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NEUMANN CONDITION IN THE SCHRÖDINGER–MAXWELL SYSTEM

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Abstract. We study a system of (nonlinear) Schrödinger and Maxwell equation in a bounded domain, with a Dirichelet boundary condition for the wave function ψ and a nonhomogeneous Neumann datum for the electric potential ϕ. Under a suitable compatibility condition, we establish the existence of infinitely many static solutions $\psi = u(x)$ in equilibrium with a purely electrostatic field $\mathbf{E} = -\nabla\phi$. Due to the Neumann condition, the same electric field is in equilibrium with stationary solutions $\psi = e^{-i\omega t}u(x)$ of arbitrary frequency ω.

1. Introduction

This paper is concerned with a system of Schrödinger and Maxwell equations. The unknown are the wave function $\psi = \psi(x,t)$ and the gauge potentials $\mathbf{A}$ and ϕ related to the electromagnetic field $(\mathbf{E},\mathbf{B})$. In other words the electromagnetic field is not assigned, but it is generated by the charged particle whose wave function is ψ. This topic has been introduced by Benci and Fortunato in [3] and it has been developed in several papers (see references below).

We consider a stationary wave

$$\psi = u(x)\, e^{-i\omega t}, \tag{1.1}$$

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with $u(x) \in \mathbb{R}$ and $\omega \in \mathbb{R}$, in equilibrium with a purely electrostatic field

$$\mathbf{E} = -\nabla\phi \tag{1.2}$$

with $\phi = \phi(x)$.

Using the minimal coupling rule (see [6] or [20]), after some calculations, which can be found in [3], we get the following system

$$-\frac{1}{2}\Delta u + q\phi u = \omega u, \tag{1.3}$$
$$-\Delta\phi = 4\pi q u^2. \tag{1.4}$$

We point out that (1.4) is the Gauss equation with charge density $\rho = qu^2$ and q is the charge of the particle whose wave function is ψ. Indeed, if we assume the usual normalizing condition for the wave function

$$\int_\Omega u^2\,dx = 1, \tag{1.5}$$

we have

$$\int_\Omega \rho\,dx = \int_\Omega q\,u^2\,dx = q.$$

The system (1.3)–(1.4) will be studied in a open bounded regular set $\Omega \subset \mathbb{R}^3$ with the following boundary conditions

$$u(x) = 0, \tag{1.6}$$
$$\frac{\partial\phi}{\partial\mathbf{n}}(x) = g(x), \tag{1.7}$$

where $\mathbf{n}$ denotes the exterior unit normal on $\partial\Omega$.

The Neumann condition on ϕ has an interesting physical interpretation: g prescribes the charge of the particle. Indeed we have the well known compatibility condition between (1.4) and (1.7):

$$-4\pi q \int_\Omega u^2\,dx = \int_{\partial\Omega} g\,ds.$$

Taking into account the normalizing condition (1.5), we deduce the necessary condition

$$q = -\frac{1}{4\pi}\int_{\partial\Omega} g\,ds. \tag{1.8}$$

Of course the equations (1.3) and (1.4) are really coupled if $q \neq 0$.

In the paper of Benci and Fortunato [3] the same system (1.3)–(1.4) was studied with the Dirichlet condition also on ϕ. Here the Neumann condition has another interesting consequence.

The "eigenvalue" ω plays no role in the existence of solutions of (1.3)–(1.4). Indeed, suppose $q \neq 0$, for every $\omega \in \mathbb{R}$, the pair (u, ϕ) is a solution of

$$\begin{cases} -\dfrac{1}{2}\Delta u + q\phi u = 0 & \text{in } \Omega, \\ \Delta\phi + 4\pi q u^2 = 0 & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \\ \dfrac{\partial\phi}{\partial\mathbf{n}}(x) = g(x) & \text{on } \partial\Omega, \end{cases}$$

if and only if $(u, \phi - \omega/q)$ is a solution of (1.3)–(1.4) (1.6)–(1.7). In other words, if we find $u(x)$, static solution of the Schrödinger equation, then we have also the stationary solutions (1.1) with any frequency $\omega \in \mathbb{R}$. All these solutions are associated to the same electric field, indeed the change of variable $\phi \mapsto \phi - \omega/q$ has no effect on $\mathbf{E}$ defined by (1.2).

We recall that in the Dirichlet problem we have infinitely many solutions of (1.3)–(1.4), for every value of the charge, but with discrete values of the frequency ω.

As it is usual in this kind of problems, we can perturb the Schrödinger equation (1.3) with a nonlinear term $|u|^{p-2}u$. So our main result can be stated as follows.

