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SOLVABILITY IN WEIGHTED SPACES OF THE THREE-DIMENSIONAL NAVIER–STOKES PROBLEM IN DOMAINS WITH CYLINDRICAL OUTLETS TO INFINITY

Konstantin Pileckas

Abstract. The nonstationary Navier–Stokes problem is studied in a three-dimensional domain with cylindrical outlets to infinity in weighted Sobolev function spaces. The unique solvability of this problem is proved under natural compatibility conditions either for a small time interval or for small data. Moreover, it is shown that the solution having prescribed fluxes over cross-sections of outlets to infinity tends in each outlet to the corresponding time-dependent Poiseuille flow.

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1. Introduction

In this paper we study the three-dimensional Navier–Stokes problem in domains with cylindrical outlets to infinity, i.e. we study the following initial-boundary value problem for the Navier–Stokes system

$$
\tag{1.1}
\begin{cases}
\mathbf{u}_t - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f}, \\
\operatorname{div}\mathbf{u} = 0, \\
\mathbf{u}(x,t)|_{\partial\Omega} = 0, \quad \mathbf{u}(x,0) = \mathbf{u}_0(x), \\
\displaystyle\int_{\sigma_j} \mathbf{u}(x,t)\cdot\mathbf{n}(x)\,dS = F_j(t) & \text{for } j = 1,\dots,J, \\
\displaystyle\sum_{j=1}^{J} F_j(t) = 0 & \text{for all } t\in[0,T],
\end{cases}
$$

in a domain $\Omega\subset\mathbb{R}^3$ with J cylindrical outlets to infinity. It is assumed that for sufficiently large $|x|$ the domain Ω splits into J disconnected components Ω_j (outlets to infinity) that in certain coordinate systems $x^{(j)}$ have the form

$$
\tag{1.2}
\Omega_j = \{x^{(j)}\in\mathbb{R}^3 : x^{(j)\prime} = (x_1^{(j)}, x_2^{(j)}) \in \sigma_j,\ 0 < x_3^{(j)} < \infty\},
$$

where cross-sections $\sigma_j \subset \mathbb{R}^2$ are bounded domains. The condition (1.1_4) prescribes fluxes of the velocity vector $\mathbf{u}(x,t)$ over cross-sections σ_j of outlets to infinity Ω_j and the condition (1.1_5) means that the total flux is equal to zero for all $t\in[0,\ T]$.

We assume that $F_j \in W_2^1(0,T)$, $j=1,\dots,J$, and that the initial velocity $\mathbf{u}_0(x)$ and the external force $\mathbf{f}(x,t)$ admit the representations

$$
\begin{aligned}
\mathbf{u}_0(x) &= \sum_{j=0}^{J} \zeta(x_3^{(j)})\mathbf{u}_0^{(j)}(x^{(j)\prime}) + \widehat{\mathbf{u}}_0(x), \\
\mathbf{f}(x,t) &= \sum_{j=0}^{J} \zeta(x_3^{(j)})\mathbf{f}^{(j)}(x^{(j)\prime},t) + \widehat{\mathbf{f}}(x,t),
\end{aligned}
$$

where $\zeta(\tau)$ is a smooth cut-off function with $\zeta(\tau)=0$ for $\tau\le 1$ and $\zeta(\tau)=1$ for $\tau\ge 2$, $\mathbf{u}_0^{(j)}\in W_2^1(\sigma_j)$, $\mathbf{f}^{(j)}\in L_2(\sigma_j^T)$, $\sigma_j^T=\sigma_j\times(0,T)$, and $\widehat{\mathbf{f}}$, $\widehat{\mathbf{u}}_0$ belong to certain weighted spaces of vanishing at infinity functions. Moreover, we suppose that there hold the compatibility conditions

$$
\begin{gathered}
\operatorname{div}\mathbf{u}_0^{(j)}(x^{(j)\prime}) = 0, \quad \operatorname{div}\mathbf{u}_0(x) = 0, \quad \mathbf{u}_0(x)|_{\partial\Omega} = 0, \\
F_j(0) = \int_{\sigma_j} u_{03}^{(j)}(x^{(j)\prime})\,dx^{(j)\prime}, j=1,\dots,J.
\end{gathered}
$$

Under these assumptions we prove the local existence (i.e. either for small data or for a small time interval $[0,T]$) of the solution to problem (1.1). The obtained solution $\mathbf{u}(x,t)$ tends in each outlet to infinity Ω_j to a time-dependent Poiseuille type flow $\mathbf{U}^{(j)}(x^{(j)\prime}, t)$ related to the pipe $\Pi_j = \{x^{(j)} \in \mathbb{R}^3 : x^{(j)\prime} \in \sigma_j, -\infty < x_3^{(j)} < \infty\}$. The decay rate of the difference $\mathbf{u}(x,t) - \mathbf{U}^{(j)}(x^{(j)\prime}, t)$ is conditioned by the decay rate of the external force and the initial data. In particular, if $\widehat{\mathbf{f}}(x,t) = 0$, $\widehat{\mathbf{u}}_0(x) = 0$ (or, if $\widehat{\mathbf{f}}(x,t)$ and $\widehat{\mathbf{u}}_0(x)$ vanish exponentially), then the solution $\mathbf{u}(x,t)$ tends as $|x| \to \infty, x \in \Omega_j$ to $\mathbf{U}^{(j)}(x^{(j)\prime}, t)$ exponentially. The uniqueness of the solution to problem (1.1) is proved in a class of functions that are bounded (in certain sense) at infinity. In particular, from this follows the uniqueness of the time-dependent Poiseuille flow in a straight pipe. Note that for the steady case the uniqueness of Poiseuille flow is not known.

The analogous results for the linearized nonstationary Stokes system were obtained in [8]. The problem (1.1) in a two-dimensional domain Ω with strip-like outlets to infinity was studied in [9], [10] where the global unique existence of the solution to problem (1.1) was proved (i.e. the unique solution exists for arbitrary data and the infinite time interval, in particular, for arbitrary fluxes). In [12] the problem (1.1) was studied in an infinite three-dimensional straight cylinder. In [12], assuming the norms of $\widehat{\mathbf{u}}_0(x)$, $\widehat{\mathbf{f}}(x,t)$ to be sufficiently small, the existence of a unique global in time solution to the nonstationary Navier–Stokes system with prescribed nonzero flux $F(t)$ was proved (the values of the flux $F(t)$, $\mathbf{u}_0(x')$ and $\mathbf{f}(x', t)$ could be arbitrary large). The obtained solution remains close to the corresponding time dependent Poiseuille type flow and it converges to the Poiseuille flow as $|x_3| \to \infty$. The existence of time-dependent Poiseuille type solutions in a straight pipe is studied in [6], [11], [7].

The nonstationary Navier–Stokes system in general domains with outlets to infinity was studied in [5], [14]–[16] where the local existence of solutions with prescribed fluxes $F_j(t)$ was proved. These solutions have finite or infinite energy integral, dependent on the geometry of the outlets to infinity. In particular, if outlets are cylindrical (or strip-like), the energy integral is infinite. We have to mention also the paper [1] where the time-dependent perturbation of the steady Poiseuille flow is studied in weighted spaces with polynomial weights. In [1] it is assumed that the flux F is independent of t and it is proved that for sufficiently small $|F|$ the solution of the nonstationary Navier–Stokes problem tends as $|x| \to \infty$ to the steady Poiseuille solution.

The paper is organized as follows. In Section 2 we define the function spaces used in the paper and recall necessary multiplicative inequalities. In Section 3 we present results proved in [7] concerning the existence of the time-dependent Poiseuille type flow in a straight pipe and we construct a divergence free flux carrier $\mathbf{V}(x,t)$ coinciding in each outlet to infinity Ω_j with the corresponding to this outlet time-dependent Poiseuille flow. Finally, we reduce problem (1.1) to a problem for the perturbation of the constructed flux carrier (see (3.14)). Note that in (3.14) all fluxes vanish and that there appear additional linear terms (containing the flux carrier $\mathbf{V}(x,t)$) in the Navier–Stokes equations (1.1_1). The results from [8] concerning the linear nonstationary Stokes problem are collected in Section 4. In Section 5 we prove estimate in weighted $W_2^{2,1}$-spaces of the nonlinear and perturbation terms contained in equations (3.14). Finally, in Sections 6 and 7 we prove the existence of a unique solution to problem (3.14) and the uniqueness of a solution to problem (1.1). In Section 8 we discuss (without proofs) the behavior as $|x| \to \infty$ of weak Hopf's solutions to problem (3.14).

2. Function spaces and auxiliary results

2.1. Notations, function spaces and multiplicative inequalities. The norm of an element u in a Banach space V is denoted by $\|u;V\|$. Vector-valued functions are denoted by bold letters and the spaces of scalar and vector-valued functions are not distinguished in notations. The vector-valued function $\mathbf{u} = (u_1,\dots,u_n)$ belongs to the space V, if $u_i \in V$, $i = 1,\dots,n$ and $\|\mathbf{u};V\| = \sum_{i=1}^n \|u_i;V\|$.

Let G be an arbitrary domain in $\mathbb{R}^n$, $n \geq 1$, with the boundary ∂G. As usual, denote by $C^\infty(G)$ the set of all infinitely many times differentiable in G functions and by $C_0^\infty(G)$ the subset of functions from $C^\infty(G)$ with compact supports in G. For a given nonnegative integer l and $q > 1$, $W_q^l(G)$ indicates the Sobolev space of functions with the finite norm

$$\|u;W_q^l(G)\| = \Big(\sum_{|\alpha|=0}^{l} \int_G |D^\alpha u(x)|^q\,dx\Big)^{1/q}, \tag{2.1}$$

where $D_x^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1}\dots\partial x_n^{\alpha_n}$, $|\alpha| = \alpha_1 + \dots + \alpha_n$, $W_q^0(G) = L_q(G)$ and $\mathring{W}_q^l(G)$ is the closure of $C_0^\infty(G)$ in the norm (2.1). We shall write $u \in W_{q,\mathrm{loc}}^l(G)$, if $u \in W_q^l(G')$ for any bounded subdomain G' with $\overline{G'} \subset G$. $L_\infty(G)$ is a linear space of real Lebesgue measurable functions defined in G with the norm

$$\|u;L_\infty(G)\| = \operatorname*{ess\,sup}_{x\in G} |u(x)| < \infty.$$

Consider now functions which depend on $x \in G$ and $t \in (0,T)$. Let $G^T = G \times (0,T)$, $T \in (0,\infty]$. $W_2^{2l,l}(G^T)$ is a Hilbert space of functions that have generalized derivatives $D_t^r D_x^\alpha$ with every r and α such that $2r + |\alpha| \leq 2l$. The norm in $W_2^{2l,l}(G^T)$ is defined by the formula

$$\|u;W_2^{2l,l}(G^T)\| = \Big(\sum_{j=0}^{2l} \sum_{2r+|\alpha|=j} \int_0^T \int_G |D_t^r D_x^\alpha u(x,t)|^2\,dx\,dt\Big)^{1/2}.$$

$W_2^{1,1}(G^T)$ and $W_2^{1,0}(G^T)$ are spaces of functions with the finite norms

$$\|u;W_2^{1,1}(G^T)\| = \Big(\int_0^T \int_G (|u_t(x,t)|^2 + |u(x,t)|^2 + |\nabla u(x,t)|^2)\,dx\,dt\Big)^{1/2}$$

and

$$\|u; W_2^{1,0}(G^T)\| = \left(\int_0^T \int_G (|u(x,t)|^2 + |\nabla u(x,t)|^2)\,dx\,dt\right)^{1/2},$$

respectively. $\overset{\circ}{W}{}_2^{1,0}(G^T)$ and $\overset{\circ}{W}{}_2^{1,1}(G^T)$ are subspaces of functions from $W_2^{1,0}(G^T)$ and $W_2^{1,1}(G^T)$ satisfying the condition $u(x,t)|_{\partial G} = 0$.

Let us recall well-known multiplicative inequalities (e.g. [4], [3]).

Lemma 2.1. *Let $G \subset \mathbb{R}^3$ be a bounded domain.*

(a) *If $u \in \overset{\circ}{W}{}_2^1(G)$, then*

$$\|u; L_4(G)\|^4 \le (4/3)^{3/2}\|u; L_2(G)\|\|\nabla u; L_2(G)\|^3.$$

(b) *If $u \in W_2^1(G)$, then*

$$\|u; L_3(G)\| \le c\|u; W_2^1(G)\|^{1/2}\|u; L_2(G)\|^{1/2}. \tag{2.2}$$

(c) *If $u \in W_2^2(G)$, then*

$$\begin{aligned} \|u; L_\infty(G)\| &\le c\|\nabla u; L_6(G)\|^{1/2}\|u; L_6(G)\|^{1/2} \\ &\le c\|\nabla u; W_2^1(G)\|^{1/2}\|u; W_2^1(G)\|^{1/2}. \end{aligned} \tag{2.3}$$

The constants in (2.2) *and* (2.3) *depend only on G.*

2.2. Weighted function spaces in domains with outlets to infinity. Let $\Omega \subset \mathbb{R}^3$ be a domain with J cylindrical outlets to infinity Ω_j having the form (1.2). We introduce the following notations:

$$\begin{aligned} \Omega_{jk} &= \{x \in \Omega_j : x_3^{(j)} < k\}, \quad \omega_{jk} = \Omega_{jk+1} \setminus \Omega_{jk},\ j = 1,\dots,J,\ k \ge 0, \\ \widehat{\omega}_{jk} &= \omega_{jk-1} \cup \omega_{jk} \cup \omega_{jk+1}, \quad j = 1,\dots,J,\ k \ge 1, \\ \Omega_{(k)} &= \Omega_{(0)} \bigcup \Big(\bigcup_{j=1}^J \Omega_{jk}\Big), \quad \Omega_{(0)} = \Omega \setminus \Big(\bigcup_{j=1}^J \Omega_j\Big). \end{aligned}$$

Denote $\beta = (\beta_1,\dots,\beta_J)$ and let $E_{\beta_j}(x) = E_{\beta_j}(x_3^{(j)})$ be a smooth monotone weight-function in Ω_j such that

$$E_{\beta_j}(x) > 0, \quad a_1 \le E_{-\beta_j}(x)E_{\beta_j}(x) \le a_2 \quad \text{for all } x \in \Omega_j, \quad E_{\beta_j}(0) = 1, \tag{2.4}$$

$$b_1 E_{\beta_j}(k) \le E_{\beta_j}(x) \le b_2 E_{\beta_j}(k) \quad \text{for all } x \in \omega_{jk}, \tag{2.5}$$

$$|\nabla E_{\beta_j}(x)| \le b_3\gamma_* E_{\beta_j}(x) \quad \text{for all } x \in \Omega_j, \tag{2.6}$$

$$\lim_{x_3^{(j)} \to \infty} E_{\beta_j}(x) = \infty, \quad \text{if } \beta_j > 0, \tag{2.7}$$

where the constants a_1, a_2, b_1, b_2 are independent of k and b_3 is independent of β_j. Simple examples of such weight-functions are

$$E_{\beta_j}(x) = (1 + \delta|x_3^{(j)}|^2)^{\beta_j} \quad \text{and} \quad E_{\beta_j}(x) = \exp(2\beta_j x_3^{(j)}).$$

The conditions (2.4), (2.5) and (2.7) for these functions are obvious. The inequality (2.6) holds for the first weight-function with $\gamma_* = |\beta_j|\delta$, and for the second one with $\gamma_* = |\beta_j|$. Below, in proofs of solvability for the Navier–Stokes system, we will require γ_* to be "sufficiently small".

