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EXISTENCE AND CONCENTRATION OF NODAL SOLUTIONS TO A CLASS OF QUASILINEAR PROBLEMS

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Abstract. The existence and concentration behavior of nodal solutions are established for the equation $-\varepsilon^p \Delta_p u + V(z)|u|^{p-2}u = f(u)$ in Ω, where Ω is a domain in $\mathbb{R}^N$, not necessarily bounded, V is a positive Hölder continuous function and $f \in C^1$ is a function having subcritical growth.

1. Introduction

In this paper we are concerned with the existence and concentration of nodal solutions for the problem

$$(\mathrm{P}_\varepsilon) \qquad \begin{cases} -\varepsilon^p \Delta_p u + V(z)|u|^{p-2}u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\varepsilon > 0$, $1 < p < N$, $\Delta_p u$ is the p-Laplacian operator defined as

$$\Delta_p u = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\bigg(|\nabla u|^{p-2} \frac{\partial u}{\partial x_i}\bigg),$$

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Ω is a domain in $\mathbb{R}^N$ containing the origin, not necessarily bounded, with empty or smooth boundary, $V:\mathbb{R}^N \to \mathbb{R}$ is a Hölder continuous function satisfying:

(V_1) The set $M = \{z \in \mathbb{R}^N : V(z) = V_0\}$ is bounded and compactly contained in Ω, where

$$V_0 = \inf_{z\in\mathbb{R}^N} V(z) > 0.$$

(V_2) There exists an open and bounded domain Λ, compactly contained in Ω, such that

$$V_0 = \inf_{z\in\Lambda} V(z) < \min_{z\in\partial\Lambda} V(z).$$

With relation the function f, we assume the following conditions:

(f_1) $f \in C^1(\mathbb{R})$ and $\lim_{s\to 0}|f(s)|/|s|^{p-1} = 0$.

(f_2) There exists $q \in (p, p^*)$, where $p^* = Np/(N-p)$ such that

$$\lim_{|s|\to\infty} \frac{|f(s)|}{|s|^{q-1}} = 0.$$

(f_3) There exists $\theta \in (p, q)$ such that $0 < \theta F(s) \leq sf(s)$, for all $s \neq 0$, where $F(s) = \int_0^s f(t)\,dt$.

(f_4) The function $s \mapsto f(s)/|s|^{p-1}$ is an increasing in $\mathbb{R}\setminus\{0\}$.

For the case $p = 2$, equation (P_ε) is equal to

$$(P_*) \qquad -\varepsilon^2\Delta u + V(z)u = f(u) \quad \text{in } \Omega,$$

and this type of equation arises in some important mathematical models. For example, when $f(s) = |s|^{q-2}s$, $2 < q < 2^* = 2N/(N-2)$, these equations are related with the existence of standing waves of the nonlinear Schrödinger equation

$$(\text{NLS}) \qquad ih\,\frac{\partial\Psi}{\partial t} = -h^2\Delta\Psi + (V(z)+E)\Psi - f(\Psi) \quad \text{for all } z \in \Omega.$$

A standing wave of (NLS) is a solution of the form $\Psi(z,t) = \exp(-iEt/h)u(z)$, where u is a solution of (P_*).

Still for the case $p = 2$, existence of nodal solutions for general semilinear elliptic equations with superlinear nonlinearity have been established by Bartsch and Wang in [7], [8], by Bartsch, Chang and Wang in [9] and by Bartsch, Weth and Willem in [10]. Moreover, the existence and concentration of positive solutions for (P_*) have been extensively studied in recent years, see for example, Floer and Weinstein [11], Oh [18], [19], Rabinowitz [20], Wang [22], Alves and Souto [1], del Pino and Felmer [13], Alves, do Ó and Souto [4], and the references therein. For the case involving nodal solutions we cite the works of Noussair and Wei [16], [17] and Alves and Soares [5].

For the general case $p \geq 2$, some results of existence and concentration for (P_ε) were proved by Alves and Figueiredo [2], [3].

In this work, motivated by paper [5] and by some ideas employed in [2] and [3], we prove the existence and concentration of nodal solutions to (P_ε). To get these nodal solutions we adapt some arguments developed in [10], [13] and in [16], [17], and to prove the concentration of the solutions, we use the same type of ideas found in [5]. However, for our case, we ought to use a different approach of that explored in [5], [13] and [16], [17], since we are working with the p-Laplacian operator and some estimates for this type of operator can not be obtained using the same type of ideas explored for the case $p = 2$. For example, results involving convergences in the C^2 sense do not hold for the p-Laplacian. To overcome these difficulties, we use the same type of arguments developed by the authors in the papers [2], [3] and make a careful analysis of the estimates proved in [5].

Our main result is the following:

Theorem 1.1. *Suppose that f and V satisfy* (f_1)–(f_4) *and* (V_1)–(V_2), *respectively. Then, there exists $\varepsilon_0 > 0$ such that* (P_ε) *possesses a nodal solution $u_\varepsilon \in W_0^{1,p}(\Omega)$, for every $\varepsilon \in (0, \varepsilon_0)$. Moreover, if $P_\varepsilon^1 \in \Omega$ is a positive global maximum point of u_ε and $P_\varepsilon^2 \in \Omega$ is a negative global minimum point of u_ε, we have that $P_\varepsilon^i \in \Lambda$ and $V(P_\varepsilon^i) \to V_0$, for $i = 1, 2$.*

This paper is organized as follows: in Section 2, we work with an auxiliary problem, which is used to show the existence of the nodal solution to (P_ε). In Section 3, we state some lemmas and propositions used in the proof of main result. In Section 4, we prove Theorem 1.1. In Section 5 we prove the technical lemmas and propositions stated in the Section 3.

2. An auxiliary problem

In this section we will work with an auxiliary problem, which is related in some sense with problem (P_ε).