THEOREM 1.1. *Consider the system*

$$\begin{cases} -\dfrac{1}{2}\Delta u + q\phi u - \alpha|u|^{p-2}u = 0 & \textit{in } \Omega, \\ \Delta\phi + 4\pi q u^2 = 0 & \textit{in } \Omega, \\ u(x) = 0 & \textit{on } \partial\Omega, \\ \displaystyle\int_\Omega u^2(x)\,dx = 1 & \\ \dfrac{\partial\phi}{\partial\mathbf{n}}(x) = g(x) & \textit{on } \partial\Omega, \end{cases} \tag{1.9}$$

with $q, \alpha \in \mathbb{R}$, $p \in (2, 10/3)$, $g \in H^{1/2}(\partial\Omega)$. *If* (1.8) *holds true with* $q \neq 0$, *then there exist* $\{u_k\} \subset H_0^1(\Omega)$, $\chi \in H^1(\Omega)$, $\{\varphi_k\} \subset H^1(\Omega)$, $\{\mu_k\} \subset \mathbb{R}$ *with*

$$\int_\Omega \chi\,dx = \int_\Omega \varphi_k\,dx = 0 \quad \textit{and} \quad \int_\Omega |\nabla u_k|^2\,dx \to \infty, \quad \mu_k \to \infty,$$

as $k \to \infty$, *such that* $(u_k, \chi + \varphi_k + \mu_k)$ *are solutions of* (1.9).

REMARK 1.2. By our choice of the functional spaces, whose definitions we recall below, the solutions we find have finite energy.

REMARK 1.3. We point out that the perturbation $|u|^{p-2}u$ does not influence the existence of solutions, indeed we can consider also $\alpha = 0$. If $\alpha < 0$, the same result holds with $p \in (2, 6)$.

Remark 1.4. If (1.8) holds true with $q = 0$ (uncoupled case), there are infinitely many solutions of (1.3)–(1.4), (1.5)–(1.7) with discrete frequencies ω, even if we perturb (1.3) with $|u|^{p-2}u$.

We recall that the case with assigned electromagnetic field has been studied in [1], [2] and [11].

On the other hand, after the quoted paper [3], a large literature on systems of Schrödinger–Maxwell equations and Klein–Gordon–Maxwell equations has been developed. The existence and non-existence of solitary waves in $\mathbb{R}^n$ for Schrödinger–Maxwell systems has been studied in [8]–[10], [14], [18]. Other papers are concerned with the semiclassical limit in the Schrödinger equation, i.e. they consider

$$-\frac{\hbar^2}{2}\Delta u + q\phi u = \omega u + f(u)$$

(coupled with (1.4)) and study the asymptotic behavior of solutions as $\hbar \to 0$ (see [12], [13], [16], [17], [23]). Multiplicity results and non-existence theorems on Klein–Gordon–Maxwell systems can be found in [4], [5], [7], [14], [15], [19], [21].

2. Variational setting

In (1.8), for the sake of simplicity we assume

$$q = -\frac{1}{4\pi}\int_{\partial\Omega} g\,ds = -1. \tag{2.1}$$

Moreover, we shall consider the more interesting case $\alpha \geq 0$.

Taking into account (2.1), we have to solve

$$\begin{cases} -\dfrac{1}{2}\Delta u - \phi u - \alpha|u|^{p-2}u = 0 & \text{in } \Omega, \\ \Delta\phi - 4\pi u^2 = 0 & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \\ \displaystyle\int_\Omega u^2(x)\,dx = 1 & \\ \dfrac{\partial\phi}{\partial\mathbf{n}}(x) = g(x) & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

First we make a change of variables to deal with an homogeneous boundary datum. So we consider the problem

$$\begin{cases} \Delta\chi = K & \text{in } \Omega, \\ \displaystyle\int_\Omega \chi\,dx = 0, & \\ \dfrac{\partial\chi}{\partial\mathbf{n}} = g & \text{on } \partial\Omega, \end{cases} \tag{2.3}$$

where $K = 4\pi/|\Omega|$ and $|\Omega|$ is the measure of Ω. By (2.1), the problem (2.3) has a unique weak solution of class $C(\overline{\Omega})$ (see [25]).

