FIXED POINT THEOREMS
AND DENJOY–WOLFF THEOREMS
FOR HILBERT'S PROJECTIVE
METRIC IN INFINITE DIMENSIONS

Roger D. Nussbaum

ABSTRACT. Let K be a closed, normal cone with nonempty interior $\operatorname{int}(K)$ in a Banach space X. Let $\Sigma = \{x \in \operatorname{int}(K) : q(x) = 1\}$ where $q : \operatorname{int}(K) \to (0, \infty)$ is continuous and homogeneous of degree 1 and it is usually assumed that Σ is bounded in norm. In this framework there is a complete metric d, *Hilbert's projective metric*, defined on Σ and a complete metric $\overline{d}$, *Thompson's metric*, defined on $\operatorname{int}(K)$. We study primarily maps $f : \Sigma \to \Sigma$ which are nonexpansive with respect to d, but also maps $g : \operatorname{int}(K) \to \operatorname{int}(K)$ which are nonexpansive with respect to $\overline{d}$. We prove under essentially minimal compactness assumptions, fixed point theorems for f and g. We generalize to infinite dimensions results of A. F. Beardon (see also A. Karlsson and G. Noskov) concerning the behaviour of Hilbert's projective metric near $\partial\Sigma := \overline{\Sigma} \setminus \Sigma$. If $x \in \Sigma$, $f : \Sigma \to \Sigma$ is nonexpansive with respect to Hilbert's projective metric, f has no fixed points on Σ and f satisfies certain mild compactness assumptions, we prove that $\omega(x; f)$, the omega limit set of x under f in the norm topology, is contained in $\partial\Sigma$; and there exists $\eta \in \partial\Sigma$, η independent of x, such that $(1-t)y + t\eta \in \partial K$ for $0 \le t \le 1$ and all $y \in \omega(x; f)$. This generalizes results of Beardon and of Karlsson and Noskov. We give some evidence for the conjecture that $\operatorname{co}(\omega(x; f))$, the convex hull of $\omega(x; f)$, is contained in ∂K.

1. Introduction

Let $B = \{x+iy \mid x^2+y^2 < 1\}$ denote the open unit ball in $\mathbb{C}$ and let $f\colon B \to B$ be an analytic map which has no fixed points in B. The classical Denjoy–Wolff theorem asserts that there exists $\zeta \in \partial B$ such that $f^k(z) \to \zeta$ as $k \to \infty$ for every $z \in B$. Here f^k denotes the k-fold composition of f with itself. See [4] and [5] for references to the classical theorems. Extensions to several complex variables and references to the literature can be found in [1]. A generalization to the infinite dimensional case is given in [24].

A. F. Beardon [5] has observed that the key idea in the original Denjoy–Wolff theorem concerns the behaviour of the Poincaré metric on B near ∂B. He shows that a generalization of the Denjoy–Wolff theorem can be proved for suitable locally compact metric spaces (G, ρ). In particular, Beardon considers bounded, strictly convex open subsets G of $\mathbb{R}^n$ equipped with Hilbert's projective metric d, and he proves a variant of the following theorem.

THEOREM 1.1 (Compare A. F. Beardon [5]). *Let G be a bounded, open, strictly convex subset of $\mathbb{R}^n$ and let d denote Hilbert's projective metric on G. Suppose that $f\colon G \to G$ is nonexpansive with respect to d (so $d(f(x), f(y)) \le d(x, y)$ for all $x, y \in G$) and f has no fixed points in G. Then there exists $\zeta \in \partial G$ such that $f^k(x) \to \zeta$ for all $x \in G$.*

Almost all applications in analysis of which this author is aware involve open sets G which are *not* strictly convex. Often $G = H \cap (\mathrm{int}(K^n))$, where $K^n = \{x \in \mathbb{R}^n \mid x_i \ge 0 \text{ for } 1 \le i \le n\}$ and $H = \{x \in \mathbb{R}^n \mid \Sigma_{k=1}^n x_i = 1\}$, so G is an open subset of the hyperplane H. When G is not strictly convex, one can easily give examples (see [32]) which show that Theorem 1.1 must be modified. In general, it is natural to consider the omega limit set $\omega(x; f)$ of x under f,

$$\omega(x; f) = \bigcap_{n \ge 1} c\ell(\{f^k(x) : k \ge n\}),$$

where $c\ell(A)$ denotes the norm closure of a set $A \subset G$. If G is a bounded open convex subset of $\mathbb{R}^n$ and $f\colon G \to G$ is nonexpansive with respect to d and fixed point free, Karlsson and Noskov [27] prove that for each $x \in G$, there exists $\zeta \in \omega(x; f)$ such that $(1 - t)\zeta + ty \in \partial G$ for all $y \in \omega(x; f)$ and all t with $0 \le t \le 1$. As we shall discuss later, it is likely that much stronger results are true. Note especially Conjectures 4.21–4.23 and the remarks following Conjectures 4.22 and 4.23 in Section 4.

For applications in analysis (see, for example, [37]) it is useful to study maps in infinite dimensional Banach spaces which are nonexpansive with respect to d. However, arguments in [5] do not generalize in a straightforward way to infinite dimensions. In particular, the analysis given in [5] of the behaviour of d near

∂G uses compactness assumptions which fail in infinite dimensions; and fixed point theorem arguments used in [5], while straightforward in finite dimensions, become nontrivial in the infinite dimensional case.

In this paper we begin in Section 3 by proving some new fixed point theorems. These theorems allow us to extend some results of [37, Chapter 4] to what appears the proper level of generality. The fixed point theorems also play a role in generalizing the Beardon and Karlsson–Noskov results to infinite dimensions (Section 4). We also present some conjectures (see Conjectures 4.21–4.23 in Section 4) that a much sharper result should be true, and in Sections 4 and 5 we give some evidence for these conjectures. Note that very recent results of B. C. Lins (see [33]) provide strong evidence for these conjectures in the finite dimensional case.

This paper is long, so a guide may be in order. Section 2 recalls some basic terminology and results from the literature. It may be safely skipped by experts familiar with the definition of Hilbert's projective metric in terms of a partial ordering induced by a cone C in a Banach space. In finite dimensions the cone approach is equivalent to studying Hilbert's projective metric on bounded, open convex subsets of $\mathbb{R}^n$.

In Sections 3–5 we usually consider a closed, normal cone C with nonempty interior $\mathrm{int}(C)$ in a Banach space $(X, \|\cdot\|)$. See Section 2 for definitions. We denote by $q: \mathrm{int}(C) \to (0, \infty)$ a continuous, homogeneous of degree one map, we write $\Sigma := \Sigma_q := \{x \in \mathrm{int}(C) \mid q(x) = 1\}$, and we usually assume that Σ is bounded in norm. We often take $q(x) = \|x\|$, the point being that there may not exist a continuous linear functional q which is positive on $\mathrm{int}(C)$ and which satisfies $\sup\{\|x\| \mid x \in \Sigma_q\} < \infty$. If $D \subset \Sigma$, we say that D is quasi-convex if, whenever $x, y \in D$, $[(1-t)x + ty]/q((1-t)x + y)$ is an element of D for $0 \le t \le 1$. In Section 3 we prove the following fixed point theorem.

THEOREM 1.2 (see Theorem 3.10). *Suppose that $D \subset \Sigma$ is quasi-convex and closed and bounded in Σ with respect to Hilbert's projective metric d. Suppose that $f: \Sigma \to \Sigma$ is nonexpansive with respect to d and $f(D) \subset D$. Assume that there exists an integer n such that $f^n|D$ is a condensing map in the norm topology. Then f has a fixed point in D.*

A more general version of Theorem 3.10 is given in Theorem 3.3'. Corollaries 3.6 and 3.6' give natural conditions under which a set like D exists and a nonexpansive map $f: \Sigma \to \Sigma$ has a fixed point in Σ. Theorem 3.10 and subsequent corollaries concern analogues of these theorems for maps $g: \mathrm{int}(C) \to \mathrm{int}(C)$ which are nonexpansive with respect to *Thompson's metric $\overline{d}$*. If $f: \Sigma \to \Sigma$ is nonexpansive with respect to d and f has no fixed points in Σ and if f satisfies mild compactness assumptions, it is proved in Theorem 3.14 that given $y \in \Sigma$

and $R > 0$, $d(f^j(y), y) > R$ for all but finitely many integers $j \geq 0$. This gener-
alizes an old result of [37, Chapter 4]. In a future paper we shall show that these
fixed point theorems can also be used to generalize results of [37, Chapter 4]
concerning the structure of the fixed point sets of maps which are nonexpansive
with respect to d.

Section 4 presents direct generalizations to infinite dimensions of the Beardon
and Karlsson–Noskov theorems: see Theorem 4.14 and 4.17. Basically, we prove
that if $f: \Sigma \to \Sigma$ is nonexpansive with respect to d, has no fixed points in Σ
and satisfies mild compactness conditions, then there exists $\zeta \in \partial\Sigma$ such that
for every $z \in \bigcup_{x \in \Sigma} \omega(x; f)$, $(1 - t)\zeta + tz \in \partial C$ for $0 \leq t \leq 1$. In fact, the
proofs of Theorems 4.14 and 4.17 are different, and their exact relationship
remains unclear. A key tool in proving Theorems 4.14 and 4.17 is provided
by Theorem 4.3, which describes the behaviour of d near $\partial\Sigma$ and generalizes
corresponding results in finite dimensions.

In Conjectures 4.21–4.23 we give variants of the conjecture that, under the
hypotheses of Theorem 4.14 or 4.17,

$$\mathrm{co}\left(\bigcup_{x \in \Sigma} \omega(x; f) \right) \subset \partial C,$$

where $\mathrm{co}(A)$ denotes the convex hull of a set A. In Sections 4 and 5 we present
some evidence for these conjectures.

2. Preliminaries

We recall here for the reader's convenience some definitions and well-known
results.

By a *closed cone* (with vertex at 0) in a Banach space X, we mean a closed,
convex set $C \subset X$ such that $C \cap (-C) = \{0\}$ and $\lambda C \subset C$ for all $\lambda \geq 0$. A closed
cone C in a Banach space X induces a partial ordering $\leq_C$ on X by $x \leq_C y$ if
and only if $y - x \in C$. If C is obvious, we shall write $\leq$ instead of $\leq_C$. A closed
cone C in a Banach space $(X, \| \cdot \|)$ is called *normal* if there exists a constant
A such that $\|x\| \leq A\|y\|$ whenever $0 \leq x \leq y$. It is known (see [21] or [48])
that any closed cone in a finite dimensional Banach space $(X, \| \cdot \|)$ is normal.
Furthermore (see [21] or [48]), if C is a closed, normal cone in a Banach space
$(X, \| \cdot \|)$, there exists an equivalent norm $\| | \cdot | \|$ on X such that $\| | x | \| \leq \| | y | \|$
whenever $0 \leq x \leq y$. In general, a map $f: D \subset X \to \mathbb{R}$ is called *order preserving*
(with respect to the partial ordering induced by a closed cone C) if $f(x) \leq f(y)$
whenever $x, y \in D$ and $x \leq y$. Thus the map $x \to f(x) := \| | x | \|$ with domain
$D := C$ is order-preserving. More generally, if C is a closed cone in a Banach
space X, a map $f: D \subset X \to X$ is called order-preserving (with respect to the

partial ordering $\leq$ induced by C) if whenever $x, y \in D$ and $x \leq y$ it follows that $f(x) \leq f(y)$.

If C is a closed cone in a Banach space X and $x \in C - \{0\}$ and $y \in X$, we shall say that x dominates y if there exists a real number b such that $y \leq bx$. The ordering $\leq$ is that induced by C. If x dominates y, we shall follow Bushell's notation in [10] and define

$$M(y/x; C) := M(y/x) := \inf\{b \in \mathbb{R} \mid y \leq bx\}.$$

By using the fact that $x \in C - \{0\}$ and C is a closed cone, it is easy to see that $M(y/x) > -\infty$. If $x \in C - \{0\}$, $y \in X$ and there exists a real number a with $ax \leq y$, we define

$$m(y/x; C) := m(y/x) := \sup\{a \in \mathbb{R} \mid a\, x \leq y\}.$$

One easily proves that $m(y/x) < \infty$ and, if $m(y/x)$ and $M(y/x)$ are defined, $m(y/x) \leq M(y/x)$. If $x, y \in C - \{0\}$ and x dominates y and y dominates x, we write $x \sim y$ and say that x is comparable to y. Comparability gives an equivalence relation on $C - \{0\}$. If $u \in C - \{0\}$ we define

(2.1) $$C_u = \{x \in C - \{0\} \mid x \sim u\}.$$

Note that if $\text{int}(C)$, the interior of C, is nonempty and $u \in \text{int}(C)$, then $\text{int}(C) = C_u$.

More generally, if C is a closed cone in a Banach space X and $u \in C - \{0\}$, we can define (see [21] or [48]) a normed linear space X_u by

$$X_u = \{x \in X \mid \text{ there exists } \alpha \geq 0 \text{ with } -\alpha u \leq x \leq \alpha u\}$$

and $|x|_u = \inf\{\alpha \geq 0 \mid -\alpha u \leq x \leq \alpha u\}$ for $x \in X_u$. If C is a closed normal cone, X_u is a Banach space, $C \cap X_u$ is a closed, normal cone in X_u and the interior of $C \cap X_u$ in X_u is C_u. By this observation, our theorems, which will generally apply to normal cones with nonempty interiors, can be applied to $C \cap X_u$ in X_u. If $y \in C - \{0\}$ and $x \in C_u$, then $\alpha := m(y/x)$ and $\beta = M(y/x)$ are defined and $0 < \alpha \leq \beta$; and we define Hilbert's projective metric d on C_u by

$$d(x, y) := \log\left(\frac{\beta}{\alpha}\right).$$

If $x, y, z \in C_u$ and λ and μ are positive scalars, one easily derives from this definition that

$$d(x, y) = d(y, x), \quad d(x, z) \leq d(x, y) + d(y, z) \quad \text{and} \quad d(\lambda x, \mu y) = d(x, y).$$

For $x, y \in C_u, d(x, y) = 0$ if and only if $y = tx$ for some $t > 0$. It follows that d is a true metric on the space of rays in C_u. For proofs and references to the literature, we refer the reader to [10], [37, Chapter 1], [39, Section 1],

[11] and [21]. An equivalent definition of d in terms of *cross products* will be described later and is usually preferred by geometers (see [5], [6], [25] and [26]) but we find the above definition more suitable for applications in analysis.

If $u \in C - \{0\}$ and $\Sigma = \{x \in C_u \mid \|x\| = 1\}$, the above remarks show that Hilbert's projective metric d, restricted to Σ, makes (Σ, d) a metric space. If $\theta \in C^*$ and $\theta(u) > 0$, we shall also want to consider $\Sigma := \{x \in C_u \mid \theta(x) = 1\}$. Note, however, that in infinite dimensions it may not be possible to choose $\theta \in C^* \setminus \{0\}$ such that $\Sigma_\theta := \{x \in C_u \mid \theta(x) = 1\}$ is bounded in norm, and it is partly for this reason that we shall also consider $\Sigma = \{x \in C_u \mid \|x\| = 1\}$.

If C is a closed cone with nonempty interior in a Banach space $(X, \|\cdot\|)$ and $q \colon \operatorname{int}(C) \to (0, \infty)$ is a norm continuous map which is homogeneous of degree 1, we shall always write

$$(2.2) \qquad \Sigma := \Sigma_q := \{x \in \operatorname{int}(C) \mid q(x) = 1\},$$

and we shall denote the closure in the norm topology by $\overline{\Sigma}_q$. If $q_j \colon \operatorname{int}(C) \to (0, \infty)$ is a norm continuous map which is homogeneous of degree one, note that there is a norm continuous homeomorphism $\Phi \colon \Sigma_{q_1} \to \Sigma_{q_2}$ defined by

$$\Phi(x) = \left(\frac{x}{q_2(x)} \right) \quad \text{and} \quad \Phi^{-1}(y) = \left(\frac{y}{q_1(y)} \right).$$

The map Φ is also an isometry of (Σ_{q_1}, d) onto (Σ_{q_2}, d), so theorems about (Σ_{q_1}, d) yield corresponding theorems about $(\Sigma_{q_2,d})$ by means of Φ.

It is important to know when (Σ_q, d) is a complete metric space. Results in this direction, in varing degrees of generality, have been obtained by several authors. We refer, for example, to G. Birkhoff [7], Zabreĭko, Krasnosel'skiĭ and Pokornyĭ [52] and the book [29]. Further references to the literature can be found in [37, p. 12–18], and in Section 1 of [39]. One should note that Zabreĭko, Krasnosel'skiĭ and Pokornyĭ were apparently unaware of closely related work of G. Birkhoff [7], E. Hopf [23], A. C. Thompson [51] and others.

LEMMA 2.1 (see [52]). *Let C be a closed, normal cone in a Banach space X. For $u \in C \setminus \{0\}$, suppose that $q \colon C_u \to (0, \infty)$ is continuous in the norm topology and homogeneous of degree one and let $\Sigma_q := \{x \in C_u \mid q(x) = 1\}$. If d denotes Hilbert's projective metric restricted to Σ_q, (Σ_q, d) is a complete metric space.*

Hilbert's projective metric d and the norm $\|\cdot\|$ on X give the same topology when restricted to Σ_q in Lemma 2.1, but, roughly speaking, d puts the points in $\overline{\Sigma} \setminus \Sigma$ at infinity.

If C is a closed, normal cone with nonempty interior in a Banach space X and, by definition of normality, A is a constant such that

$$\|x\| \le A\|y\| \quad \text{for } 0 \le x \le y,$$

equation (1.14) in [39] implies that for all $x, y \in \Sigma := \{z \in \text{int}(C) \mid \|z\| = 1\}$,

$$(2.3) \qquad \|x - y\| \le 2A[\exp(d(x, y)) - 1].$$

If $y \in \text{int}(C)$ and $\rho = \rho(y) > 0$ is such that $\{z \in X \mid \|z - y\| < \rho\} \subset \text{int}(C)$, equation (1.17) in [39] implies that for all x with $\|x - y\| < \rho$ we have

$$(2.4) \qquad d(x, y) \le \log \left[\frac{\rho + \|x - y\|}{\rho - \|x - y\|} \right].$$

The following standard lemma follows easily from equations (2.3) and (2.4).