Set

$$E_\beta(x) = \begin{cases} 1 & \text{for } x \in \Omega_{(0)}, \\ E_{\beta_j}(x_3^{(j)}) & \text{for } x \in \Omega_j,\ j = 1, \dots, J, \end{cases}$$

and define in Ω weighted function spaces. Let $C_0^\infty(\overline{\Omega})$ be the set of all functions from $C^\infty(\Omega)$ that are equal to zero for large $|x|$. Denote by $\mathcal{W}^l_{2,\beta}(\Omega)$, $l \geq 0$, the space of functions obtained as a closure of $C_0^\infty(\overline{\Omega})$ in the norm

$$\|u; \mathcal{W}^l_{2,\beta}(\Omega)\| = \Big(\sum_{|\alpha|=0}^{l} \int_\Omega E_\beta(x) |D^\alpha u(x)|^2 \, dx \Big)^{1/2}$$

and let $\mathcal{L}_{2,\beta}(\Omega) = \mathcal{W}^0_{2,\beta}(\Omega)$. If $\beta_j > 0$, weight-indices β_j show the decay rate of elements $u \in \mathcal{W}^l_{2,\beta}(\Omega)$ and their derivatives as $|x| \to \infty$, $x \in \Omega_j$. If $\beta_j < 0$, elements $u \in \mathcal{W}^l_{2,\beta}(\Omega)$ may grow as $|x| \to \infty$, $x \in \Omega_j$. Obviously,

$$\begin{aligned} \mathcal{W}^l_{2,\beta}(\Omega) &\subset W^l_2(\Omega) \subset \mathcal{W}^l_{2,-\beta}(\Omega) \quad && \text{for } \beta_j \geq 0,\ j = 1, \dots, J, \\ \mathcal{W}^l_{2,\beta}(\Omega) &= W^l_2(\Omega) && \text{for } \beta_j = 0,\ j = 1, \cdots, J. \end{aligned}$$

Analogously, $\mathcal{W}^{2l,l}_{2,\beta}(\Omega^T)$ ($l \geq 0$ is an integer), $\mathcal{W}^{1,1}_{2,\beta}(\Omega^T)$ and $\mathcal{W}^{1,0}_{2,\beta}(\Omega^T)$ are the spaces of functions obtained as closures of the set of all infinitely many times differentiable with respect to x and t functions equal to zero for large $|x|$ in the norms

$$\begin{aligned} \|u; \mathcal{W}^{2l,l}_{2,\beta}(\Omega^T)\| &= \Big(\sum_{j=0}^{2l} \sum_{2r+|\alpha|=j} \int_0^T \int_\Omega E_\beta(x) |D_t^r D_x^\alpha u(x,t)|^2 \, dx \, dt \Big)^{1/2}, \\ \|u; \mathcal{W}^{1,1}_{2,\beta}(\Omega^T)\| &= \Big(\int_0^T \int_\Omega E_\beta(x) (|u_t(x,t)|^2 + |u(x,t)|^2 + |\nabla u(x,t)|^2) \, dx \, dt \Big)^{1/2}, \\ \|u; \mathcal{W}^{1,0}_{2,\beta}(\Omega^T)\| &= \Big(\int_0^T \int_\Omega E_\beta(x) (|u(x,t)|^2 + |\nabla u(x,t)|^2) \, dx \, dt \Big)^{1/2}, \end{aligned}$$

respectively. Finally, $\mathcal{L}_{2,\beta}(\Omega^T)$ is the space of functions with the finite norm

$$\|u; \mathcal{L}_{2,\beta}(\Omega^T)\| = \Big(\int_0^T \int_\Omega E_\beta(x) |u(x,t)|^2 \, dx \, dt \Big)^{1/2}.$$

We will need also a "step" weight-function

$$E^{(k)}_\beta(x) = \begin{cases} 1 & \text{for } x \in \Omega_{(0)}, \\ E_{\beta_j}(x_3^{(j)}) & \text{for } x \in \Omega_{jk},\ j = 1, \dots, J, \\ E_{\beta_j}(k) & \text{for } x \in \Omega_j \setminus \Omega_{jk},\ j = 1, \dots, J. \end{cases}$$

Obviously,

$$E_\beta^{(k)}(x) = E_\beta(x) \quad \text{for } x \in \Omega_{(k)}, \quad |\nabla E_\beta^{(k)}(x)| \le b_3 \gamma_* E_\beta^{(k)}(x),$$

and, if $\beta_j \ge 0$, then

$$E_{\beta_j}^{(k)}(x) \le E_{\beta_j}(x) \quad \text{for all } x \in \Omega.$$

Lemma 2.2. *For any function $u \in \mathcal{W}^1_{2,\beta}(\Omega)$ which is equal to zero on $\partial\Omega$ there holds the following weighted Poincaré inequality*

$$\int_\Omega E_\beta(x)|u(x)|^2\,dx \le c \int_\Omega E_\beta(x)|\nabla u(x)|^2\,dx. \tag{2.8}$$

Proof. Since $E_\beta^{(k)}(x)$ depends only on $x_3^{(j)}$ in Ω_j and it is equal to 1 in $\Omega_{(0)}$, we get (2.8) applying classical Poincaré inequality in the domain $\Omega_{(0)}$ and on the cross-sections σ_j. □

Remark 2.3. Obviously, in (2.8) one may take the bounded domain $\Omega_{(l)}$ and the "step" weight-function $E_\beta^{(k)}(x)$ instead of Ω and $E_\beta(x)$. The constant in the obtained weighted Poincaré inequality is the same as in (2.8) and does not depend on k and l.

3. Reduction of problem (1.1) to a problem with zero fluxes

Let $\partial\Omega \in C^2$. Consider in the domain Ω problem (1.1). Assume that $F_j \in W_2^1(0,T)$, $j = 1, \dots, J$, and that the initial velocity $\mathbf{u}_0(x,t)$ and the external force $\mathbf{f}(x,t)$ admit the representations

$$\begin{aligned} \mathbf{u}_0(x) &= \sum_{j=0}^{J} \zeta(x_3^{(j)})\mathbf{u}_0^{(j)}(x^{(j)\prime}) + \widehat{\mathbf{u}}_0(x), \\ \mathbf{f}(x,t) &= \sum_{j=0}^{J} \zeta(x_3^{(j)})\mathbf{f}^{(j)}(x^{(j)\prime}, t) + \widehat{\mathbf{f}}(x,t), \end{aligned} \tag{3.1}$$

where $\zeta(\tau)$ is a smooth cut-off function with $\zeta(\tau) = 0$ for $\tau \le 1$ and $\zeta(\tau) = 1$ for $\tau \ge 2$,

$$\begin{aligned} \mathbf{u}_0^{(j)}(x^{(j)\prime}) &= (u_{01}^{(j)}(x^{(j)\prime}), u_{02}^{(j)}(x^{(j)\prime}), u_{03}^{(j)}(x^{(j)\prime})), \\ \mathbf{f}^{(j)}(x^{(j)\prime}, t) &= (f_1^{(j)}(x^{(j)\prime}, t), f_2^{(j)}(x^{(j)\prime}, t), f_3^{(j)}(x^{(j)\prime}, t)), \end{aligned} \tag{3.2}$$

$$\begin{aligned} &\mathbf{u}_0^{(j)} \in \overset{\circ}{W}{}_2^1(\sigma_j), && \mathbf{f}^{(j)} \in L_2(\sigma_j^T), && j = 1, \dots, J, \\ &\widehat{\mathbf{u}}_0 \in \mathcal{W}^1_{2,\beta}(\Omega) \cap \overset{\circ}{W}{}_2^1(\Omega), && \widehat{\mathbf{f}} \in \mathcal{L}_{2,\beta}(\Omega^T), \quad \beta_j \ge 0, && j = 1, \dots, J, \end{aligned}$$

Moreover, suppose that there hold the compatibility conditions

$$\operatorname{div}'\mathbf{u}_0^{(j)\prime}(x^{(j)\prime}) = 0, \quad F_j(0) = \int_{\sigma_j} u_{03}^{(j)}(x^{(j)\prime})\,dx^{(j)\prime}, \; j = 1, \dots, J. \tag{3.3}$$

Then (see [7]) in each cylinder $\Pi_j = \sigma_j \times (0, T)$ there exists a Poiseuille type solution $(\mathbf{U}^{(j)}(x^{(j)\prime}, t), P^{(j)}(x^{(j)}, t))$ having the form

$$
\begin{aligned}
\mathbf{U}^{(j)}(x^{(j)\prime}, t) &= (U_1^{(j)}(x^{(j)\prime}, t), U_2^{(j)}(x^{(j)\prime}, t), U_3^{(j)}(x^{(j)\prime}, t)),\\
P^{(j)}(x^{(j)}, t) &= \widetilde{p}^{(j)}(x^{(j)\prime}, t) - q^{(j)}(t) x_3^{(j)} + p_0^{(j)}(t),
\end{aligned}
\tag{3.4}
$$

where $(\mathbf{U}^{(j)\prime}(x^{(j)\prime}, t), \widetilde{p}^{(j)}(x^{(j)\prime}, t)) = ((U_1^{(j)}(x^{(j)\prime}, t), U_2^{(j)}(x^{(j)\prime}, t)), \widetilde{p}^{(j)}(x^{(j)\prime}, t))$ is a solution of the two-dimensional Navier–Stokes problem on the cross-section σ_j:

$$
\begin{cases}
\mathbf{U}_t^{(j)\prime} - \nu \Delta' \mathbf{U}^{(j)\prime} + (\mathbf{U}^{(j)\prime} \cdot \nabla') \mathbf{U}^{(j)\prime} + \nabla' \widetilde{p}^{(j)} = \mathbf{f}^{(j)\prime},\\
\operatorname{div}' \mathbf{U}^{(j)\prime}(x^{(j)\prime}, t) = 0,\\
\mathbf{U}^{(j)\prime}(x^{(j)\prime}, t)|_{\partial\sigma_j} = 0, \quad \mathbf{U}^{(j)\prime}(x^{(j)\prime}, 0) = \mathbf{u}^{(j)}{}_0^{\prime}(x^{(j)\prime}),
\end{cases}
\tag{3.5}
$$

with

$$
\mathbf{f}^{(j)\prime}(x^{(j)\prime}, t) = (f_1^{(j)}(x^{(j)\prime}, t), f_2^{(j)}(x^{(j)\prime}, t)), \mathbf{u}_0^{(j)\prime}(x^{(j)\prime}) = (u_{01}^{(j)}(x^{(j)\prime}), u_{02}^{(j)}(x^{(j)\prime}))
$$

and $(U_3^{(j)}(x^{(j)\prime}, t),\ q^{(j)}(t))$ is the solution of the inverse parabolic problem:

$$
\begin{cases}
U_{3t}^{(j)} - \nu \Delta' U_3^{(j)} + (\mathbf{U}^{(j)\prime} \cdot \nabla') U_3^{(j)} = q^{(j)}(t) + f_3^{(j)},\\
U_3^{(j)}(x^{(j)\prime}, t)|_{\partial\sigma} = 0, \quad U_3^{(j)}(x^{(j)\prime}, 0) = u_{03}^{(j)}(x^{(j)\prime}),\\
\displaystyle\int_\sigma U_3^{(j)}(x^{(j)\prime}, t)\, dx^{(j)\prime} = F_j(t).
\end{cases}
\tag{3.6}
$$

Note that in (3.6) $F_j(t)$, $f_3^{(j)}(x^{(j)\prime}, t)$ and $u_{03}^{(j)}(x^{(j)\prime})$ are given while $U_3^{(j)}(x^{(j)\prime}, t)$ and $q^{(j)}(t))$ has to be found. The function $p_0^{(j)}(t)$ in (3.4) is arbitrary.

The following result is well known (see [3, Chapter VI]).