Now we set

$$\mu = \frac{1}{|\Omega|}\int_\Omega \phi\,dx, \quad \varphi = \phi - \chi - \mu.$$

With the new variables (u, φ, μ), our problem (2.2) becomes

$$-\frac{1}{2}\Delta u - (\chi+\varphi)u - \alpha|u|^{p-2}u = \mu u \quad \text{in } \Omega, \tag{2.4a}$$

$$\Delta\varphi + K - 4\pi u^2 = 0 \quad \text{in } \Omega, \tag{2.4b}$$

$$u(x) = 0 \quad \text{on } \partial\Omega, \tag{2.4c}$$

$$\int_\Omega u^2(x)\,dx = 1, \tag{2.4d}$$

$$\frac{\partial\varphi}{\partial\mathbf{n}} = 0 \quad \text{on } \partial\Omega, \tag{2.4e}$$

$$\int_\Omega \varphi(x)\,dx = 0. \tag{2.4f}$$

The problem (2.4) has a variational characterization as eigenvalue problem.

Consider the Sobolev space $H^1(\Omega)$ endowed with norm

$$\|\xi\|_{H^1} = \|\xi\|_2 + \|\nabla\xi\|_2,$$

where, hereafter, $\|\cdot\|_p$ denotes the usual L^p norm. We have

$$H^1(\Omega) = \widetilde{H} \oplus \mathbb{R}$$

where $\widetilde{H} = \{\eta \in H^1(\Omega) : \int_\Omega \eta\,dx = 0\}$.

On $\widetilde{H}$ we have the equivalent norm $\|\eta\|_{\widetilde{H}} = \|\nabla\eta\|_2$.

We recall that $H_0^1(\Omega)$ denotes the completion of $C_0^\infty(\Omega)$ in $H^1(\Omega)$. In $H_0^1(\Omega)$ we shall use the equivalent norm $\|v\|_{H_0^1} = \|\nabla v\|_2$.

Consider the functional $F\colon H_0^1(\Omega)\times H^1(\Omega) \to \mathbb{R}$ defined as follows

$$F(u,\phi) = \int_\Omega \left(\frac{1}{4}|\nabla u|^2 - \frac{1}{2}\chi u^2 - \frac{\alpha}{p}|u|^p\right)dx - \frac{1}{2}\int_\Omega u^2\phi\,dx - \int_\Omega\left(\frac{1}{16\pi}|\nabla\phi|^2\,dx - \frac{1}{2|\Omega|}\phi\right)dx. \tag{2.5}$$

This functional is C^1 and we have, for every $v \in H_0^1(\Omega)$ and $\xi \in H^1(\Omega)$,

$$\langle F'_u(u,\phi), v\rangle = \int_\Omega\left(\frac{1}{2}\nabla u\nabla v - (\phi+\chi)uv - \alpha|u|^{p-2}uv\right)dx, \tag{2.6}$$

$$\langle F'_\phi(u,\phi), \xi\rangle = \int_\Omega\left(-\frac{1}{8\pi}\nabla\phi\nabla\xi - \frac{1}{2}u^2\xi + \frac{1}{2|\Omega|}\xi\right)dx. \tag{2.7}$$

Let

$$S = \left\{u \in H_0^1(\Omega) : \int_\Omega u^2\,dx = 1\right\}.$$

Such S is a smooth manifold of codimension 1; for every $u \in S$ the tangent space to S is

$$T_u S = \left\{ v \in H_0^1(\Omega) : \int_\Omega uv\, dx = 0 \right\}. \tag{2.8}$$

Now we can characterize the solutions of (2.4).

Theorem 2.1. *The triplet (u, φ, μ), $u \in H_0^1(\Omega)$, $\varphi \in H^1(\Omega)$, $\mu \in \mathbb{R}$, is a solution of (2.4) if and only if (u, φ) is a critical point of F constrained on $S \times \widetilde{H}$ and*

$$\mu = \langle F'_u(u, \varphi), u \rangle.$$

We recall that (u, φ) critical point of F constrained on $S \times \widetilde{H}$ means that $(u, \varphi) \in S \times \widetilde{H}$ and

$$\langle F'_u(u, \varphi), v \rangle = 0, \quad \text{for all } v \in T_u S, \tag{2.9}$$

$$\langle F'_\phi(u, \varphi), \eta \rangle = 0, \quad \text{for all } \eta \in T_\varphi \widetilde{H} = \widetilde{H}. \tag{2.10}$$

Proof. The "only if" part is obvious.

Suppose that (u, φ) is a critical point of F constrained on $S \times \widetilde{H}$. The normalizing condition (2.4d) and (2.4f) are satisfied.

From (2.9) and $\langle F'_u(u, \varphi), u \rangle = \mu$, by standard arguments, we obtain (2.4a) and (2.4c).