LEMMA 2.2. *Let C be a closed, normal cone with nonempty interior in a Banach space $(X, \|\cdot\|)$. Suppose that $q: \text{int}(C) \to (0, \infty)$ is homogeneous of degree one and continuous in the norm topology and define $\Sigma_q = \{x \in \text{int}(C) \mid q(x) = 1\}$. If $\langle x_k \mid k \ge 1 \rangle$ is a sequence in Σ_q and $x \in \Sigma_q$, then $\lim_{k \to \infty} d(x_k, x) = 0$ if and only if $\lim_{k \to \infty} \|x_k - x\| = 0$. If $K \subset \Sigma_q$, then K is compact in the topology from d if and only if K is compact in the norm topology.*

PROOF. If $q(x) := q_1(x) := \|x\|$, Lemma 2.2 follows easily from equations (2.3) and (2.4). For general q, define $\Phi: \Sigma_{q_1} \to \Sigma_q$ by $\Phi(x) = (x/q(x))$ and note that Φ is a homeomorphism onto Σ_q in the norm topology and an isometry in the d-topology. $\qquad \square$

REMARK 2.3. Part of Lemma 2.2 only depends on equation (2.4) and thus only requires that C have nonempty interior and not that C be normal. Let C be a closed cone with nonempty interior in a Banach space X and let Σ_q be as defined in Lemma 2.2. If $K \subset \Sigma_q$ is compact in the norm topology, then it follows that K is compact in the topology induced by Hilbert's projective metric d. The proof is left to the reader.

The assumption that a closed cone is normal may seem unrestrictive, especially since any closed cone in a finite dimensional Banach space is necessarily normal. However, in infinite dimensions, even in discussing linear operators, it may be necessary to consider non-normal cones: see [40], where non-normal cones play a central role in the treatment of linear Perron–Frobenius operators.

Usually we shall be assuming that C is a closed, normal cone with nonempty interior in a Banach space X, so it may be useful to mention some examples. If C is a closed cone in a finite dimensional Banach space X, then C is normal; and if Y is the smallest linear subspace of X which contains C, C is a closed, normal cone with nonempty interior in Y. In particular, if $X = \mathbb{R}^n$, a simple example of a closed cone is given by

$$(2.5) \qquad K^n = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid x_i \ge 0 \text{ for } 1 \le i \le n\}.$$

If S is a compact Hausdorff space, let $C(S)$ denote the Banach space of real-valued continuous maps $f\colon S \to \mathbb{R}$, with $\|f\| := \sup\{|f(t)| \mid t \in S\}$. If we define $C \subset X := C(S)$ by

$$C = \{f \in X \mid f(t) \geq 0 \text{ for all } t \in S\},$$

then C is a closed, normal cone with nonempty interior in X. For $S = \{1, \dots, n\}$ we can identify $C(S)$ with $\mathbb{R}^n$ and C with K^n. If H is a Hilbert space, X is the Banach space of bounded, self-adjoint linear operators $L\colon H \to H$ and C is the cone of positive semi-definite operators $L \in X$, then C is a closed, normal cone with nonempty interior in X.

   Finally, we mention a class of examples which provides the connection between the geometer's definition of d in terms of "cross-ratio" and the definition above. Suppose that G is a bounded, open convex set in a Banach space Y. By translating G we can assume that $0 \in G$, and we let $p\colon Y \to \mathbb{R}$ denote the Minkowski functional of G (see [13, p. 108] or [17, p. 411]) so

$$p(y) := \inf\{s > 0 \mid y \in sG\}, \quad y \in Y.$$

Let $X = Y \times \mathbb{R}$, a Banach space, and for a fixed positive constant λ, define $C \subset X$ by

$$C = \{x := (y, t) \in X \mid p(y) \leq \lambda t\}.$$

One can prove that C is a closed, normal cone with nonempty interior in X. If we define $\theta((y, t)) = \lambda^{-1} t$ for $(y, t) \in X$, then $\theta \in C^* \setminus \{0\}$ and

$$\Sigma := \{x \in \text{int}(C) \mid \theta(x) = 1\} = G \times \{\lambda\}.$$

Lemmas 2.1 and 2.2 imply that $(\Sigma, d) = (G \times \{\lambda\}, d)$ is a complete metric space and that the topology induced by d on $G \times \{\lambda\}$ is the same as that induced by the norm on X. We can identify $G \times (\lambda)$ with G and abusing notation slightly, consider d as a metric on G via the identification, so for $y_1, y_2 \in G$,

$$d(y_1, y_2) := d((y_1, \lambda), (y_2, \lambda)).$$

   If $y_1, y_2 \in G$ with $y_1 \neq y_2$ we can also follow Hilbert [22] and consider the straight line ℓ passing through y_1 and y_2. The line ℓ intersects ∂G in precisely two points, a and b, and we can assume that the points a, y_1, y_2, b appear in that order on ℓ. We define $[a, y_1, y_2, b]$, the cross-ratio of a, y_1, y_2, b, by

$$[a, y_1, y_2, b] = \frac{\|y_2 - a\|\|y_1 - b\|}{\|y_1 - a\|\|y_2 - b\|},$$

and we define

$$\widehat{d}(y_1, y_2) = \log([a, y_1, y_2, b])$$

and $\widehat{d}(y_1, y_2) = 0$ if $y_1 = y_2$. It is straightforward calculation (see [37, p. 31–32]) to show that

$$d(y_1, y_2) = \widehat{d}(y_1, y_2) \quad \text{for all } y_1, y_2 \in G.$$

As we have already noted, geometers prefer the above definition in terms of cross-ratio. Typically, analysts have been less interested in Hilbert's projective metric *per se* than in the fact that many important classes of maps are either *contractions with respect to d* or *nonexpansive with respect to d*. We refer, for example, to the beautiful classical theory of positive linear operators (see [7], [23] and [47]); and we refer to [19] and [20] for further references to the literature. Many maps of interest in analysis arise naturally as cone-preserving operators.

If C is a closed, normal cone with nonempty interior in a Banach space X, $q: \text{int}(C) \to (0, \infty)$ is norm continuous and homogeneous of degree one, Σ_q is given by equation (2.2) and $\overline{\Sigma}_q$ denotes the norm closure of Σ_q, we shall be interested in maps $f: \Sigma \to \Sigma$ such that for all $x, y \in \Sigma$.

(2.6) $$d(f(x), f(y)) \leq d(x, y).$$

Here, of course, d denotes Hilbert's projective metric on $\text{int}(C)$. If $f: \Sigma \to \Sigma$ satisfies (2.6), we shall call f *nonexpansive with respect to d* or *d-nonexpansive*. If $g: \text{int}(C) \to C$ we shall say that g is *homogeneous of degree one* if $g(tx) = tg(x)$ for all $t > 0$ and $x \in \text{int}(C)$. If $g: \text{int}(C) \to \text{int}(C)$ is order-preserving and homogeneous of degree one, one can define $f: \Sigma \to \Sigma$ by

(2.7) $$f(x) = g(x)/q(g(x))$$

and it is easy to prove (see [37, Chapter 1]) that f is d-nonexpansive. More generally, $g: \text{int}(C) \to C$ is called *subhomogeneous* if $g(tx) \geq tg(x)$ whenever $x \in \text{int}(C)$ and $0 < t \leq 1$. If $g: \text{int}(C) \to \text{int}(C)$ is order-preserving and sub-homogeneous, q is order-preserving and $f: \Sigma \to \Sigma$ is defined by (2.7), one can easily prove that f is d-nonexpansive.

In general, if C, Σ_q and $\overline{\Sigma}_q$ are as above, $f: \Sigma_q \to \Sigma_q$ is a map and $y \in \Sigma_q$, we shall need to study the *omega limit set of y under f*. However, it is necessary to distinguish carefully between the omega limit set taken in the norm topology on $\overline{\Sigma}_q$ and the omega limit set taken with respect to the metric d on Σ_q. We define

(2.8) $$\omega(y; f, \|\cdot\|) = \bigcap_{n \geq 1} (\text{norm closure in } \overline{\Sigma}_q \text{ of } \{f^j(y) \mid j \geq n\}).$$

Alternatively, it is known that $z \in \omega(y; f, \|\cdot\|)$ if and only if $z \in \overline{\Sigma}_q$ and there exists a sequence of integers $n_j \to \infty$ such that

$$\lim_{j \to \infty} \|f^{n_j}(y) - z\| = 0.$$

If $S \subset \Sigma_q$, we can take the closure of S in Σ with respect to the metric d, and shall call this the *d-closure of S*. We define

(2.9) $$\omega(y; f, d) = \bigcap_{n \geq 1} (d - \text{closure in } \Sigma_q \text{ of } \{f^j(y) \mid j \geq n\}).$$

It is known that $z \in \omega(y; f, d)$ if and only if $z \in \Sigma_q$ and there exists a sequence of integers $n_j \to \infty$ such that

$$\lim_{j \to \infty} d(f^{n_j}(y), z) = 0.$$

Lemma 2.2 implies that

$$\omega(y; f, d) = \omega(y; f, \|\cdot\|) \bigcap \Sigma_q.$$

However, the eventual focus of this paper will be to specify the location of $\omega(y; f, \|\cdot\|)$ in $\overline{\Sigma}_q$ when $\omega(y; f, d)$ is empty. If $\omega(y; f, d)$ is nonempty and $f: \Sigma_q \to \Sigma_q$ is d-nonexpansive, a result of Dafermos and Slemrod (see [14] and [37, remarks, p. 111–112]) implies that $f|\omega(y; f, d)$ is an isometry in the d metric of $\omega(y; f, d)$ onto $\omega(y; f, d)$. Furthermore, for all $z \in \omega(y; f, d)$, $\omega(y; f, d) = \omega(z; f, d)$; and there exists a sequence of positive integers $\sigma_k \to \infty$ such that $\lim_{k \to \infty} d(f^{\sigma_k}(z), z) = 0$ for all $z \in \omega(y; f, d)$. Note also closely related results of Edelstein [18].

3. Omega limit sets and fixed points of maps $f: \Sigma \to \Sigma$

In this section, we shall give generalizations of some fixed point theorems for cone mappings, in particular, generalizations of Theorems 4.1 and 4.2 of [37, p. 114]. It is convenient first to recall Kuratowski's measure of noncompactness [31]. If (Y, ρ) is a metric space and $S \subset Y$, we define $\text{diam}(S)$, the diameter of S, by

$$\text{diam}(S) = \sup\{\rho(s, t) \mid s, t \in S\},$$

and we say that S is *bounded* or of *finite diameter* if $\text{diam}(S) < \infty$. If $S \subset Y$ is bounded, $\alpha(S)$, the Kuratowski measure of noncompactness of S, is defined by

$$\alpha(S) = \inf \left\{ \delta > 0 \ \middle| \ S = \bigcup_{i=1}^{n} S_i, \text{ with } n < \infty \text{ and } \text{diam}(S_i) \leq \delta \text{ for } 1 \leq i \leq n \right\}.$$

For any bounded sets S, T in Y, one easily verifies that $\alpha(\overline{S}) = \alpha(S)$ and $\alpha(S \cup T) = \max\{\alpha(S), \alpha(T)\}$. If (Y, ρ) is a complete metric space and $S_n, n \geq 1$, is a decreasing sequence of closed, bounded nonempty sets in Y and $\lim_{n \to \infty} \alpha(S_n) = 0$, then Kuratowski [31] proved the $S_\infty := \cap S_n$ is a nonempty, compact subset of Y. Furthermore, if U is any open set with $S_\infty \subset U$, then there exists an integer $N(U) = N$ with $S_n \subset U$ for all $n \geq N$.

It was G. Darbo [15] who first realized the usefulness of the Kuratowski measure of noncompactness in fixed point theory. If $(Y, \| \cdot \|)$ is a Banach space (so the metric ρ on Y is given by $\rho(x,y) = \|x - y\|$) and S and T are bounded subsets of Y, define $\mathrm{co}(S)$, the *convex hull of* S, to be the smallest convex set which contains S and define $S + T := \{s + t \mid s \in S, \ t \in T\}$. Darbo proved that

$$\alpha(\mathrm{co}(S)) = \alpha(S) \quad \text{and} \quad \alpha(S + T) \leq \alpha(S) + \alpha(T).$$

If λ is a scalar and $\lambda S := \{\lambda s \mid s \in S\}$, one also easily verifies that $\alpha(\lambda S) = |\lambda| \alpha(S)$. We shall denote by $\overline{\mathrm{co}}(S)$ the closure in the norm topology of $\mathrm{co}(S)$, and we shall call $\overline{\mathrm{co}}(S)$ the closed, convex hull of S.

Suppose that $D \subset (Y, \| \cdot \|)$, $F \colon D \to Y$ is a continuous map and $c \geq 0$. We shall call f a c-set-contraction (with respect to the Kuratowski measure of noncompactness) if $\alpha(f(C)) \leq c\alpha(S)$ for all bounded sets $S \subset D$. Darbo [15] used these ideas to give an elegant generalization of the Schauder fixed point theorem: If G is a closed, bounded, convex set in a Banach space and $f \colon G \to G$ is a c-set-contraction, $c < 1$, f has a fixed point in G. If D is a subset of a Banach space Y, $f \colon D \to Y$ is a continuous map and $\alpha(f(S)) < \alpha(S)$ whenever $0 < \alpha(S) < \infty$, f is called a condensing map. If G is a closed, bounded convex set in a Banach space and $f \colon G \to G$ is a condensing map, Sadovskiĭ [46] observed that f has a fixed point. Sadovskiĭ's theorem directly generalizes Darbo's result, although one should note that Sadovskiĭ actually worked with a measure of noncompactness different from Kuratowski's. Conversely, one can easily use Darbo's theorem and a limiting argument to obtain Sadovskiĭ's theorem, although this was not Sadovskii's original approach.

If Σ is a convex subset of a Banach space X and $f \colon \Sigma \to \Sigma$ is a map, we say that f *satisfies the fixed point property on* Σ (see [37, p. 113]) if, for every norm closed, norm bounded, convex set $D \subset \Sigma$ such that $f(D) \subset D$, f has a fixed point in D. If $f \colon \Sigma \to \Sigma$ is a condensing map, f satisfies the fixed point property on Σ.

If C is a closed, normal cone with nonempty interior in a Banach space $(X, \| \cdot \|)$ and if $q \colon \mathrm{int}(C) \to (0, \infty)$ is a norm continuous map which is homogeneous of degree one, we shall always be interested in $\Sigma := \Sigma_q$ defined by

$$\Sigma_q := \{x \in \mathrm{int}(C) \mid q(x) = 1\}.$$

If d denotes Hilbert's projective metric on $\mathrm{int}(C)$, $z \in \mathrm{int}(C)$ and $R > 0$, we shall always write

(3.1) $$V_R(z) := \{x \in \mathrm{int}(C) \mid d(x, z) \leq R\}.$$

Note, (see [37, Lemma 4.1, p. 112]) that $V_R(z)$ is convex and, in fact, one can prove that $V_R(z) \cup \{0\}$ is a closed cone in X. If $D \subset \Sigma_q$, we shall say that D is

quasi-convex if whenever $x, y \in D$ and $0 < \alpha < 1$,

$$((1 - \alpha)x + \alpha y)/q((1 - \alpha)x + \alpha y) \in D.$$

The term *quasi-convex* has a different meaning in other areas of mathematics, but no confusion should result. If $D \subset \Sigma_q$, we shall say that D is *bounded in* (Σ_q, d) if there exist $R > 0$ and $z \in \Sigma_q$ with $D \subset V_R(z)$. If D is bounded in (Σ_q, d) and C is normal, D is closed in (Σ_q, d) or closed in the d-topology if and only if D is closed in the norm topology.

We want to generalize the definition that $f : \Sigma \to \Sigma$ satisfies the fixed point property on Σ.

DEFINITION 3.1. Let C, X, Σ_q and q be as above and assume that $f : \Sigma_q \to \Sigma_q$ is a continuous map in the norm topology. We say that f *satisfies the fixed point property on Σ_q with respect to d* if for every quasi-convex set $D \subset \Sigma_q$ such that D is closed and bounded in (Σ_q, d) and $f(D) \subset D$, f has a fixed point in D.

If $D_1 \subset \Sigma_q$ and $D_2 \subset \Sigma_q$ are quasi-convex, it is easy to see that $D_1 \cap D_2$ is quasi-convex. Thus, any intersection of quasi-convex sets is quasi-convex (though possibly empty). The reader can verify that for $z \in \operatorname{int}(C)$ and $R > 0$, $V_R(z) \cap \Sigma_q$ is quasi-convex. It follows that if $R > 0$ and $\omega \subset \operatorname{int}(C)$, then

$$D := \left(\bigcap_{z \in \omega} V_R(z) \right) \cap \Sigma_q$$

is quasi-convex.

We need conditions which insure that $f : \Sigma_q \to \Sigma_q$ satisfies the fixed point property on Σ_q with respect to d. We begin by recalling a special case of Lemma 7.5 in [3]. Note that the proof of Lemma 7.5 in [3] is closely related to the proof of Lemma 2.1 of [37, p. 45].

LEMMA 3.2 (see [3, Lemma 7.5] and [37, Lemma 2.1, p. 45]). *Let C be a closed cone with nonempty interior $\operatorname{int}(C)$ in a Banach space $(X, \| \cdot \|)$ and let d denote Hilbert's projective metric on $\operatorname{int}(C)$. Let $q : C \to [0, \infty)$ be a norm continuous map which is order-preserving (with respect to the partial ordering from C), homogeneous of degree one and strictly positive on $\operatorname{int}(C)$ and define $\Sigma = \{x \in \operatorname{int}(C) \mid q(x) = 1\}$. Given $u \in \operatorname{int}(C)$, define $\Phi_u : \operatorname{int}(C) \to \operatorname{int}(C)$ and $\Psi_u : \operatorname{int}(C) \to \operatorname{int}(C) \cap \Sigma$ by*

$$\Phi_u(x) = x + q(x)u \quad and \quad \Psi_u(x) = \frac{\Phi_u(x)}{q(\Phi_u(x))}.$$

Then, for all $v \in \operatorname{int}(C)$ and $R > 0$, there exists a constant $c = c(u, v, q, R)$ such that $0 \le c < 1$ and

$$d(\Phi_u(x), \Phi_u(y)) = d(\Psi_u(x), \Psi_u(y)) \le c\,d(x, y) \quad for\ all\ x, y \in V_R(v),$$

where $V_R(v)$ is as in (3.1). If Σ is bounded in norm, then $\{\Psi_u(z) \mid z \in \text{int}(C)\}$ is a bounded set in (Σ, d); and this fact does not depend on the assumption that q is order preserving.

With the aid of Lemma 3.2 we can prove a fixed point theorem which will later play a crucial role.