2.1. Preliminaries and notations. In this section we fix some notations and prove some lemmas which are key points in our arguments.

Hereafter, let us omit the symbol "dx" in all integrals.

We recall that the weak solutions of (P_ε) are the critical points of the functional

$$J_\varepsilon(u) = \frac{1}{p}\int_\Omega \varepsilon^p |\nabla u|^p + \frac{1}{p}\int_\Omega V(z)|u|^p - \int_\Omega F(u)$$

where $F(t) = \int_0^t f(s)\,ds$, and $J_\varepsilon(u)$ is defined for u in the Banach space

$$W = \left\{ u \in W_0^{1,p}(\Omega) : \int_\Omega V(z)|u|^p < \infty \right\}$$

endowed with the norm

$$\|u\| = \left(\int_\Omega |\nabla u|^p + \int_\Omega V(z)|u|^p \right)^{1/p}.$$

To establish the existence of nodal solution, we will adapt for our case, an argument explored by del Pino and Felmer [13] (see also [3] and [5]), which consists in considering a modified problem. To this end, we need to fix some notations.

Let θ be the number given in (f_3), and $a, k_1, k_2 > 0$ be constants satisfying $k_1, k_2 > (\theta/\theta - p)$, $(f(a)/a^{p-1}) = (V_0/k_1)$ and $(f(-a)/a^{p-1}) = -(V_0/k_2)$, where V_0 appears in (V_1). Using the above numbers, let us define the functions

$$\widetilde{f}(s) = \begin{cases} f(s) & \text{if } |s| \le a, \\ \dfrac{V_0}{k_1}|s|^{p-2}s & \text{if } s > a, \\ \dfrac{V_0}{k_2}|s|^{p-2}s & \text{if } s < -a, \end{cases}$$

$$g(z,s) = \chi_\Lambda(z) f(s) + (1 - \chi_\Lambda(z))\widetilde{f}(s),$$

and the auxiliary problem

$$(\mathrm{P}_\varepsilon)_a \qquad \begin{cases} -\varepsilon^p \Delta_p u + V(z)|u|^{p-2}u = g(z,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where χ_Λ is the characteristic function of the set Λ. It is easy to check that (f_1)–(f_4) imply that g is a Carathéodory function and for $z \in \Omega$, the function $s \to g(z,s)$ satisfies the following conditions, uniformly for $z \in \Omega$:

$$(\mathrm{g}_1) \qquad \lim_{|s| \to 0} \frac{g(z,s)}{|s|^{p-1}} = 0,$$

$$(\mathrm{g}_2) \qquad \lim_{|s| \to \infty} \frac{g(z,s)}{|s|^{q-1}} = 0,$$

$$(g_3)_i \qquad 0 \le \theta G(z,s) \le g(z,s)s,$$

for all $z \in \Lambda$ and all $s \ne 0$, and

$$(g_3)_{ii} \qquad 0 < pG(z,s) \le g(z,s)s \le \frac{1}{k} V(z)|s|^p,$$

for all $z \notin \Lambda$ and all $s \ne 0$, where $k = \min\{k_1, k_2\}$ and $G(z,s) = \int_0^s g(z,t)\,dt$.

The function

$$(g_4) \qquad s \to \frac{g(z,s)}{|s|^{p-1}} \quad \text{is nondecreasing for each } z \in \Omega \text{ and all } s \ne 0.$$

Remark 2.1. Note that if u is a nodal solution of $(\mathrm{P}_\varepsilon)_a$ with $|u(z)| \le a$ for every $z \in \Omega \setminus \Lambda$, then u is also a nodal solution of (P_ε).

2.2. Existence of ground state nodal solution for $(\mathrm{P}_\varepsilon)_a$. In this section, we adapt some arguments found in Bartsch, Weth and Willem [10], Alves and Figueiredo [3] and Alves and Soares [5] to establish the existence of ground state nodal solution for problem $(\mathrm{P}_\varepsilon)_a$.

Hereafter, let us denote by I_ε the functional

$$I_\varepsilon(u) = \frac{1}{p}\int_\Omega \varepsilon^p |\nabla u|^p + \frac{1}{p}\int_\Omega V(z)|u|^p - \int_\Omega G(z,u)$$

and by $\mathcal{M}_\varepsilon$ the set

$$\mathcal{M}_\varepsilon = \{u \in W : u^\pm \neq 0 \text{ and } I'_\varepsilon(u^\pm)u^\pm = 0\},$$

where

$$u^+(z) = \max\{u(z),0\} \quad \text{and} \quad u^-(z) = \min\{u(z),0\}.$$

It is easy to check that there exists $\mu^* > 0$ such that

$$\int_\Lambda |u^\pm|^q > \mu^* \quad \text{for all } u \in \mathcal{M}_\varepsilon. \tag{2.1}$$

Hereafter, we will denote by c_ε the following real number

$$c_\varepsilon = \inf_{\mathcal{M}_\varepsilon} I_\varepsilon.$$

Theorem 2.2. *c_ε is achieved by some $u_\varepsilon \in \mathcal{M}_\varepsilon$. Moreover, u_ε is a nodal solution of* $(\mathrm{P}_\varepsilon)_a$.

Proof. It is easy to check that I_ε is bounded from below on $\mathcal{M}_\varepsilon$ since, using the definition of $\mathcal{M}_\varepsilon$, there exists $C > 0$ such that

$$I_\varepsilon(u) \geq C\|u\|^p \quad \text{for all } u \in \mathcal{M}_\varepsilon. \tag{2.2}$$

Thus, there exists a sequence $\{v_n\} \subset \mathcal{M}_\varepsilon$ verifying $I_\varepsilon(v_n) \to c_\varepsilon$ as $n \to \infty$ which is bounded by (2.2). Since W is reflexive, there exists $v \in W$ such that $v_n \rightharpoonup v$ in W. By (2.1) and by the Sobolev imbedding, we have

$$\int_\Lambda |v^\pm|^q > \mu^*,$$

showing that $v^\pm \neq 0$. By the weak convergence of the sequence and condition (g_3), it follows that

$$\|v^\pm\|^p \leq \int_\Omega g(z,v^\pm)v^\pm.$$

The above inequalities imply that there exist $t^+, t^- \in (0,1]$ such that

$$\|t^\pm v^\pm\|^p = g(z, t^\pm v^\pm),$$

which implies that $w = t^+v^+ + t^-v^-$ is an element of $\mathcal{M}_\varepsilon$.