Now consider $\xi \in H^1(\Omega)$. We have $\xi = \eta + \lambda$ with $\eta \in \widetilde{H}$ and $\lambda \in \mathbb{R}$, hence

$$\langle F'_\phi(u, \varphi), \xi \rangle = \langle F'_\phi(u, \varphi), \eta \rangle + \langle F'_\phi(u, \varphi), \lambda \rangle = 0. \tag{2.11}$$

Indeed

(a) $\langle F'_\phi(u, \varphi), \eta \rangle = 0$ by (2.10).

(b) On the other hand

$$\langle F'_\phi(u, \varphi), \lambda \rangle = -\frac{1}{2}\lambda \int_\Omega u^2\, dx + \frac{\lambda}{2|\Omega|} \int_\Omega dx = \frac{\lambda}{2}\Big(\int_\Omega u^2\, dx - 1\Big) = 0.$$

From (2.11) we deduce (2.4b) and (2.4e). □

The functional F constrained on $S \times \widetilde{H}$ is unbounded from above and from below, even modulo compact perturbations. So the next step is to characterize the critical points of F constrained on $S \times \widetilde{H}$ as critical point of a functional defined on S and bounded from below.

The following result follows from the Sobolev embedding and the Riesz representation theorems.

PROPOSITION 2.2. *For every $w \in L^{6/5}(\Omega)$ there exists a unique $L(w) \in \widetilde{H}$ such that, for every $\eta \in \widetilde{H}$,*

$$\int_\Omega \nabla L(w) \nabla \eta \, dx + \int_\Omega w \eta \, dx = 0.$$

The map $L: L^{6/5}(\Omega) \to \widetilde{H}$ is linear and continuous, hence C^∞.

The next result follows from well known properties of the Nemytsky operator (see e.g. [26]).

PROPOSITION 2.3. *The map $u \in L^6(\Omega) \mapsto 4\pi u^2 - K \in L^{6/5}(\Omega)$ is of class C^1.*

Taking into account the previous propositions, we can define the C^1 map

$$\Phi: H_0^1(\Omega) \to \widetilde{H}, \qquad \Phi(u) = L(4\pi u^2 - K).$$

For every $(u, \varphi) \in H_0^1(\Omega) \times \widetilde{H}$, we have $\varphi = \Phi(u)$ if and only if, for every $\eta \in \widetilde{H}$,

$$\int_\Omega \nabla \varphi \nabla \eta \, dx + \int_\Omega (4\pi u^2 - K) \eta \, dx = 0. \tag{2.12}$$

If we take $\eta = \Phi(u)$ in (2.12), we obtain

$$\int_\Omega |\nabla \Phi(u)|^2 dx + \int_\Omega (4\pi u^2 - K) \Phi(u) \, dx = 0,$$

that is

$$\int_\Omega |\nabla \Phi(u)|^2 dx + \int_\Omega 4\pi u^2 \Phi(u) \, dx = 0, \tag{2.13}$$

from which we deduce

$$\|\nabla \Phi(u)\|_2^2 \le 4\pi \|u^2\|_2 \|\Phi(u)\|_2 \le c_1 \|u\|_4^2 \|\nabla \Phi(u)\|_2$$

(where, from now on, c_i, $i = 1, 2, \ldots$ stand for suitable positive constants). So

$$\|\nabla \Phi(u)\|_2 \le c_2 \|\nabla u\|_2^2$$

and we have proved the following lemma.

LEMMA 2.4. *The map Φ is bounded, i.e. it maps bounded sets of $H_0^1(\Omega)$ in bounded sets of $\widetilde{H}$.*

REMARK 2.5. We notice that, for every $(u, \varphi) \in H_0^1(\Omega) \times \widetilde{H}$, (2.12) can be written as $\langle F'_\phi(u, \varphi), \eta \rangle = 0$, hence the map $\varphi = \Phi(u)$ is implicitly defined by

$F'_\phi(u,\varphi)=0$, as cotangent vector on $\widetilde{H}$. Moreover, if $u\in S$, then (2.11) holds true for every $\xi\in H^1(\Omega)$, so we deduce that $\Phi(u)\in\widetilde{H}$ is the unique solution of

$$\begin{cases} \Delta\varphi+K-4\pi u^2=0 & \text{in } \Omega,\\ \dfrac{\partial\varphi}{\partial\mathbf{n}}=0 & \text{on } \partial\Omega,\\ \displaystyle\int_\Omega \varphi\,dx=0. \end{cases} \tag{2.14}$$

On the other hand, if $u\notin S$, the problem (2.14) has no solutions. In this sense $\Phi_{|S}$ is implicitly defined by (2.14). Since Φ is C^1, we have proved that the solution of (2.14) has a C^1 dependence on $u\in S$.