THEOREM 3.3. *Let C be a closed, normal cone with nonempty interior in a Banach space $(X, \|\cdot\|)$. Let $q\colon \text{int}(C) \to (0, \infty)$ be a positive, norm continuous map which is homogeneous of degree one and let $\Sigma_q = \{x \in \text{int}(C) \mid q(x) = 1\}$. Let $f\colon \Sigma_q \to \Sigma_q$ be a map which is nonexpansive with respect to d. For every $D \subset \Sigma_q$ which is bounded in (Σ_q, d), assume that there is a positive integer $n = n(D)$ such that $f^n|D$ is a condensing map (in the norm topology). Then (see Definition 3.1) f satisfies the fixed point property on Σ_q with respect to d.*

PROOF. Since C is normal, there exists an equivalent norm $|\cdot|$ on X whose restriction to C is order-preserving, We first assume that $q(x) = |x|$ and prove the theorem in this case. Equation (2.3) implies that for all $x, y \in \Sigma := \{x \in \text{int}(C) \mid |x| = 1\}$,

$$(3.2) \qquad |x - y| \le 2[\exp(d(x, y)) - 1].$$

Let $D \subset \Sigma$ be a quasi-convex set which is closed and bounded in (Σ, d) and such that $f(D) \subset D$. Select a fixed element $u \in D$ and a number $R > 0$ such that $D \subset V_R(u)$. For $0 < \varepsilon < 1$, define maps Φ_ε and Ψ_ε of $\text{int}(C)$ into itself by

$$\Phi_\varepsilon(x) = x + \varepsilon|x|u \quad \text{and} \quad \Psi_\varepsilon(x) = \Phi_\varepsilon(x)/|\Phi_\varepsilon(x)|.$$

If $x \in D$, $|x| = 1$ and, for $\alpha := \varepsilon/(1 + \varepsilon)$, we have

$$\Psi_\varepsilon(x) = [(1 - \alpha)x + \alpha u]/|(1 - \alpha)x + \alpha u|.$$

Since D is quasi-convex, it follows that $\Psi_\varepsilon(D) \subset D$. Lemma 3.2 implies that there is a constant $k < 1$ (k dependent on D, u and ε) with

$$(3.3) \qquad d(\Psi_\varepsilon(x), \Psi_\varepsilon(y)) \le kd(x, y) \quad \text{for all } x, y \in D.$$

If we define $f_\varepsilon\colon D \to D$ by $f_\varepsilon = \Psi_\varepsilon \circ f$, equation (3.3) implies that

$$(3.4) \qquad d(f_\varepsilon(x), f_\varepsilon(y)) \le k\, d(x, y) \quad \text{for all } x, y \in D.$$

Since D is a normal cone, (D, d) is a complete metric space, and the contraction mapping principle implies that there exists $x_\varepsilon \in D$ with $f_\varepsilon(x_\varepsilon) = x_\varepsilon$.

We need to prove that $d(f(x_\varepsilon), x_\varepsilon) \to 0$ as $\varepsilon \to 0^+$. Since $|y| = 1$ for all $y \in \Sigma_q$, the equation $f_\varepsilon(x_\varepsilon) = x_\varepsilon$ gives, for $\lambda_\varepsilon := |f(x_\varepsilon) + \varepsilon u|$,

$$f(x_\varepsilon) + \varepsilon u = |f(x_\varepsilon) + \varepsilon u|x_\varepsilon = \lambda_\varepsilon x_\varepsilon.$$

If $x \in D$, $d(x, u) \le R$, and there exist $\alpha > 0$ and $\beta > 0$ with $\alpha x \le u \le \beta x$ and $(\beta/\alpha) \le \exp(R)$. Because $y \to |y|$ is order-preserving on C, we know that $\alpha|x| = \alpha \le |u| = 1 \le \beta|x| = \beta$, so $\beta \le (\beta/\alpha) \le \exp(R)$. Combining these inequalities, we see that for $x \in D$,

$$x \le x + \varepsilon u \le x + \varepsilon \exp(R)x.$$

For λ_ε as above, we also see that

$$1 = |f(x_\varepsilon)| \le \lambda_\varepsilon = |f(x_\varepsilon) + \varepsilon u| \le |f(x_\varepsilon) + \varepsilon \exp(R)f(x_\varepsilon)| = 1 + \varepsilon \exp(R).$$

We have that

$$(3.5) \quad d(f(x_\varepsilon), x_\varepsilon) = d(f(x_\varepsilon), \lambda_\varepsilon x_\varepsilon) = d(\lambda_\varepsilon x_\varepsilon - \varepsilon u, \lambda_\varepsilon x_\varepsilon) = d(x_\varepsilon - (\varepsilon/\lambda_\varepsilon)u, x_\varepsilon)$$

We know that

$$(\varepsilon/\lambda_\varepsilon)u \le (\varepsilon/\lambda_\varepsilon)\exp(R)x_\varepsilon \le \varepsilon \exp(R)x_\varepsilon,$$

so we obtain that

$$(3.6) \qquad\qquad (1 - \varepsilon \exp(R))x_\varepsilon \le x_\varepsilon - (\varepsilon/\lambda_\varepsilon)x_\varepsilon \le x_\varepsilon.$$

If we assume, as we can, that $\varepsilon \exp(R) < 1$, equations (3.5) and (3.6) give

$$(3.7) \qquad\qquad d(f(x_\varepsilon), x_\varepsilon) \le -\log(1 - \varepsilon \exp(R)).$$

Since f is nonexpansive in Hilbert's projective metric, we obtain from (3.7) that (assuming $\varepsilon \exp(R) < 1$)

$$(3.8) \qquad d(f^n(x_\varepsilon), x_\varepsilon) \le \sum_{j=1}^{n} d(f^j(x_\varepsilon), f^{j-1}(x_\varepsilon)) \le -n\log(1 - \varepsilon \exp(R)),$$

and equations (3.2) and (3.8) give

$$(3.9) \quad |f^n(x_\varepsilon) - x_\varepsilon| \le 2[\exp(d(f^n(x_\varepsilon), x_\varepsilon)) - 1] \le 2[(1 - \varepsilon \exp(R))^{-n} - 1].$$

We now let $\varepsilon_k > 0$ be a sequence which approaches 0 and for notational convenience we write $y_k = x_{\varepsilon_k}$ and $r_k = y_k - f^n(y_k)$. Since $|\cdot|$ and $\|\cdot\|$ are equivalent norms, equation (3.9) implies that $\|r_k\| \to 0$ as $k \to \infty$. We let $A = \{y_k \mid k \ge 1\}$ and $B = \{r_k \mid k \ge 1\}$ so B has compact closure in the norm topology and

$$(3.10) \qquad\qquad\qquad A \subset f^n(A) + B.$$

Using the Kuratowski measure of noncompactness derived from $\|\cdot\|$, (3.10) implies

$$\alpha(A) \le \alpha(f^n(A)) + \alpha(B) = \alpha(f^n(A)).$$

Since $f^n|D$ is condensing, the latter equation implies that $\alpha(A) = 0$ and A has compact norm closure. Thus, by selecting a subsequence, we can assume that $\|y_k - y\| \to 0$ for some $y \in D$. Equations (3.2) and (3.7) imply that

$\|f(y_k) - y_k\| \to 0$, so the continuity of f in the norm topology on D implies that $f(y) = y$.

We now let $q \colon \text{int}(C) \to (0, \infty)$ denote a general map as in the statement of Theorem 3.3. We define $\Sigma_2 = \{y \in \text{int}(C) \mid q(y) = 1\}$ and $\Sigma_1 = \{x \in \text{int}(C) \mid |x| = 1\}$. Assume that $f \colon \Sigma_2 \to \Sigma_2$ is norm continuous and d-nonexpansive. Let $D_2 \subset \Sigma_2$ be quasi-convex, closed and bounded in (Σ_2, d) and such that $f(D_2) \subset D_2$. Assume that $f^n | D_2$ is a condensing map (in the norm topology) for some $n = n(D_2) \geq 1$. Define a homeomorphism $\vartheta \colon \Sigma_1 \to \Sigma_2$ by $\vartheta(x) = x/q(x)$, and note that $\vartheta^{-1}(y) = y/|y|$ and $\vartheta \colon (\Sigma_1, d) \to (\Sigma_2, d)$ is an isometry. It follows that $\vartheta^{-1}(D_2) := D_1$ is closed and bounded in (Σ_1, d). We claim that D_1 is also quasi-convex. If $x, y \in D_2$ and $0 < \alpha < 1$, define $x_1 = \vartheta^{-1}(x)$ and $y_1 = \vartheta^{-1}(y)$. We must prove that $[(1-\alpha)x_1 + \alpha y_1]/|(1-\alpha)x_1 + \alpha y_1|$ is an element of D_1; or, equivalently, we must prove that for

$$z := (1-\alpha)(x/|x|) + \alpha(y/|y|), \quad z/q(z) \in D_2.$$

Define $\nu^{-1} = [((1-\alpha)/|x|) + (\alpha/|y|)]$ and $\beta = (\alpha \nu)/|y|$, so $(1-\beta) = ((1-\alpha)\nu)/|x|$, and observe that, for $w := (1-\beta)x + \beta y$,

$$z/q(z) = w/q(w) \in D_2.$$

If we define $g = \vartheta^{-1} f \vartheta$, it is easily verified that $g(D_1) \subset D_1$ and $g \colon \Sigma_1 \to \Sigma_1$ is d-nonexpansive. Since D is quasi-convex and closed and bounded in (Σ_1, d), the argument of the first part of the proof shows that there exists $x_\varepsilon \in D_1$, such that $\lim_{\varepsilon \to 0^+} d(g(x_\varepsilon), x_\varepsilon) = 0 : x_\varepsilon$ is just the fixed point of $\Psi_\varepsilon \circ g$. If we define $y_\varepsilon = \vartheta(x_\varepsilon) \in D_2$, it follows that $\lim_{\varepsilon \to 0^+} d(f(y_\varepsilon), y_\varepsilon) = 0$. Since f is d-nonexpansive, we have

$$d(f^n(y_\varepsilon), y_\varepsilon) \leq \sum_{j=1}^{n} d(f^j(y_\varepsilon), f^{j-1}(y_\varepsilon)) \leq n\, d(f(y_\varepsilon), y_\varepsilon) := n\psi(\varepsilon),$$

so $\lim_{\varepsilon \to 0^+} d(f^n(y_\varepsilon), y_\varepsilon) = 0$. Equation (3.2) now implies that

$$\lim_{\varepsilon \to 0^+} \|f^n(y_\varepsilon) - y_\varepsilon\| = 0.$$

As in the special case $q(x) = |x|$, we now let $\varepsilon_k > 0$ be a sequence which approaches 0, and we write $y_k = y_{\varepsilon_k}$ and $r_k = y_k - f^n(y_k)$. Since $f^n | D_2$ is a condensing map, the same argument used previously shows that $A := \{y_k \mid k \geq 1\}$ has compact closure and that, if $k_i \to \infty$ is a sequence with $y_{k_i} \to y \in D_2$, we have $f(y) = y$. $\qquad \square$

REMARK 3.4. If G is a closed, bounded convex set in a Banach space $(X, \|\cdot\|)$ and $f \colon G \to G$ is a continuous map such that f^n is a c-set-contraction, $c < 1$, for some $n \geq 1$, it is an old conjecture that f has a fixed point. More generally, if $G \subset X$ is a finite union of closed, bounded convex sets and $f \colon G \to G$ is

a continuous map such that f^n is a c-set-contraction, $c < 1$, for some $n \geq 1$, and $L(f)$, the Lefschetz number of f, is nonzero, one may conjecture that f has a fixed point in G. In general, a number of algebraic topology tools like the fixed point index (see [8] or [16]) and the Lefschetz fixed point theorem have played a role in studying this conjecture. A variety of partial results and references to the literature can be found in [42]–[44]; but even if G is the closed unit ball in Hilbert space and f^2 is compact, the general question remains unresolved. In the context of Theorem 3.3, one may conjecture that f satisfies the fixed point property on Σ_q with respect to d even if $f: \Sigma_q \to \Sigma_q$ is not assumed nonexpansive with respect to d, but clearly the given proof depends very strongly on d-nonexpansivity.

The proof of Theorem 3.3 actually yields the following cleaner and more general version.

THEOREM 3.3'. *Let C be a closed, normal cone with nonempty interior in a Banach space $(X, \| \cdot \|)$. Let $q: \mathrm{int}(C) \to (0, \infty)$ be a norm continuous map which is homogeneous of degree one and let $\Sigma_q = \{x \in \mathrm{int}(C) \mid q(x) = 1\}$. Let $f: \Sigma_q \to \Sigma_q$ be a map which is nonexpansive with respect to d. For every sequence $\langle x_k \mid k \geq 1 \rangle \subset \Sigma_q$ such that $\langle x_k \mid k \geq 1 \rangle$ is bounded in (Σ_q, d) and $d(f(x_k), x_k) \to 0$, assume that there exist a subsequence $x_{k_i}, k_i \uparrow \infty$, and $\xi \in \Sigma_q$ with $\lim_{i \to \infty} d(x_{k_i}, \xi) = 0$. Then f satisfies the fixed point property on Σ_q with respect to d.*

Our next result is a generalization of Theorem 4.1 in [37, p. 114], but given the assumption of the fixed point property on Σ_q with respect to d, the proof is the same. We sketch the proof for the reader's convenience.

THEOREM 3.5 (comp. [37, Theorem 4.1, p. 114]). *Let C be a closed, normal cone with nonempty interior in a Banach space $(X, \| \cdot \|)$. Let $q: \mathrm{int}(C) \to (0, \infty)$ be a norm continuous map which is homogeneous of degree 1 and write $\Sigma := \Sigma_q := \{x \in \mathrm{int}(C) \mid q(x) = 1\}$. Let $f: \Sigma \to \Sigma$ be a map which is nonexpansive with respect to Hilbert's projective metric d and which satisfies the fixed point property on Σ with respect to d (see Definition 3.1, Theorem 3.3 and Theorem 3.3'). If there exists $x_* \in \Sigma$ such that $\omega(x_*; f, d)$ is nonempty and bounded in (Σ, d) (see (2.9) and Lemma 2.2), then f has a fixed point in Σ.*

PROOF. Let $\omega = \omega(x_*; f, d)$. Since (Σ, d) is a complete metric space, a result of Dafermos and Slemrod (see Section 2) implies that $f(\omega) = \omega$. Select $R \geq \sup\{d(x, y) \mid x, y \in \omega\}$, so $V_R(z) \cap \Sigma \supset \omega$ for all $z \in \omega$ and $D := (\bigcap_{z \in \omega} V_R(z)) \cap \Sigma \supset \omega$ is nonempty. If $x \in D$, then $d(x, z) \leq R$ for all $z \in \omega$, so $d(f(x), f(z)) \leq R$ for all $z \in \omega$. Since $f(\omega) = \omega, d(f(x), \zeta) \leq R$ for all $\zeta \in \omega$, so $f(x) \in D$. Because f satisfies the fixed property on Σ and because, as noted previously, D is quasi-convex and closed and bounded in (Σ, d), f has a fixed point in D. $\square$

A more easily applicable version of Theorem 3.5 is provided by the following corollary.

COROLLARY 3.6. *Let* $C, (X, \|\cdot\|), q, \Sigma_q$ *and* $f : \Sigma_q \to \Sigma_q$ *be as in Theorem* 3.3. *If there exists* $x_* \in \Sigma_q$ *and* $R_* > 0$ *such that*

$$\gamma(x_*; f) := \{f^i(x) \mid k \geq 0\} \subset V_{R_*}(x_*),$$

then f *has a fixed point in* Σ_q.

PROOF. If we write $D = \gamma(x_*; f)$, D is bounded in (Σ_q, d) and $f(D) \subset D$. By assumption there exists $n = n(D) \geq 1$ such that $f^n \mid D$ is a condensing map. Let α denote Kuratowski's measure of noncompactness. If $\alpha(D) > 0$, the equation

$$D = f^n(D) \cup \{f^j(x_*) \mid 0 \leq j < n\}$$

would imply that $\alpha(D) = \alpha(f^n(D)) < \alpha(D)$, a contradiction. Thus $\alpha(D) = 0$ and $\gamma(x_*; f)$ has compact closure B in the norm topology. Lemma 2.2 implies that $B \subset V_R(x_*) \cap \Sigma_q$, that B equals the closure of $\gamma(x_*; f)$ in the d-topology and that B is compact in the d-topology. Because $f(\gamma(x_*; f)) \subset \gamma(x_*; f)$ we have $f(B) \subset B$, so $\bigcap_{k \geq 1} f^k(B)$ is the intersection of a decreasing sequence of nonempty, compact sets and $B_\infty := \bigcap_{k \geq 1} f^k(B)$ is compact and nonempty in (Σ_q, d). However, one can easily see that $B_\infty = \omega(x_*; f, d)$, so Theorem 3.5 implies that f has a fixed point in Σ_q. $\qquad\square$

If assumptions are as in Theorem 3.3' and one assumes that $\gamma(x_*; f)$ is bounded in (Σ_q, d) and has compact closure in the norm topology, Theorems 3.3' and 3.5 give the following variant of Corollary 3.6.

COROLLARY 3.6'. *Let* C, $(X, \|\cdot\|)$, q, Σ_q *and* $f : \Sigma_q \to \Sigma_q$ *be as in Theorem* 3.3'. *If there exists* $x_* \in \Sigma_q$ *such that* $\gamma(x_*; f) = \{f^k(x_*) \mid k \geq 0\}$ *is bounded in* (Σ_q, d) *and has compact closure in the norm topology, then* f *has a fixed point in* Σ_q.

A. C. Thompson [51] has defined a useful variant of Hilbert's projective metric which we shall call Thompson's metric. If C is a closed cone in a Banach space and $u \in C\{0\}$, we define Thompson's metric $\overline{d}$ on C_u (see (2.1)) by

$$\overline{d}(x, y) = \max(\log(M(y/x)), \log(M(x/y)))$$

Thompson [51] has proved that $\overline{d}$ is indeed a metric on C_u and that, if C is a normal cone, $(C_u, \overline{d})$ is a complete metric space. If $f : C_u \to C_u$ is order-preserving (in the partial ordering from C) and subhomogeneous, then

(3.11) $$\overline{d}(f(x), f(y)) \leq \overline{d}(x, y)$$

for all $x, y \in C_u$. In general, if $f \colon D \subset C_u \to C_u$ is a map such that (3.11) is satisfied for all $x, y \in D$, we shall say that f is nonexpansive with respect to $\overline{d}$ or $\overline{d}$-nonexpansive.

