_i^+(\sigma_i t)\} \cup \{s_t e_1 + w : w \in \sigma_i t \Sigma_i^+(e)\}, \\ \Delta_i^+ &= \{z = w + \sigma e : w \in H_i^\perp, \sigma \geq 0, \|z\|_{H_0^1} \leq \sigma_i t\}. \end{aligned}$$

PROOF. Since $(\alpha, \beta) \in F_i$, by (2.9) (see also Lemma 7.1 (d) in [14]) it follows that there exists $\rho_i > 0$ small enough such that $N_i(\rho_i, \alpha, \beta) < n_i(\alpha, \beta) \leq 0$ so that if we choose $\varepsilon > 0$ small enough and $t > 0$ large enough in (6.1), (6.2) we obtain

$$(6.11) \quad \sup_{v \in S_i^-(\rho_i s_t)} \{I_{N,t}^{(m)}(s_t e_1 + v) - I_{N,t}^{(m)}(s_t e_1)\} < 0,$$

$$(6.12) \quad \sup_{v \in S_i^-(\rho_i s_t)} I_{N,t}^{(m)}(s_t e_1 + v) < \inf_{w \in H_i^\perp} I_{N,t}^{(m)}(s_t e_1 + w).$$

By (6.3), (6.11) and (6.12) we also obtain

$$(6.13) \quad \sup_{v \in S_i^-(\rho_i s_t)} I_{N,t}^{(m)}(s_t e_1 + v) < \inf_{\Sigma_i^+} I_{N,t}^{(m)} \quad \text{for all } m \in \mathbb{N}.$$

By Lemma 4.2 we infer that there exists $\bar{t} > 0$ such that for any $t > \bar{t}$ and any $m \in \mathbb{N}$ the functional $I_{N,t}^{(m)}$ satisfies the Palais–Smale condition. We may choose $t_0 \geq \bar{t}$ large enough so that also (6.13) holds true. Now the existence of two critical points and the estimates in (6.10) follows immediately from Theorem 8.2 in [14]. \square

Since in Theorem 2.4 we assume that $\alpha \in (\lambda_i, \lambda_{i+1}]$ and $\alpha > \beta$ then $(\alpha, \beta) \in F_i$ (see Lemma 6.2 in [14]) and hence Lemma 6.2 applies. The proof of the theorem now follows by Lemma 4.6 repeating the procedure introduced in Subsection 4.1.

7. Proof of Theorem 2.6

We give here only an idea of the proof since it easily follows using the procedure introduced in the proof of Theorem 2.3.

LEMMA 7.1. *Let $k \geq j \geq 2$ be such that $\lambda_{j-1} < \lambda_j = \dots = \lambda_k < \lambda_{k+1}$. For any $t > 0$ let $s_t = t/(\alpha - \lambda_1)$. If $(\alpha, \beta) \in F_k \cap F_{j-1}$ and $\beta < \nu_{j-1}(\alpha)$ ($\Rightarrow \alpha > \beta$) then there exist $\rho_k > 0$, $\rho_{j-1} > 0$, $t_0 > 0$, $e \in H_{j-1} \setminus \{0\}$, $\bar{\sigma}_k > \rho_k/(\alpha - \lambda_1)$, $\bar{\sigma}_{j-1} > \rho_{j-1}/(\alpha - \lambda_1)$ all independent of m such that if $\sigma_k > \bar{\sigma}_k$, $\sigma_{j-1} > \bar{\sigma}_{j-1}$, $\sigma_k \leq \sigma_{j-1}$, $t > t_0$ and $m \in \mathbb{N}$ then the functional $I_{N,t}^{(m)}$ admits three critical points $w_{1,m}$, $w_{2,m}$, $w_{3,m}$ at levels $c_{1,m}$, $c_{2,m}$, $c_{3,m}$ with*

$$(7.1) \quad \begin{aligned} \inf_{\Delta_{j-1}^+} I_{N,t}^{(m)} &\leq c_{3,m} \leq \sup_{z \in S_{j-1}^-(\rho_{j-1}s_t)} I_{N,t}^{(m)}(s_t e_1 + z) < \inf_{\Sigma_{j-1}^+} I_{N,t}^{(m)} \\ &\leq \inf_{\Delta_k^+} I_{N,t}^{(m)} \leq c_{2,m} \leq \sup_{z \in S_k^-(\rho_k s_t)} I_{N,t}^{(m)}(s_t e_1 + z) \\ &< \inf_{\Sigma_k^+} I_{N,t}^{(m)} \leq c_{1,m} \leq \sup_{z \in B_k^-(\rho_k s_t)} I_{N,t}^{(m)}(s_t e_1 + z) \end{aligned}$$