Since

$$I_\varepsilon(t^\pm v^\pm) = \int_\Omega \bigg[\frac{1}{p} g(z,t^\pm v^\pm)t^\pm v^\pm - G(z,t^\pm v^\pm)\bigg],$$

using Fatou's lemma, it follows that

$$I_\varepsilon(t^{\pm}v^{\pm}) \leq \liminf_{n\to\infty} \int_\Omega \bigg[\frac{1}{p}g(z,t^{\pm}v_n^{\pm})t^{\pm}v_n^{\pm} - G(z,t^{\pm}v_n^{\pm})\bigg].$$

For each n, conditions (f_4) and (g_4) imply that the function

$$t \mapsto \int_\Omega \bigg[\frac{1}{p}g(z,tv_n^{\pm})tv_n^{\pm} - G(z,tv_n^{\pm})\bigg]$$

is increasing and thus

$$I_\varepsilon(t^{\pm}v^{\pm}) \leq \liminf_{n\to\infty} \int_\Omega \bigg[\frac{1}{p}g(z,v_n^{\pm})v_n^{\pm} - G(z,v_n^{\pm})\bigg] = \liminf_{n\to\infty} I_\varepsilon(v_n^{\pm}).$$

From the last inequality, it follows that

$$I_\varepsilon(w) = I_\varepsilon(t^+v^+) + I_\varepsilon(t^-v^-) \leq \liminf_{n\to\infty} I_\varepsilon(v_n) = c_\varepsilon.$$

Since $w \in \mathcal{M}_\varepsilon$, the above inequality implies that $I_\varepsilon(w) = c_\varepsilon$, and thus c_ε is achieved. Now, from [3, Proposition 3.1] (see also [14]) the functional I_ε satisfies the Palais–Smale at all $c \in \mathbb{R}$ and hence, we can repeat the same arguments found in [10, Proposition 3.1] to conclude that c_ε is a critical level, that is, there exists $u_\varepsilon \in \mathcal{M}_\varepsilon$ such that

$$I_\varepsilon(u_\varepsilon) = c_\varepsilon \quad \text{and} \quad I'_\varepsilon(u_\varepsilon) = 0. \qquad \square$$

Remark 2.3. The nodal solution found in Theorem 2.2 satisfies the following properties:

(a) If Ω is bounded, the function u_ε belongs to $C(\overline{\Omega})$ (see [12]).
(b) If Ω is unbounded, the function u_ε is a continuous function and verifies $\lim_{|z|\to\infty} u_\varepsilon(z) = 0$ (see [15]).

3. Statement of lemmas and propositions

Hereafter, let us denote by $w \in W^{1,p}(\mathbb{R}^N)$ a least energy solution of the problem

$$-\Delta_p w + V_0|w|^{p-2}w = f(w).$$

Consequently, w satisfies

$$c_{V_0} = J_{V_0}(w) = \inf_{\substack{v\in W^{1,p}(\mathbb{R}^N)\\ v\neq 0}} \sup_{\tau\geq 0} J_{V_0}(\tau v),$$

where J_{V_0} is defined as

$$J_{V_0}(v) = \frac{1}{p}\int_{\mathbb{R}^N} (|\nabla v|^p + V_0|v|^p)dx - \int_{\mathbb{R}^N} F(v)\,dx,$$

for all $v \in W^{1,p}(\mathbb{R}^N)$ (see [1]).

Let us denote by u_ε the nodal solution of $(\mathrm{P}_\varepsilon)_a$ obtained in Theorem 2.2. Adapting the same type of argument explored in [5, Lemma 2.1], it is easy to check that

$$\limsup_{\varepsilon\to 0} \varepsilon^{-N} I_\varepsilon(u_\varepsilon) \le 2c_{V_0}. \tag{3.1}$$

The next lemma shows a situation where we have convergence on compacts sets.

Lemma 3.1. *Let $\{x_n\} \subset \overline{\Lambda}$ and let $\{\varepsilon_n\}$ be a sequence of positive numbers such that $\varepsilon_n \to 0$ as $n \to \infty$. If we set $v_n(x) = u_{\varepsilon_n}(\varepsilon_n x + x_n)$, then $\{v_n\}$ converges uniformly on compact subsets of $\mathbb{R}^N$.*

Proof. See Section 5. □

The next result establishes a first information about the localization of maximum and minimum points of u_ε.

Lemma 3.2. *Let $\{\varepsilon_n\} \subset (0,\infty)$ be a sequence with $\varepsilon_n \to 0$ and P_n^1, P_n^2 be, respectively, a maximum and minimum point of the function u_{ε_n}. Then, there exist $\delta^* > 0$ and a subsequence, still denoted by $\{u_{\varepsilon_n}\}$, such that $\{P_n^i\}$, $i = 1,2$, are convergent sequences, $u_{\varepsilon_n}(P_n^1) \ge \delta^*$ and $u_{\varepsilon_n}(P_n^1) \le -\delta^*$.*

Proof. See Section 5. □

The next result is a key point to prove the existence of nodal solution to the original problem. A version of this result for $p = 2$ can be found in [5].