So we can define the C^1 functional

$$J\colon H_0^1(\Omega)\to\mathbb{R},\qquad J(u)=F(u,\Phi(u)). \tag{2.15}$$

From the previous remark we deduce that for every $u,v\in H_0^1(\Omega)$

$$\langle J'(u),v\rangle=\langle F'_u(u,\Phi(u)),v\rangle.$$

Then we easily get the following result.

Theorem 2.6. *The pair $(u,\varphi)\in S\times\widetilde{H}$ is a critical point of F constrained on $S\times\widetilde{H}$ if and only if u is a critical point of $J_{|S}$ and $\varphi=\Phi(u)$.*

Taking into account Theorem 2.1 and Theorem 2.6, we have to prove that there exists a sequence $\{u_k\}$ of critical points of J constrained on S. Then we shall set

$$\varphi_k=\Phi(u_k),$$
$$\mu_k=\langle F'_u(u_k,\varphi_k),u_k\rangle=\langle F'_u(u_k,\Phi(u_k)),u_k\rangle=\langle J'(u_k),u_k\rangle.$$

Finally, in order to complete the proof of Theorem 1.1, we are going to show that

$$\|\nabla u_k\|_2\to\infty,\qquad \langle J'(u_k),u_k\rangle\to\infty.$$

3. Proof of Theorem 1.1

The functional J has been defined in (2.15). Taking into account (2.13), we obtain

$$J(u)=\int_\Omega\Big(\frac{1}{4}|\nabla u|^2-\frac{1}{2}\chi u^2-\frac{\alpha}{p}|u|^p\Big)\,dx+\frac{1}{16\pi}\int_\Omega|\nabla\Phi(u)|^2\,dx.$$

First we prove that J constrained on S is bounded from below.

Let D a regular domain in $\mathbb{R}^n$. Using the Sobolev Embedding Theorem, we can prove a simple lemma about the Sobolev space

$$W^{1,s}(D)=\{u\in L^s(D):\nabla u\in L^s(D)\}.$$

LEMMA 3.1. *Assume*

$$1 \le s < n, \tag{3.1}$$

$$s < p < s^* = \frac{ns}{n-s}, \tag{3.2}$$

$$0 < r \le n\left(1 - \frac{p}{s^*}\right). \tag{3.3}$$

There exists $C > 0$ such that, for every $u \in W^{1,s}(D)$,

$$\|u\|_p^p \le C\|u\|_{W^{1,s}}^{p-r}\|u\|_s^r. \tag{3.4}$$

PROOF. By (3.2) and the Sobolev embeddings, we have

$$W^{1,s}(D) \hookrightarrow L^p(D). \tag{3.5}$$

For every $u \in W^{1,s}(D)$, we have $|u|^r \in L^{s/r}(D)$ and $|u|^{p-r} \in L^{s/(s-r)}(D)$, indeed the left hand side of (3.2) and the right hand side of (3.3) imply that

$$\frac{s(p-r)}{s-r} \in (s, s^*). \tag{3.6}$$

So, by the Hölder inequality, we deduce

$$\|u\|_p^p = \int_D |u|^{p-r}|u|^r\,dx \le \||u|^{p-r}\|_{s/(s-r)}\||u|^r\|_{s/r} = \|u\|_{s(p-r)/(s-r)}^{p-r}\|u\|_s^r. \tag{3.7}$$

By (3.6), we get

$$\|u\|_{s(p-r)/(s-r)} \le C\|u\|_{W^{1,s}} \tag{3.8}$$

with C independent of u. Substituting (3.8) in (3.7), we get the thesis. □

REMARK 3.2. If D is bounded, then (3.4) is trivially true also in the case $p \in [1, s]$, with $r < p$. Moreover, if $u \in W_0^{1,s}(D)$, then we can write

$$\|u\|_p^p \le C\|\nabla u\|_s^{p-r}\|u\|_s^r. \tag{3.9}$$

PROPOSITION 3.3. *The functional*

$$J_1(u) = \int_\Omega \left(\frac{1}{4}|\nabla u|^2 - \frac{1}{2}\chi u^2 - \frac{\alpha}{p}|u|^p\right) dx, \tag{3.10}$$

constrained on S, is bounded from below and coercive, that is for every sequence $\{u_n\} \subset S$, if $\|\nabla u_n\|_2 \to \infty$ then $J_1(u_n) \to \infty$. Hence the functional

$$J(u) = J_1(u) + \frac{1}{16\pi}\|\nabla\Phi(u)\|_2^2,$$

constrained on S, is bounded from below and coercive.