where

$$\begin{aligned} \Sigma_{j-1}^+ &= \{s_t e_1 + w : w \in B_{j-1}^+(\sigma_{j-1} t)\} \cup \{s_t e_1 + w : w \in \sigma_{j-1} t \Sigma_{j-1}^-(e)\}, \\ \Sigma_k^+ &= \{s_t e_1 + w : w \in B_k^+(\sigma_k t)\} \cup \{s_t e_1 + w : w \in \sigma_k t \Sigma_k^+(e_k)\}, \\ \Delta_{j-1}^+ &= \{z = w + \sigma e : w \in H_{j-1}^\perp, \sigma \geq 0, \|z\|_{H_0^1} \leq \sigma_{j-1} t\}, \\ \Delta_k^+ &= \{z = w + \sigma e_k : w \in H_k^\perp, \sigma \geq 0, \|z\|_{H_0^1} \leq \sigma_k t\}. \end{aligned}$$

PROOF. Since $(\alpha, \beta) \in F_{j-1}$ and $\beta < \nu_{j-1}(\alpha)$ arguing as in the proof of Lemma 6.2 with $j-1$ in place of i we deduce that there exist $t_1 > 0$ such that

$$(7.2) \quad \sup_{z \in S_{j-1}^-(\rho_{j-1}s_t)} I_{N,t}^{(m)}(s_t e_1 + z) < \inf_{\Sigma_{j-1}^+} I_{N,t}^{(m)}$$

for $t_1 > 0$ large enough.

On the other hand, since $(\alpha, \beta) \in F_k$ then by (6.1) and (6.2), arguing as in the proof of Lemma 6.2, we deduce that there exist $\rho_k > 0$ and $t_2 > 0$ such that

$$(7.3) \quad \sup_{v \in S_k^-(\rho_k s_t)} \{I_{N,t}^{(m)}(s_t e_1 + v) - I_{N,t}^{(m)}(s_t e_1)\} < 0$$

for all $t > t_2$ and all $m \in \mathbb{N}$ and

$$(7.4) \quad \sup_{v \in S_k^-(\rho_k s_t)} I_{N,t}^{(m)}(s_t e_1 + v) < \inf_{w \in H_k^\perp} I_{N,t}^{(m)}(s_t e_1 + w)$$

for all $t > t_2$ and all $m \in \mathbb{N}$. Since $(\alpha, \beta) \in F_{j-1}$ then by Lemma 7.1 (b_1) in [14] we have $c = \inf_{w \in S_{j-1}^+(1)} Q_{\alpha, \beta}(w) > 0$ and in particular $\inf_{w \in \Sigma_k^+(e_k)} Q_{\alpha, \beta}(w) > 0$.

Arguing as in the proof of Lemma 6.1(c), we infer that there exist $\bar{\sigma}_k > 0$ and $t_3 > 0$ such that for any $\sigma_k > \bar{\sigma}_k$ and $t > t_3$ we have

$$(7.5) \quad \sup_{w \in \sigma_k t \Sigma_k^+(e_k)} \{I_{N,t}^{(m)}(s_t e_1 + w) - I_{N,t}^{(m)}(s_t e_1)\} \geq 0 \quad \text{for all } m \in \mathbb{N}.$$

Combining (7.3)–(7.5) and choosing $t_4 > \max\{t_2, t_3\}$ we obtain

$$(7.6) \quad \sup_{z \in S_k^-(\rho_k s_t)} I_{N,t}^{(m)}(s_t e_1 + z) < \inf_{\Sigma_k^+} I_{N,t}^{(m)} \quad \text{for all } t > t_4 \text{ and all } m \in \mathbb{N}.$$

By Lemma 4.2 we infer that there exists $\bar{t} > 0$ such that for any $t > \bar{t}$ and any $m \in \mathbb{N}$ the functional $I_{N,t}^{(m)}$ satisfies the Palais–Smale condition. We may choose $t_0 > \max\{t_1, t_4, \bar{t}\}$ so that also (7.2) and (7.6) hold true.