Proposition 3.3. *If P_ε^1 is a maximum point of u_ε^+ and P_ε^2 is a minimum point of u_ε^-, then*

$$\left|\frac{P_\varepsilon^1 - P_\varepsilon^2}{\varepsilon}\right| \to \infty \quad \textit{as } \varepsilon \to 0.$$

Proof. See Section 5. □

In order to use the Remark 2.1, the lemmas below are important in our arguments to prove that the family $\{u_\varepsilon\}$ satisfies the estimate $|u_\varepsilon(z)| \le a$, for $z \in \Omega \setminus \Lambda$ when ε is sufficiently small.

Lemma 3.4. *Let $\{\varepsilon_n\}$ be a sequence of positive number with $\varepsilon_n \to 0$ as $n \to \infty$ and let $\{z_n^i\} \subset \overline{\Lambda}$, $i = 1,2$, be a sequence such that $u_{\varepsilon_n}(z_n^1) \ge b > 0$ and $u_{\varepsilon_n}(z_n^2) \le -b < 0$. Then,*

$$\lim_{n\to\infty} V(z_n^i) = V_0 \quad \textit{for } i = 1,2.$$

Proof. See Section 5. □

Lemma 3.5. *If $m_\varepsilon^+ = \max_{\partial\Lambda} u_\varepsilon$ and $m_\varepsilon^- = \min_{\partial\Lambda} u_\varepsilon$, then $\lim_{\varepsilon\to 0} m_\varepsilon^\pm = 0$.*

Proof. This Lemma follows directly from Lemma 3.4. □

4. Proof of Theorem 1.1

By Theorem 2.2, we have that problem $(\mathrm{P}_\varepsilon)_a$ has a nodal solution u_ε, for all $\varepsilon \in (0,\overline{\varepsilon})$. By Lemma 3.5, $m_\varepsilon^+ < a$, for all $\varepsilon \in (0,\overline{\varepsilon})$ and then $(u_\varepsilon - a)^+(z) = 0$ for a neighbourhood of $\partial\Lambda$. Hence $(u_\varepsilon - a)^+ \in W_0^{1,p}(\Omega \setminus \Lambda)$ and the function Ψ given by $\Psi(z) = 0$, if $z \in \Lambda$ and $\Psi(z) = (u_\varepsilon - a)^+(z)$, if $z \in \Omega \setminus \Lambda$ belongs to $W_0^{1,p}(\Omega)$. Using Ψ as a test function, we have

$$\int_{\Omega\setminus\Lambda} \varepsilon^p |\nabla\Psi(z)|^p + \int_{\Omega\setminus\Lambda} \bigg[V(z)|u_\varepsilon|^{p-2} - \frac{g(z,u_\varepsilon)}{u_\varepsilon}\bigg]|\Psi(z)|^2 + \int_{\Omega\setminus\Lambda} \bigg[V(z)|u_\varepsilon|^{p-2} - \frac{g(z,u_\varepsilon)}{u_\varepsilon}\bigg] t_0\Psi(z) = 0.$$

The last equality implies

$$\Psi(z) = 0, \quad \text{a.e. in } z \in \Omega \setminus \Lambda.$$

Hence $u_\varepsilon(z) \le a$, for $z \in \Omega \setminus \Lambda$. Since we can assume $m_\varepsilon^- \ge -a$ for $\varepsilon \in (0,\overline{\varepsilon})$, working with the function $(u_\varepsilon + a)^-$, it is possible to prove that $u_\varepsilon(z) \ge -a$ for $z \in \Omega \setminus \Lambda$. This fact implies that $|u_\varepsilon(z)| \le a$ for $z \in \Omega \setminus \Lambda$, and the existence of a nodal solution follows from Remark 2.1. The concentration of the nodal solutions follows from Lemmas 3.2, 3.4 and 3.5 □

5. Proofs of lemmas and propositions

In this section, we will prove the lemmas and propositions established in Section 3.

Proof of Lemma 3.1. First of all, note that the sequence $\{v_n\}$ satisfies the following problem

$$(\mathrm{P}_n) \qquad \begin{cases} -\Delta_p v_n + V(x_n + \varepsilon_n x)|v_n|^{p-2}v_n = g(x_n + \varepsilon_n x, v_n) & \text{in } \widehat{\Omega}_{\varepsilon_n}, \\ v_n = 0 & \text{on } \partial\widehat{\Omega}_{\varepsilon_n}, \end{cases}$$

where $\widehat{\Omega}_{\varepsilon_n} = \varepsilon_n^{-1}\{\Omega - x_n\}$. From (3.1), it follows that $\{v_n\}$ is bounded in $W^{1,p}(\mathbb{R}^N)$ and adapting some arguments explored in [3, Lemma 4.1], we have that the sequences v_n^+ and v_n^- are bounded in $L^\infty(\mathbb{R}^N)$. Thus, there exists a subsequence, still denoted by $\{v_n\}$, such that for each compact subset K of $\mathbb{R}^N$, there exists a constant $C_K > 0$ with $|\nabla v_n|_{\infty,K} \le C_K$, for n sufficiently large. Therefore $\{v_n\}$ converges uniformly on compact subsets of $\mathbb{R}^N$. □

Proof of Lemma 3.2. Firstly, we will show that there exist $\delta^* > 0$ and $n_0 \in \mathbb{N}$ such that

$$u_{\varepsilon_n}(P_n^1) \ge \delta^* \quad \text{and} \quad u_{\varepsilon_n}(P_n^2) \le -\delta^* \quad \text{for } n \ge n_0.$$

Assume, by contradiction, that there exists a subsequence, still denoted by $\{u_{\varepsilon_n}\}$, such that

$$u_{\varepsilon_n}(P_n^1) = \|u_{\varepsilon_n}^+\|_{L^\infty(\mathbb{R}^N)} \to 0.$$