PROOF. For every $u \in S$ we have

$$\left|\int_\Omega \chi u^2\,dx\right| \le \|\chi\|_\infty\|u^2\|_1 = \|\chi\|_\infty\|u\|_2^2 = \|\chi\|_\infty. \tag{3.11}$$

Now we apply Lemma 3.1 with $s = 2$ and $n = 3$. Since $2 < p < 10/3$, we have

$$p - 2 < 3\left(1 - \frac{p}{6}\right) < 2,$$

and we can choose r such that

$$p - 2 < r \leq 3\left(1 - \frac{p}{6}\right).$$

So, by (3.9),

$$\int_\Omega |u|^p \, dx \leq c_5 \|\nabla u\|_2^{p-r}.$$

Therefore we obtain

$$J_1(u) \geq \frac{1}{4}\|\nabla u\|_2^2 - c_6 \|\nabla u\|_2^{p-r} - \frac{1}{2}\|\chi\|_\infty,$$

which implies the thesis about J_1 (indeed $p - r < 2$). Since $J(u) \geq J_1(u)$, the same conclusions follow for J. □

Theorem 3.4. *The functional $J_{|S}$ satisfies the Palais–Smale condition, i.e. every $\{u_n\} \subset S$ such that $\{J(u_n)\}$ is bounded and $J'_{|S}(u_n) \to 0$ has a convergent subsequence.*

Proof. Let $\{u_n\} \subset S$ such that

$$\{J(u_n)\} \text{ is bounded,} \tag{3.12}$$

$$J'_{|S}(u_n) \to 0. \tag{3.13}$$

Since J is coercive, from (3.12) we deduce that $\{u_n\}$ is bounded in $H_0^1(\Omega)$, hence, up to subsequence,

$$u_n \rightharpoonup u \quad \text{in } H_0^1(\Omega). \tag{3.14}$$

Since J_1, defined in (3.10), is bounded from below, from (3.12) we deduce that $\{\|\nabla\Phi(u_n)\|_2\}$ is bounded; then, up to subtracting a subsequence, we have also

$$\varphi_n = \Phi(u_n) \rightharpoonup \varphi \quad \text{in } \widetilde{H}.$$

By the compact embedding of $H_0^1(\Omega)$ in $L^2(\Omega)$, we have

$$u_n \to u \quad \text{in } L^2(\Omega) \tag{3.15}$$

and of course $u \in S$. We have to prove that

$$u_n \to u \quad \text{in } H_0^1(\Omega). \tag{3.16}$$

For simplicity, we set $T_n = T_{u_n}S$. (3.13) means that for every sequence $\{\xi_n\} \subset H_0^1(\Omega)$ with $\xi_n \in T_n$, it results

$$|\langle J'(u_n), \xi_n\rangle| = \varepsilon_n \|\xi_n\| \quad \text{with } \varepsilon_n \to 0. \tag{3.17}$$

Let P_n be the projection on T_n. Consider

$$\xi_n = P_n(u_n - u) = (\int_\Omega u\, u_n\, dx)u_n - u \in T_n.$$

By (3.15), we have

$$a_n = \int_\Omega u\, u_n\, dx \to 1 \tag{3.18}$$

and, in virtue of (3.14),

$$\xi_n \rightharpoonup 0 \quad \text{in } H_0^1(\Omega). \tag{3.19}$$

By (3.17) and (3.19), we have

$$\langle J'(u_n) - J'(u), \xi_n \rangle \to 0. \tag{3.20}$$

Expanding the bracket, one reads

$$\langle J'(u_n) - J'(u), \xi_n \rangle = \frac{1}{2} \int_\Omega |\nabla u_n - \nabla u|^2\, dx + A_n + B_n + C_n \tag{3.21}$$

where,

$$\begin{aligned} A_n &= \frac{1}{2}(a_n - 1) \int_\Omega (\nabla u_n - \nabla u)\nabla u_n\, dx, \\ B_n &= \int_\Omega [u(\chi + \varphi) - u_n(\chi + \varphi_n)]\xi_n\, dx, \\ C_n &= \alpha \int_\Omega (|u|^{p-2}u - |u_n|^{p-2}u_n)\xi_n\, dx. \end{aligned}$$

(3.16) will follow from (3.20) and (3.21) if we prove that $A_n, B_n, C_n \to 0$.