Consider first the inequality (7.2). Then Theorem 8.2 in [14] applies and hence for any $t > t_0$ and any $m \in \mathbb{N}$, the functional $I_{N,t}^{(m)}$ admits a critical point $w_{3,m}$ at level $c_{3,m}$ with

$$(7.7) \quad \inf_{\Delta_{j-1}^+} I_{N,t}^{(m)} \leq c_{3,m} \leq \sup_{z \in S_{j-1}^-(\rho_{j-1} s_t)} I_{N,t}^{(m)}(s_t e_1 + z) < \inf_{\Sigma_{j-1}^+} I_{N,t}^{(m)}.$$

On the other hand if we consider the inequality (7.6) then applying again Theorem 8.2 in [14] we infer that for any $t > t_0$ admits two critical points $w_{1,m}, w_{2,m}$ respectively at levels $c_{1,m}, c_{2,m}$ with

$$(7.8) \quad \inf_{\Delta_k^+} I_{N,t}^{(m)} \leq c_{2,m} \leq \sup_{z \in S_k^-(\rho_k s_t)} I_{N,t}^{(m)}(s_t e_1 + z) \\ < \inf_{\Sigma_k^+} I_{N,t}^{(m)} \leq c_{1,m} \leq \sup_{z \in B_k^-(\rho_k s_t)} I_{N,t}^{(m)}(s_t e_1 + z).$$

Then (7.1) follows by (7.7)–(7.8) since $\sigma_k \leq \sigma_{j-1}$ and in turn $\Delta_k^+ \subset \Sigma_{j-1}^+$. \square

The proof of the theorem now follows by Lemma 4.6 and Lemma 7.1 repeating the procedure introduced in Subsection 5.1.

8. An alternative proof of Theorems 2.1 and 2.6 for $\mu \geq 0$

In this section we give an alternative proof to our multiplicity results when μ is a nonnegative Radon measure. We assume that the nonlinearity $g(x, s) = \alpha s^+ - \beta s^-$. We consider again the functional

$$I_{N,t}(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \int_{\Omega} H_N(x, u) dx + t \int_{\Omega} e_1 u dx - \int_{\Omega} f_N u dx$$

for all $u \in H_0^1(\Omega)$, introduced in Section 3 and we look for its critical points. We show that $I_{N,t}$ satisfies the Palais–Smale condition under the restriction $\mu \geq 0$.

LEMMA 8.1. *Let $n \geq 2$ and assume that $g(x, s) = \alpha s^+ - \beta s^-$ with $\alpha \neq \beta$. Moreover, suppose that at least one of the two alternatives occur*

- (a) $\beta > \alpha$ and $\beta > \lambda_1$
- (b) $\alpha > \beta$ and $\alpha > \lambda_1$.

If μ is a nonnegative Radon measure then there exists $\bar{t} > 0$ such that for any $t > \bar{t}$ the functional $I_{N,t}$ satisfies the Palais–Smale condition.

PROOF. Since μ is a nonnegative Radon measure, by (3.2)–(3.3) and the weak comparison principle (see Lemma 3 in [7]) we deduce that v_k is nonnegative for any $k \geq 1$. In particular the function $\gamma_N = \sum_{i=1}^N v_i$ is nonnegative. In view of Theorem 2.5 in [14], in order to prove the Palais–Smale condition, it is enough to verify the two conditions

$$(8.1) \quad 2[H_N(x, s) + f_N s] - [h_N(x, s) + f_N]s \leq a_0(x)|s|$$

for almost every $x \in \Omega$, for all $s \in \mathbb{R}$, if $\beta > \alpha$ and $\beta > \lambda_1$ and

$$(8.2) \quad 2[H_N(x, s) + f_N s] - [h_N(x, s) + f_N]s \geq -a_0(x)|s|$$

for almost every $x \in \Omega$, for all $s \in \mathbb{R}$, if $\alpha > \beta$ and $\alpha > \lambda_1$ where the function $a_0 \in L^q(\Omega)$ with $q > 1$ if $n = 2$ and $q \geq 2n/(n+2)$ if $n > 2$.

Since $\gamma_N \geq 0$, by direct computation we obtain

$$(8.3) \quad 2[H_N(x, s) + f_N s] - [h_N(x, s) + f_N]s = f_N s \quad \text{if } s + \gamma_N \geq 0,$$

$$(8.4) \quad 2[H_N(x, s) + f_N s] - [h_N(x, s) + f_N]s = (\beta - \alpha)\gamma_N(s + \gamma_N) + f_N s \\ \text{if } s + \gamma_N < 0.$$

By (8.3), (8.4) we obtain (8.1), (8.2) respectively in the cases $\beta > \alpha$ and $\alpha > \beta$ with $a_0(x) = |f_N(x)| \in L^p(\Omega)$, $p > n/2$. \square

In view of Lemma 8.1 and (3.12) we may apply Theorems 4.14 and 4.17 in [14] respectively under the assumptions of Theorems 2.1 and 2.3 and obtain existence of two and three solutions for (3.11). In the same way, applying Theorems 6.7 and 6.10 in [14] respectively, under the assumptions of Theorems 2.4 and 2.6 we obtain again the existence of two and three solutions for (3.11).

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