If C is a closed cone with nonempty interior in a Banach space $(X, \|\cdot\|)$, $z \in \operatorname{int}(C)$ and $R > 0$, we shall always write

$$(3.12) \qquad\qquad B_R(z) := \{x \in \operatorname{int}(C) : \overline{d}(x, z) \le R\}.$$

DEFINITION 3.7. Let C be a closed cone with nonempty interior $\operatorname{int}(C)$ in a Banach space X and let $f \colon \operatorname{int}(C) \to \operatorname{int}(C)$ be a continuous map. Assume that whenever $D \subset \operatorname{int}(C)$ is convex and closed and bounded with respect to Thompson's metric $\overline{d}$ and $f(D) \subset D$ it follows that f has a fixed point in D. Then we shall say that f *satisfies the fixed point property on* $\operatorname{int}(C)$ *with respect to* $\overline{d}$.

Our previous results for Hilbert's projective metric have direct analogues for Thompson's metric. We begin with a simple calculus lemma.

LEMMA 3.8. *If* $1/2 \le k < 1$, *it follows that*

$$[\exp(k\rho) - 1][\exp(\rho) - \exp(k\rho)]^{-1} \ge (k/(1-k)) \exp(-k\rho) \quad \text{for all } \rho > 0.$$

PROOF. If $k = 1/2$, the inequality becomes an equality, so we assume that $1/2 < k < 1$. A calculation shows that proving the inequality in the lemma is equivalent to showing that

$$(1 - k)e^{k\rho}(e^{k\rho} - 1) \ge k(e^{\rho} - e^{k\rho})$$

for all $\rho > 0$. Multiplying by $e^{-k\rho}$, the latter inequalithy is equivalent to proving that

$$g(\rho) := (1 - k)e^{k\rho} - ke^{(1-k)\rho} + (2k - 1) \ge 0$$

for all $\rho > 0$. We have $g(0) = 0$, and since $1/2 < k < 1$,

$$g'(\rho) = k(1 - k)e^{k\rho} - k(1 - k)e^{(1-k)\rho} > 0$$

for all $\rho > 0$, so $g(\rho) > 0$ for all $\rho > 0$. $\qquad\qquad \square$

LEMMA 3.9. *Let* C *be a closed cone with nonempty interior in a Banach space* $(X, \|\cdot\|)$ *and let* $\overline{d}$ *denote Thompson's metric on* $\operatorname{int}(C)$. *For a fixed* $u \in \operatorname{int}(C)$ *and* $0 < \varepsilon < 1$, *define* $\vartheta_\varepsilon \colon \operatorname{int}(C) \to \operatorname{int}(C)$ *by*

$$\vartheta_\varepsilon(x) = (1 - \varepsilon)x + \varepsilon u.$$

If $z \in \operatorname{int}(C), R > 0$ *and* $u \in B_R(z), \vartheta_\varepsilon(B_R(z)) \subset B_R(z)$. *Furthermore, there exists* $k, 0 < k < 1$, *such that*

$$\overline{d}(\theta_\varepsilon(x), \vartheta_\varepsilon(y)) \le k\overline{d}(x, y) \quad \text{for all } x, y \in B_R(z).$$

PROOF. We leave to the reader the simple argument that $B_R(z)$ is convex. If $u \in B_R(z)$, it follows that $\vartheta_\varepsilon(x) \in B_R(z)$ for all $x \in B_R(z)$. If $x, y \in B_R(z)$ and $\overline{d}(x, y) = \rho > 0$, we know that $e^{-\rho} x \leq y \leq e^\rho x$. We have to find a number k, $0 < k < 1$, k independent of $x, y \in B_R((z)$ such that

$$(1 - \varepsilon)y + \varepsilon u \leq e^{k\rho}((1 - \varepsilon)x + \varepsilon u) \quad \text{and} \quad (1 - \varepsilon)x + \varepsilon u \leq e^{k\rho}((1 - \varepsilon)y + \varepsilon u).$$

Since $y \leq e^\rho x$ and $x \leq e^\rho y$, it suffices to find $k, 0 < k < 1$, such that

(3.13)
$$\begin{aligned}
(1 - \varepsilon)e^\rho x + \varepsilon u &\leq e^{k\rho}((1 - \varepsilon)x + \varepsilon u), \\
(1 - \varepsilon)e^\rho y + \varepsilon u &\leq e^{k\rho}((1 - \varepsilon)y + \varepsilon u).
\end{aligned}$$

Since $x \leq e^{2R}u$ and $y \leq e^{2R}u$ for all $x, y \in B_R(z)$, (3.13) will be satisfied if

$$(1 - \varepsilon)(e^\rho - e^{k\rho})e^{2R}u \leq \varepsilon(e^{k\rho} - 1)u$$

for $0 < \rho \leq 2R$, i.e. if

$$1 \leq \left(\frac{\varepsilon}{1 - \varepsilon}\right)e^{-2R}\left(\frac{e^{k\rho} - 1}{e^\rho - e^{k\rho}}\right) \quad \text{for } 0 < \rho \leq 2R.$$

Lemma 3.8 implies that the latter inequality will be satisfied if

(3.14)
$$1 \leq \left(\frac{\varepsilon}{1 - \varepsilon}\right)e^{-4R}\left(\frac{k}{1 - k}\right)$$

and (3.14) is satisfied if

$$k = \frac{e^{4R}(1 - \varepsilon)}{e^{4R}(1 - \varepsilon) + \varepsilon}. \qquad \square$$

We can now give a direct analogue for Thompson's metric of Theorem 3.3 for Hilbert's projective metric.

THEOREM 3.10. *Let C be a closed, normal cone with nonempty interior in a Banach space X and let $f : \text{int}(C) \to \text{int}(C)$ be a map which is nonexpansive with respect to Thompson's $\overline{d}$-metric. For every set $D \subset \text{int}(C)$ which is bounded in $(\text{int}(C), \overline{d})$ and satisfies $f(D) \subset D$, assume that there is an integer $n = n_D$ such that $f^n|D$ is a condensing map in the norm topology. Then f satisfies the fixed point property on $\text{int}(C)$ with respect to $\overline{d}$. The assumption that $f^n|D$ is condensing can be replaced by the weaker assumption that whenever $D \subset \text{int}(C)$ is closed and bounded in the $\overline{d}$ topology, $f(D) \subset D$ and $\overline{d}(f(x_k), x_k) \to 0$ for some sequence $\langle x_k \mid k \geq 1 \rangle$ in D, then there exists $\xi \in D$ with $f(\xi) = \xi$.*

PROOF. Suppose that $D \subset \text{int}(C)$ is closed and bounded in $(\text{int}(C), \overline{d})$, D is convex and $f(D) \subset D$. D is also closed and bounded in the norm topology because C is normal. If $f|D$ is condensing, Sadovskiĭ's theorem implies that f has a fixed point without the assumption that f is $\overline{d}$-nonexpansive. In general, select $u \in D$ and ε with $0 < \varepsilon < 1$ and define $\theta_\varepsilon(x) = (1 - \varepsilon) + \varepsilon u$. Lemma 3.10 implies that $\theta_\varepsilon(D) \subset D$ and that there is a constant k, $0 < k < 1$, with $\overline{d}(\theta_\varepsilon(x), \theta_\varepsilon(y)) \leq$

$k\overline{d}(x,y)$ for all $x,y \in D$. Define $f_\varepsilon = \theta_\varepsilon \circ f$, so $\overline{d}(f_\varepsilon(x), f_\varepsilon(y)) \le k\overline{d}(x,y)$ for all $x,y \in D$.

Because D is bounded in $(\mathrm{int}(C), \overline{d})$, there exists R so $\overline{d}(x,y) \le R$ for all $x,y \in D$, so $\varepsilon u \le \varepsilon e^R f(x_\varepsilon)$ and

$$x_\varepsilon = (1-\varepsilon)f(x_\varepsilon) + \varepsilon u \le \big((1-\varepsilon) + \varepsilon e^R\big)f(x_\varepsilon).$$

We also have that

$$f(x_\varepsilon) \le (1-\varepsilon)^{-1}x_\varepsilon,$$

so

$$\overline{d}(f(x_\varepsilon), x_\varepsilon) \le \max(-\log(1-\varepsilon), \log(1 - \varepsilon + \varepsilon\,e^R)) := \psi(\varepsilon).$$

If $n = n_D$ is as in the statement of the theorem,

$$(3.15) \qquad \overline{d}(f^n(x_\varepsilon), x_\varepsilon) \le \sum_{j=1}^{n} \overline{d}(f^j(x_\varepsilon), f^{j-1}(x_\varepsilon)) \le n\psi(\varepsilon).$$

If $\|\cdot\|$ denotes the norm on X, then because C is normal, there exists a constant M_1, such that

$$\|x\| \le M_1\|y\|$$

whenever $0 \le x \le y$. If $x,y \in D$ and $\overline{d}(x,y) = \rho$, we find that $0 \le y - e^{-\rho}x \le (e^\rho - e^{-\rho})x$, so

$$\|y - x\| \le \|y - e^{-\rho}x\| + \|e^{-\rho}x - x\| \le M_1(e^\rho - e^{-\rho})\|x\| + (1 - e^{-\rho})\|x\|.$$

Since $x \le e^R u$ for all $x \in D$, we see that $\|x\| \le M_1 e^R\|u\|$ and

$$(3.16) \qquad \|y - x\| \le M_1 e^R\|u\|(1 - e^{-\rho} + M_1^2(e^\rho - e^{-\rho})\|u\|, \rho := \overline{d}(x,y).$$

Applying (3.16) to $x = x_\varepsilon$ and $y = f^n(x_\varepsilon)$ and using (3.15), we see that there is a function $\psi_1(\varepsilon)$ with $\lim_{\varepsilon \to 0+} \psi_1(\varepsilon) = 0$ and

$$(3.17) \qquad \|f^n(x_\varepsilon) - x_\varepsilon\| \le \psi_1(\varepsilon)$$

for $0 < \varepsilon < 1$. Taking a sequence $\varepsilon_j \to 0+$ and defining $y_j = x_{\varepsilon_j}$, (3.17) shows that the sequence $\langle y_j - f^n(y_j) \mid j \ge 1 \rangle$ converges to zero in norm; and because f^n is condensing, the same argument used in Theorem 3.3 shows that $\{y_j \mid j \ge 1\}$ has compact closure in the norm topology and the $\overline{d}$-topology. It follows that there is a subsequence $j_i \to \infty$ with $y_{j_i} \to y \in D$, and the continuity of f implies that $f(y) = y$.

In general, even if f^n is not assumed condensing, we have shown that

$$\overline{d}(f(y_j), y_j) \to 0,$$

and the more general final statement of the theorem then implies that there exists $\eta \in D$ with $f(\eta) = \eta$. $\qquad\qquad\square$

With the aid of Theorem 3.10 we can give generalizations of Theorem 4.3 in [37, p. 117].

THEOREM 3.11 (comp. [37, Theorem 4.3, p. 117]). *Let C be a closed, normal cone with nonempty interior in a Banach space $(X, \|\cdot\|)$. Let $f\colon \mathrm{int}(C) \to \mathrm{int}(C)$ be a continuous map which is nonexpansive with respect to Thompson's metric $\overline{d}$ and which satisfies the fixed point property on $\mathrm{int}(C)$ with respect to $\overline{d}$ (see Definition 3.7 and Theorem 3.10). Define $\omega(x_*; f, \overline{d})$ by*

$$\omega(x_*; f, \overline{d}) = \bigcap_{n \geq 1} (\overline{d} - closure\ of\ \{f^j(x_*) : j \geq n\}),$$

and assume that, for some $x_* \in \mathrm{int}(C)$, $\omega(x_*; f, \overline{d})$ is nonempty and bounded in the $\overline{d}$ metric. Then f has a fixed point in $\mathrm{int}(C)$.

PROOF. Let $\omega := \omega(x_*; f, \overline{d})$. Because $(\mathrm{int}(C), \overline{d})$ is a complete metric space and f is $\overline{d}$-nonexpansive, a result of Dafermos and Slemrod (see Section 2) implies that $f(\omega) = \omega$. Select $R \geq \sup\{\overline{d}(x, y) \mid x, y \in \omega\}$ and define $D = C \bigcap_{z \in \omega} B_R(z) \supset \omega$. Note that D is convex and D is closed and bounded in $(\mathrm{int}(C), \overline{d})$. The same argument as in Theorem 3.8 shows that $f(D) \subset D$, so f has a fixed point in D. $\qquad\square$

The following result follows from Theorem 3.11 but is often more easily applicable. The proof is exactly analogous to that of Corollary 3.6 and is left to the reader.

COROLLARY 3.12. *Let C be a closed, normal cone with nonempty interior in a Banach space X. Let $f\colon \mathrm{int}(C) \to \mathrm{int}(C)$ be a map which is nonexpansive in the $\overline{d}$-metric. For each set $D \subset \mathrm{int}(C)$ which is bounded in $(\mathrm{int}(C), \overline{d})$ and satisfies $f(D) \subset D$, assume that there exists an integer $n = n_D$ such that $f^n|D$ is a condensing map in the norm metric. Assume that there exists $x_* \in \mathrm{int}(C)$ and $R_* > 0$ such that $\gamma(x_*; f) := \{f^k(x_*) \mid k \geq 0\} \subset B_{R_*}(x_*)$. Then f has a fixed point in $\mathrm{int}(C)$.*

We shall now use our previous results in order to generalize Theorem 4.2 in [37, p. 114] to the case of infinite dimensional cones. The original proof of Theorem 4.2 in [37] used a result of Roehrig and Sine [45], and we could prove the generalization here by again using the Roehrig–Sine result. However, we prefer to use a theorem of A. Calka [12]. We are indebted to Simeon Reich for informing us of Calka's paper.

Recall that a metric space (M, ρ) is *finitely totally bounded* if each bounded subset of M is *totally bounded*, i.e. if, for each bounded set $S \subset M$ and each $\varepsilon > 0$, S can be covered by a finite number of balls of radius ε. A map $f\colon M \to M$ is nonexpansive (with respect to ρ) if $\rho(f(x), f(y)) \leq \rho(x, y)$ for all $x, y \in M$. Calka [12] has proved the following theorem.

THEOREM 3.13 (see [12, Theorem 5.6]). *Let (M, ρ) be a metric space which is finitely totally bounded and let $f \colon M \to M$ be a nonexpansive map. If, for some $z_0 \in M$, the sequence $\langle f^j(z_0) \mid j \geq 0 \rangle$ contains a bounded subsequence, then for every $z \in M$ the sequence $\langle f^j(z) \mid j \geq 0 \rangle$ is bounded.*

With the aid of Theorem 3.13, we can give a direct generalization of Theorem 4.2 in [37].

THEOREM 3.14 (comp. [37, Theorem 4.2, p. 114]). *Let C, X, q, Σ_q and $f \colon \Sigma_q \to \Sigma_q$ be as in Theorems 3.3 or 3.3'. Assume that f has no fixed points in Σ_q. Assume also that there exists $y_* \in \Sigma_q$ such that $\gamma(y_*; f) := \{ f^j(y_*) \mid j \geq 0 \}$ has compact closure in the norm topology on $c\ell(\Sigma_q)$, the norm closure of Σ_q. Then for each $R > 0$ and every $y \in \Sigma_q$, there are at most finitely many integers j with $d(f^j(y), y) \leq R$.*

PROOF. We first claim that for each $R > 0$, there are at most finitely many j with $f^j(y_*) \in V_R(y_*)$. If not, there exist $R > 0$ and a sequence $n_i \to \infty$ with $f^{n_i}(y_*) \in V_R(y_*)$ for all $i \geq 1$. We define $M = \gamma(y_*; f)$, so $c\ell(M)$, the norm closure of M, is compact in the norm topology. For any $r > 0$, Lemma 2.2 implies that $c\ell(M) \cap V_r(y_*)$ is compact in the d-topology, so $M \cap V_r(y_*)$ is totally bounded in the d-topology. This shows that (M, d) is finitely totally bounded, so Calka's theorem implies that M is bounded in the d-metric. Corollary 3.6 or Corollary 3.6' now implies that f has a fixed point in Σ_q, a contradiction.

If $y \in \Sigma_q$ and $R > 0$, define $R_* = d(y_*, y)$ and select N so that $d(f^j(y_*), y_*) > R + 2R_*$ for all $j \geq N$. Since $d(f^j(y_*) f^j(y)) \leq R_*$, it follows that for all $j \geq N$,

$$d(f^j(y), y) \geq d(f^j(y_*), y_*) - d(y_*, y) - d(f^j(y_*), f^j(y)) > R,$$

which completes the proof.   □

REMARK 3.15. If Σ_q in Theorem 3.14 is bounded in norm and if there exists an integer n such that $f^n \colon \Sigma_q \to \Sigma_q$ is a condensing map in the norm metric, then $\gamma(y; f)$ has compact closure in the norm topology for all $y \in \Sigma_q$. In particular if Σ_q is bounded in norm and X is finite dimensional, $\gamma(y; f)$ has compact norm closure for all $u \in \Sigma_q$. In infinite dimensions, caution is necessary. Edelstein [18] has given an example of a fixed point free, affine linear map $f \colon H \to H$, H a separable Hilbert space, such that f is nonexpansive (with respect to the Hilbert space norm), $\{ f^{n_i}(0) \}$ is bounded for a sequence $n_i \to \infty$ but $\{ f^n(0) \colon n \geq 0 \}$ is unbounded.

As was noted in Section 2, the case of Hilbert's projective metric on a bounded, open convex set in a Banach space is subsumed by the cone case. Thus Theorem 3.14 implies the following.

COROLLARY 3.16. *Let G be a bounded, open convex set in a Banach space $(Y, \|\cdot\|)$ and let d denote Hilbert's projective metric on G. Assume that $f : G \to G$ is nonexpansive with respect to d and that f has no fixed points in G. For each set $D \subset G$ which is bounded in Hilbert's projective metric and satisfies $f(D) \subset D$ assume that there is an integer $n = n_D$ such that $f^n | D$ is a condensing map in the norm metric. Assume also that there exists $x_* \in G$ such that $\gamma(x_*; f) := \{f^j(x_* \mid j \geq 0\}$ has compact closure in the norm topology (as will be true if there exists N such that $f^N : G \to G$ is a condensing map). Then for every $x \in G$ and every $R > 0$, $d(f^j(x), x) > R$ except for finitely many j.*

By using Theorem 3.11 or Corollary 3.12 in conjunction with Calka's result (Theorem 3.13), we can also give the direct analogue for Thompson's metric $\overline{d}$ of Theorem 3.14.