Theorem 3.1. *Let $\partial\sigma_j \in C^2$, $\mathbf{u}_0^{(j)\prime} \in \overset{\circ}{W}{}_2^1(\sigma_j)$, $\operatorname{div}' \mathbf{u}^{(j)}{}_0^{\prime}(x^{(j)\prime}) = 0$, $\mathbf{f}^{(j)\prime} \in L_2(\sigma_j^T)$. Then for arbitrary $T \in (0, \infty]$ problem* (3.5) *admits a unique solution $(\mathbf{U}^{(j)\prime}, \widetilde{p}^{(j)})$ such that $\mathbf{U}^{(j)\prime} \in W_2^{2,1}(\sigma_j^T)$, $\nabla' \widetilde{p}^{(j)} \in L_2(\sigma_j^T)$ and there holds the estimate*

$$
\|\mathbf{U}^{(j)\prime}; W_2^{2,1}(\sigma_j^T)\|^2 + \|\nabla' \widetilde{p}^{(j)}; L_2(\sigma_j^T)\|^2 \le A_0^{(j)},
\tag{3.7}
$$

The constant $A_0^{(j)}$ in (3.7) *depends on the norms $\|\mathbf{u}_0^{(j)\prime}; W_2^1(\sigma_j)\|$, $\|\mathbf{f}^{(j)\prime}; L_2(\sigma_j^T)\|$. If $T = \infty$, then*

$$
\|\mathbf{U}^{(j)\prime}(\,\cdot\,, t); L_2(\sigma_j)\|^2 + \|\nabla' \mathbf{U}^{(j)\prime}(\,\cdot\,, t); L_2(\sigma_j)\|^2 \to 0 \quad \textit{as } t \to \infty.
$$

Problem (3.6) is studied in [7]:

Theorem 3.2. *Let $\partial\sigma_j \in C^2$, $u_{03}^{(j)} \in \overset{\circ}{W}{}_2^1(\sigma_j)$, $f_3^{(j)} \in L_2(\sigma_j^T)$, $F_j \in W_2^1(0, T)$ and let there holds the compatibility condition*

$$
F_j(0) = \int_\sigma u_{03}^{(j)}(x^{(j)\prime})\, dx^{(j)\prime}.
$$

Then for arbitrary $T \in (0, \infty]$ *problem* (3.6) *admits a unique solution* $(U_3^{(j)}, q^{(j)}) \in W_2^{2,1}(\sigma_j^T) \times L_2(0,T)$. *There holds the estimate*

$$\|U_3^{(j)}; W_2^{2,1}(\sigma_j^t)\|^2 + \|q^{(j)}; L_2(0,t)\|^2 \le c(A_0^{(j)})(\|f_3^{(j)}; L_2(\sigma_j^t)\|^2 + \|u_{03}^{(j)}; W_2^1(\sigma_j)\|^2 + \|F_j; W_2^1(0,t)\|^2), \tag{3.8}$$

for all $t \in (0, T]$, *with a constant* $c(A_0^{(j)})$ *independent of* t *and* T.

We define

$$\mathbf{U}(x,t) = \sum_{j=1}^{J} \zeta(x_3^{(j)}) \mathbf{U}^{(j)}(x^{(j)\prime}, t), \quad P(x,t) = \sum_{j=1}^{J} \zeta(x_3^{(j)}) P^{(j)}(x^{(j)}, t). \tag{3.9}$$

Let

$$g(x,t) = -\operatorname{div} \mathbf{U}(x,t) = -\sum_{j=1}^{J} \frac{\partial \zeta(x_3^{(j)})}{\partial x_3^{(j)}} U_3^{(j)}(x^{(j)\prime}, t).$$

Then $\operatorname{supp}_x g(x,t) \subset \overline{\Omega_{(2)} \setminus \Omega_{(1)}}$, and from the condition $\sum_{j=1}^{J} F_j(t) = 0$ we obtain that

$$\int_{\Omega_{(2)}} g(x,t)\, dx = 0 \quad \text{for all } t \in [0, T].$$

Moreover, in virtue of (3.8)

$$\begin{aligned} &\int_0^T \|g(\cdot\,,t); W_2^1(\Omega_{(2)})\|^2\, dt + \int_0^T \|g_t(\cdot\,,t); L_2(\Omega_{(2)})\|^2\, dt \\ &\le c \sum_{j=1}^{J} \left(\int_0^T \|U_3^{(j)}(\cdot\,,t); W_2^1(\sigma_j)\|^2\, dt + \int_0^T \|U_{3t}^{(j)}(\cdot\,,t); L_2(\sigma_j)\|^2\, dt \right) \\ &\le \sum_{j=1}^{J} c(A_0^{(j)})(\|f_3^{(j)}; L_2(\sigma_j^T)\|^2 + \|u_{03}^{(j)}; W_2^1(\sigma_j)\|^2 + \|F_j; W_2^1(0,T)\|^2). \end{aligned} \tag{3.10}$$

Since $U_3^{(j)}(\cdot\,,t) \in \overset{\circ}{W}{}_2^1(\sigma_j)$, we get $g(\cdot\,,t) \in \overset{\circ}{W}{}_2^1(\Omega_{(3)})$. Therefore, there exits a vector-field $\mathbf{W}(\cdot\,,t) \in \overset{\circ}{W}{}_2^2(\Omega_{(3)})$ such that (see [2])

$$\operatorname{div} \mathbf{W}(x,t) = g(x,t),$$

and the following estimate

$$\|\mathbf{W}(\cdot\,,t); W_2^2(\Omega_{(3)})\|^2 \le c\|g(\cdot\,,t); W_2^1(\Omega_{(3)})\|^2 \tag{3.11}$$

holds. Note that for the proof of this result an explicit representation formula for the solution of the divergence equation was used (see [2]). This formula admits the differentiation with respect to t, i.e. for $\mathbf{W}_t(x,t)$ holds the same representation formula as for $\mathbf{W}(x,t)$ with $g(x,t)$ changed to $g_t(x,t)$ and it is easy to see that

$$\mathbf{W}_t(\cdot\,,t) \in \overset{\circ}{W}{}_2^1(\Omega_{(3)}), \quad \operatorname{div} \mathbf{W}_t(x,t) = g_t(x,t),$$

and

$$\|\mathbf{W}_t(\cdot,t);W_2^1(\Omega_{(3)})\|^2 \le c\|g_t(\,\cdot\,,t);L_2(\Omega_{(3)})\|^2. \tag{3.12}$$

Integrating (3.11), (3.12) with respect to t and using (3.10), we get

$$\begin{aligned}\int_0^T \|\mathbf{W}(\,\cdot\,,t);W_2^2(\Omega_{(3)})\|^2\,dt &+ \int_0^T \|\mathbf{W}_t(\,\cdot\,,t);W_2^1(\Omega_{(3)})\|^2\,dt\\ &\le c\left(\int_0^T \|g(\,\cdot\,,t);W_2^1(\Omega_{(3)})\|^2\,dt + \int_0^T \|g_t(\,\cdot\,,t);L_2(\Omega_{(3)})\|^2\,dt\right)\\ &\le \sum_{j=1}^J c(A_0^{(j)})(\|f_3^{(j)};L_2(\sigma_j^T)\|^2 + \|u_{03}^{(j)};W_2^1(\sigma_j)\|^2 + \|F_j;W_2^1(0,T)\|^2).\end{aligned}$$

Let $\mathbf{V}(x,t)=\mathbf{U}(x,t)+\mathbf{W}(x,t)$. Then,

$$\begin{gathered}\operatorname{div}\mathbf{V}(x,t)=0, \quad \mathbf{V}(x,t)|_{\partial\Omega}=0,\\ \int_{\sigma_j}\mathbf{V}(x,t)\cdot\mathbf{n}(x)\,ds = F_j(t), \quad j=1,\dots,J,\end{gathered}$$

and for $x\in\Omega_j\setminus\Omega_{j3}$, $j=1,\dots,J$, the vector-field $\mathbf{V}(x,t)$ coincides with the velocity part $\mathbf{U}^{(j)}(x^{(j)\prime},t)$ of the corresponding Poiseuille solution.

We look for the solution $(\mathbf{u}(x,t),p(x,t))$ of problem (1.1) in the form

$$\mathbf{u}(x,t)=\mathbf{v}(x,t)+\mathbf{V}(x,t), \quad p(x,t)=\widetilde{p}(x,t)+P(x,t), \tag{3.13}$$

where $P(x,t)$ is defined by the formula (3.9). Then we obtain for $(\mathbf{v}(x,t),\widetilde{p}(x,t))$ the following problem

$$\begin{cases}\mathbf{v}_t-\nu\Delta\mathbf{v}+(\mathbf{v}\cdot\nabla)\mathbf{v}+(\mathbf{V}\cdot\nabla)\mathbf{v}+(\mathbf{v}\cdot\nabla)\mathbf{V}+\nabla\widetilde{p}=\widetilde{\mathbf{f}},\\ \operatorname{div}\mathbf{v}(x,t)=0,\\ \mathbf{v}(x,t)|_{\partial\Omega}=0, \quad \mathbf{v}(x,0)=\widetilde{\mathbf{u}}_0(x),\\ \displaystyle\int_{\sigma_j}\mathbf{v}(x,t)\cdot\mathbf{n}(x)\,ds=0 \quad \text{for } j=1,\dots,J,\end{cases} \tag{3.14}$$

with

$$\begin{aligned}\widetilde{\mathbf{f}}(x,t)&=\widehat{\mathbf{f}}(x,t)+\mathbf{f}_{(1)}(x,t)+\mathbf{f}_{(2)}(x,t),\\ \widetilde{\mathbf{u}}_0(x)&=\widehat{\mathbf{u}}_0(x)-\mathbf{W}(x,0),\\ \mathbf{f}_{(1)}(x,t)&=(\mathbf{f}^{(1)\prime}(x,t),f_3^{(1)}(x,t)),\end{aligned}$$

$$
\begin{aligned}
\mathbf{f}_{(1)}{}'(x,t) = {} & \sum_{j=1}^{J} (\nu\zeta''(x_3^{(j)})\mathbf{U}^{(j)\prime}(x^{(j)\prime},\, t) \\
& - \zeta(x_3^{(j)})\zeta'(x_3^{(j)})U_3^{(j)}(x^{(j)\prime},\, t)\mathbf{U}^{(j)\prime}(x^{(j)\prime},\, t) \\
& - \zeta'(x_3^{(j)})(\zeta'(x_3^{(j)}) - 1)(\mathbf{U}^{(j)\prime}(x^{(j)\prime},\, t)\cdot\nabla')\mathbf{U}^{(j)\prime}(x^{(j)\prime},\, t)), \\
f_{(1)3}(x,t) = {} & \sum_{j=1}^{J} (\nu\zeta''(x_3^{(j)})U_3^{(j)}(x^{(j)\prime},\, t) - \zeta(x_3^{(j)})\zeta'(x_3^{(j)})|U_3^{(j)}(x^{(j)\prime},\, t)|^2 \\
& - \zeta'(x_3^{(j)})(\zeta'(x_3^{(j)}) - 1)(\mathbf{U}^{(j)\prime}(x^{(j)\prime},\, t)\cdot\nabla')U_3^{(j)}(x^{(j)\prime}, t) \\
& - \zeta'(x_3^{(j)})x_3^{(j)}q^{(j)}(t)), \\
\mathbf{f}_{(2)}(x,t) = {} & -\mathbf{W}_t(x,t) + \nu\Delta\mathbf{W}(x,t) - (\mathbf{W}(x,t)\cdot\nabla)\mathbf{W}(x,t) \\
& - (\mathbf{U}(x,t)\cdot\nabla)\mathbf{W}(x,t) - (\mathbf{W}(x,t)\cdot\nabla)\mathbf{U}(x,t).
\end{aligned}
$$

It easy to see that

$$
\operatorname{supp}_x(\mathbf{f}_{(1)}(x,t) + \mathbf{f}_{(2)}(x,t)) \subset \Omega_{(3)}.
$$

Using multiplicative inequalities (see Lemma 2.1), estimates (3.7), (3.8) for $\mathbf{U}^{(j)}(x^{(j)\prime}, t)$ and estimates (3.10)–(3.12) for $\mathbf{W}(x,t)$, we obtain the inequalities

$$
\begin{aligned}
(3.15)\quad \int_0^t\int_\Omega |\mathbf{f}_{(1)}(x,\tau)|^2\,dx\,d\tau \le {} & c\sum_{j=1}^{J}\int_0^t\int_{\Omega_{j2}} (|\mathbf{U}^{(j)}(x^{(j)\prime},\tau)|^2 + |\mathbf{U}^{(j)}(x^{(j)\prime},\tau)|^4 \\
& + |\mathbf{U}^{(j)}(x^{(j)\prime},\tau)|^2|\nabla'\mathbf{U}^{(j)}(x^{(j)\prime},\tau)|^2 + |q^{(j)}(\tau)|^2)\,dx\,d\tau \\
\le {} & c\sum_{j=1}^{J}\int_0^t\int_{\sigma_j} (|\mathbf{U}^{(j)}(x^{(j)\prime},\tau)|^2 + |q^{(j)}(\tau)|^2)\,dx^{(j)\prime}\,d\tau \\
& + c\sum_{j=1}^{J}\int_0^t \sup_{x^{(j)\prime}\in\overline{\sigma}_j} (|\mathbf{U}^{(j)}(x^{(j)\prime},\tau)|^2)\|\mathbf{U}^{(j)}(\,\cdot\,,\tau); W_2^1(\sigma_j)\|^2\,d\tau \\
\le {} & c(A_0 + A_1) + c\sum_{j=1}^{J}\sup_{t\in[0,T]} (\|\mathbf{U}^{(j)}(\,\cdot\,,\tau); W_2^1(\sigma_j)\|^2) \\
& \cdot\int_0^t \|\mathbf{U}^{(j)}(\,\cdot\,,\tau); W_2^2(\sigma_j)\|^2\,d\tau \\
\le {} & c\,(A_0 + A_1)(1 + A_0 + A_1) := c\,A_2,
\end{aligned}
$$

$$
\begin{aligned}
(3.16)\quad \int_0^t\int_\Omega |\mathbf{f}_{(2)}(x,\tau)|^2\,dx\,d\tau \le {} & c\int_0^t\int_{\Omega_{(3)}} \Big(|\mathbf{W}_t(x,\tau)|^2 + |\Delta\mathbf{W}(x,\tau)|^2 \\
& + (|\mathbf{W}(x,\tau)|^2 + \mathbf{U}(x,\tau)|^2)|\nabla\mathbf{W}(x,\tau)|^2 + |\mathbf{W}(x,\tau)|^2|\nabla\mathbf{U}(x,\tau)|^2\Big)\,dx\,d\tau \\
\le {} & c\,A_1 + c\int_0^t \Big\{\sup_{x\in\overline{\Omega}_{(3)}} (|\mathbf{W}(x,\tau)|^2 + |\mathbf{U}(x,\tau)|^2)\int_{\Omega_{(3)}} (|\nabla\mathbf{W}|^2 + |\nabla\mathbf{U}|^2)\,dx\Big\}\,d\tau
\end{aligned}
$$

$$\leq cA_1 + c\sup_{t\in[0,T]}\bigg[\|\mathbf{W}(\,\cdot\,,t);W_2^1(\Omega_{(3)})\|^2 + \sum_{j=1}^J \|\mathbf{U}^{(j)}(\,\cdot\,,t);W_2^1(\sigma_j)\|^2\bigg]$$

$$\cdot\int_0^t\bigg(\|\mathbf{W}(\,\cdot\,,\tau);W_2^2(\Omega_{(3)})\|^2 + \sum_{j=1}^J \|\mathbf{U}^{(j)}(\,\cdot\,,t);W_2^1(\sigma_j)\|^2\bigg)\,d\tau \leq c\,A_2.$$

In (3.15), (3.16)

$$A_0 = \sum_{j=1}^J A_0^{(j)}, \tag{3.17}$$

$$A_1 = \sum_{j=1}^J c(A_0^{(j)})(\|F_j;W_2^1(0,T)\|^2 + \|u_{03}^{(j)};W_2^1(\sigma_j)\|^2 + \|f_3^{(j)};W_2^1(\sigma_j^T)\|^2). \tag{3.18}$$

4. Linear problem

Consider in Ω^T the nonstationary Stokes problem assuming that all fluxes $F_j(t)$, $j=1,\ldots,J$, are equal to zero, i.e. consider the problem

$$\begin{cases} \mathbf{v}_t - \nu\Delta\mathbf{v} + \nabla p = \mathbf{f}, \\ \operatorname{div}\mathbf{v} = 0, \\ \mathbf{v}(x,t)|_{\partial\Omega} = 0, \quad \mathbf{v}(x,0) = \mathbf{v}_0(x), \\ \displaystyle\int_{\sigma_j}\mathbf{v}(x,t)\cdot\mathbf{n}(x)\,ds = 0, \quad \text{for } j=1,\ldots,J. \end{cases} \tag{4.1}$$

The following theorem is proved in [8].