Defining $h_n(x) = u_{\varepsilon_n}(\varepsilon_n x)$, we have that $\|h_n^+\|_{L^\infty(\mathbb{R}^N)} \to 0$. For a fixed $r = V_0/2$, it follows by (f_1) that there exists $n_0 \in \mathbb{N}$ such that

$$\frac{f(\|h_n^+\|_{L^\infty(\mathbb{R}^N)})}{\|h_n^+\|_{L^\infty(\mathbb{R}^N)}^{p-1}} < r \quad \text{for } n \geq n_0.$$

Hence,

$$\int_{\mathbb{R}^N} |\nabla h_n^+|^p + \int_{\mathbb{R}^N} V_0 |h_n^+|^p \leq \int_{\mathbb{R}^N} \frac{f(\|h_n^+\|_{L^\infty(\mathbb{R}^N)})}{\|h_n^+\|_{L^\infty(\mathbb{R}^N)}^{p-1}} |h_n^+|^p \leq r \int_{\mathbb{R}^N} |h_n^+|^p,$$

thus $\|h_n^+\|_{W^{1,p}(\mathbb{R}^N)} = 0$ for $n \geq n_0$, which is impossible, because $h_n^+ \neq 0$, for all $n \in \mathbb{N}$. Then, there exists $\delta^* > 0$ and $n_1 \in \mathbb{N}$ such that $\|u_{\varepsilon_n}^+\|_{L^\infty(\mathbb{R}^N)} \geq \delta^*$, for all $n \geq n_1$. Repeating these arguments, we find $n_2 \in \mathbb{N}$ such that $\|u_{\varepsilon_n}^-\|_{L^\infty(\mathbb{R}^N)} \geq \delta^*$, for all $n \geq n_2$. Choosing $n_0 = \max\{n_1, n_2\}$, we have that

$$u_{\varepsilon_n}(P_n^1) \geq \delta^* \quad \text{and} \quad u_{\varepsilon_n}(P_n^2) \leq -\delta^*, \quad \text{for all } n \geq n_0.$$

On the other hand, the function h_n satisfies the following problem

$$(\mathrm{P}_n^1) \qquad \begin{cases} -\Delta_p h_n + V(\varepsilon_n x)|h_n|^{p-2} h_n = g(\varepsilon_n x, h_n) & \text{in } \Omega_{\varepsilon_n}, \\ h_n = 0 & \text{on } \partial\Omega_{\varepsilon_n} \end{cases}$$

where $\Omega_{\varepsilon_n} = \Omega/\varepsilon_n$. Since $V_0 = \inf_{z \in \mathbb{R}^N} V(z)$ and $\limsup_{\varepsilon_n \to 0}(\varepsilon_n^{-N} I_{\varepsilon_n}(u_{\varepsilon_n})) \leq 2c_{V_0}$, defining

$$J_n^1(v) = \frac{1}{p} \int_{\Omega_{\varepsilon_n}} (|\nabla v|^p + V(\varepsilon_n x)|v|^p)\, dx - \int_{\Omega_{\varepsilon_n}} G(\varepsilon_n x, v)\, dx$$

we have that $J_n^1(h_n) = \varepsilon_n^{-N} I_{\varepsilon_n}(u_{\varepsilon_n})$ and consequently

$$J_n^1(h_n^+) \to c_{V_0} \quad \text{with } h_n^+ \in \mathcal{N}_{\varepsilon_n},$$

where $\mathcal{N}_{\varepsilon_n} = \{u \in W_n \setminus \{0\} : {J_n^1}'(u)u = 0\}$.

Using [2, Lemma 4.3] and [3, Proposition 3.3], there exists a sequence $\{\widetilde{y}_n\} \subset \mathbb{R}^N$, such that $\widetilde{h}_n(x) = h_n^+(x + \widetilde{y}_n)$ converges strongly to a function $h \in W^{1,p}(\mathbb{R}^N) \setminus \{0\}$. Moreover, the sequence $\{y_n\}$ given by $y_n = \varepsilon_n \widetilde{y}_n$, has a subsequence, still denoted by y_n, such that $y_n \to y \in M$. Now, to adapt some arguments explored in [2, Lemma 4.5], let us consider, for each $n \in \mathbb{N}$ and $L > 0$, the functions

$$v_{L,n}(x) = \begin{cases} \widetilde{h}_n(x) & \text{if } \widetilde{h}_n(x) \leq L, \\ L & \text{if } \widetilde{h}_n(x) \geq L, \end{cases}$$

and

$$z_{L,n} = \eta^p v_{L,n}^{p(\beta-1)} \widetilde{h}_n \quad \text{and} \quad w_{L,n} = \eta \widetilde{h}_n v_{L,n}^{\beta-1}$$

with $\beta > 1$ to be picked up later. Remarking that the function $\widehat{h}_n(x) = h_n(x+\widetilde{y}_n)$ is a critical point of the functional

$$T(v) = \frac{1}{p}\int_{\widetilde{\Omega}_{\varepsilon_n}} (|\nabla v|^p + V(\varepsilon_n x + y_n)|v|^p)\,dx - \int_{\widetilde{\Omega}_{\varepsilon_n}} G(\varepsilon_n x + y_n, v)\,dx$$

where $\widetilde{\Omega}_{\varepsilon_n} = \varepsilon_n^{-1}\{\Omega - y_n\}$. Taking $z_{L,n}$ as a test function, it is possible to prove that, for a fixed $\gamma \in (0,\delta)$, there exists $R > 0$ such that

$$\|\widetilde{h}_n\|_{L^\infty(|x|\geq R)} < \gamma \quad \text{for all } n \in \mathbb{N}.$$

Consequently,

$$u_{\varepsilon_n}^+(\varepsilon_n x + y_n) < \gamma, \quad \text{for all } n \in \mathbb{N} \text{ and } |x| \geq R. \tag{5.1}$$

If P_n^1 denotes a maximum point of u_{ε_n}, we have that

$$u_{\varepsilon_n}(P_n^1) \geq \delta^* \quad \text{for all } n \in \mathbb{N}. \tag{5.2}$$