By (3.18) we have

$$|A_n| \le \frac{1}{2}|a_n - 1| \|\nabla u_n - \nabla u\|_2 \|\nabla u_n\|_2 \to 0.$$

Furthermore (3.19) yields $\|\xi_n\|_3 \to 0$, so

$$|B_n| \le (\|u_n\|_3 \|\chi + \varphi_n\|_3 + \|u\|_3 \|\chi + \varphi\|_3)\|\xi_n\|_3 \to 0.$$

Since $2 < 2(p-1) < 14/3 < 6$, and $\{u_n\}$ is bounded in $H_0^1(\Omega)$, we deduce that $\{\|u_n\|_{2(p-1)}\}$ is bounded. Then, since $\|\xi_n\|_2 \to 0$, we obtain

$$|C_n| \le \alpha \|\xi_n\|_2 \||u|^{p-1} - |u_n|^{p-1}\|_2 \le \alpha \|\xi_n\|_2 (\|u\|_{2(p-1)}^{p-1} + \|u_n\|_{2(p-1)}^{p-1}) \to 0. \quad \square$$

Finally we can prove the existence of a sequence of critical points of J constrained on S.

Since J is even, we can use the Krasnoselskii *genus* index theory. Let us recall the basic definition: for every $A \subset S$ closed and symmetric subset of S, the *genus* of A, denoted by $\gamma(A)$, is defined as the smallest integer $k \in \mathbb{N}$ for which there exists an odd and continuous map $h\colon A \to \mathbb{R}^k \setminus \{0\}$. If there is no finite such k we set $\gamma(A) = +\infty$ and, finally, $\gamma(\emptyset) = 0$.

For every $b \in \mathbb{R}$ the sublevel

$$J^b = \{u \in S : J(u) \le b\}$$

has finite *genus* (see e.g. [3, Lemma 9]). So, for every $k \in \mathbb{N}$, we can set

$$n_k = \gamma(J^k)$$

and $I_{n_k+1} = \{A \subset S : A \text{ closed}, A = -A \text{ and } \gamma(A) \ge n_k + 1\}$.

We know that $I_{n_k+1} \ne \emptyset$ (see Lemma 8 of [3]), so we can consider

$$b_k = \inf_{I_{n_k+1}} \sup_A J.$$

It is well known (see e.g. [24]) that b_k is a critical value for $J_{|S}$. So there exists $u_k \in S$ critical point of $J_{|S}$ such that

$$J(u_k) = b_k \ge k. \tag{3.22}$$

In order to complete the proof of Theorem 1.1, we have to show that, up to a subsequence,

$$\|\nabla u_k\|_2 \to \infty, \tag{3.23}$$

$$\langle J'(u_k), u_k\rangle \to \infty. \tag{3.24}$$

Arguing by contradiction, we deduce that $\{J_1(u_k)\}$ (defined in (3.10)) is bounded; indeed

$$|J_1(u_k)| \le \frac{1}{4}\|\nabla u_k\|_2^2 + \frac{1}{2}\|\chi\|_\infty + c_7\|\nabla u_k\|_2^{p-r},$$

where r is the same constant used in the proof of Proposition 3.3. Moreover, by Lemma 2.4, $\{\|\nabla\Phi(u_k)\|_2\}$ is also bounded. Then the sum

$$J(u_k) = J_1(u_k) + \frac{1}{16\pi}\|\nabla\Phi(u_k)\|_2^2$$

is bounded and this contradicts (3.22).

On the other hand we have

$$\langle J'(u_k), u_k\rangle = \frac{1}{2}\int_\Omega |\nabla u_k|^2 dx - \int_\Omega u_k^2(\chi + \Phi(u_k))\,dx - \alpha\int_\Omega |u_k|^p\,dx.$$

By (2.13),

$$\begin{aligned}\langle J'(u_k), u_k\rangle &= \int_\Omega \left(\frac{1}{2}|\nabla u_k|^2 - \alpha|u_k|^p - u_k^2\chi\right)dx + \frac{1}{4\pi}\int_\Omega |\nabla\Phi(u_k)|^2\,dx \\ &\ge \int_\Omega \left(\frac{1}{2}|\nabla u_k|^2 - \alpha|u_k|^p - u_k^2\chi\right)dx \\ &\ge \frac{1}{2}\|\nabla u_k\|_2^2 - c_8\|\nabla u_k\|_2^{p-r} - \|\chi\|_\infty.\end{aligned}$$

Then, by (3.23), we deduce $\langle J'(u_k), u_k\rangle \to \infty$. The proof of Theorem 1.1 is thereby complete. □

References

[1] J. Avron, I. Herbst and B. Simon, *Schrödinger operators with magnetic fields* I. *General interaction*, Duke Math. J. **45** (1978), 847–883.