Theorem 4.1. *Let* $\partial\Omega\in C^2$, $\mathbf{f}\in\mathcal{L}_{2,\beta}(\Omega^T)$, $\mathbf{v}_0\in\mathcal{W}_{2,\beta}^1(\Omega)\cap\overset{\circ}{W}{}_2^1(\Omega), \beta_j\geq 0$, $j=1,\ldots,J$, $T\in(0,\infty]$, *and let there hold the compatibility conditions*

$$\operatorname{div}\mathbf{v}_0 = 0, \quad \int_{\sigma_j}\mathbf{v}_0(x)\cdot\mathbf{n}(x)\,dS = 0, \quad j=1,\ldots,J. \tag{4.2}$$

If the number γ_* *in the inequality* (2.6) *for the weight-function* $E_\beta(x)$ *is sufficiently small, then there exists a unique solution* $(\mathbf{v}(x,t),p(x,t))$ *of problem* (4.1) *such that* $\mathbf{v}\in\mathcal{W}_{2,\beta}^{2,1}(\Omega^T)$, $\nabla p\in\mathcal{L}_{2,\beta}(\Omega^T)$ *and there holds the estimate*

$$\sup_{t\in[0,T]}\|\mathbf{v}(\,\cdot\,,t);\mathcal{W}_{2,\beta}^1(\Omega)\| + \|\mathbf{v};\mathcal{W}_{2,\beta}^{2,1}(\Omega^T)\| + \|\nabla p;\mathcal{L}_{2,\beta}(\Omega^T)\| \leq c(\|\mathbf{v}_0;\mathcal{W}_{2,\beta}^1(\Omega)\| + \|\mathbf{f};\mathcal{L}_{2,\beta}(\Omega^T)\|). \tag{4.3}$$

The constant in (4.3) *is independent of* T.

Remark 4.2. Estimate (4.2) is proved in [8] for positive β_j. However, it is easy to see from the proofs in [8] that this estimate remains valid for negative β_j. More precisely, if the number γ_* in the inequality (2.6) for the weight-function $E_\beta(x)$ is sufficiently small and if there exists a solution $(\mathbf{v}(x,t),p(x,t))$

to problem (4.1) such that $\mathbf{v} \in \mathcal{W}^{2,1}_{2,-\beta}(\Omega^T)$, $\nabla p \in \mathcal{L}_{2,-\beta}(\Omega^T)$, $\beta_j > 0$, $j = 1,\dots,J$,

$$\int_{\sigma_j} \mathbf{v}(x,t)\cdot\mathbf{n}(x)\,dS = 0, \quad j = 1,\dots,J,$$

then for this solution holds the estimate

$$\begin{aligned}(4.4)\qquad &\sup_{\tau\in[0,t]} \|\mathbf{v}(\,\cdot\,,\tau);\mathcal{W}^1_{2,-\beta}(\Omega)\| + \|\mathbf{v};\mathcal{W}^{2,1}_{2,-\beta}(\Omega^t)\| + \|\nabla p;\mathcal{L}_{2,-\beta}(\Omega^t)\| \\ &\qquad \le c(\|\mathbf{v}_0;\mathcal{W}^1_{2,-\beta}(\Omega)\| + \|\mathbf{f};\mathcal{L}_{2,-\beta}(\Omega^t)\|),\end{aligned}$$

for all $t \in [0,T]$, where the constant c is independent of t and T.

5. Estimates of the nonlinear terms

Lemma 5.1. *Let $\mathbf{v} \in \mathcal{W}^{2,1}_{2,\beta}(\Omega^T)$, $\beta_j \ge 0$, $j = 1,\dots,J$, $T \in (0,\infty]$. Then $(\mathbf{v}\cdot\nabla)\mathbf{v} \in \mathcal{L}_{2,\beta}(\Omega^T)$ and*

$$\begin{aligned}(5.1)\qquad &\int_0^t \|(\mathbf{v}(\cdot,\tau)\cdot\nabla)\mathbf{v}(\,\cdot\,,\tau);\mathcal{L}_{2,\beta}(\Omega)\|^2\,d\tau \\ &\qquad \le c\min\{1,T^{1/2}\}\|\mathbf{v};\mathcal{W}^{2,1}_{2,\beta}(\Omega\times(0,t))\|^4,\end{aligned}$$

for all $t \in [0,T]$, where the constant c is independent of $t \in [0,T]$ and T.

Proof. First of all we mention that the condition $\mathbf{v} \in W^{2,1}_2(\Omega^T)$ implies $\mathbf{v}(\,\cdot\,,t) \in W^1_2(\Omega)$. Denote $I_{js}(t) = \|(\mathbf{v}(\,\cdot\,,t)\cdot\nabla)\mathbf{v}(\,\cdot\,,t);L_2(\omega_{js})\|^2$. Then by Hölder inequality and by multiplicative inequalities (2.2), (2.3)

$$\begin{aligned}(5.2)\quad I_{js}(t) &\le \|\mathbf{v}(\,\cdot\,,t);L_6(\omega_{js})\|^2\|\nabla\mathbf{v}(\,\cdot\,,t);L_3(\omega_{js})\|^2 \\ &\le c\|\mathbf{v}(\,\cdot\,,t);W^1_2(\omega_{js})\|^2\|\nabla\mathbf{v}(\,\cdot\,,t);W^1_2(\omega_{js})\|\|\nabla\mathbf{v}(\,\cdot\,,t);L_2(\omega_{js})\|.\end{aligned}$$

Therefore,

$$\begin{aligned}(5.3)\qquad \int_0^t I_{js}(\tau)\,d\tau &\le c\sup_{\tau\in(0,t)}\|\mathbf{v}(\,\cdot\,,\tau);W^1_2(\omega_{js})\|^2\int_0^t\|\nabla\mathbf{v}(\,\cdot\,,\tau);W^1_2(\omega_{js})\|^2\,d\tau \\ &\le c\sup_{\tau\in(0,t)}\|\mathbf{v}(\,\cdot\,,\tau);W^1_2(\Omega)\|^2\int_0^t\|\mathbf{v}(\,\cdot\,,\tau);W^2_2(\omega_{js})\|^2\,d\tau \\ &\le c\|\mathbf{v};W^{2,1}_2(\Omega\times(0,t))\|^2\|\mathbf{v};W^{2,1}_2(\omega_{js}\times(0,t))\|^2.\end{aligned}$$

On the other hand,

$$\begin{aligned}&\int_0^t \|\nabla\mathbf{v}(\,\cdot\,,\tau);W^1_2(\omega_{js})\|\|\nabla\mathbf{v}(\,\cdot\,,\tau);L_2(\omega_{js})\|\,d\tau \\ &\quad\le \sup_{\tau\in(0,t)}\|\mathbf{v}(\,\cdot\,,\tau);W^1_2(\omega_{js})\|\,t^{1/2}\bigg(\int_0^t\|\mathbf{v}(\,\cdot\,,\tau);W^2_2(\omega_{js})\|^2\,d\tau\bigg)^{1/2} \\ &\quad\le c\,T^{1/2}\|\mathbf{v};W^{2,1}_2(\omega_{js}\times(0,t))\|^2\end{aligned}$$

and we derive from (5.2)

$$(5.4)\quad \int_0^t I_{js}(\tau)\,d\tau \le c\,T^{1/2}\sup_{\tau\in(0,t)}\|\mathbf{v}(\,\cdot\,,\tau);W_2^1(\Omega)\|^2\|\mathbf{v};W_2^{2,1}(\omega_{js}\times(0,t))\|^2$$
$$\le c\,T^{1/2}\|\mathbf{v};W_2^{2,1}(\Omega\times(0,t))\|^2\|\mathbf{v};W_2^{2,1}(\omega_{js}\times(0,t))\|^2.$$

Inequalities (5.3) and (5.4) yield

$$(5.5)\quad \int_0^t \|(\mathbf{v}(\,\cdot\,,\tau)\cdot\nabla)\mathbf{v}(\,\cdot\,,\tau);L_2(\omega_{js})\|^2\,d\tau$$
$$\le c\min\{1,T^{1/2}\}\|\mathbf{v};W_2^{2,1}(\Omega\times(0,t))\|^2\|\mathbf{v};W_2^{2,1}(\omega_{js}\times(0,t))\|^2.$$

Obviously, the constant c in (5.5) does not depend on s. Multiplying inequalities (5.5) by $E_{\beta_j}(s)$ and summing obtained relations over s from 0 to ∞, we get in virtue of properties (2.4)–(2.7) of the weight-function $E_\beta(x)$

$$(5.6)\quad \int_0^t \|(\mathbf{v}(\,\cdot\,,\tau)\cdot\nabla)\mathbf{v}(\,\cdot\,,\tau);\mathcal{L}_{2,\beta_j}(\Omega_j)\|^2\,d\tau$$
$$\le c\min\{1,T^{1/2}\}\|\mathbf{v};W_2^{2,1}(\Omega\times(0,t))\|^2\|\mathbf{v};\mathcal{W}_{2,\beta_j}^{2,1}(\Omega_j\times(0,t))\|^2$$
$$\le c\min\{1,T^{1/2}\}\|\mathbf{v};\mathcal{W}_{2,\beta}^{2,1}(\Omega\times(0,t))\|^4.$$

Analogously, could be proved that

$$(5.7)\quad \int_0^t \|(\mathbf{v}(\,\cdot\,,\tau)\cdot\nabla)\mathbf{v}(\,\cdot\,,\tau);L_2(\Omega_{(3)})\|^2\,d\tau$$
$$\le c\min\{1,T^{1/2}\}\|\mathbf{v};W_2^{2,1}(\Omega\times(0,t))\|^2\|\mathbf{v};W_2^{2,1}(\Omega_{(3)}\times(0,t))\|^2.$$

Inequality (5.1) follows from (5.6), (5.7). □

Lemma 5.2. *Let* $\mathbf{v},\mathbf{u}\in\mathcal{W}_{2,\beta}^{2,1}(\Omega^T)$, $\beta_j\ge 0$, $j=1,\dots,J$, $T\in(0,\infty]$. *Then* $(\mathbf{u}\cdot\nabla)\mathbf{v}\in\mathcal{L}_{2,\beta}(\Omega^T)$ *and there hold the estimates*

$$(5.8)\quad \int_0^t \|(\mathbf{u}(\,\cdot\,,\tau)\cdot\nabla)\mathbf{v}(\,\cdot\,,\tau);\mathcal{L}_{2,\beta}(\Omega)\|^2\,d\tau \le c_\varepsilon\int_0^t\|\mathbf{u};\mathcal{W}_{2,\beta}^{2,1}(\Omega\times(0,\tau))\|^2\,d\tau$$
$$+\varepsilon c\,\|\mathbf{v};\mathcal{W}_{2,\beta}^{2,1}(\Omega\times(0,t))\|^4\|\mathbf{u};\mathcal{W}_{2,\beta}^{2,1}(\Omega\times(0,t))\|^2,$$

and

$$(5.9)\quad \int_0^t \|(\mathbf{u}(\,\cdot\,,\tau)\cdot\nabla)\mathbf{v}(\,\cdot\,,\tau);\mathcal{L}_{2,\beta}(\Omega)\|^2\,d\tau \le c_\varepsilon\int_0^t\|\mathbf{v};\mathcal{W}_{2,\beta}^{2,1}(\Omega\times(0,\tau))\|^2\,d\tau$$
$$+\varepsilon c\,\|\mathbf{u};\mathcal{W}_{2,\beta}^{2,1}(\Omega\times(0,t))\|^4\|\mathbf{v};\mathcal{W}_{2,\beta}^{2,1}(\Omega\times(0,t))\|^2.$$

Constants in (5.8), (5.9) *are independent of* $t\in[0,T]$ *and* T.