THEOREM 3.17. *Let C be a closed, normal cone with nonempty interior in a Banach space X and let $f : \mathrm{int}(C) \to \mathrm{int}(C)$ be a map which is nonexpansive with respect to Thompson's metric $\overline{d}$ and has no fixed points in $(\mathrm{int}(C))$. For each set $D \subset \mathrm{int}(C)$ which is bounded in $(\mathrm{int}(C), \overline{d})$ and satisfies $f(D) \subset D$, assume that there exists an integer $n = n_D$ such that $f^n | D$ is a condensing map. Assume also that there exists $x_* \in \mathrm{int}(C)$ such that $\{f^k(x_*) \mid k \geq 0\} := \gamma(x_*; f)$ has compact closure in the norm topology. Then for every $x \in \mathrm{int}(C)$ and $R > 0$, $\overline{d}(f^j(x), x) > R$ except for finitely many j.*

PROOF. The same argument as in the proof of Theorem 3.14 shows that for $M := \gamma(x_*; f), (M, \overline{d})$ is finitely totally bounded. If there exists $R > 0$ such that $\overline{d}(f^j(x_*), x_*) \leq R$ for infinitely many j, Calka's theorem implies that $\{f^j(x_*) \mid j \geq 0\}$ is bounded in $(\mathrm{int}(C), \overline{d})$, and Corollary 3.16 then implies that f has a fixed point in $\mathrm{int}(C)$, a contradiction. The assertion of Theorem 3.17 for general $x \in \mathrm{int}(C)$ now follows exactly as in the proof of Theorem 3.14. $\square$

4. Denjoy–Wolff theorems and Hilbert's projective metric

If f is a map and $f^k(x)$ is defined for all $k \geq 0$, we shall continue using the notation of Section 3 and write

$$(4.1) \qquad \gamma(x; f) = \{f^k(x) \mid k \geq 0\}.$$

As before, $V_R(z)$ will denote the ball of radius R about z for Hilbert's projective metric d (see (3.1)) and $B_R(z)$ the corresponding ball for Thompson's metric $\overline{d}$ (see (3.12)). We shall denote by $c\ell(A)$ the closure in the norm topology of a subset A of a Banach space.

We need some further results about the maps f_ε (see (3.4)) constructed in the proof of Theorem 3.3.

LEMMA 4.1. *Let C be a closed, normal cone with nonempty interior in a Banach space $(X, \|\cdot\|)$. Let d denote Hilbert's projective metric on $\mathrm{int}(C)$ and let $|\cdot|$ denote an equivalent norm on X such that $x \to |x|$ is order-preserving on C. Let $q\colon \mathrm{int}(C) \to (0, \infty)$ be a norm continuous map which is homogeneous of degree 1; and if Σ_q is defined by*

$$\Sigma_q = \{x \in \mathrm{int}(C) \mid q(x) = 1\},$$

assume that

$$(4.2) \qquad\qquad \sup\{|x| \mid x \in \Sigma_q\} = b < \infty.$$

Further assume that for any sequence $\langle b_j \mid j \geq 1\rangle$ in $\mathrm{int}(C)$ with $\lim_{j\to\infty} \|b_j\| = 0$ one has that

$$(4.3) \qquad\qquad \lim_{j\to\infty} \sup\{|q(x + b_j) - 1| \mid x \in \Sigma_q\} = 0$$

For $u \in \mathrm{int}(C)$ and for $\varepsilon > 0$ define maps $\Phi_{\varepsilon u}$ and $\Psi_{\varepsilon u}$ on $C - \{0\}$ by

$$(4.4) \qquad \Phi_{\varepsilon u}(y) = y + \varepsilon|y|u \quad and \quad \Psi_{\varepsilon u}(y) = \Phi_{\varepsilon u}(y)/q(\Phi_{\varepsilon u}(y)).$$

Let $f\colon \Sigma_q \to \Sigma_q$ be nonexpansive with respect to d and define $f_{\varepsilon u}\colon \Sigma_q \to \Sigma_q$ by

$$f_{\varepsilon u}(x) = \Psi_{\varepsilon u}(f(x))$$

Then there exists $R > 0$ (continuously dependent on u and ε) with $f_{\varepsilon u}(\Sigma_q) \subset V_R(u)$. For all $x, y \in \Sigma_q$ with $x \neq y$, $d(f_{\varepsilon u}(x), f_{\varepsilon u}(y)) < d(x, y)$ and $f_{\varepsilon u}$ has a unique fixed point $x(\varepsilon, u) \in \Sigma_q$. The map $(\varepsilon, u) \to x(\varepsilon, u)$ for $\varepsilon > 0$ and $u \in \mathrm{int}(C)$ is continuous in the norm topology. If $\langle \varepsilon_j \mid j \geq 1\rangle$ is a sequence of positive reals with $\lim_{j\to\infty} \varepsilon_j = 0$ and $\langle u_j \mid j \geq 1\rangle \subset \mathrm{int}(C)$ is a norm bounded sequence and $f_j := f_{\varepsilon_j u_j}$, then

$$(4.5) \qquad\qquad \lim_{j\to\infty} \left(\sup\{\|f_j(x) - f(x)\| \mid x \in \Sigma_q\}\right) = 0.$$

If, in addition, $\lim_{j\to\infty} \|x(\varepsilon_j, u_j) - \zeta\| = 0$ for some $\zeta \in \Sigma_q$, then we have that $f(\zeta) = \zeta$.

PROOF. Lemma 3.2 implies that $d(\Phi_{\varepsilon u}(x), \Phi_{\varepsilon u}(y)) < d(x, y)$ for all $x, y \in \Sigma_q$ with $x \neq y$, so $d(\Psi_{\varepsilon u}(x), \Psi_{\varepsilon u}(y)) = d(\Phi_{\varepsilon u}(x), \Phi_{\varepsilon u}(y)) < d(x, y)$ and if $f(x) \neq f(y)$, $d(f_{\varepsilon u}(x), f_{\varepsilon u}(y)) < d(f(x), f(y)) \leq d(x, y)$. Of course if $f(x) = f(y)$, we have $0 = d(f_{\varepsilon u}(x), f_{\varepsilon u}(y)) < d(x, y)$. Since $\{x \in \mathrm{int}(C) : |x| = 1\}$ is bounded in norm, Lemma 3.2 also implies that there exists $R > 0$ with $\Psi_{\varepsilon u}(\Sigma_q) \subset V_R(u) \cap \Sigma_q$. In fact, if $\{z \mid |z - u| \leq \delta\} \subset C$, one easily checks that for $y \in \Sigma_q$ (so $|y| \leq b$) one has

$$\varepsilon u \leq y + \varepsilon u \leq \left(\frac{b + \varepsilon\delta}{\delta}\right)u,$$

so $\Psi_{\varepsilon u}(\Sigma_q) \subset V_R(u)$ for any R with $R \geq \log((b + \varepsilon\delta)/\delta\varepsilon) := R_0$; and it follows that $f_{\varepsilon u}(\Sigma_q) \subset f(V_R(u) \cap \Sigma_q)$. Because f is d-nonexpansive, it follows that

$$f(V_R(u) \cap \Sigma_q) \subset V_{R_1}(u) \cap \Sigma_q,$$

where $R_1 \geq R + d(f(u/q(u)), u)$. By Lemma 3.2 again, there exists $c < 1$ with $d(\Psi_{\varepsilon u}(x), \Psi_{\varepsilon u}(y)) \leq cd(x, y)$ for all $x, y \in V_{R_1}(u)$. If we take $R \geq 2R_0$ and $R_1 \geq 2(R + d(f(u/q(u)), u))$, one can see that there exists $\eta > 0$ with $\Psi_{\varepsilon' u'}(\Sigma_q) \subset V_R(u)$ for all (ε', u') with $|\varepsilon' - \varepsilon| < \eta$ and $|u' - u| < \eta$. The proof of Lemma 3.2 also shows that for $\eta > 0$ sufficiently small, $d(\Psi_{\varepsilon' u'}(x), \Psi_{\varepsilon' u'}(y)) \leq ((c+1)/2)d(x, y)$ for all $x, y \in V_{R_1}(u)$ and for all (ε', u') with $|\varepsilon' - \varepsilon| < \eta$ and $|u' - u| < \eta$. Because f is d-nonexpansive, we conclude that for all (ε', u') with $|\varepsilon' - \varepsilon| < \eta$ and $|u' - u| < \eta$, $f_{\varepsilon' u'}(\Sigma_q) \subset V_R(u) \cap \Sigma_q$ and

$$d(f_{\varepsilon' u'}(x), f_{\varepsilon' u'}(y)) \leq \left(\frac{c+1}{2}\right)d(x, y)$$

for all $x, y \in V_R(u) \cap \Sigma_q$. By the contraction mapping theorem, $f_{\varepsilon' u'}$ has a unique fixed point in $V_R(u) \cap \Sigma_q$ and hence a unique fixed point in Σ_q.

It remains to show that if $x(\varepsilon', u')$ is the fixed point of $f_{\varepsilon' i'}$, then $(\varepsilon', u') \to x(\varepsilon', u')$ is continuous. This is a standard argument in the proof of the contraction mapping principle. In the notation above, select ρ with $0 < \rho < R_0$, write $y = x(\varepsilon, u)$ and note that for $|\varepsilon' - \varepsilon| < \eta$ and $|u' - u| < \eta$, $f_{\varepsilon' u'}|V_\rho(y) \cap \Sigma_q$ is a Lipschitz map in the d-metric with Lipschitz constant $((c+1)/2)$. By decreasing η, we can also arrange that

$$d(f_{\varepsilon' u'}(y), y) \leq \left(\frac{1-c}{2}\right)\rho,$$

so for $x \in V_\rho(y) \cap \Sigma_q$ we have

$$d(f_{\varepsilon' u'}(x), y) \leq d(f_{\varepsilon' u'}(x), f_{\varepsilon' u'}(y)) + d(f_{\varepsilon' u'}(y), y) \leq \left(\frac{1+c}{2}\right)\rho + \left(\frac{1-c}{2}\right)\rho = \rho.$$

It follows that $f_{\varepsilon' u'}(V_\rho(y) \cap \Sigma_q) \subset V_\rho(y) \cap \Sigma_q$, so $x(\varepsilon', u')$, the fixed point of $f_{\varepsilon' u'}$ lies in $V_\rho(y) \cap \Sigma_q$. This proves the continuity of $(\varepsilon', u') \to x(\varepsilon', u')$, since the norm topology and the d-topology are equivalent on Σ_q.

Up to this point we have not used (4.3). If $\langle \varepsilon_j \mid j \geq 1 \rangle$ is a sequence of positive reals approaching zero and $\langle u_j \mid j \geq 1 \rangle \subset \text{int}(C)$ is a norm bounded sequence and $f_j := f_{\varepsilon_j u_j}$, we must prove that (4.5) is satisfied. Recall that $|y| \leq b$ for all $y \in \Sigma_q$, so $|f(x)| \leq b$ for all $x \in \Sigma_q$. If $x \in \Sigma_q$, an application of the triangle inequality gives

(4.6) $\quad |f_j(x) - f(x)| \leq \varepsilon_j |f(x)| \, |u_j| + [|f(x)|$
$$+ \varepsilon_j |f(x)||u_j|]\left[\frac{|q(f(x) + \varepsilon_j |f(x)|u_j) - 1|}{q(f(x) + \varepsilon_j |f(x)|u_j)}\right].$$

Since $|f(x)| \le b < \infty$ for all $x \in \Sigma_q$, (4.5) will follow from (4.6) if we can prove that

$$\lim_{j \to \infty} (\sup\{|q(y + \varepsilon_j|y|u_j) - 1| \mid y \in \Sigma_q\}) = 0.$$

If the latter equality fails, there exist a sequence $j_i \to \infty$, a number $\alpha > 0$ and points $y_i \in \Sigma_q$, $i \ge 1$, with

$$|q(y_i + \varepsilon_{j_i}|y_i|u_{j_i}) - 1| \ge \alpha.$$

If we define $b_i = \varepsilon_{j_i}|y_i|u_{j_i} \in \text{int}(C)$, then $\lim_{i \to \infty} \|b_i\| = 0$, and

$$|q(y_i + b_i) - 1| \ge \alpha$$

for all $i \ge 1$, which contradicts (4.3). Thus we have proved (4.5).

The final statement of Lemma 4.1 follows because continuity of f in the d-topology implies, for normal cones, continuity of f in the norm topology. $\square$

REMARK 4.2. The same argument used in Lemma 4.1 also allows more general maps. Suppose that $q_* \colon \text{int}(C) \to (0, \infty)$ is continuous, order-preserving and homogeneous of degree one. Assume also that q_* is bounded on Σ_q and that Σ_{q_*} is bounded in norm. If, for $\varepsilon > 0$ and $u \in \text{int}(C)$, one defines maps $\Phi_{\varepsilon u}$ and $\Psi_{\varepsilon u}$ by

$$\Phi_{\varepsilon u}(x) = x + \varepsilon\, q_*(x)u \quad \text{and} \quad \Psi_{\varepsilon u}(x) = \Phi_{\varepsilon u}(x)/q(\Phi_{\varepsilon u}(x)),$$

and if one defines $f_{\varepsilon u}(x) = \Psi_{\varepsilon u}(f(x))$, then the conclusions of Lemma 4.1 remain true.

If $B \subset \text{int}(C)$ and $\delta > 0$, define, in the notation of Lemma 4.1, a set $\Gamma(\delta, B)$ by

$$(4.7) \qquad\qquad \Gamma(\delta, B) = c\ell\{x(\varepsilon, u) \mid 0 < \varepsilon \le \delta, u \in B\}$$

and define $\Gamma(B)$ by

$$(4.8) \qquad\qquad \Gamma(B) = \bigcap_{\delta > 0} \Gamma(\delta, B).$$

Some of our subsequent work will involve $\Gamma(B)$, so it is of interest that $\Gamma(B)$ has more structure than is immediately apparent.

THEOREM 4.3. *Let assumptions and notation be as in Lemma 4.1. Assume also that f satisfies the following compactness assumption: If $\langle x_k \mid k \ge 1\rangle \subset \Sigma_q$ is any sequence such that $\|x_k - f(x_k)\| \to 0$, then there exists a norm convergent subsequence $\langle x_{k_i} \mid i \ge 1\rangle$. If B is a compact, connected, nonempty subset of $\text{int}(C)$ and if $\Gamma(B)$ is defined by (4.8), then $\Gamma(B)$ is a compact, connected nonempty subset of $c\ell(\Sigma_q)$. If f has no fixed points in Σ_q, then $\Gamma(B) \subset \partial C$. If*

f has a norm continuous extension $F: \Gamma(B) \cup \Sigma_q \to c\ell(\Sigma_q)$, then $F(y) = y$ for all $y \in \Gamma(B)$.

PROOF. The map $(\varepsilon, u) \to x(\varepsilon, u)$ is continuous, so $\{x(\varepsilon, u) \mid 0 < \varepsilon \le \delta, \ u \in B\}$ is the continuous image of a connected set and hence connected. It follows that $\Gamma(\delta, B)$, the closure of a connected set, is connected.

To show that $\Gamma(\delta, B)$ is compact, it suffices to show that if $y_j = x(\varepsilon_j, u_j)$ is a sequence of points in $\{x(\varepsilon, u) \mid 0 < \varepsilon \le \delta, \ u \in B\}$, then y_j has a convergent subsequence. By taking a subsequence, we can assume that $\varepsilon_j \to \varepsilon$ and $u_j \to u \in B$. If $\varepsilon > 0$, the continuity of $(\varepsilon', u') \to x(\varepsilon', u')$ on $(0, \delta] \times B$ implies that $x(\varepsilon_j, u_j) \to x(\varepsilon, u)$. If $\varepsilon = 0$, Lemma 4.1 implies that

$$\|x(\varepsilon_j, u_j) - f(x(\varepsilon_j, u_j))\| \to 0,$$

so our assumptions on f imply that a subsequence of $\langle x(\varepsilon_j, u_j) \mid j \ge 1 \rangle$ converges in the norm topology. Thus we have proved that $\Gamma(\delta, B)$ is compact.

Since $\Gamma(\delta, B)$ is nonempty, compact and connected for all $\delta > 0$ and $\Gamma(\delta_1, B) \subset \Gamma(\delta_2 B)$ whenever $0 < \delta_1 \le \delta_2$, it follows that $\Gamma(B) := \bigcap_{\delta > 0} \Gamma(\delta, B)$ is nonempty, compact and connected. If $y \in \Gamma(B)$, then there exist a sequence $\varepsilon_j \to 0^+$ and $u_j \in B$ for $j \ge 1$ with $\lim_{j \ge \infty} \|x(\varepsilon_j, u_j) - y\| = 0$. Lemma 4.1 implies that $\lim_{j \to \infty} \|f(x(\varepsilon_j, u_j)) - x(\varepsilon_j, u_j)\| = 0$. If $y \in \Sigma_q$, Lemma 4.1 implies that $f(y) = y$; so if f has no fixed points in Σ_q, it must be the case that $\Gamma(B) \subset \partial C$. If f has a norm continuous extension F defined on $\Gamma(B) \cup \Sigma_q$, then

$$0 = \|F(y) - y\| = \lim_{j \to \infty} \|f(x(\varepsilon_j, u_j)) - x(\varepsilon_j, u_j)\|,$$

and the proof is complete. $\qquad\qquad\square$

In general, it is unclear whether f extends continuously to $\Gamma(B)$. However, if f comes from a continuous, homogeneous of degree one and order-preserving map $g: C \to C$, this problem does not arise. If C is a closed cone in a Banach space and $g: C \to C$ is continuous, order-preserving and homogeneous of degree one, recall (see [34]–[36]) that one can define $r_C(g)$, *the cone spectral radius of g*, by

(4.9) $$r_C(g) := \sup\{\mu(x) \mid x \in C\}$$

where

$$\mu(x) := \limsup_{j \to \infty} \|g^j(x)\|^{(1/j)}.$$

COROLLARY 4.4. *Let C be a closed, normal cone with nonempty interior in a Banach space $(X, \|\cdot\|)$. Let $q: C - \{0\} \to (0, \infty)$ be a norm continuous map which is homogeneous of degree one, let $\Sigma_q = \{x \in \mathrm{int}(C) : q(x) = 1\}$ and assume that q satisfies the conditions of Lemma 4.1 (equations (4.2) and (4.3)). Let $g: C \to C$ be a continuous, order-preserving map which is homogeneous of degree*

one and satisfies $g(\text{int}(C)) \subset \text{int}(C)$. Define $f \colon \Sigma_q \to \Sigma_q$ by $f(x) = g(x)/q(g(x))$, and for $\varepsilon > 0$ and $u \in \text{int}(C)$, let $f_{\varepsilon u}$ and $x(\varepsilon, u)$ be as defined in Lemma 4.1 and, for B a compact, connected nonempty subset of $\text{int}(C)$, let $\Gamma(\delta, B)$ and $\Gamma(B)$ be given by (4.7) and (4.8). Let $r := r_C(g)$, the cone spectral radius of g. Assume that g satisfies the following compactness condition:

- *If $\rho \geq r$ and if $\langle x_k \mid k \geq 1 \rangle$ is any norm bounded sequence in $\text{int}(C)$ with $\|g(x_k) - \rho x_k\| \to 0$, then there exists a subsequence $\langle x_{k_i} \mid i \geq 1 \rangle$ which is convergent in the norm topology.*

Then it follows that $\Gamma(B)$ is compact, connected and nonempty, $r > 0$ and $g(y) = ry$ for all $y \in \Gamma(B)$. In particular, if g has no eigenvector in $\text{int}(C)$, then $\Gamma(B) \subset \partial C$.