So, if $P_n^1 = \varepsilon_n Q_n + y_n$ it follows from (5.1) and (5.2) that $|Q_n| \leq R$. Using the fact that $\{y_n\}$ converges to $y^1 \in M$, we can conclude that $\{P_n^1\}$ also converges to $y^1 \in M$ and therefore

$$\lim_{n\to\infty} V(P_n^1) = V_0.$$

Using similar arguments we can prove that the sequence $\{P_n^2\}$ also converges to some $y^2 \in M$. □

Proof of Proposition 3.3. Assume by contradiction that there exists a sequence $\{\varepsilon_n\}$ with $\varepsilon_n \to 0$ verifying

$$\left|\frac{P_n^1 - P_n^2}{\varepsilon_n}\right| \to \beta < \infty,$$

where $P_n^1 = P_{\varepsilon_n}^1$ and $P_n^2 = P_{\varepsilon_n}^2$.

Defining $v_n(x) = u_{\varepsilon_n}(P_n^1 + \varepsilon_n x)$, it is easy to check that v_n is a solution of the problem

$$(\mathrm{P'}_n)\quad -\Delta_p v_n + V(P_n^1 + \varepsilon_n x)|v_n|^{p-2}v_n = g(P_n^1 + \varepsilon_n x, v_n) \quad \text{in } \Omega_{\varepsilon_n}^1,\ v_n = 0 \partial\Omega_{\varepsilon_n}^1$$

where $\Omega_{\varepsilon_n}^1 = \varepsilon_n^{-1}\{\Omega - P_n^1\}$, that the function v_n is a critical point of the functional

$$J(v) = \frac{1}{p}\int_{\Omega_{\varepsilon_n}^1} (|\nabla v|^p + V(\varepsilon_n x + P_n^1)|v|^p)\,dx - \int_{\Omega_{\varepsilon_n}^1} G(\varepsilon_n x + P_n^1, v)\,dx$$

and that $J(v_n) = \varepsilon_n^{-N} I_{\varepsilon_n}(u_{\varepsilon_n})$. Using the fact that $J'(v_n)(v_n) = 0$, from (3.1), we have that $\{v_n\}$ is a bounded sequence in $W^{1,p}(\mathbb{R}^N)$. Using the same type of arguments explored in the proof of Lemma 3.1, there exists a function $v \in W^{1,p}(\mathbb{R}^N)$ such that $\{v_n\}$ converges uniformly on compact subsets of $\mathbb{R}^N$. For each fixed compact subset $K \subset \mathbb{R}^N$, there exists $C_K > 0$ such that $|\nabla v_n|_{\infty,K} \leq C_K$, for all $x \in K$. Moreover, the sequence of functions $\chi_n(y) = \chi_\Lambda(\varepsilon_n y + P_n^1)$ can be assumed to converge weakly on compacts subsets in any $L^r(\mathbb{R}^N)$ to a function $0 \le \chi \le 1$ and the function v satisfies the problem

$$\text{(PL)} \qquad -\Delta_p v + V(\overline{P})|v|^{p-2}v = \overline{g}(x,v) \quad \text{in } \mathbb{R}^N,$$

where

$$\overline{g}(z,s) = \chi(z)f(s) + (1-\chi(z))\widetilde{f}(s) \quad \text{and} \quad \overline{P} = \lim_{n\to\infty} P_n^1.$$

Suppose that $\beta = 0$. By Lemma 3.2 and the Mean Value Inequality, we have

$$2\delta \le |v_n(0) - v_n(Z_\varepsilon)| \le |\nabla v_\varepsilon(Q_n)||Z_n|, \tag{5.3}$$

where $Z_n = (P_n^1 - P_n^2)/\varepsilon_n$ and $Q_n = t_n Z_n$, for some $t_n \in [0,1]$. Using the hypothesis that $\beta = 0$, we have that the sequence $\{Q_n\} \subset \overline{B}_1(0)$. Hence, there exists $C > 0$ such that $|\nabla v_n(Q_n)| \le C$, for all $n \in \mathbb{N}$. Using the last inequality in (5.3), we obtain the inequality

$$2\delta \le |v_n(0) - v_n(Z_\varepsilon)| \le C|Z_n|, \tag{5.4}$$

leading to a contradiction with the fact that $\lim_{n\to\infty} Z_n = 0$. Therefore, $\beta > 0$ and $P = \lim_{n\to\infty} Z_n \ne 0$. To conclude the proof, taking a subsequence if necessary, v_n converges on compacts to a function $v \in W^{1,p}(\mathbb{R}^N)$. As $v_n(0) = u_{\varepsilon_n}(P_n^1) \ge \delta^*$ and $v_n(Z_n) = u_{\varepsilon_n}(P_n^2) \le -\delta^* < 0$, it follows that $v(0) > 0$ and $v(P) < 0$, i.e. v is a nodal function of (PL). Associated to (PL) we have the functional $\overline{J}\colon W^{1,p}(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$\overline{J}(u) = \frac{1}{p}\int_{\mathbb{R}^N} (|\nabla u|^p + V(\overline{P})|u|^p)\,dx - \int_{\mathbb{R}^N} \overline{G}(z,u)\,dx, \quad u \in W^{1,p}(\mathbb{R}^N)$$

where $\overline{G}(z,s) = \int_0^s \overline{g}(z,t)\,dt$ and $\overline{P} = \lim_{n\to\infty} P_n^1$. Then, v is a critical point of $\overline{J}$. Thus, $\overline{J}'(v)\varphi = 0$ for every $\varphi \in W^{1,p}(\mathbb{R}^N)$. In particular, $\overline{J}'(v)v^\pm = 0$, which implies $v \in M_{V(\overline{z})}$, where

$$M_{V(\overline{z})} = \bigg\{u \in W_0^{1,p}(\Omega) : u^\pm \not\equiv 0, \\ \int_\Omega (|\nabla u^\pm|^p + V(\overline{z})|u^\pm|^p)\,dx = \int_\Omega \overline{g}(z,u^\pm)u^\pm\,dx\bigg\}.$$

Since $J(v_n) = \varepsilon_n^{-N} I_{\varepsilon_n}(u_{\varepsilon_n})$, it follows that

$$J((v_n)^+), J((v_n)^-) \to c_{V_0} \quad \text{as } J'((v_n)^+)((v_n)^+) = J'((v_n)^-)((v_n)^-) = 0,$$

thus, applying the same type of arguments found in [3], we can conclude that $\{(v_n)^+\}$ and $\{(v_n)^-\}$ are convergent sequences in $W^{1,p}(\mathbb{R}^N)$.