Proof. As in estimates (5.2), (5.3) we obtain that

$$\begin{aligned} N_{js}(t) &= \int_0^t \int_{\omega_{js}} |\mathbf{u}(x,\tau)|^2 |\nabla \mathbf{v}(x,\tau)|^2 \, dx \, d\tau \\ &\le c \int_0^t \|\mathbf{u}(\,\cdot\,,\tau); W_2^1(\omega_{js})\|^2 \|\mathbf{v}(\,\cdot\,,\tau); W_2^1(\omega_{js})\| \|\mathbf{v}(\,\cdot\,,\tau); W_2^2(\omega_{js})\| \, d\tau. \end{aligned}$$

Therefore,

$$\begin{aligned} N_{js}(t) &\le c \sup_{\tau\in(0,t)} \|\mathbf{v}(\,\cdot\,,\tau); W_2^1(\omega_{js})\| \sup_{\tau\in(0,t)} \|\mathbf{u}(\,\cdot\,,\tau); W_2^1(\omega_{js})\| \\ &\quad \cdot \int_0^t \|\mathbf{u}(\,\cdot\,,\tau); W_2^1(\omega_{js})\| \|\mathbf{v}(\,\cdot\,,\tau); W_2^2(\omega_{js})\| \, d\tau \\ &\le c \|\mathbf{v}; W_2^{2,1}(\Omega^T)\| \|\mathbf{u}; W_2^{2,1}(\omega_{js} \times (0,t))\| \\ &\quad \cdot \bigg(\int_0^t \|\mathbf{u}(\,\cdot\,,\tau); W_2^1(\omega_{js})\|^2 \, d\tau \bigg)^{1/2} \bigg(\int_0^t \|\mathbf{v}(\,\cdot\,,\tau); W_2^2(\omega_{js})\|^2 \, d\tau \bigg)^{1/2} \\ &\le c \|\mathbf{v};\, W_2^{2,1}(\Omega^T)\|^2 \| \|\mathbf{u}; W_2^{2,1}(\omega_{js} \times (0,t))\| \\ &\quad \cdot \bigg(\int_0^t \|\mathbf{u}; W_2^{2,1}(\omega_{js} \times (0,\tau))\|^2 \, d\tau \bigg)^{1/2} \\ &\le \varepsilon c \|\mathbf{v};\, \mathcal{W}_{2,\beta}^{2,1}(\Omega^T)\|^4 \|\mathbf{u}; W_2^{2,1}(\omega_{js} \times (0,t))\|^2 \\ &\quad + c_\varepsilon \int_0^t \|\mathbf{u}; W_2^{2,1}(\omega_{js} \times (0,\tau))\|^2 \, d\tau \end{aligned}$$

and

$$\begin{aligned} N_{js}(t) &\le c \sup_{\tau\in(0,t)} \|\mathbf{u}(\,\cdot\,,\tau); W_2^1(\omega_{js})\|^2 \\ &\quad \cdot \int_0^t \|\mathbf{v}(\,\cdot\,,\tau); W_2^1(\omega_{js})\| \|\mathbf{v}(\,\cdot\,,\tau);\, W_2^2(\omega_{js})\| \, d\tau \\ &\le \varepsilon c \|\mathbf{u}; \mathcal{W}_{2,\beta}^{2,1}(\Omega^T)\|^4 \|\mathbf{v}; W_2^{2,1}(\omega_{js} \times (0,t))\|^2 \\ &\quad + c_\varepsilon \int_0^t \|\mathbf{v}; W_2^{2,1}(\omega_{js} \times (0,\tau))\|^2 \, d\tau. \end{aligned}$$

Multiplying these relations by $E_{\beta_j}(s)$ and then summing over s furnishes

$$\begin{aligned} \int_0^t \int_{\Omega_j} E_{\beta_j}(x) |\mathbf{u}(x,\tau)|^2 |\nabla \mathbf{v}(x,\tau)|^2 \, dx \, d\tau & \\ \le \varepsilon c \|\mathbf{v}; \mathcal{W}_{2,\beta}^{2,1}(\Omega^T)\|^4 \|\mathbf{u}; \mathcal{W}_{2,\beta}^{2,1}(\Omega_j \times (0,t))\|^2 & \\ + c_\varepsilon \int_0^t \|\mathbf{u};\, \mathcal{W}_{2,\beta}^{2,1}(\Omega_j \times (0,\tau))\|^2 \, d\tau, & \end{aligned}$$

$$\int_0^t \int_{\Omega_j} E_{\beta_j}(x)|\mathbf{u}(x,\tau)|^2|\nabla \mathbf{v}(x,\tau)|^2\,dx\,d\tau$$
$$\le \varepsilon c\|\mathbf{u}; \mathcal{W}^{2,1}_{2,\beta}(\Omega^T)\|^4\|\mathbf{v}; \mathcal{W}^{2,1}_{2,\beta}(\Omega_j\times(0,t))\|^2 + c_\varepsilon\int_0^t \|\mathbf{v};\, \mathcal{W}^{2,1}_{2,\beta}(\Omega_j\times(0,\tau))\|^2\,d\tau.$$

Analogously could be proved that

$$\int_0^t \int_{\Omega_{(3)}} |\mathbf{u}(x,\tau)|^2|\nabla \mathbf{v}(x,\tau)|^2\,dx\,d\tau$$
$$\le \varepsilon c\|\mathbf{v}; \mathcal{W}^{2,1}_{2,\beta}(\Omega^T)\|^4\|\mathbf{u}; W^{2,1}_2(\Omega_{(3)}\times(0,t))\|^2 + c_\varepsilon\int_0^t \|\mathbf{u}; W^{2,1}_2(\Omega_{(3)}\times(0,\tau))\|^2\,d\tau,$$
$$\int_0^t \int_{\Omega_{(3)}} |\mathbf{u}(x,\tau)|^2|\nabla \mathbf{v}(x,\tau)|^2\,dx\,d\tau$$
$$\le \varepsilon c\|\mathbf{u}; \mathcal{W}^{2,1}_{2,\beta}(\Omega^T)\|^4\|\mathbf{v}; W^{2,1}_2(\Omega_{(3)}\times(0,t))\|^2 + c_\varepsilon\int_0^t \|\mathbf{v}; W^{2,1}_2(\Omega_{(3)}\times(0,\tau))\|^2\,d\tau$$

and, therefore, we get (5.8) and (5.9). □

Lemma 5.3. *Let* $\mathbf{v}\in \mathcal{W}^{2,1}_{2,\beta}(\Omega^T)$, $T\in(0,\infty]$. *Then* $(\mathbf{V}\cdot\nabla)\mathbf{v}\in\mathcal{L}_{2,\beta}(\Omega^T)$, $(\mathbf{v}\cdot\nabla)\mathbf{V}\in\mathcal{L}_{2,\beta}(\Omega^T)$ *and*

$$\int_0^t \|(\mathbf{V}(\,\cdot\,,\tau)\cdot\nabla)\mathbf{v}(\,\cdot\,,\tau);\mathcal{L}_{2,\beta}(\Omega)\|^2\,d\tau + \int_0^t \|(\mathbf{v}(\,\cdot\,,\tau)\cdot\nabla)\mathbf{V}(\,\cdot\,,\tau);\mathcal{L}_{2,\beta}(\Omega)\|^2\,d\tau \le c\min\{1,T^{1/2}\}(A_0+A_1)\|\mathbf{v};\mathcal{W}^{2,1}_{2,\beta}(\Omega\times(0,t))\|^2 \tag{5.10}$$

for all $t\in[0,T]$, *where the constant* c *is independent of* $t\in[0,T]$ *and* T.

Proof. Arguing as in Lemma 5.1, we get for

$$J_{js}(t)=\|(\mathbf{V}(\,\cdot\,,t)\cdot\nabla)\mathbf{v}(\,\cdot\,,t);L_2(\omega_{js})\|^2$$

the estimates

$$\int_0^t J_{js}(\tau)\,d\tau \le c\sup_{\tau\in(0,t)}\|\mathbf{V}(\,\cdot\,,\tau);W^1_2(\omega_{js})\|^2 \cdot\int_0^t \|\nabla\mathbf{v}(\,\cdot\,,\tau);W^1_2(\omega_{js})\|\|\nabla\mathbf{v}(\,\cdot\,,\tau);L_2(\omega_{js})\|\,d\tau$$
$$\le c\sup_{\tau\in(0,t)}\|\mathbf{V}(\,\cdot\,,\tau);W^1_2(\omega_{js})\|^2\|\mathbf{v};W^{2,1}_2(\omega_{js}\times(0,t))\|^2$$

and

$$\int_0^t J_{js}(\tau)\,d\tau \le c \sup_{\tau\in(0,t)} \|\mathbf{V}(\,\cdot\,,\tau);W_2^1(\omega_{js})\|^2 \sup_{\tau\in(0,t)} \|\mathbf{v}(\,\cdot\,,\tau);W_2^1(\omega_{js})\|\, t^{1/2} \cdot \biggl(\int_0^t \|\nabla\mathbf{v}(\,\cdot\,,\tau);W_2^1(\omega_{js})\|^2\,d\tau\biggr)^{1/2} \le c\,T^{1/2} \sup_{\tau\in(0,t)} \|\mathbf{V}(\,\cdot\,,\tau);W_2^1(\omega_{js})\|^2 \|\mathbf{v};W_2^{2,1}(\omega_{js}\times(0,t))\|^2.$$

Therefore,

$$\int_0^t J_{js}(\tau)\,d\tau \le c\min\{1,T^{1/2}\} \cdot \sup_{\tau\in(0,t)} \|\mathbf{V}(\,\cdot\,,\tau);W_2^1(\omega_{js})\|^2 \|\mathbf{v};W_2^{2,1}(\omega_{js}\times(0,t))\|^2. \tag{5.11}$$

It follows from (3.7), (3.8) and from (3.10)–(3.12) that

$$\begin{aligned}\sup_{\tau\in(0,t)} &\|\mathbf{V}(\,\cdot\,,\tau);W_2^1(\omega_{js})\|^2 \\ &\le c\bigl(\sup_{\tau\in(0,t)} \|\mathbf{U}^{(j)}(\,\cdot\,,\tau);W_2^1(\omega_{js})\|^2 + \sup_{\tau\in(0,t)} \|\mathbf{W}(\,\cdot\,,\tau);W_2^1(\omega_{js})\|^2\bigr) \\ &\le c\bigl(\sup_{\tau\in(0,t)} \|\mathbf{U}^{(j)}(\,\cdot\,,\tau);W_2^1(\sigma_j)\|^2 + \sup_{\tau\in(0,t)} \|\mathbf{W}(\,\cdot\,,\tau);W_2^1(\Omega_{(3)})\|^2\bigr) \\ &\le c(\|\mathbf{U}^{(j)};W_2^{2,1}(\sigma_j^T)\|^2 + \|\mathbf{W};W_2^{2,1}(\Omega_{(3)}^T)\|^2\bigr) \le c(A_0+A_1).\end{aligned}$$

Thus, (5.11) yields

$$\int_0^t J_{js}(\tau)\,d\tau \le c\min\{1,T^{1/2}\}(A_0+A_1)\,\|\mathbf{v};W_2^{2,1}(\omega_{js}\times(0,t))\|^2. \tag{5.12}$$

Multiplying (5.12) by $E_{\beta_j}(s)$ and then summing over s, we derive

$$\int_0^t \|(\mathbf{V}(\,\cdot\,,\tau)\cdot\nabla)\mathbf{v}(\,\cdot\,,\tau);\mathcal{L}_{2,\beta_j}(\Omega_j)\|^2\,d\tau \le c\min\{1,T^{1/2}\}(A_0+A_1)\|\mathbf{v};\mathcal{W}_{2,\beta_j}^{2,1}(\Omega_j\times(0,t))\|^2. \tag{5.13}$$

Analogously,

$$\int_0^t \|(\mathbf{V}(\,\cdot\,,\tau)\cdot\nabla)\mathbf{v}(\,\cdot\,,\tau);L_2(\Omega_{(3)})\|^2\,d\tau \le c\min\{1,T^{1/2}\}(A_0+A_1)\|\mathbf{v};W_2^{2,1}(\Omega_{(3)}\times(0,t))\|^2. \tag{5.14}$$

It follows from (5.13) and (5.14) that

$$\int_0^t \|(\mathbf{V}(\,\cdot\,,\tau)\cdot\nabla)\mathbf{v}(\,\cdot\,,\tau);\mathcal{L}_{2,\beta}(\Omega)\|^2\,d\tau \le c\min\{1,T^{1/2}\}(A_0+A_1)\|\mathbf{v};\mathcal{W}_{2,\beta}^{2,1}(\Omega\times(0,t))\|^2.$$

Consider now the integrals $K_{js}(t) = \|(\mathbf{v}(\,\cdot\,,t)\cdot\nabla)\mathbf{V}(\,\cdot\,,t); L_2(\omega_{js})\|^2$. Using (2.2), we get

$$\begin{aligned} K_{js}(t) &\leq \|\mathbf{v}(\,\cdot\,,t); L_\infty(\omega_{js})\|^2 \|\nabla\mathbf{V}(\,\cdot\,,t); L_2(\omega_{js})\|^2 \\ &\leq c\,\|\nabla\mathbf{v}(\,\cdot\,,t); W_2^1(\omega_{js})\|\|\mathbf{v}(\,\cdot\,,t); W_2^1(\omega_{js})\|\|\nabla\mathbf{V}(\,\cdot\,,t); L_2(\omega_{js})\|^2. \end{aligned}$$

Hence,

$$\begin{aligned} \int_0^t K_{js}(\tau)\,d\tau &\leq c \sup_{\tau\in(0,t)} \|\nabla\mathbf{V}(\,\cdot\,,\tau); L_2(\omega_{js})\|^2 \\ &\quad\cdot \int_0^t \|\nabla\mathbf{v}(\,\cdot\,,\tau); W_2^1(\omega_{js})\|\|\mathbf{v}(\,\cdot\,,\tau); W_2^1(\omega_{js})\|\,d\tau \\ &\leq c(A_0+A_1)\int_0^t \|\mathbf{v}(\,\cdot\,,\tau); W_2^2(\omega_{js})\|^2\,d\tau \\ &\leq c\,(A_0+A_1)\|\mathbf{v}; W_2^{2,1}(\omega_{js}\times(0,t))\|^2. \end{aligned}$$