REMARK 4.5. Since $g(\text{int}(C)) \subset \text{int}(C)$, it is known (see [37]) that $r_C(g) > 0$ and $r_C(g) = \mu(x)$ for any $x \in \text{int}(C)$. In particular, if $g(x) = \lambda x$ for some $x \in \text{int}(C), \lambda = r$.

PROOF OF COROLLARY 4.4. If $Z := \{x \in C \mid g(x) = 0\}$ and if $f(x) = g(x)/q(g(x))$ for $x \in C - Z$, then f is norm continuous and $f(\Sigma_q) \subset \Sigma_q$.

For $\delta > 0$, let $y_j := x(\varepsilon_j, u_j)$ be a sequence of points in $\{x(\varepsilon, u) \mid 0 < \varepsilon \leq \delta, u \in B\}$. If we can prove that $\langle y_j \mid y \geq 1 \rangle$ has a convergent subsequence, then the same argument used in Theorem 4.3 implies that $\Gamma(\delta, B)$ and $\Gamma(B)$ are compact, connected and nonempty. As in the proof of Theorem 4.3, we can assume that $\varepsilon_j \to 0^+$ and $u_j \to u \in B$. The defining equation for y_j implies that, for $s_j := q(f(y_j) + \varepsilon_j |f(y_j)| u_j)$,

$$f(y_j) \leq f(y_j) + \varepsilon_j |f(y_j)| u_j = s_j y_j,$$

which implies that for $t_j := q(g(y_j))s_j$,

$$g(y_j) \leq t_j y_j.$$

Since $y_j \in \text{int}(C)$, the latter equation implies that $t_j \geq r$. Equation (4.4) implies that $s_j \to 1$, so Lemma 4.1 implies that $\|f(y_j) - s_j y_j\| \to 0$. Our assumptions on q and g imply that $q(g(y_j))$, $j \geq 1$, is a bounded sequence, so

$$\|g(y_j) - t_j y_j\| \to 0.$$

By taking a subsequence, we can assume that $t_j \to \rho$; and since $t_j \geq r$, we have $\rho \geq r$ and

$$(4.10) \qquad\qquad\qquad \|g(y_j) - \rho y_j\| \to 0.$$

Our compactness assumption on g now implies that there exists a convergent subsequence of $\langle y_j \mid j \geq 1 \rangle$, so $\Gamma(\delta, B)$ and $\Gamma(B)$ are compact, connected and nonempty.

If $y \in \Gamma(B)$, the argument of Theorum 4.3 shows that there exists $\varepsilon_j \to 0^+$ and $u_j \in B, u_j \to u$, such that $y_j \to y$, $y_j := x(\varepsilon_j, u_j)$. The same argument used above shows that (4.10) holds; and since g is continuous on C, $g(y) = \rho y$. We know that $\rho \geq r$, and every eigenvalue λ of g satisfies $\lambda \leq r$, so $\rho = r$ and $g(y) = ry$. Since f is defined and continuous on $C - Z$ we also have that $f(y) = y$ for all $y \in \Gamma(B)$. $\qquad\square$

REMARK 4.6. In applications, one is frequently given a closed, normal cone C with nonempty interior in a Banach space X and a map $g: \text{int}(C) \to \text{int}(C)$ which is continuous, order-preserving, homogeneous of degree one and maps norm bounded sets to norm bounded sets. If X is finite dimensional and C is polyhedral, g always has a norm continuous extension $G: C \to C$, see [9]. If X is infinite dimensional or C is not polyhedral, such a continuous extension may fail to exist. An important and instructive example is provided by a class of maps which arise in studying so-called "DAD theorems", see [41]. Nevertheless, one can still define $r_C(g)$ by

$$r_C(g) = \sup\{\mu(x) \mid x \in \text{int}(C)\},$$

and this definition is consistent with (4.9). The compactness condition on g in Corollary 4.4 may still hold (again, consider operators from DAD theorems); and the proof of Corollary 4.4. still applies and implies that $\Gamma(\delta, B)$ and $\Gamma(B)$ are compact, connected and nonempty, although one can no longer assert that $g(y) = ry$ for all $y \in \Gamma(B)$.

REMARK 4.7. One can argue that the compactness condition on g in Corollary 4.4 is a natural one. For example, suppose that C and X are as in Corollary 4.4 and that $g: X \to X$ is as bounded linear operator with $g(\text{int}(C)) \subset \text{int}(C)$. If $\rho(g)$ denotes the essential spectral radius of g (see [34], [35] or [37] for definitions) and $r(g)$ denotes the spectral radius of g and if $\rho(g) < r(g)$, then it is proved in [35] that $r(g) = r_C(g)$. Furthermore, one can prove (we omit the proof) that g satisfies the compactness condition of Corollary 4.4.

REMARK 4.8. If B is a compact subset of Σ_q in Lemma 4.1, the set $\Gamma(B)$ (see (4.8)) superficially is strongly dependent on B. Compare remarks in this vein in [25, p. 1451]. However, Theorem 4.3 and Corollary 4.4 show this dependency is partly illusory. Suppose, for example, that the hypotheses of Theorem 4.3 hold and that for any compact $B \subset \Sigma_q$, f has a continuous extension $F: \Sigma_q \cup \Gamma(B) \to c\ell(\Sigma_q)$. If B_1 and B_2 are any compact subsets of Σ_q, let B be a compact, connected subset of Σ_q with $B_1 \cup B_2 \subset B$ (such a set B always exists). Then Theorem 4.3 implies that $\Gamma(B_1) \cup \Gamma(B_2)$ is contained in a connected component of $\{x \in \Sigma_q \cup \Gamma(B) \mid F(x) = x\}$. In the framework of Corollary 4.4, $\Gamma(B_1) \cup \Gamma(B_2)$ is contained in a connected component of $\{x \in c\ell(\Sigma_q) \mid g(x) = rx\}$, where $r = r_C(g)$.

REMARK 4.9. If $q(x) = \|x\|$, equations (4.2) and (4.3) are satisfied. More generally, suppose that C is a closed, normal cone with nonempty interior in a Banach space $(X, \|\cdot\|)$ and that $q: C \to [0, \infty)$ is continuous, homogeneous of all degree one and satisfies $q(x) > 0$ for all $x \in C \setminus \{0\}$. If X is finite dimensional, equations (4.2) and (4.3) are automatically satisfied. If X is infinite dimensional and q is also Lipschitzian in the norm metric, equation (4.3) is satisfied, but equation (4.2) may fail even if $q \in C^*$.

The exact form of the approximating functions f_ε in Lemma 4.1 will be irrelevant for much of our work, and it is convenient to rephrase Lemma 4.1.

LEMMA 4.10. *Let C be a closed, normal cone with nonempty interior in a Banach space X and let q and Σ_q be as in Lemma 4.1. Assume that $f: \Sigma_q \to \Sigma_q$ is nonexpansive with respect to Hilbert's projective metric d. There exists a sequence of functions $f_j: \Sigma_q \to \Sigma_q$, $j \geq 1$, such that for all $j \geq 1$*

    (a) *f_j is nonexpansive with respect to d,*
    (b) *f_j has a fixed point $a_j \in \Sigma_q$,*
    (c) *$\lim_{j \to \infty}(\sup\{\|f(x) - f_j(x)\| \mid x \in \Sigma\}) = 0$, and*
    (d) *$\lim_{j \to \infty} d(f_j(x), f(x)) = 0$ for all $x \in \Sigma_q$.*

PROOF. Select $u \in \mathrm{int}(C)$ and let $\varepsilon_j \to 0^+$ be a sequence of positive reals. For $\Psi_{\varepsilon u}$ as in equation (4.4), define $f_j(x) = \Psi_{\varepsilon_j u}(f(x))$, and apply Lemma 4.1.$\square$

If C in Lemma 4.10 is finite dimensional, then it is immediate that $\{a_j \mid j \geq 1\}$ has compact norm closure, and the same is true if f is a condensing map in the norm topology. However, in general $c\ell(\{a_j \mid j \geq 1\})$ may not be compact. If f is as in Lemma 4.1, $u \in \mathrm{int}(C)$, $\Psi_j := \Psi_{\varepsilon_j u}$ is as in (4.4), and n is a positive integer, one can define $h = f^n$ and $h_j = \Psi_j \circ h$ and ask whether $c\ell(\{b_j \mid j \geq 1\})$ is compact, where b_j denotes the (unique) fixed point of h_j. It may happen that $c\ell(\{b_j \mid j \geq 1\})$ is compact, although $c\ell(\{a_j \mid j \geq 1\})$ is not compact, and the compactness of $c\ell(\{b_j \mid j \geq 1\})$ will suffice for our later work.

LEMMA 4.11. *Let C, X, q and Σ_q be as in Lemma 4.1. Assume that $f: \Sigma_q \to \Sigma_q$ is nonexpansive with respect to Hilbert's projective metric d. For a given integer $n \geq 1$, let $h = f^n$. Then there exists a sequence of functions $h_j: \Sigma_q \to \Sigma_q$, $j \geq 1$, such that for all $j \geq 1$*

    (a) *h_j is nonexpansive with respect to d,*
    (b) *h_j has a fixed point $b_j \in \Sigma_q$,*
    (c) *$\lim_{j \to \infty}(\sup\{\|h(x) - h_j(x)\| \mid x \in \Sigma_q\}) = 0$, and*
    (d) *$\lim_{j \to \infty} d(h(x), h_j(x)) = 0$ for all $x \in \Sigma_q$.*

If f satisfies the fixed point property on Σ_q with respect to d (see Definition 3.1, Theorem 3.3 and Theorem 3.3') and f has no fixed points in Σ_q, then any limit

point of $\{b_j \mid j \geq 1\}$ in the norm topology lies in the boundary of C. If f^n is a condensing map in the norm metric, then

(e) f satisfies the fixed point property on Σ_q with respect to d,
(f) for any sequence $\langle x_k \mid k \geq 1 \rangle \subset \Sigma_q$ such that $\lim_{k \to \infty} \|f^n(x_k) - x_k\| = 0$, $cl(\{x_k | k \geq 1\})$ is compact and
(g) $cl(\{b_k \mid k \geq 1\})$ is compact.

More generally, if f satisfies condition (f), conditions (e) and (g) are satisfied.

PROOF. The existence of $h_j, j \leq 1$, follows by applying Lemma 4.10 to $h = f^n$. If f satisfies the fixed point property and f has no fixed points in Σ_q, then f^p has no fixed points in Σ_q for any $p \geq 1$. For suppose to the contrary that $f^p(x_*) = x_*$ for some $p \geq 1$ and $x_* \in \Sigma_q$. Define $\omega = \{f^j(x_*) \mid 0 \leq j < p\}$ and note that $f(\omega) = \omega$. Because f is d-nonexpansive, if

$$R := \sup\{d(f^j(x_*), f^k(x_*)) : 0 \leq j < k < p\},$$

then $(\bigcap_{z \in \omega} V_R(z)) \cap \Sigma_q := D \supset \omega$ and $f(D) \subset D$, so the fixed point property implies that f has a fixed point in D, contrary to our assumption that f is fixed point free. If now $h_j(b_j) = b_j$ and $b_{j_i} \to b \in \Sigma_q$ for some sequence $j_i \to \infty$, property (c) in Lemma 4.11 implies that $\|h(b_{j_i}) - b_{j_i}\| \to 0$. By the continuity of h on Σ_q, we deduce that $h(b) = b$, which contradicts the fact that f^n has no fixed points in Σ_q. Thus we must have that $b \in \partial C$.

Assume next that $f : \Sigma_q \to \Sigma_q$ is d-nonexpansive and satisfies condition (f). Suppose that $\langle y_k \mid k \geq 1 \rangle \subset \Sigma_q$ is a sequence such that $\langle y_k \mid k \geq 1 \rangle$ is bounded in (Σ_q, d) and $d(y_k, f(y_k)) \to 0$. Because f is d-nonexpansive, it follows as in the proof of Theorem 3.3 that $d(f^n(y_k), y_k) \to 0$ as $k \to \infty$ and $\lim_{k \to \infty} \|f^n(y_k) - y_k\| = 0$. Condition (f) implies that there is a subsequence $k_i \to \infty$ with $y_{k_i} \to y$ in the norm topology. Because $\langle y_k \mid k \geq 1 \rangle$ is bounded in $(\Sigma_q, d), y \in \Sigma_q$ and $d(y_{k_i}, y) \to 0$ and $f(y) = y$. Theorem 3.3' now implies that f satisfies the fixed point property on Σ_q with respect to d, i.e. f satisfies condition (e). Of course condition (g) follows immediately from condition (f) because $\lim_{k \to \infty} \|f^n(b_k) - b_k\| = 0$.

It remains to show that if $f^n : \Sigma_q \to \Sigma_q$ is a condensing map, then f satisfies condition (f). Suppose that $\langle x_k \mid k \geq 1 \rangle \subset \Sigma_q$ is a sequence in Σ_q such that $\lim_{j \to \infty} \|x_j - f^n(x_j)\| = 0$. Define $r_j = x_j - f^n(x_j)$, $R = \{r_j \mid j \geq 1\}$ and $B = \{x_j \mid j \geq 1\}$, so $cl(R)$ is compact, B is bounded in norm and

$$B \subset f^n(B) + R.$$

If α denotes Kuratowski's measure of noncompactness, it follows that $\alpha(B) \leq \alpha(f^n(B))$. However, if $\alpha(B) > 0$, the fact that f^n is condensing would imply that $\alpha(f^n(B)) < \alpha(B)$, a contradiction. Thus we must have that $\alpha(B) = 0$, so $cl(B)$ is compact. $\square$

If assumptions and notation are as in Lemma 4.11 and $j_i \uparrow \infty$ is a sequence of integers such that $\lim_{i \to \infty} b_{j_i} = b$, then we shall write $J = \{j_i \mid i \geq 1\}$, and we shall use the notation

$$\lim_{j \in J} b_j := \lim_{i \to \infty} b_{j_i}$$

We need to generalize slightly Beardon's construction of *horoballs* in [5]. With our preliminaries, the argument in the following lemma is similar to that in [5].

LEMMA 4.12. *Let C, X, q and Σ_q be as in Lemma 4.1. Assume that $f: \Sigma_q \to \Sigma_q$ is nonexpansive with respect to d, f satisfies the fixed point property on Σ_q with respect to d and f has no fixed points in Σ_q. For a given integer n, let $h = f^n$ and let h_j be as in Lemma 4.11. If $b_j \in \Sigma_q$ is a fixed point of h_j, assume that there is a sequence $j_i \to \infty$ and $b \in c\ell(\Sigma_q)$ with $\lim_{i \to \infty} b_{j_i} = b$ and define $J = \{j_i \mid i \geq 1\}$. For a fixed $y \in \Sigma_q$ and $j \in J$, define V_j by*

$$(4.11) \qquad V_j = \{x \in \Sigma_q \mid d(x, b_j) \leq d(y, b_j)\}.$$

Define the horoball H by

$$(4.12) \qquad H = \Big\{z \in \Sigma_q \mid \text{there exists } z_j \in V_j \text{ for } j \in J \text{ with } \lim_{j \in J} d(z_j, z) = 0\Big\}.$$

Then we have that $y \in H$ and $f^n(H) \subset H$.

PROOF. Lemma 4.11 implies that $b \in \partial C$; and this in turn implies that $\lim_{j \in J} d(b_j, y) = \infty$.

The fact that $y \in H$ is immediate. If $z \in H$, then $z \in \Sigma_q$ and there exists a sequence $z_j \in V_j$ for $j \in J$ with $\lim_{j \in J} d(z_j, z) = 0$. Because $z_j \in V_j$, $d(z_j, b_j) \leq d(y, b_j)$, so

$$d(h_j(z_j), h_j(b_j)) = d(h_j(z_j), b_j) \leq d(z_j, b_j) \leq d(y, b_j),$$

and $h_j(z_j) \in V_j$. We have that

$$d(h(z), h_j(z_j)) \leq d(h(z), h_j(z)) + d(h_j(z), h_j(z_j))$$
$$\leq d(h(z), h_j(z)) + d(z, z_j).$$

Because $z \in \Sigma_q$, Lemma 4.3 implies that

$$\lim_{j \in J} d(h(z), h_j(z)) = 0 \quad \text{and} \quad \lim_{j \in J} d(z_j, z) = 0$$

by assumption. This proves that $h(z) = f^n(z) \in H$. $\square$

We also need to generalize results of Beardon [5] and Karlsson and Noskov [27] concerning the geomery of Hilbert's projective metric. The argument in [5] uses "intersecting chord theorems" and involves compactness assumptions which fail in our generality. However, the proof of Theorem 5.2 in [27], although stated for the finite dimensional case, actually generalizes to the infinite dimensional case. Recall (see [37, Proposition 1.9, p. 25]) that if x and y are comparable

elements of a closed cone C in a Banach space X and if $x_t = (1-t)x + ty$ for $0 \le t \le 1$, then for $0 \le t \le 1$ we have

$$d(x,y) = d(x, x_t) + d(x_t, y).$$

In other words, $t \to x_t$ is a minimal geodesic from x to y.