Considering the sequences $\{t_n^i\} \subset \mathbb{R}, w_n^1(z) = t_n^1(v_n)^+(z)$ and $w_n^2(z) = t_n^2(v_n)^-(z)$ such that $J'_{V_0}(w_n^i)w_n^i = 0$ where

$$J_{V_0}(u) = \frac{1}{p}\int_{\mathbb{R}^N} (|\nabla u|^p + V_0|u|^p)\,dx - \int_{\mathbb{R}^N} F(u)\,dx, \quad u \in W^{1,p}(\mathbb{R}^N)$$

it is possible to show that the sequences $\{t_n^i\}$ are also convergent in $\mathbb{R}$. Using this information, we have that w_n^i converges to $w^i \in W^{1,p}(\mathbb{R}^N) \setminus \{0\}$ with

$$J'_{V_0}(w^i) = 0 \quad \text{and} \quad J_{V_0}(w^i) \geq c_{V_0}.$$

Thus

$$\varepsilon_n^{-N} I_{\varepsilon_n}(u_{\varepsilon_n}) \geq J_{V_0}(t_n^1(v_n^1)^+) + J_{V_0}(t_n^2(v_n^2)^-)$$

and so

$$\varepsilon_n^{-N} I_{\varepsilon_n}(u_{\varepsilon_n}) \geq J_{V_0}(w_n^1) + J_{V_0}(w_n^2).$$

Taking the limit as $n \to \infty$ in the last inequality, we get

$$2c_{V_0} \geq J_{V_0}(w^1) + J_{V_0}(w^2) \geq 2c_{V_0}.$$

The last inequality implies that $J_{V_0}(w^i) = c_{V_0}$. By Theorem 4.3 in [23], $J'_{V_0}(w^i) = 0$ for $i = 1, 2$, and by Maximum Principle implies $w^1 > 0$ and $w^2 < 0$ in $\mathbb{R}^N$ (see [21]), consequently $v^+ > 0$ and $v^- < 0$ in $\mathbb{R}^N$, obtaining this way an absurd. □

Proof of Lemma 3.4. In the next, we will argue by contradiction. Assume, passing to a subsequence if necessary, that $z_n^i \to \overline{z_i} \in \overline{\Lambda}$, $i = 1, 2$ with $V(\overline{z}_1) > V_0$ and $V(\overline{z}_2) \geq V_0$. As in the above lemmas, we consider the sequences $v_n^i(z) = u_{\varepsilon_n}(z_n^i + \varepsilon_n z)$, $i = 1, 2$ and study their behaviors as n goes to infinity. For $i = 1, 2$, the function v_n^i satisfies the problem

$$(\mathrm{P}_n)_i \qquad \begin{cases} -\Delta_p v_n^i + V(\varepsilon_n z + z_n^i)|v_n^i|^{p-2} v_n^i = g(\varepsilon z + z_n^i, v_n^i) & \text{in } \Omega_n^i, \\ v_n^i = 0 & \text{on } \partial\Omega_n^i, \end{cases}$$

where $\Omega_n^i = \varepsilon^{-N}\{\Omega - z_n^i\}$. Again, the sequence v_n^i is bounded in $W^{1,p}(\mathbb{R}^N)$, thus it can be assumed to converge uniformly on compacts subsets of $\mathbb{R}^N$ to a function $v^i \in W^{1,p}(\mathbb{R}^N)$. Now, the sequence of functions $\chi_n^i(z) \equiv \chi_\Lambda(\varepsilon_n z + z_n^i)$ can be assumed to converge weakly in any $L^p(\mathbb{R}^N)$ on compacts to a function $0 \leq \chi^i \leq 1$. Therefore, v^i satisfies the limiting problem

$$(\mathrm{PL})_i \qquad -\Delta_p v^i + V(\overline{z_i})|v^i|^{p-2} v^i = \overline{g}_i(z, v^i) \quad \text{in } \mathbb{R}^N$$

where $\overline{g_i}(z, s) = \chi^i(z) f(s) + (1 - \chi^i(z))\widetilde{f}(s)$ and $\overline{z}_i = \lim_{n\to\infty} z_n^i$. We claim that v^i does not change sign, for $i = 1, 2$. More precisely, $v^1 \geq 0$ and $v^2 \leq 0$. In fact, suppose, by contradiction, that v^1 changes sign. Let $r > 0$ be such that v^1 changes sign on the closed ball $B[0, r]$ and that there exist $Q_{1,n}^+, Q_{1,n}^- \in B[0, r]$ such that

$$(v_n^1)^+(Q_{1,n}^+) = \max_{z\in\mathbb{R}^N} (v_n^1)^+(z) \quad \text{and} \quad (v_n^1)^-(Q_{1,n}^-) = \min_{z\in\mathbb{R}^N} (v_n^1)^-(z).$$