On the other hand,

$$\begin{aligned} \int_0^t K_{js}(\tau)\,d\tau &\leq c(A_0+A_1) \\ &\quad\cdot \sup_{\tau\in(0,t)} \|\mathbf{v}(\,\cdot\,,\tau); W_2^1(\omega_{js})\| \int_0^t \|\mathbf{v}(\,\cdot\,,\tau); W_2^2(\omega_{js})\|\,d\tau \\ &\leq c\,(A_0+A_1)\|\mathbf{v}; W_2^{2,1}(\omega_{js}\times(0,t))\|T^{1/2} \\ &\quad\cdot\left(\int_0^t \|\mathbf{v}(\,\cdot\,,\tau); W_2^2(\omega_{js})\|^2\,d\tau\right)^{1/2} \\ &\leq c\,(A_0+A_1)\,T^{1/2}\|\mathbf{v}; W_2^{2,1}(\omega_{js}\times(0,t))\|^2. \end{aligned}$$

Therefore,

$$\begin{aligned} &\int_0^t \|(\mathbf{v}(\,\cdot\,,\tau)\cdot\nabla)\mathbf{V}(\,\cdot\,,\tau); L_2(\omega_{js})\|^2\,d\tau \\ &\qquad\leq c\,(A_0+A_1)\min\{1,\,T^{1/2}\}\|\mathbf{v}; W_2^{2,1}(\omega_{js}\times(0,t))\|^2 \end{aligned}$$

and, as above, we derive

$$\begin{aligned} &\int_0^t \|(\mathbf{v}(\,\cdot\,,\tau)\cdot\nabla)\mathbf{V}(\,\cdot\,,\tau); \mathcal{L}_{2,\beta_j}(\Omega_j)\|^2\,d\tau \\ &\qquad\leq c\,(A_0+A_1)\min\{1,\,T^{1/2}\}\|\mathbf{v}; \mathcal{W}_{2,\beta_j}^{2,1}(\Omega_j\times(0,t))\|^2. \end{aligned}$$

It could be proved analogously that

$$\begin{aligned} &\int_0^t \|(\mathbf{v}(\,\cdot\,,\tau)\cdot\nabla)\mathbf{V}(\,\cdot\,,\tau); L_2(\Omega_{(3)})\|^2\,d\tau \\ &\qquad\leq c\,(A_0+A_1)\min\{1,\,T^{1/2}\}\|\mathbf{v}; W_2^{2,1}(\Omega_{(3)}\times(0,t))\|^2 \end{aligned}$$

and, therefore,

$$\int_0^t \|(\mathbf{v}(\,\cdot\,,\tau)\cdot\nabla)\mathbf{V}(\,\cdot\,,\tau);\mathcal{L}_{2,\beta}(\Omega)\|^2\,d\tau \le c\,(A_0+A_1)\min\{1,\,T^{1/2}\}\|\mathbf{v};\mathcal{W}^{2,1}_{2,\beta}(\Omega\times(0,t))\|^2. \qquad \square$$

By considerations similar to those of Lemmats 5.2 and 5.3, we prove also the following

Lemma 5.4. *Let $\mathbf{v}\in\mathcal{W}^{2,1}_{2,\beta}(\Omega^T)$, $T\in(0,\infty]$. Then there holds the estimate*

$$\int_0^t \|(\mathbf{V}(\,\cdot\,,\tau)\cdot\nabla)\mathbf{v}(\,\cdot\,,\tau);\mathcal{L}_{2,\beta}(\Omega)\|^2\,d\tau+\int_0^t \|(\mathbf{v}(\,\cdot\,,\tau)\cdot\nabla)\mathbf{V}(\,\cdot\,,\tau);\mathcal{L}_{2,\beta}(\Omega)\|^2\,d\tau \le \varepsilon c(A_0^2+A_1^2)\|\mathbf{v};\mathcal{W}^{2,1}_{2,\beta}(\Omega\times(0,t))\|^2+c_\varepsilon\int_0^t\|\mathbf{v};\mathcal{W}^{2,1}_{2,\beta}(\Omega\times(0,\tau))\|^2d\tau,$$

where the constant c is independent of $t\in[0,T]$ and T.

6. Solvability of problem (3.14)

Theorem 6.1. *Let $\partial\Omega\in C^2$, $F_j\in W^1_2(0,\,T)$, $\operatorname{div}\mathbf{u}_0=0$ and let the initial data $\mathbf{u}_0(x)$ and the external force $\mathbf{f}(x,t)$ are represented in the form* (3.1) *and* (3.2) *with*

$$\begin{aligned}
\widehat{\mathbf{u}}_0 &\in \mathcal{W}^2_{2,\beta}(\Omega)\cap\overset{\circ}{W}{}^1_2(\Omega), &\quad \widehat{\mathbf{f}}&\in\mathcal{L}_{2,\beta}(\Omega^T), &\quad \beta_j\ge 0,\ j=1,\dots,J,\\
\mathbf{u}_0^{(j)} &\in \overset{\circ}{W}{}^1_2(\sigma_j), & \mathbf{f}^{(j)}&\in L_2(\sigma_j^T), & j=1,\dots,J.
\end{aligned}$$

Moreover, assume that there hold compatibility conditions (3.4) *and that the number γ_* in inequality* (2.6) *for the weight function $E_\beta(x)$ is sufficiently small* (*such that the conditions of Theorem* 4.1 *are valid*). *If*

$$\begin{aligned}
&c_2\min\{1,T^{1/2}\}(A_0+A_1)\,<1,\\
&4c_1c_2\sum_{k=0}^{3}A_k\min\{1,T^{1/2}\}\,<(1-c_2\min\{1,T^{1/2}\}(A_0+A_1))^2,
\end{aligned}\tag{6.1}$$

where A_0, A_1, A_2 are defined in (3.15), (3.17), (3.18),

$$A_3=\|\widehat{\mathbf{f}};\mathcal{L}_{2,\beta}(\Omega^T)\|^2+\|\widehat{\mathbf{u}_0};\mathcal{W}^1_{2,\beta}(\Omega)\|^2,$$

c_1, c_2 are absolute constants defined below, then problem (3.14) *admits a unique solution $\mathbf{v}\in\mathcal{W}^{2,1}_{2,\beta}(\Omega^T)$, $\nabla\widetilde{p}\in\mathcal{L}_{2,\beta}(\Omega^T)$. There holds the estimate*

$$\|\mathbf{v};\mathcal{W}^{2,1}_{2,\beta}(\Omega^T)\|^2+\|\nabla\widetilde{p};\mathcal{L}_{2,\beta}(\Omega^T)\|^2\le cr_0,\tag{6.2}$$

with

$$
(6.3) \qquad \begin{aligned}
r_0 &= \frac{2\alpha_0}{1-\alpha_1+\sqrt{(1-\alpha_1)^2-4\alpha_0\alpha_2}},\\
\alpha_0 &= c_1\sum_{k=0}^{3} A_k,\\
\alpha_1 &= c_2\min\{1, T^{1/2}\}(A_0+A_1),\\
\alpha_2 &= c_2\min\{1, T^{1/2}\}.
\end{aligned}
$$

PROOF. We prove the existence of the solution to problem (3.14) by the method of successive approximations using the scheme proposed by V. A. Solonnikov [16]. We consider problem (3.14) as a linear time dependent Stokes problem

$$
\begin{cases}
\mathbf{v}_t(x,t) - \nu\Delta\mathbf{v}(x,t) + \nabla\widetilde{p}(x,t) = \mathbf{g}(x,t;\mathbf{v}(x,t)),\\
\operatorname{div}\mathbf{v}(x,t) = 0,\\
\mathbf{v}(x,t)|_{\partial\Omega} = 0, \quad \mathbf{v}(x,0) = \widetilde{\mathbf{u}}_0(x),\\
\displaystyle\int_{\sigma_j} \mathbf{v}(x,t)\cdot\mathbf{n}(x)\,ds = 0, \quad j = 1,\dots,J,
\end{cases}
$$

with

$$
\begin{aligned}
\mathbf{g}(x,t,\mathbf{v}(x,t)) = {}&\widetilde{\mathbf{f}}(x,t) - (\mathbf{v}(x,t)\cdot\nabla)\mathbf{v}(x,t)\\
&- (\mathbf{V}(x,t)\cdot\nabla)\mathbf{v}(x,t) - (\mathbf{v}(x,t)\cdot\nabla)\mathbf{V}(x,t).
\end{aligned}
$$

Let us put $\mathbf{v}^{(0)}(x,t) = 0$, $\widetilde{p}^{(0)}(x,t) = 0$, and define the successive approximations recurrently as solutions of linear problems

$$
\begin{cases}
\mathbf{v}_t^{(l+1)}(x,t) - \nu\Delta\mathbf{v}^{(l+1)}(x,t) + \nabla\widetilde{p}^{(l+1)}(x,t) = \mathbf{g}(x,t;\mathbf{v}^{(l)}(x,t)),\\
\operatorname{div}\mathbf{v}^{(l+1)}(x,t) = 0,\\
\mathbf{v}^{(l+1)}(x,t)|_{\partial\Omega} = 0, \quad \mathbf{v}^{(l+1)}(x,0) = \widetilde{\mathbf{u}}_0(x),\\
\displaystyle\int_{\sigma_j} \mathbf{v}^{(l+1)}(x,t)\cdot\mathbf{n}(x)\,ds = 0, \quad j = 1,\dots,J.
\end{cases}
$$

In virtue of (3.15), (3.16), (5.1) (5.10) and (3.10)–(3.12), the right-hand sides $\mathbf{g}(x,t,\mathbf{v}^{(l)}(x,t))$ and $\widetilde{\mathbf{u}}_0(x)$ admit the estimates

$$
\begin{aligned}
\|\mathbf{g}(x,t;&\mathbf{v}^{(l)}(x,t));\mathcal{L}_{2,\beta}(\Omega^T)\|^2 \le c\,\|\widetilde{\mathbf{f}};\mathcal{L}_{2,\beta}(\Omega^T)\|^2\\
&+ c\min\{1,T^{1/2}\}(\|\mathbf{v}^{(l)};\mathcal{W}^{2,1}_{2,\beta}(\Omega^T)\|^4 + (A_0+A_1)\|\mathbf{v}^{(l)};\mathcal{W}^{2,1}_{2,\beta}(\Omega^T)\|^2)\\
\le{}& c\,(\|\widehat{\mathbf{f}};\mathcal{L}_{2,\beta}(\Omega^T)\|^2 + \|\mathbf{f}^{(1)};\mathcal{L}_{2,\beta}(\Omega^T)\|^2 + \|\mathbf{f}^{(2)};\mathcal{L}_{2,\beta}(\Omega^T)\|^2)\\
&+ c\min\{1,T^{1/2}\}(\|\mathbf{v}^{(l)};\mathcal{W}^{2,1}_{2,\beta}(\Omega^T)\|^4 + (A_0+A_1)\|\mathbf{v}^{(l)};\mathcal{W}^{2,1}_{2,\beta}(\Omega^T)\|^2)\\
\le{}& c(A_3+A_2)\\
&+ c\min\{1,\, T^{1/2}\}(\|\mathbf{v}^{(l)};\mathcal{W}^{2,1}_{2,\beta}(\Omega^T)\|^4 + (A_0+A_1)\|\mathbf{v}^{(l)};\mathcal{W}^{2,1}_{2,\beta}(\Omega^T)\|^2)
\end{aligned}
$$

and

$$\|\widetilde{\mathbf{u}}_0;\mathcal{W}^1_{2,\beta}(\Omega)\|^2 \le \|\widehat{\mathbf{u}}_0;\mathcal{W}^1_{2,\beta}(\Omega)\|^2 + \|\mathbf{W}(\,\cdot\,,0);\mathcal{W}^1_{2,\beta}(\Omega)\|^2$$
$$\le c(\|\widehat{\mathbf{u}}_0;\mathcal{W}^1_{2,\beta}(\Omega)\|^2 + \|\mathbf{W};W^{2,1}_2(\Omega^T_{(3)})\|^2) \le c(A_3+A_1+A_0).$$

Notice that we have used the fact that $\operatorname{supp}_x(\mathbf{f}_{(1)}(x,t)+\mathbf{f}_{(2)}(x,t)) \subset \overline{\Omega}_{(3)}$, $\operatorname{supp}_x\mathbf{W}(x,t)\subset\overline{\Omega}_{(3)}$, and, therefore,

$$\|\mathbf{f}_{(1)}+\mathbf{f}_{(2)};\mathcal{L}_{2,\beta}(\Omega^T)\|^2 \le c\|\mathbf{f}_{(1)}+\mathbf{f}_{(2)};L_2(\Omega^T)\|^2.$$

By Theorem 4.1, all approximations $(\mathbf{v}^{(l+1)}(x,t),\widetilde{p}^{(l+1)}(x,t))$ are well defined and satisfy the estimates

$$\begin{aligned}
(6.4)\qquad &\|\mathbf{v}^{(l+1)};\mathcal{W}^{2,1}_{2,\beta}(\Omega^T)\|^2 + \|\nabla\widetilde{p}^{(l+1)};\mathcal{L}_{2,\beta}(\Omega^T)\|^2\\
&\le c(\|\mathbf{g}(\,\cdot\,,\mathbf{v}^{(l)});\mathcal{L}_{2,\beta}(\Omega^T)\|^2 + \|\widetilde{\mathbf{u}}_0;\mathcal{W}^1_{2,\beta}(\Omega)\|)\\
&\le c_1\sum_{k=0}^{3}A_k + c_2\min\{1,\,T^{1/2}\}\\
&\quad\cdot(\|\mathbf{v}^{(l)};\mathcal{W}^{2,1}_{2,\beta}(\Omega^T)\|^4 + (A_0+A_1)\|\mathbf{v}^{(l)};\mathcal{W}^{2,1}_{2,\beta}(\Omega^T)\|^2)\\
&=\alpha_0+\alpha_1\|\mathbf{v}^{(l)};\mathcal{W}^{2,1}_{2,\beta}(\Omega^T)\|^2 + \alpha_2\|\mathbf{v}^{(l)};\mathcal{W}^{2,1}_{2,\beta}(\Omega^T)\|^4.
\end{aligned}$$

If

$$(6.5)\qquad \alpha_1<1,\quad 4\alpha_0\alpha_2<(1-\alpha_1)^2,$$

then the quadratic equation $\alpha_2\rho^2+(\alpha_1-1)\rho+\alpha_0=0$ has two positive roots and the smaller one r_0 is given by formula (6.3). Conditions (6.5) are satisfied because of assumptions (6.1). From (6.4), (6.5) it follows that, if $\|\mathbf{v}^{(l)};W^{2,1}_2(\Omega^T)\|^2\le r_0$, then also

$$(6.6)\qquad \|\mathbf{v}^{(l+1)};\mathcal{W}^{2,1}_{2,\beta}(\Omega^T)\|^2 + \|\nabla\widetilde{p}^{(l+1)};\mathcal{L}_{2,\beta}(\Omega^T)\|^2 \le r_0.$$

Since, obviously $\|\mathbf{v}^{(0)};\mathcal{W}^{2,1}_{2,\beta}(\Omega^T)\|^2\le r_0$, we conclude that (6.6) is valid for all $l\ge 0$.