[2] ———, *Schrödinger operators with magnetic fields. Atoms in homogeneous magnetic fields*, Comm. Math. Phys. **79** (1981), 529–883.

[3] V. Benci and D. Fortunato, *An eigenvalue problem for the Schrödinger–Maxwell equations*, Topol. Methods Nonlinear Anal. **11** (1998), 283–293.

[4] ———, *Solitary waves of the nonlinear Klein–Gordon field equation coupled with the Maxwell equations*, Rev. Math. Phys. **14** (2002), 409–420.

[5] ———, *The nonlinear Klein–Gordon equation coupled with the Maxwell equations*, Nonlinear Anal. **47** (2001), 6065–6072.

[6] D. Bleecker, *Gauge Theory and Variational Principles*, Dover Publications, 2005.

[7] D. Cassani, *Existence and non-existence of solitary waves for the critical Klein–Gordon equation coupled with Maxwell's equations*, Nonlinear Anal. **58** (2004), 733–747.

[8] G. M. Coclite, *A multiplicity result for the nonlinear Schrödinger-Maxwell equations*, Commun. Appl. Anal. **7** (2003), no. 2–3, 417–423.

[9] ———, *A multiplicity result for the Schrödinger–Maxwell equations*, Ann. Polon. Math. **79** (2002), 21–30.

[10] G. M. Coclite and V. Georgiev, *Solitary Waves for Schrödinger–Maxwell equations*, Electron. J. Differential Equations **2004** (2004), no. 94, 1–31.

[11] J. M. Combes, R. Schrader and R. Seiler, *Classical bounds and limits for energy distributions of Hamiltonian operators in electromagnetic fields*, Ann. Phys. **111** (1978), 1–18.

[12] T. D'Aprile, *An eigenvalue semiclassical problem for the Schrödinger operator with an electrostatic field*, Topol. Methods Nonlinear Anal. **27** (2006), 145–176.

[13] ———, *Semiclassical states for the nonlinear Schrödinger equation with the electromagnetic field*, NoDEA Nonlinear Differential Equations Appl. (to appear).

[14] T. D'Aprile and D. Mugnai, *Solitary waves for nonlinear Klein–Gordon–Maxwell and Schrödinger–Maxwell equations*, Proc. Roy. Soc. Edinburgh Sect. A **134** (2004), 1–14.

[15] ———, *Non-existence results for the coupled Klein-Gordon-Maxwell equations*, Adv. Nonlinear Stud. **4** (2004), 307–322.

[16] T. D'Aprile and J. Wei, *On bound states concentrating on spheres for the Maxwell–Schrödinger equation*, SIAM J. Math. Anal. **37** (2005), 321–342.

[17] ———, *Standing waves in the Maxwell–Schrödinger equation and an optimal configuration problem*, Calc. Var. Partial Differential Equations **25** (2006), 105–137.

[18] P. D'Avenia, *Non-radially symmetric solutions of nonlinear Schrödinger equation coupled with Maxwell equations*, Adv. Nonlinear Stud. **2** (2002), 177–192.

[19] P. D'Avenia and L. Pisani, *Nonlinear Klein–Gordon equations coupled with Born–Infeld type equations*, Electron. J. Differential Equations **2002** (2002), no. 26, 1–13.

[20] B. Felsager, *Geometry, Particles and Fields*, Springer, 1998.

[21] V. Giorgiev and N. Visciglia, *Solitary waves for Klein-Gordon-Maxwell systems with external Coulomb potential*, J. Math. Pures Appl. (9) **84** (2005), 957–983.

[22] D. Mugnai, *Coupled Klein–Gordon and Born–Infeld type equations: looking for solitary waves*, R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci. **460** (2004), 1519–1528.

[23] D. Ruiz, *Semiclassical states for coupled Schrödinger-Maxwell equations: concentration around a sphere,*, Math. Models Methods Appl. Sci. **15** (2005), 141–164.

[24] M. Struwe, *Variational Methods and Their Applications to Non-linear Partial Differential Equations and Hamiltonian System*, Springer, 1990.

[25] M. E. Taylor, *Partial Differential Equations* I, Springer, 1996.

[26] M. M. Vainberg, *Variational Methods for the Study of Nonlinear Operators*, Holden–Day, Inc., 1964.

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