Since v_n^1 converges uniformly to v^1 on compacts subsets of $\mathbb{R}^N$, there exists n_0 such that

$$(v_n^1)^+(Q_{1,n}^+) \geq C > 0 \quad \text{and} \quad (v_n^1)^-(Q_{1,n}^-) \leq -C < 0, \quad \text{for all } n \geq n_0$$

for some positive constant C. Thus, for

$$(v_n^1)^+(Q_{1,n}^+) = v_n^1(Q_{1,n}^+) = u_{\varepsilon_n}(\varepsilon_n Q_{1,n}^+ + z_n^1),$$
$$(v_n^1)^-(Q_{1,n}^-) = v_n^1(Q_{1,n}^-) = u_{\varepsilon_n}(\varepsilon_n Q_{1,n}^- + z_n^1),$$

considering

$$\widetilde{P}_n^+ = \varepsilon_n Q_{1,n}^+ + z_n^1 \quad \text{and} \quad \widetilde{P}_n^- = \varepsilon_n Q_{1,n}^- + z_n^1$$

it follows that $\widetilde{P}_n^+$ and $\widetilde{P}_n^-$ are maximum and minimum points of u_{ε_n} in $\mathbb{R}^N$, respectively, with

$$u_{\varepsilon_n}(\widetilde{P}_n^+) \geq C \quad \text{and} \quad u_{\varepsilon_n}(\widetilde{P}_n^-) \leq -C.$$

Applying the same arguments employed in Proposition 3.1, we have

$$\left|\frac{\widetilde{P}_n^+ - \widetilde{P}_n^-}{\varepsilon_n}\right| \to \infty.$$

But this is impossible because

$$\left|\frac{\widetilde{P}_n^+ - \widetilde{P}_n^-}{\varepsilon_n}\right| = |Q_{1,n}^+ - Q_{1,n}^-| \leq 2r.$$

Hence, since $v_n^1(0) > 0$ and $v_n^1 \to v^1$ uniformly on compacts subsets of $\mathbb{R}^N$, it follows that $v^1 \geq 0$. Moreover, using the Maximum Principle it follows that $v^1 > 0$ in $\mathbb{R}^N$ (see [21]). A similar argument implies that $v^2 < 0$ in $\mathbb{R}^N$.

Associated to the limiting problem $(\mathrm{PL})_i$, we have the functional

$$\overline{J_i}\colon H^1(\mathbb{R}^N) \to \mathbb{R}$$

defined by

$$\overline{J_i}(u) = \frac{1}{p}\int_{\mathbb{R}^N} (|\nabla u|^p + V(\overline{z_i})|u|^p)\,dx - \int_{\mathbb{R}^N} \overline{G_i}(z,u)\,dx, \quad u \in W^{1,p}(\mathbb{R}^N)$$

where $\overline{G_i}(z,s) = \int_0^s \overline{g_i}(z,t)\,dt$. Then, v^i is clearly a critical point of $\overline{J_i}$. Also associated to problem $(\mathrm{P}_n)_i$ we have the functional

$$J_n^i(u) = \frac{1}{p}\int_{\Omega_n^i} (|\nabla u|^p + V(\varepsilon_n z + z_n^i)|u|^p)\,dx - \int_{\Omega_n^i} \overline{G}_i(\varepsilon_n z + z_n^i, u)\,dx,$$

for $u \in W_0^{1,p}(\Omega_n^i)$, which satisfies the following equality involving u_{ε_n} and v_n

$$\varepsilon_n^{-N} J_{\varepsilon_n}(u_{\varepsilon_n}) = \varepsilon_n^{-N}(J_{\varepsilon_n}(u_{\varepsilon_n}^+) + J_{\varepsilon_n}(u_{\varepsilon_n}^-)) = J_n^1((v_n^1)^+) + J_n^2((v_n^2)^-), \tag{5.5}$$

so, repeating the idea explored in the proof of Proposition 3.1, the sequences $\{(v_n^1)^+\}$, $\{(v_n^2)^-\}$ are convergent in $W^{1,p}(\mathbb{R}^N)$.

Considering again sequences $\{t_n^i\} \subset \mathbb{R}$, $w_n^1(z) = t_n^1(v_n^1)^+(z)$ and $w_n^2(z) = t_n^2(v_n^2)^-(z)$ verifying $\widetilde{J}'_{n,i}(w_n^i)w_n^i = 0$ where

$$\widetilde{J}_{n,i}(u) = \frac{1}{p}\int_{\mathbb{R}^N} (|\nabla u|^p + V(\varepsilon_n z + z_n^i)|u|^p)\,dx - \int_{\mathbb{R}^N} F(u)\,dx,$$

for $u \in W^{1,p}(\mathbb{R}^N)$, it is possible to prove that the sequences $\{t_n^i\}$ are also convergent in $(0,\infty)$. Using this information, we have that w_n^i converges to $w^i \in W^{1,p}(\mathbb{R}^N) \setminus \{0\}$ with

$$J'_{V(z^i)}(w^i) = 0 \quad \text{and} \quad J_{V(z^i)}(w^i) \geq c_{V(z^i)}.$$

From (5.5),

$$\varepsilon_n^{-N} J_{\varepsilon_n}(u_{\varepsilon_n}) \geq \widetilde{J}_n^1(t_n^1(v_n^1)^+) + \widetilde{J}_n^2(t_n^2(v_n^2)^-)$$

consequently

$$\varepsilon_n^{-N} J_{\varepsilon_n}(u_{\varepsilon_n}) \geq \widetilde{J}_{n,1}(w_n^1) + \widetilde{J}_{n,2}(w_n^2).$$

Taking the limit as $n \to \infty$ in the last inequality, we get

$$2c_{V_0} \geq J_{V(z_1)}(w^1) + J_{V(z_1)}(w^1) \geq c_{V(z^1)} + c_{V(z^2)}. \tag{5.6}$$

Since by hypothesis, let us assume $V(z^1) > V(0)$ and $V(z^2) \geq V(0)$, it follows the inequality

$$c_{V(z^1)} + c_{V(z^2)} > 2c_{V(0)}$$

which contradicts (5.6). □

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