Let us show that the sequence $\{(\mathbf{v}^{(l)}(x,t),\widetilde{p}^{(l)}(x,t))\}$ converges to the solution $(\mathbf{v}(x,t),\widetilde{p}(x,t))$ of problem (3.14). The differences

$$\mathbf{w}^{(l)}(x,t)=\mathbf{v}^{(l+1)}(x,t)-\mathbf{v}^{(l)}(x,t),\quad q^{(l)}(x,t)=\widetilde{p}^{(l+1)}(x,t)-\widetilde{p}^{(l)}(x,t)$$

are the solutions of the following linear problems

$$\begin{cases}
\mathbf{w}^{(l)}_t(x,t)-\nu\Delta\mathbf{w}^{(l)}(x,t)+\nabla q^{(l)}(x,t)=\mathcal{G}^{(l)}(x,t),\\
\operatorname{div}\mathbf{w}^{(l)}(x,t)=0,\\
\mathbf{w}^{(l)}(x,t)|_{\partial\Omega}=0,\quad \mathbf{w}^{(l)}(x,0)=0,\\
\displaystyle\int_{\sigma_j}\mathbf{w}^{(l)}(x,t)\cdot\mathbf{n}(x)\,ds=0,\quad j=1,\dots,J,
\end{cases}$$

where

$$\mathcal{G}^{(l)}(x,t) = -(\mathbf{V}(x,t)\cdot\nabla)\mathbf{w}^{(l)}(x,t) - (\mathbf{w}^{(l)}(x,t)\cdot\nabla)\mathbf{V}(x,t) \\ -(\mathbf{w}^{(l)}(x,t)\cdot\nabla)\mathbf{v}^{(l)}(x,t) - (\mathbf{v}^{(l)}(x,t)\cdot\nabla)\mathbf{w}^{(l)}(x,t).$$

Let

$$X^{(l+1)}(t) = \|\mathbf{w}^{(l+1)}; \mathcal{W}^{2,1}_{2,\beta}(\Omega\times(0,t))\|^2 + \|\nabla q^{(l+1)}; \mathcal{L}_{2,\beta}(\Omega\times(0,t))\|^2.$$

Using Theorem 4.1, Lemmas 5.2, 5.4 and estimate (6.6), we derive the inequality

$$\begin{aligned}
X^{(l+1)}(t) &\le c\,\|\mathcal{G}^{(l)}(x,t); \mathcal{L}_{2,\beta}(\Omega\times(0,t))\|^2 \\
&\le c\,(\|(\mathbf{V}\cdot\nabla)\mathbf{w}^{(l)}; \mathcal{L}_{2,\beta}(\Omega\times(0,t))\|^2 \\
&\quad + \|(\mathbf{w}^{(l)}\cdot\nabla)\mathbf{V}; \mathcal{L}_{2,\beta}(\Omega\times(0,t))\|^2 \\
&\quad + \|(\mathbf{w}^{(l)}\cdot\nabla)\mathbf{v}^{(l)}; \mathcal{L}_{2,\beta}(\Omega\times(0,t))\|^2 \\
&\quad + \|(\mathbf{v}^{(l)}\cdot\nabla)\mathbf{w}^{(l)}; \mathcal{L}_{2,\beta}(\Omega\times(0,t))\|^2) \\
&\le \varepsilon c(A_0^2 + A_1^2 + \|\mathbf{v}^{(l)}; \mathcal{W}^{2,1}_{2,\beta}(\Omega^T)\|^4)\|\mathbf{w}^{(l)}; \mathcal{W}^{2,1}_{2,\beta}(\Omega\times(0,t))\|^2 \\
&\quad + c_\varepsilon \int_0^t \|\mathbf{w}^{(l)}; \mathcal{W}^{2,1}_{2,\beta}(\Omega\times(0,\tau))\|^2\,d\tau \\
&\le \varepsilon c_*(A_0^2 + A_1^2 + r_0^2)\|\mathbf{w}^{(l)}; \mathcal{W}^{2,1}_{2,\beta}(\Omega\times(0,t))\|^2 \\
&\quad + c_\varepsilon \int_0^t \|\mathbf{w}^{(l)}; \mathcal{W}^{2,1}_{2,\beta}(\Omega\times(0,\tau))\|^2\,d\tau.
\end{aligned} \tag{6.7}$$

Let us fix $\varepsilon = 1/(2c_*(A_0^2 + A_1^2 + r_0^2))$ and sum relations (6.7) by l from 1 to M. This yields

$$\sum_{m=2}^{M+1} X^{(l)}(t) \le \frac{1}{2}\sum_{m=1}^{M} X^{(l)}(t) + c\int_0^t \sum_{m=1}^{M} X^{(l)}(\tau)\,d\tau. \tag{6.8}$$

Setting $Y^{(M)}(t) = \sum_{m=1}^{M} X^{(l)}(t)$, from (6.8) we get

$$Y^{(M)}(t) \le 2X^{(1)}(t) + c\int_0^t Y^{(M)}(\tau)\,d\tau$$

and, by Gronwall inequality,

$$Y^{(M)}(t) \le 2e^{ct}X^{(1)}(t).$$

Therefore, the series $\sum_{l=1}^{\infty} X^{(l)}(t)$ converges for any $t\in[0,T]$ and, consequently, $\{\mathbf{v}^{(l+1)}(x,t), \nabla\widetilde{p}^{(l+1)}(x,t)\}$ converges in the norm of $\mathcal{W}^{2,1}_{2,\beta}(\Omega^T)\times\mathcal{L}_{2,\beta}(\Omega^T)$ to the solution $(\mathbf{v}(x,t), \widetilde{p}(x,t))$ of problem (3.14). Obviously, for $(\mathbf{v}(x,t), \widetilde{p}(x,t))$ remains valid inequality (6.6).

Let us prove the uniqueness of the solution to problem (3.14). The difference of two solutions $\mathbf{w}(x,t) = \mathbf{v}^{[1]}(x,t) - \mathbf{v}^{[2]}(x,t)$, $q(x,t) = \widetilde{p}^{[1]}(x,t) - \widetilde{p}^{[2]}(x,t)$ satisfies the equations

$$\begin{cases} \mathbf{w}_t(x,t) - \nu\Delta\mathbf{w}(x,t) + \nabla q(x,t) = -(\mathbf{V}(x,t)\cdot\nabla)\mathbf{w}(x,t) \\ -(\mathbf{w}(x,t)\cdot\nabla)\mathbf{V}(x,t) - (\mathbf{w}(x,t)\cdot\nabla)\mathbf{v}^{[1]}(x,t) - (\mathbf{v}^{[2]}(x,t)\cdot\nabla)\mathbf{w}(x,t), \\ \operatorname{div}\mathbf{w}(x,t) = 0, \\ \mathbf{w}(x,t)|_{\partial\Omega} = 0, \quad \mathbf{w}(x,0) = 0, \\ \displaystyle\int_{\sigma_j} \mathbf{w}(x,t)\cdot\mathbf{n}(x)\,ds = 0, \quad j = 1,\dots,J. \end{cases}$$

Therefore, for $(\mathbf{w}(x,t), q(x,t))$ holds the estimate

$$\|\mathbf{w}; \mathcal{W}^{2,1}_{2,\beta}(\Omega\times(0,t))\|^2 + \|\nabla q; \mathcal{L}_{2,\beta}(\Omega\times(0,t))\|^2 \\ \leq c\int_0^t \|\mathbf{w}; \mathcal{W}^{2,1}_{2,\beta}(\Omega\times(0,\tau))\|^2\,d\tau,$$

from which it follows that $\mathbf{w}(x,t) = 0$, $\nabla q(x,t) = 0$. □

Remark 6.2. If data are sufficiently small, i.e. if

$$\begin{gathered} c_2(A_0 + A_1) < 1, \\ 4c_1c_2(A_0 + A_1 + A_2 + A_3) < (1 - c_2(A_0 + A_1))^2, \end{gathered} \tag{6.9}$$

then it follows from Theorem 6.1 that the solution $(\mathbf{v}(x,t), \widetilde{p}(x,t))$ of problem (3.14) exists for any finite time interval $[0,T]$. Moreover, the constant in estimate (6.2) does not depend on T and, therefore, the solution exists also for the infinite time interval $(0,\infty)$. If the data are "large" ((6.9) is not valid), then the solution of problem (3.14) exists for "small" time intervals $[0,T]$, where the bound for T is given by (6.1).

7. Uniqueness of the solution to problem (1.1)

In Section 6 we proved the existence of the unique solution to problem (1.1) which has the representation (3.13). In this section we show that problem (1.1) cannot have other solutions in a class of functions that are "bounded at infinity". In particular, from this result it follows the uniqueness of the time-dependent Poiseuille flow in a straight pipe.

Denote by $\mathbb{W}^{2,1}_2(\Omega^T)$ the space of functions having the finite norm

$$\|u; \mathbb{W}^{2,1}_2(\Omega^T)\| = \left(\|u; W^{2,1}_2(\Omega^T_{(3)})\|^2 + \sum_{j=1}^{J}\sup_{s\geq 0}\|u; W^{2,1}_2(\omega^T_{js})\|^2\right)^{1/2}.$$

Obviously, the solution $\mathbf{u}$ of problem (1.1) having the representation (3.13) belongs to $\mathbb{W}^{2,1}_2(\Omega^T)$.

Theorem 7.1. *Problem* (1.1) *cannot have two different solutions in the space* $\mathbb{W}_2^{2,1}(\Omega^T)$.

Proof. Let $(\mathbf{u}^{[1]}(x,t), p^{[1]}(x,t))$ and $(\mathbf{u}^{[2]}(x,t), p^{[2]}(x,t))$ be two solutions to problem (1.1). The differences $\mathbf{w}(x,t) = \mathbf{u}^{[1]}(x,t) - \mathbf{u}^{[2]}(x,t)$, $q(x,t) = p^{[1]}(x,t) - p^{[2]}(x,t)$ satisfy the linear system

$$\begin{cases} \mathbf{w}_t - \nu\Delta\mathbf{w} + \nabla q = -(\mathbf{w}\cdot\nabla)\mathbf{u}^{[1]} - (\mathbf{u}^{[2]}\cdot\nabla)\mathbf{w}, \\ \operatorname{div}\, \mathbf{w} = 0, \\ \mathbf{w}(x,t)|_{\partial\Omega} = 0, \quad \mathbf{w}(x,0) = 0, \\ \displaystyle\int_{\sigma_j} \mathbf{w}(x,t)\cdot\mathbf{n}(x)\, ds = 0, \quad j = 1,\dots,J. \end{cases}$$

Denote

$$\mathcal{E}_{-\gamma_*}(x) = \begin{cases} 1 & \text{for } x \in \Omega_{(0)}, \\ e^{-\gamma_* x_n^{(j)}} & \text{for } x \in \Omega_j,\ j = 1,\dots,J, \end{cases}$$

where $\gamma_* = (\gamma_*,\dots,\gamma_*)$, $\gamma_* > 0$. It is evident that $\mathbb{W}_2^{2,1}(\Omega^T) \subset \mathcal{W}_{2,-\gamma_*}^{2,1}(\Omega^T)$. Therefore, we may treat $\mathbf{w} \in \mathcal{W}_{2,-\gamma_*}^{2,1}(\Omega^T)$ as a solution of linear problem (4.1) with the right hand side $\mathbf{f}(x,t) = -(\mathbf{u}^{[2]}(x,t)\cdot\nabla)\mathbf{w}(x,t) - (\mathbf{w}(x,t)\cdot\nabla)\mathbf{u}^{[1]}(x,t)$ and $\mathbf{u}_0(x) = 0$. Let us show that $\mathbf{f} \in \mathcal{L}_{2,-\gamma_*}(\Omega^T)$. We have

$$\begin{aligned} &\int_0^t \int_\Omega \mathcal{E}_{-\gamma_*}(x) |\mathbf{u}^{[2]}(x,\tau)|^2 |\nabla\mathbf{w}(x,\tau)|^2\, dx\, d\tau \\ &\quad\le \int_0^t \int_{\Omega_{(3)}} |\mathbf{u}^{[2]}(x,\tau)|^2 |\nabla\mathbf{w}(x,\tau)|^2\, dx\, d\tau \\ &\qquad + \sum_{j=1}^J \sum_{s=0}^\infty \int_0^t \int_{\omega_{js}} e^{-\gamma_* x_3^{(j)}} |\mathbf{u}^{[2]}(x,\tau)|^2 |\nabla\mathbf{w}(x,\tau)|^2\, dx\, d\tau. \end{aligned} \tag{7.1}$$