THEOREM 4.13 (comp. [5, Sections 5 and 6], [27, Theorem 5.2]). *Let C be a closed cone with nonempty interior in a Banach space X. Let $\langle x_k \mid k \ge 1 \rangle$ and $\langle y_k \mid k \ge 1 \rangle$ be sequences in the interior of C such that $\lim_{k \to \infty} x_k = \zeta$ and $\lim_{k \to \infty} y_k = \eta$, where the limits are taken in the norm topology. Assume that*

(a) *$\zeta \in \partial C$,*
(b) *$\eta \in \partial C$, and*
(c) *$(\zeta + \eta)/2 \in \mathrm{int}(C)$.*

If $w \in \mathrm{int}(C)$ and d denotes Hilbert's projective metric on $\mathrm{int}(C)$, we have that

$$(4.13) \qquad \lim_{k \to \infty} d(x_k, w) = \lim_{k \to \infty} d(y_k, w) = \infty,$$

$$(4.14) \qquad \limsup_{k \to \infty} [d(x_k, w) + d(y_k, w) - d(x_k, y_k)] < \infty,$$

$$(4.15) \qquad \liminf_{k \to \infty} [d(x_k, y_k) - \max(d(x_k, w), d(y_k, w))] = \infty.$$

PROOF. Equation (4.13) is well-known, and we omit the proof. Because $(\zeta + \eta)/2 := u$ is in the interior of C and $(x_k + y_k)/2 := u_k$ approaches u in the norm topology, we have that $\lim_{k \to \infty} d(u_k, u) = 0$ and

$$(4.16) \qquad \lim_{k \to \infty} d(u_k, w) = d(u, w) < \infty.$$

On the other hand, the triangle inequality gives

$$d(w, x_k) \le d(w, u_k) + d(x_k, u_k) \text{ and } d(w, y_k) \le d(w, u_k) + d(u_k, y_k).$$

Adding these inequalities and recalling that

$$d(x_k, y_k) = d(x_k, u_k) + d(u_k, y_k),$$

we see that

$$(4.17) \qquad d(w, x_k) + d(w, y_k) - d(x_k, y_k) \le 2d(w, u_k).$$

Combining equations (4.16) and (4.17) yields (4.14).

Equation (4.17) implies that

$$(4.18) \quad d(x_k, y_k) - \max(d(x_k, w), d(y_k w)) \ge \min(d(x_k, w), d(y_k, w)) - 2d(w, u_k).$$

If we use (4.18) and recall (4.13) and (4.16), we obtain (4.15). $\qquad \square$

We can now present our first Denjoy–Wolff theorem. In the following we shall write (see (2.8))

$$\omega(y; f) := \omega(y; f, \| \cdot \|).$$

Theorem 4.14 is a direct generalization of results of Beardon (compare [5, Sections 5 and 6]). It is also related to a theorem of Karlsson and Noskov (see [27, Theorem 5.5]), but the exact connection remains unclear even in finite dimensions.

THEOREM 4.14 (comp. [5], [27]). *Let C be a closed, normal cone with nonempty interior in a Banach space X. Let $q: \text{int}(C) \to (0, \infty)$ be a norm continuous map which is homogeneous of degree one and satisfies the conditions given by equations (4.2) and (4.3). Define $\Sigma_q = \{x \in \text{int}(C) \mid q(x) = 1\}$ and let $f: \Sigma_q \to \Sigma_q$ be a map which is nonexpansive with respect to Hilbert's projective metric d and which has no fixed points in Σ_q. For a fixed integer $n \geq 1$, let $h := f^n$ and let $h_j: \Sigma_q \to \Sigma_q$, $j \geq 1$, be a sequence of approximating functions as in Lemma 4.11, so h_j has a fixed point $b_j \in \Sigma_q$. Assume that f satisfies the following compactness conditions:*

(a) *If $\langle x_j \mid j \geq 1 \rangle$ is any sequence in Σ_q such that $\lim_{j \to \infty} \|x_j - f^n(x_j)\| = 0$, then $c\ell(\{x_j \mid j \geq 1\})$ is compact.*
(b) *For all $x \in \Sigma_q$, $c\ell(\gamma(x; f))$ is compact, where $\gamma(x; f)$ is given by (4.1).*

Then, for all $x \in \Sigma_q$, $\omega(x; f)$ is compact and nonempty and $\omega(x; f) \subset \partial C$. If $B = \{b_j \mid j \geq 1\}$, $c\ell(B)$ is compact; and if j_i is an increasing sequence of integers such that $\lim_{i \to \infty} \|b_{j_i} - b\| = 0$, then $b \in \partial C$. If $y \in \omega(x; f)$ for some $x \in \Sigma_q$ and b is as above, then $(1 - t)b + ty \in \partial C$ for $0 \leq t \leq 1$.

REMARK 4.15. If $f^n: \Sigma_q \to \Sigma_q$ is a condensing map in the norm topology, we have already proved that conditions (a) and (b) of Theorem 4.14 hold: see Lemma 4.11 and the proof of Corollary 3.6.

PROOF OF THEOREM 4.14. The proof of Lemma 4.11 shows that under our assumptions, the hypotheses of Theorem 3.3' are satisfied, so Theorem 3.14 implies that $\omega(x; f) \subset \partial C$ for all $x \in \Sigma_q$. Because we assume that $c\ell(\gamma(x; f))$ is compact, the argument of Corollary 3.6 shows $\omega(x; f) \neq \emptyset$ is compact. Lemma 4.11 implies that $c\ell(B)$ is compact. If b is defined as above and $b \in \Sigma_q$, the continuity of h gives $h(b) = f^n(b) = b$. Lemma 4.11 implies that f satisfies the fixed point property on Σ_q with respect to d; and since f is fixed point free on Σ_q, it follows (see the proof of Lemma 4.11) that f^k is fixed point free on Σ_q for any $k \geq 1$. Thus we must have $b \in \partial C$.

If $x \in \Sigma_q$, it is easy to show that

$$\omega(x; f) = \bigcup_{j=0}^{n-1} \omega(f^j(x); h),$$

so it suffices to prove that $(1 - t)z + tb \in \partial C$ whenever $z \in \omega(y; h)$ for some $y \in \Sigma_q$ and $0 \leq t \leq 1$. We argue by contradiction and assume that there exists $y \in \Sigma_q$, $z \in \omega(y; h)$ and t with $0 \leq t \leq 1$ such that $(1 - t)z + tb \in \text{int}(C)$. We

know that $t \neq 0$ and $t \neq 1$, because $z \in \partial C$ and $b \in \partial C$; and a simple convexity argument implies that $(1 - s)z + sb \in \text{int}(C)$ for all $s \in (0, 1)$. For j_i as in the statement of Theorem 4.14, let $J = \{j_i \mid i \geq 1\}$ and for this y and J let H be a horoball defined by (4.11) and (4.12) and $V_j := \{x \in \Sigma_q \mid d(x, b_j) \leq d(y, b_j)\}$, $j \in J$. By definition of $\omega(y; h)$, there exists a strictly increasing sequence of integers $\sigma(i)$, $i \geq 1$, such that

$$\lim_{i \to \infty} \|h^{\sigma(i)}(y) - z\| = 0.$$

Lemma 4.12 implies that $h^{\sigma(i)}(y) \in H$ for all $i \geq 1$.

By definition of H, there exists a strictly increasing sequence of integers $j(p) \in J$ for $p \geq 1$ and points $x_{j(p)} \in V_{j(p)}$ such that

$$d(x_{j(p)}, h^{\sigma(p)}(y)) < \left(\frac{1}{p}\right) \quad \text{and} \quad \|x_{j(p)} - h^{\sigma(p)}(y)\| < \left(\frac{1}{p}\right),$$

so $\lim_{p \to \infty} \|x_{j(p)} - z\| = 0$. By (4.15) in Theorem 4.13 we obtain that

$$(4.19) \qquad \lim_{p \to \infty} [d(x_{j(p)}, b_{j(p)}) - \max\{d(x_{j(p)}, y), d(b_{j(p)}, y)\}] = \infty.$$

However, by definition of $V_{j(p)}$, $d(x_{j(p)}, b_{j(p)}) \leq d(y, b_{j(p)})$, so (4.19) gives a contradiction. $\qquad\square$

As noted in Section 2, the case of Hilbert's projective metric on a cone subsumes the case of Hilbert's projective metric on a bounded, open convex subset of a Banach space. Thus the following result follows immediately from Theorem 4.14 and Remark 4.15.

COROLLARY 4.16 (comp. [5], [27]). *Let G be a bounded, open, convex subset of a Banach space $(X, \|\cdot\|)$ and let d denote Hilbert's projective metric on G. Assume that $f \colon G \to G$ is nonexpansive with respect to d and fixed point free and assume that there exists an integer $n \geq 1$ such that $f^n := h$ is a condensing map. Then there exists a sequence of maps $h_j \colon G \to G$ for $j \geq 1$ such that*

(a) *h_j is nonexpansive with respect to d,*
(b) *h_j has a fixed point $b_j \in G$,*
(c) *$\lim_{j \to \infty}(\sup\{\|h(x) - h_j(x)\| \mid x \in G\}) = 0$, and*
(d) *$\lim_{j \to \infty} d(h(x), h_j(x)) = 0$ for all $x \in G$.*

If $\{h_j \mid j \geq 1\}$ is any collection of maps which satisfies (a)–(d) and $B = \{b_j \mid j \geq 1\}$, then $cl(B)$ is compact, and $cl(B) \setminus B$ is nonempty and contained in ∂G. If $b \in cl(B) \setminus B$ and $z \in \omega(y; f, \|\cdot\|)$ for some $y \in G$, then $(1 - t)b + tz \in \partial G$ for $0 \leq t \leq 1$.

Theorem 4.16 requires approximating f^n by d-nonexpansive maps $h_j \colon \Sigma_q \to \Sigma_q$. By directly using Theorem 4.13 and (4.12), one can give a variant of Theorem 4.16 which avoids the problem of approximating $h = f^n$ by "nicer" maps h_j, $j \geq 1$, and, in fact, also avoids the explicit use of horoballs.

THEOREM 4.17 (comp. [27, Theorem 5.5]). *Let C be a closed, normal cone with nonempty interior in a Banach space X. Let $q \colon \mathrm{int}(C) \to (0, \infty)$ be a norm continuous map which is homogeneous of degree one and let $\Sigma_q := \Sigma := \{x \in \mathrm{int}(C) \mid q(x) = 1\}$. Assume that $f \colon \Sigma \to \Sigma$ is nonexpansive with respect to Hilbert's projective metric d and has no fixed points in Σ. Assume also that f satisfies the following compactness conditions:*

(a) *If $\langle x_k \mid k \geq 1 \rangle \subset \Sigma$ is any sequence which is bounded in (Σ, d) and satisfies $\lim_{k \to \infty} d(f(x_k), x_k) = 0$, then $cl(\{x_k \mid k \geq 1\})$ is compact and*
(b) *For every $x \in \Sigma$, $cl(\gamma(x; f))$ is compact.*

Then if $y \in \Sigma$, there exists $\eta \in \omega(y; f, \|\cdot\|) := \omega(y; f)$ such that $(1-t)\eta + t\zeta \in \partial C$ for all $\zeta \in (\bigcup_{z \in \Sigma} \omega(z; f))$ and all t with $0 \leq t \leq 1$.

PROOF. Assumption (b) implies that $\omega(x; f)$ is compact and nonempty for all $x \in \Sigma$. By using assumption (a), Theorem 3.6' and Theorem 3.14, we see that $\omega(x; f) \subset \partial C$ for all $x \in \Sigma$ and $\lim_{k \to \infty} d(f^k(x), x) = \infty$ for all $x \in \Sigma$. Thus, if $y \in \Sigma$ and $r_k := d(f^k(y), y)$, $\lim_{k \to \infty} r_k = \infty$, and a well known elementary result implies that there exists a strictly increasing sequence of integers $\langle m_i \mid i \geq 1 \rangle$ such that $r_k \leq r_{m_i}$ for $1 \leq k \leq m_i$. By taking a further subsequence, we can also assume that $\lim_{i \to \infty} f^{m_i}(y) = \eta \in \omega(y; f)$. If $\zeta \in \omega(z; f)$ for some $z \in \Sigma$, there exists a strictly increasing sequence of integers $\langle n_j \mid j \geq 1 \rangle$ such that $\lim_{j \to \infty} f^{n_j}(z) = \zeta$. Define a subsequence $\langle \nu_j \mid j \geq 1 \rangle$ of $\langle m_i \mid i \geq 1 \rangle$ by $\nu_j := m_{\sigma(j)}$, where $\sigma(j) = \inf\{i \geq 1 \mid m_i \geq n_j\}$. We then have

(1) $\nu_j \geq n_j$,
(2) $\lim_{j \to \infty} f^{\nu_j}(y) = \eta$ and
(3) $r_k \leq r_{\nu_j}$ for $1 \leq k \leq \nu_j$.

If $(1-t)\zeta + t\eta \notin \partial C$ for some t with $0 < t < 1$, then $\zeta/2 + \eta/2 \in \mathrm{int}(C)$ and Theorem 4.13 implies that

$$(4.20) \qquad d(f^{n_j}(z), f^{\nu_j}(y)) - \max(d(f^{\nu_j}(y), y), d(f^{n_j}(z), y)) \to \infty.$$

However, because f is d-nonexpansive, we have

$$\begin{aligned} d(f^{n_j}(z), f^{\nu_j}(y)) &\leq d(z, f^{\nu_j - n_j}(y)) \\ &\leq d(z, y) + d(y, f^{\nu_j - n_j}(y)) \leq d(z, y) + d(y, f^{\nu_j}(y)), \end{aligned}$$

which contradicts (4.20). $\square$

REMARK 4.18. Theorem 4.17 is a direct generalization of Theorem 5.5 in [27]. There clearly is a version of Theorem 4.17 analogous to Corollary 4.16, but we

leave the details to the reader. Notice that the hypotheses of Theorem 4.14 are more restrictive than those of Theorem 4.17, e.g. if assumption (a) of Theorem 4.14 holds for any $n \geq 1$, then assumption (a) of Theorem 4.17 holds.

Theorems 4.14 and 4.17 are closely related, but their exact relationship is unclear. For example, is the point b constructed in Theorem 4.14 necessarily an element of $\bigcup_{z \in \Sigma_q} \omega(z; f)$?

REMARK 4.19. Let assumptions and notations be as in Theorem 4.17. For a given $y \in \Sigma$, say that $\eta \in \omega_*(y; f) \subset \omega(y; f)$ if there exists a sequence $m_i \to \infty$ and a constant M, dependent on y and $\langle m_i \mid i \geq 1 \rangle$ such that $f^{m_i}(y) \to \eta$ and

$$(4.21) \qquad M + d(f^{m_i}(y), y) \geq d(f^k(y), y) \quad \text{for } 1 \leq k \leq m_i.$$

The proof of Theorem 4.17 then shows that $(1-t)\zeta + t\eta \in \partial C$ for all $\zeta \in \omega(z; f)$, all $z \in \Sigma$ and all t with $0 \leq t \leq 1$. If y and m_i are as above and p is a positive integer, it is easy to derive from (4.21) that there exists a constant $M_p \geq 0$ such that

$$M_p + d(f^{m_i+p}(y), y) \geq d(f^k(y, y) \quad \text{for } 1 \leq k \leq m_i + p.$$

By taking a further subsequence, we can assume that $f^{m_i+p}(y) \to \eta_p$, so $\eta_p \in \omega_*(y; f)$ and $(1 - t)\zeta + t\eta_p \in \partial C$ for all $\zeta \in \omega(z; f)$, all $z \in \Sigma$ and all t with $0 \leq t \leq 1$. If f extends continuously to $\Sigma \cup \omega(y; f)$, we can also say that $\eta_p = f^p(\eta)$.

REMARK 4.20. In the framework of Theorem 4.14 or 4.17, we conjecture that for broad classes of analytically interesting maps (and possibly in the full generality of Theorems 4.14 and 4.17), it is true that

$$\mathrm{co}\left(\bigcup_{z \in \Sigma} \omega(z; f) \right) \subset \partial C.$$

A. Karlsson has mentioned a similar conjecture in an e-mail communication. In the presence of strict convexity assumptions, Theorems 4.14 and 4.17 imply that $\bigcup_{z \in \Sigma} \omega(z; f)$ is a single point, but if $C = K^n$ (see (2.5)) one can construct examples (see [32]) for which $\omega(y; f)$ is an infinite subset of a face of K^n.

We list below several conjectures in the spirit of Remark 4.20. Recall that $\omega(z; f)$ denotes the omega limit set of z under f in the norm topology.

CONJECTURE 4.21. *Under the hypotheses of Theorem* 4.14 *or Theorem* 4.17, *we have*

$$(4.22) \qquad\qquad \mathrm{co}\left(\bigcup_{z \in \Sigma} \omega(z; f) \right) \subset \partial C.$$

Conjecture 4.22. *Let C be a closed cone with nonempty interior in a finite dimensional Banach space X. Let $q: C \to [0, \infty]$ be a norm-continuous map which is homogeneous of degree one and satisfies $q(x) > 0$ for $x \neq 0$. Let $\Sigma = \{x \in \text{int}(C) \mid q(x) = 1\}$ and assume that $f: \Sigma \to \Sigma$ is nonexpansive with respect to Hilbert's metric d and has no fixed points in Σ. Then it follows that (4.22) is satisfied.*

Under the hypotheses of Conjectures 4.21 or 4.22, we shall prove in Section 5 that (4.22) is satisfied if there exists some $x \in \Sigma$ with $\text{co}(\omega(x; f)) \subset \partial C$. As part of his Rutgers University Ph.D. dissertation (see [32]), Brian Lins has proved that if C is a polyhedral cone and the hypotheses of Conjecture 4.22 are satisfied, then $\text{co}(\omega(x; f)) \subset \partial C$ for all $x \in \Sigma$. Thus Conjecture 4.22 is true if C is polyhedral. (Recall that a closed cone C in a finite dimensional Banach space X is polyhedral if there exist continuous linear functionals $\theta_j \in X^*$, $1 \leq j \leq N$, such that $C = \{x \in X \mid \theta_j(x) \geq 0 \text{ for } 1 \leq j \leq N\}$.) It follows that Conjecture 4.22 is true in the important case that $C = K^n$. Beardon's original theorem proves Conjecture 4.22 when C is a strictly convex cone.

For general closed cones C in a finite dimensional Banach space X, Conjecture 4.22 remains open. Brian Lins has proved the conjecture for general cones C of dimension less than or equal to 3; and (see [33]) the conjecture is also true when f is obtained by normalization from an affine linear map.