In virtue of Lemma 2.1 (see inequality (2.3)), there hold the estimates

$$\begin{aligned} &\int_0^t \int_{\omega_{js}} e^{-\gamma_* x_3^{(j)}} |\mathbf{u}^{[2]}(x,\tau)|^2 |\nabla\mathbf{w}(x,\tau)|^2\, dx\, d\tau \\ &\quad\le c\int_0^t e^{-\gamma_* s} \|\mathbf{u}^{[2]}(\,\cdot\,,\tau); L_\infty(\omega_{js})\|^2 \|\nabla\mathbf{w}(\,\cdot\,,\tau); L_2(\omega_{js})\|^2\, d\tau \\ &\quad\le c\int_0^t \|\mathbf{u}^{[2]}(\,\cdot\,,\tau); W_2^1(\omega_{js})\| \|\mathbf{u}^{[2]}(\,\cdot\,,\tau); W_2^2(\omega_{js})\| \\ &\qquad\cdot \|e^{-\gamma_* x_3^{(j)}/2} \nabla\mathbf{w}(\,\cdot\,,\tau); L_2(\omega_{js})\|^2\, d\tau \\ &\quad\le c \sup_{\tau\in[0,t]} \|\mathbf{u}^{[2]}(\,\cdot\,,\tau); W_2^1(\omega_{js})\| \sup_{\tau\in[0,t]} \|e^{-\gamma_* x_3^{(j)}/2} \nabla\mathbf{w}(\,\cdot\,,\tau); L_2(\omega_{js})\| \\ &\qquad\cdot \int_0^t \|\mathbf{u}^{[2]}(\,\cdot\,,\tau); W_2^2(\omega_{js})\| \|e^{-\gamma_* x_3^{(j)}/2} \nabla\mathbf{w}(\,\cdot\,,\tau); L_2(\omega_{js})\|\, d\tau \end{aligned}$$

$$\leq \varepsilon \sup_{\tau\in[0,t]} \|\mathbf{u}^{[2]}(\,\cdot\,,\tau); W_2^1(\omega_{js})\|^2 \int_0^t \|e^{-\gamma_* x_3^{(j)}/2}\nabla\mathbf{w}(\,\cdot\,,\tau); L_2(\omega_{js})\|^2\,d\tau$$
$$+ c_\varepsilon \int_0^t \|\mathbf{u}^{[2]}(\,\cdot\,,\tau); W_2^2(\omega_{js})\|^2\,d\tau \sup_{\tau\in[0,t]} \|e^{-\gamma_* x_3^{(j)}/2}\nabla\mathbf{w}(\,\cdot\,,\tau); L_2(\omega_{js})\|^2.$$

Therefore,

$$\begin{aligned}(7.2)\quad &\sum_{s=0}^\infty \int_0^t \int_{\omega_{js}} e^{-\gamma_* x_3^{(j)}} |\mathbf{u}^{[2]}(x,\tau)|^2 |\nabla\mathbf{w}(x,\tau)|^2\,dx\,d\tau\\
&\leq \varepsilon \sup_{s\geq 0}\Big(\sup_{\tau\in[0,T]} \|\mathbf{u}^{[2]}(\,\cdot\,,\tau); W_2^1(\omega_{js})\|^2\Big)\\
&\quad\cdot \sum_{s=0}^\infty \int_0^t \|e^{-\gamma_* x_3^{(j)/2}}\nabla\mathbf{w}(\,\cdot\,,\tau);\ L_2(\omega_{js})\|^2\,d\tau\\
&\quad+ c_\varepsilon \sup_{s\geq 0}\Big(\int_0^t \|\mathbf{u}^{[2]}(\,\cdot\,,\tau); W_2^2(\omega_{js})\|^2 d\tau\Big)\\
&\quad\cdot \sum_{s=0}^\infty \sup_{\tau\in[0,t]} \|e^{-\gamma_* x_3^{(j)}/2}\nabla\mathbf{w}(\,\cdot\,,\tau); L_2(\omega_{js})\|^2\\
&\leq c\varepsilon \|\mathbf{u}^{[2]}; \mathbb{W}_2^{2,1}(\Omega^T)\|^2 \int_0^t \|e^{-\gamma_* x_3^{(j)}/2}\nabla\mathbf{w}(\,\cdot\,,\tau);\ L_2(\Omega_j)\|^2\,d\tau\\
&\quad+ c_\varepsilon \|\mathbf{u}^{[2]}; \mathbb{W}_2^{2,1}(\Omega\times(0,t))\|^2 \sum_{s=0}^\infty \|e^{-\gamma_* x_3^{(j)}/2}\mathbf{w}; W_2^{2,1}(\omega_{js}\times(0,t))\|^2\\
&\leq c\varepsilon \|\mathbf{u}^{[2]}; \mathbb{W}_2^{2,1}(\Omega^T)\|^2 \int_0^t \|e^{-\gamma_* x_3^{(j)}/2}\nabla\mathbf{w}(\,\cdot\,,\tau);\ L_2(\Omega_j)\|^2\,d\tau\\
&\quad+ c_\varepsilon \|\mathbf{u}^{[2]}; \mathbb{W}_2^{2,1}(\Omega\times(0,t))\|^2 \|e^{-\gamma_* x_3^{(j)}/2}\mathbf{w}; W_2^{2,1}(\Omega_j\times(0,t))\|^2.\end{aligned}$$

Analogously, we get that

$$\begin{aligned}(7.3)\quad &\int_0^t \int_{\Omega_{(3)}} |\mathbf{u}^{[2]}(x,\tau)|^2 |\nabla\mathbf{w}(x,\tau)|^2\,dx\,d\tau\\
&\leq c\varepsilon \|\mathbf{u}^{[2]}; \mathbb{W}_2^{2,1}(\Omega^T)\|^2 \int_0^t \|\nabla\mathbf{w}(\,\cdot\,,\tau);\ L_2(\Omega_{(3)})\|^2\,d\tau\\
&\quad+ c_\varepsilon \|\mathbf{u}^{[2]}; \mathbb{W}_2^{2,1}(\Omega\times(0,t))\|^2 \|\mathbf{w}; W_2^{2,1}(\Omega_{(3)}\times(0,t))\|^2.\end{aligned}$$

Inequalities (7.1)–(7.3) yield

$$\begin{aligned}(7.4)\quad &\|(\mathbf{u}^{[2]}\cdot\nabla)\cdot\mathbf{w}; \mathcal{L}_{2,-\gamma_*}(\Omega\times(0,t))\|^2\\
&\leq c\varepsilon \|\mathbf{u}^{[2]};\ \mathbb{W}_2^{2,1}(\Omega^T)\|^2 \|e^{-\gamma_* x_3^{(j)/2}}\mathbf{w}; W_2^{2,1}(\Omega\times(0,t))\|^2\\
&\quad+ c_\varepsilon \|\mathbf{u}^{[2]}; \mathbb{W}_2^{2,1}(\Omega\times(0,t))\|^2 \|e^{-\gamma_* x_3^{(j)}/2}\mathbf{w}; W_2^{2,1}(\Omega\times(0,t))\|^2.\end{aligned}$$

Consider now the term $(\mathbf{w}(x,t)\cdot\nabla)\mathbf{u}^{[1]}(x,t)$. Using inequality (2.2) instead of (2.3) and arguing as above, we obtain

$$\begin{aligned}(7.5)\quad &\|(\mathbf{w}\cdot\nabla)\cdot\mathbf{u}^{[1]};\mathcal{L}_{2,-\gamma_*}(\Omega\times(0,t))\|^2\\ &\le c\varepsilon\|\mathbf{u}^{[1]};\mathbb{W}_2^{2,1}(\Omega^T)\|^2\|e^{-\gamma_* x_3^{(j)}/2}\mathbf{w};W_2^{2,1}(\Omega\times(0,t))\|^2\\ &\quad+c_\varepsilon\|\mathbf{u}^{[1]};\mathbb{W}_2^{2,1}(\Omega\times(0,t))\|^2\|e^{-\gamma_* x_3^{(j)}/2}\mathbf{w};W_2^{2,1}(\Omega\times(0,t))\|^2.\end{aligned}$$

Thus, $\mathbf{f}\in\mathcal{L}_{2,-\gamma_*}(\Omega^T)$ and, if γ_* is sufficiently small, then according to Remark 4.1 for $\mathbf{w}(x,t)$ holds estimate (4.4) which together with (7.4), (7.5) yields

$$\begin{aligned}&\|\mathbf{w};\mathcal{W}^{2,1}_{2,-\gamma_*}(\Omega\times(0,t))\|^2\le c\|\mathbf{f};\mathcal{L}_{2,-\gamma_*}(\Omega\times(0,t))\|^2\\ &\le c_1\varepsilon(\|\mathbf{u}^{[1]};\mathbb{W}_2^{2,1}(\Omega^T)\|^2+\|\mathbf{u}^{[2]};\mathbb{W}_2^{2,1}(\Omega^T)\|^2)\|\mathbf{w};\mathcal{W}^{2,1}_{2,-\gamma_*}(\Omega\times(0,t))\|^2\\ &+c_\varepsilon(\|\mathbf{u}^{[1]};\mathbb{W}_2^{2,1}(\Omega\times(0,t))\|^2+\|\mathbf{u}^{[2]};\mathbb{W}_2^{2,1}(\Omega\times(0,t))\|^2)\|\mathbf{w};\mathcal{W}^{2,1}_{2,-\gamma_*}(\Omega\times(0,t))\|^2.\end{aligned}$$

Fixing in the last inequality

$$\varepsilon\le\frac{1}{2c_1(\|\mathbf{u}^{[1]};\mathbb{W}_2^{2,1}(\Omega^T)\|^2+\|\mathbf{u}^{[2]};\mathbb{W}_2^{2,1}(\Omega^T)\|^2)}$$

furnishes

$$\begin{aligned}(7.6)\quad &\|\mathbf{w};\mathcal{W}^{2,1}_{2,-\gamma_*}(\Omega\times(0,t))\|^2\le c_2(\|\mathbf{u}^{[1]};\mathbb{W}_2^{2,1}(\Omega\times(0,t))\|^2\\ &\quad+\|\mathbf{u}^{[2]};\mathbb{W}_2^{2,1}(\Omega\times(0,t))\|^2)\|\mathbf{w};\mathcal{W}^{2,1}_{2,-\gamma_*}(\Omega\times(0,t))\|^2.\end{aligned}$$

Let t_1 be such that $c_2(\|\mathbf{u}^{[1]};\mathbb{W}_2^{2,1}(\Omega\times(0,t))\|^2+\|\mathbf{u}^{[2]};\mathbb{W}_2^{2,1}(\Omega\times(0,t))\|^2)<1$. Then relation (7.6) yields $\|\mathbf{w};\mathcal{W}^{2,1}_{2,-\gamma_*}(\Omega\times(0,t_1))\|^2=0$ and, hence, $\mathbf{w}(x,t)=0$ in $\Omega\times[0,t_1]$. If $t_1<T$, we repeat our considerations for $t\in[t_1,t_2]$, where $c_2(\|\mathbf{u}^{[1]};\mathbb{W}_2^{2,1}(\Omega\times(t_1,t_2))\|^2+\|\mathbf{u}^{[1]};\mathbb{W}_2^{2,1}(\Omega\times(t_1,t_2))\|^2)<1$, and get that $\mathbf{w}(x,t)=0$ in $\Omega\times[0,t_2]$, and so on. By finite number of steps we deduce that $\mathbf{w}(x,t)=0$ in $\Omega\times[0,T]$ for any $T<\infty$. □

8. Remark on weak Hopf's solution

The global unique solvability of problem (3.14) is proved in Section 6 assuming that the data are sufficiently small (see conditions (6.9)). On the other hand, it could be proved (e.g. [3]) that for arbitrary data there exists at least one weak global Hopf's solution $\mathbf{v}_H(x,t)$ such that $\mathbf{v}_H\in W_2^{1,0}(\Omega\times(0,\infty))$ and $\|\mathbf{v}_H(\,\cdot\,,t);L_2(\Omega)\|=\varphi(t)\in L_\infty(0,\infty)$. Therefore, for any $\varepsilon>0$ there exists such $t_*=t_*(\varepsilon)>0$ that

$$\|\mathbf{v}_H(\,\cdot\,,t_*);W_2^1(\Omega)\|\le\varepsilon.$$

For large t_* also the norm $\|\widetilde{\mathbf{f}};L_2(\Omega\times(t_*,\infty)\|$ becomes small. Moreover, since $\mathbf{U}^{(j)}\in W_2^{2,1}(\sigma_j\times(0,\infty))$, $\mathbf{W}\in W_2^{2,1}(\Omega\times(0,\infty))$, we get for sufficiently large t_*

that

$$\sum_{j=1}^{J} \|\mathbf{U}^{(j)}; W_2^{2,1}(\sigma_j \times (t_*, \infty))\| + \|\mathbf{W}; W_2^{2,1}(\Omega \times (t_*, \infty))\| \leq \varepsilon.$$

Considering $\mathbf{v}_H(\,\cdot\,, t_*) \in \overset{\circ}{W}{}_2^1(\Omega)$ as an initial data for the problem (3.14), we verify the condition (6.9) and, therefore, by Theorem 6.1 problem (3.14) admits a unique solution $\mathbf{v} \in W_2^{2,1}(\Omega \times (t_*, \infty))$ defined on the interval $[t_*, \infty)$. By uniqueness results (see [13]) $\mathbf{v}_H(x,t) = \mathbf{v}(x,t)$ for $t \in [t_*, \infty)$.

Thus, also for large data there exists at least one weak solution of problem (1.1) having the form $\mathbf{u}(x,t) = \mathbf{V}(x,t) + \mathbf{v}_H(x,t)$ (it may be not unique) and all such solutions tend in outlets to infinity Ω_j to the corresponding Poiseuille flows in the sense that

$$\int_0^\infty \int_\Omega (|\mathbf{v}_H(x,t)|^2 + |\nabla \mathbf{v}_H(x,t)|^2)\, dx\, dt < \infty.$$

Moreover, for $t \geq t_*$ these solutions become regular and

$$\int_{t_*}^\infty \left(\left\| \frac{\partial \mathbf{v}_H(\,\cdot\,,t)}{\partial t}; L_2(\Omega) \right\|^2 + \|\mathbf{v}_H(\,\cdot\,,t); W_2^2(\Omega)\|^2 \right) dx\, dt < \infty.$$

Note that we are not able to prove that for $t \geq t_*$ the solution $\mathbf{v}_H(x,t)$ belongs to certain weighted space of vanishing at infinity function, since we do not know whether the initial data $\mathbf{v}_H(\,\cdot\,, t_*) \in \overset{\circ}{W}{}_2^1(\Omega)$ belongs to some weighted space $\mathcal{W}^1_{2,\beta}(\Omega)$ with $\beta_j > 0$, $j = 1, \ldots, J$.

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Konstantin Pileckas
Faculty of Mathematics and Informatics
Vilnius University
Naugarduko Str. 24
Vilnius, 2006, LITHUANIA
and
Institute of Mathematics and Informatics
Akademijos 4
Vilnius 08663, LITHUANIA

E-mail address: pileckas@ktl.mii.lt