There is also a natural variant of Conjecture 4.22 for $\bar{d}$-nonexpansive map g.

Conjecture 4.23. *Let C be a closed cone with nonempty interior in a finite dimensional Banach space X. Let $g: \text{int}(C) \to \text{int}(C)$ be a map which is nonexpansive with respect to Thompson's metric $\bar{d}$ and has no fixed points in $\text{int}(C)$. Then it follows that*

$$\text{co}\left(\bigcup_{z \in \text{int}(C)} \omega(z; g) \right) \subset \partial C.$$

If, in the context of Conjecture 4.23, C is polyhedral, Brian Lins (see [32]) has proved that for every $z \in \text{int}(C)$, $\text{co}(\omega(z; g)) \subset \partial C$; so the results of Section 5 imply that Conjecture 4.23 is true when C is a polyhedral cone. Despite a superficial similarity between Thompson's metric and Hilbert's projective metric, the analogue of Theorem 4.13 for Thompson's metric is false; and it is not known whether Conjecture 4.23 is true for strictly convex cones.

Up to this point we have used horoballs, but we have not defined *horofunctions*. Work of Karlsson [25], Karlsson, Metz and Noskov [26] and Lins [32] shows the importance of using horofunctions in studying Conjecture 4.22. Despite the fact that our set Σ_q will not be locally compact if X is infinite dimensional, we shall now show that analogues of horofunctions can be defined and can be used

to give further evidence for Conjecture 4.21. We begin by defining a variant of horofunctions.

LEMMA 4.24. *Let C be a closed, normal cone with nonempty interior in a Banach space X. Let $q: C \to \mathbb{R}$ be a norm continuous map which is homogeneous of degree one and strictly positive on $C - \{0\}$ and define $\Sigma_q = \{x \in \mathrm{int}(C) \mid q(x) = 1\}$. Let $\{y^i \mid i \geq 1\}$ be a sequence of points in Σ_q, and for $w \in \Sigma_q$, define a sequence of functions $h_i \mid \mathrm{int}(C) \to \mathbb{R}$, $i \geq 1$, by*

$$h_i(z) = d(z, y^i) - d(y^i, w).$$

Let $M \subset C - \{0\}$ be a compact set and define N by

$$N := \{tz \mid t > 0, \ z \in \overline{\mathrm{co}}(M)\}.$$

Then there exists a sequence $m(i) \uparrow \infty$ such that $h_{m(i)}(\zeta) \to h(\zeta)$ for all $\zeta \in N \cap \mathrm{int}(C)$, and the convergence is uniform in $\zeta \in N \cap V_R(w)$ for every $R > 0$.

PROOF. It is a standard result that $\overline{\mathrm{co}}(M)$ is compact and that $\overline{\mathrm{co}}(M) \subset C - \{0\}$. If $\widehat{M} = \{z/\|z\| : z \in \overline{\mathrm{co}}(M)\}$, $\widehat{M}$ is the continuous image of a compact set, hence compact. If $z \in \mathrm{int}(C)$ and $t > 0$, $h_i(tz) = h_i(z)$. Thus it suffices to prove the lemma with N replaced by $\widehat{M}$. For a fixed $R > 0$ consider the set $\widehat{M} \cap V_R(w) := \widehat{M}_R$ and note that $\widehat{M}_R$ is compact in the d-topology, which is equivalent to the norm topology on $\widehat{M}_R$. Because d is a metric on $\widehat{M}_R$, the functions $h_i|\widehat{M}_R$ form a bounded equicontinuous family. The Ascoli–Arzela theorem implies that, for some subsequence $n_R(i) = n(i) \to \infty$, $h_{n(i)}(z) \to h(z)$ uniformly in $z \in \widehat{M}_R$. We now take a sequence $R_j \to \infty$ and use a standard Cantor diagonalization argument to fnd a sequence $m(i) \to \infty$ such that $h_{m(i)}|\widehat{M}_{R_j}$ converges to $h(z)$ uniformly on $\widehat{M}_{R_j}$ for each $j \geq 1$. $\square$

The function h is called a horofunction. The argument in Lemma 4.24 is the standard one for constructing h, the only point being that Σ_q will not be locally compact if X is infinite dimensional. If X is separable, one can arrange that $h_{m(i)}(z)$ converges for all $z \in \mathrm{int}(C)$ and the convergence is uniform on any compact subset of $\mathrm{int}(C)$.

If C and Σ_q are as in Lemma 4.24 and $f: \Sigma_q \to \Sigma_q$ is nonexpansive with respect to Hilbert's projective metric d, then it is easy to show (see [25]) that for every $x \in \Sigma_q$,

$$A = \lim_{k \to \infty} \frac{d(f^k(x), x)}{k} = \inf_{k \geq 1} \frac{d(f^k(x), x)}{k}$$

and A is independent of x. If S is a compact Hausdorff space, $X := C(S)$, the Banach space of continuous functions $x: S \to \mathbb{R}$ and C is the cone of nonnegative functions in X, then (see [37, Chapter 1]) there is an isometry Φ which maps

(Σ_q, d) onto a Banach space $(V, |\cdot|)$. It then follows from results of Kohlberg and Neyman [28] that the constant A above satisfies

$$A = \inf\{d(f(x), x) \mid x \in \Sigma_q\}.$$

(This observation is made in [26] for the case of $C = K^n$.)

If $A > 0$ and $0 < \varepsilon < A$, Karlsson [25] has made the useful observation that there is a sequence of integers $k_i \uparrow \infty$ with

$$d(f^{k_i}(x), x) - k_i(A - \varepsilon) > d(f^j(x), x) - j(A - \varepsilon)$$

for $0 \le j < k_i$, and this implies that

$$d(f^{k_i - m}(x), x) - d(f^{k_i}(x), x) \le -m(A - \varepsilon)$$

for $0 \le m < k_i$.

Our next theorem is a generalization of Theorem 17 in [26].

THEOREM 4.25 (comp. [26, Theorem 17]). *Let C, X, q and Σ_q be as in Lemma 4.24 and assume that $f \colon \Sigma_q \to \Sigma_q$ is nonexpansive with respect to d and satisfies $\lim_{k \to \infty} d(f^k(x), x)/k = A > 0$ for some $x \in \Sigma_q$ (so f has no fixed points in Σ_q). Assume also that $cl(\{f^k(x) \mid k \ge 0\})$ is compact for all $x \in \Sigma_q$. Then it follows that $\mathrm{co}(\bigcup_{z \in \Sigma_q} \omega(z; f)) \subset \partial C$.*

PROOF. A simple argument in Section 5 will show that it suffices to prove $\mathrm{co}(\omega(x; f)) \subset \partial C$ for some $x \in \Sigma_q$. Define $x^k = f^k(x)$, select $0 < \varepsilon < A$, and by the remarks preceding this theorem, let $k_i \uparrow \infty$ be a sequence such that

$$d(x^{k_i - m}, x) - d(x^{k_i}, x) \le -m(A - \varepsilon),$$

for $0 \le m < k_i$. In the notation of Lemma 4.24, define $M = cl(\gamma(x; f))$, so M is compact and define $y^i = x^{k_i}$ and

$$h_i(z) = d(z, y^i) - d(y^i, x).$$

for $z \in \mathrm{int}(C)$. By taking a further subsequence of the sequence $\langle k_i \mid i \ge 1\rangle$, we can assume that $h_i(\zeta) \to h(\zeta)$ for all $\zeta \in N \cap \mathrm{int}(C)$, where $N := \{tz \mid t > 0$ and $z \in \overline{\mathrm{co}}(M)\}$. Furthermore, the convergence is uniform on $N \cap V_R(x)$ for every $R > 0$. If $m \ge m_*$ and $k_i \ge m$, then the nonexpansiveness of f gives $h_i(x^m) = d(x^m, x^{k_i}) - d(x^{k_i}, x) \le d(x, x^{k_i - m}) - d(x^{k_i}, x) \le -m_*(A - \varepsilon)$, so $h(x^m) \le -m_*(A - \varepsilon)$ for all $m \ge m_*$. If $z \in \mathrm{co}(\{x^m \mid m \ge m_*\})$, then $z = \Sigma_{j=1}^p \lambda_j x^{m_j}$, where $\lambda_j > 0$, $\Sigma_{j=1}^p \lambda_j = 1$ and $m_j \ge m_*$ for $1 \le j \le p$. If we choose k_i so that $k_i > m_j$ for $1 \le j \le p$, then we have

$$h_i(z) \le -m_*(A - \varepsilon),$$

because $V_r(x^{k_i})$ is convex for $r = d(x^{k_i}, x) - m_*(A - \varepsilon)$. Leting $i \to \infty$, we see that $h(z) \le -m_*(A - \varepsilon)$ for all $z \in \mathrm{co}(\{x^m \mid m \ge m_*\})$, so $h(z) \le -m_*(A - \varepsilon)$ for all

$z \in \overline{\text{co}}(\{x^m \mid m \geq m_*\}) \cap (\text{int}(C))$. It follows that if $z \in \overline{\text{co}}(\omega(x; f)) \cap \text{int}(C)$, then we have $h(z) \leq -m_*(A - \varepsilon)$ for every positive integer m_*. Since $h(z) \geq -d(z, x)$ for $z \in N \cap (\text{int}(C))$, we conclude that $\overline{\text{co}}(\omega(x; f)) \cap (\text{int}(C))$ is empty. $\qquad \square$

The drawback of Theorem 4.25 is, of course, the condition that $A > 0$. If $A = 0$, the proof of Theorem 4.25 fails. However, there is a case when $A = 0$ but the conclusion of Theorem 4.25 remains true.

THEOREM 4.26. *Let C, X, q and Σ_q be as in Lemma 4.24 and assume that $f \colon \Sigma_q \to \Sigma_q$ is nonexpansive with respect to d, has no fixed points in Σ_q and satisfies conditions* (a) *and* (b) *of Theorem 4.17. Assume also that there exists $x \in \Sigma_q$ with $\lim_{k \to \infty} d(f^{k+1}(x), f^k(x)) = 0$ (so $A = 0$). Then it follows that $\text{co}(\bigcup_{z \in \Sigma_q} \omega(z; f)) \subset \partial C$.*

PROOF. A simple argument given in Section 5 shows that it suffices to prove that $\text{co}(\omega(x; f)) \subset \partial C$, where x is as in the statement of Theorem 4.26. We assume, by way of contradiction, that $\text{co}(\omega(x; f))$ is not contained in ∂C, and we select $\xi_j \in \omega(x; f)$ and positive numbers λ_j, $1 \leq j \leq m$, with $\Sigma_{j=1}^m \lambda_j = 1$ and $\Sigma_{j=1}^m \lambda_j \xi_j \in \text{int}(C)$. We know that $\omega(x; f) \subset \partial C$, so $m > 1$; and we can choose $m \geq 2$ to be minimal. As usual, we write $x^k = f^k(x)$, and for each j, $1 \leq j \leq m$, we take a strictly increasing sequence $\langle k_{ij} \mid i \geq 1 \rangle$ with $x^{k_{ij}} \to \xi_j$ as $i \to \infty$. For notational simplicity we write $k_i = k_{i1}$, $\xi_1 = \xi \in \partial C$, $\eta^i = \Sigma_{j=2}^m \mu_j x^{k_{ij}}$, where $\mu_j := (\lambda_j/(1 - \lambda_1))$ and $\eta = \Sigma_{j=2}^m \mu_j \xi_j$. Because m is minimal, we know that $\eta \in \partial C$ and $(1 - \lambda_1)\xi + \lambda_1 \eta \in \text{int}(C)$, so, using basic facts about convex sets, $(1 - t)\xi + t\eta \in \text{int}(C)$ for $0 < t < 1$.

We now use Lemma 4.24. Define $y^i = x^{k_i}$ and, for $z \in \text{int}(C)$,

$$h_i(z) = d(z, y^i) - d(y^i, x).$$

Let M and N be as in the proof of Theorem 4.25. Lemma 4.24 implies that by taking a subsequence of $\langle k_i \rangle$, which we label the same, we can assume that $h_i(z) \to h(z)$ for all $z \in N$ and that the convergence is uniform in $z \in N \cap V_R(x)$ for every $R > 0$.

If $z \in N \cap \Sigma_q$ and $h(z) \leq B$ we next claim that $\limsup_{i \to \infty} h_i(f(z)) \leq B$ and $h(f(z)) \leq B$ if $f(z) \in N \cap \Sigma_q$. To see this, observe that

$$\begin{aligned}
h_i(f(z)) &= d(f(z), f^{k_i}(x)) - d(f^{k_i}(x), x) \\
&\leq d(f(z), f^{k_i+1}(x) + d(f^{k_i+1}(x), f^{k_i}(x)) - d(f^{k_i}(x), x) \\
&\leq d(z, f^{k_i}(x)) - d(f^{k_i}(x), x) + d(f^{k_i+1}(x), f^{k_i}(x)) \\
&\leq h_i(z) + d(f^{k_i+1}(x), f^{k_i}(x)).
\end{aligned}$$

Since we assume that $\lim_{k \to \infty} d(f^{k+1}(x), f^k(x)) = 0$, we obtain the desired result. Because $h(x) = 0$, we conclude that $h(x^m) \leq 0$ for all $m \geq 0$. It follows that for each fixed $i \geq 1$, there exists an integer $p(i)$ such that for all $p \geq p(i)$,

$h_p(x^{kij}) \leq 1$ for $2 \leq j \leq m$ and $h_p(x^{k_i}) \leq 1$. Because $V_r(w)$ is convex for $w := x^{k_p}$ and $r := d(x^{k_p}, x) + 1$, it follows that $h_p(\eta^i) \leq 1$ for $p \geq p(i)$. If we define $\nu_i = k_{p(i)}$, we have arranged that $\eta^i \to \eta \in \partial C, x^{\nu_i} \to \xi \in \partial C$ and $(\eta + \xi)/2 \in \text{int}(C)$, so Theorem 4.13 implies that

$$\liminf_{i \to \infty} [d(\eta^i, x^{\nu_i}) - \max(d(\eta^i, x), d(x^{\nu_i}, x))] = \infty.$$

However, we have also arranged that $d(\eta^i, x^{\nu_i}) \leq d(x^{\nu_i}, x) + 1$, which gives a contradiction. $\qquad\square$

REMARK 4.27. Under the hypotheses of Theorem 4.26, it is easy to show that $\omega(x; f)$ is a compact, connected set. However, it may easily happen, even in finite dimensions, that $\omega(x; f)$ is an infinite set: see the example in [32].

There is an important special case of Theorems 4.14 and 4.17 in which one can obtain sharper results. We shall sketch these results here but defer detailed proofs to a later paper.

Let C be a closed cone in a Banach space $(X, \|\cdot\|)$ and assume that $g \colon C \to C$ is continuous and homogeneous of degree one. We have already defined $r_C(g)$, the *cone spectral radius of g*, see equation (4.9) and [34]–[37]. If α denotes Kuratowski's measure of noncompactness, define $\alpha_C(g)$ by

$$\alpha_C(g) := \inf\{\lambda \geq 0 \mid \alpha(g(A)) \leq \lambda\alpha(A) \text{ for all bounded } A \subset C\}.$$

We could also use a *homogeneous, generalized measure of noncompactness* as in [34] and [35]. We assume that $\alpha_C(g) < \infty$, and we define $\rho_C(g)$, the *cone essential spectral radius of g*, by

$$\rho_C(g) := \lim_{k \to \infty} (\alpha_C(g^k))^{(1/k)}.$$

The following theorem is a slight generalization of Theorem 3.3 of [34] and Theorem 2.1 of [35], but the proof is the same.

THEOREM 4.28 (comp. [34, Theorem 3.3], [35, Theorem 2.1]). *Let C and C_1 be closed cones in a Banach space X and assume that $C \subset C_1$. Let $g \colon C \to C$ be a continuous map which is homogeneous of degree one and which is order-preserving with respect to the partial ordering $\leq_{C_1}$ induced by C_1. If there exists an integer $n \geq 1$ such that $\alpha_C(g^n) < (r_C(g))^n$, then there exists $x_n \in C$, $\|x_n\| = 1$, with $g^n(x_n) = r^n x_n$, $r := r_C(g)$.*

REMARK 4.29. If $\rho_C(g) < r_C(g)$ for C and g as in Theorem 4.28, Theorem 4.28 implies that there exists $n_0 \geq 1$ such that for all $n \geq n_0$, there exists $x_n \in C, \|x_n\| = 1$, with $g^n(x_n) = r^n x_n$. We conjecture that in fact there exists $x_1 \in C, \|x_1\| = 1$, with $g(x_1) = r x_1$. If g is also a bounded linear map, the conjecture is true (see [35]), but in general the answer is unknown. The problem here is related to the question in Remark 3.4.

As noted before, we defer the proof of the following theorem to a later paper, but we present the statement here because of its close relation to Theorems 4.14 and 4.17. Note that in Theorem 4.30 below we do not necessarily assume that the cone C is normal.

THEOREM 4.30. *Let C be a closed cone with nonempty interior* $\mathrm{int}(C)$ *in a Banach space $(X, \|\cdot\|)$. Let $f\colon C \to C$ be continuous, order-preserving (in the partial ordering $\leq_C$) and homogeneous of degree one and assume that $f(\mathrm{int}(C)) \subset \mathrm{int}(C)$. Assume (see (4.9)) that $\rho_C(f) < r_C(f) := r$. Define $g(x) = (1/r)f(x)$ and assume that there exists $x_* \in \mathrm{int}(C)$ such that*

(4.23) $$\sup\{\|g^m(x_*)\| \mid m \geq 0\} < \infty.$$

Then, for every $x \in \mathrm{int}(C)$, we have that

$$0 < \inf\{\|g^m(x)\|\} \mid m \geq 0\} \leq \sup\{\|g^m(x)\| \mid m \geq 0\} < \infty.$$

There exists $m_0 \geq 1$ such that $\alpha_C(g^m) < 1$ for all $m \geq m_0$; and if $\alpha_C(g^m) < 1$, g^m has a fixed point $x_m \in C$ with $\|x_m\| = 1$. For every $x \in \mathrm{int}(C)$, $c\ell(\gamma(x; g))$ is compact and $\omega(x; g, \|\cdot\|) := \omega(x; g)$ is compact and nonempty. If either

    (a) *g^m has no fixed points in $\mathrm{int}(C)$ for some $m \geq 1$ with $\alpha_C(g^m) < 1$, or*
    (b) *C is normal and g has no fixed points in $\mathrm{int}(C)$,*

then $\omega(x; g) \subset \partial C$ for all $x \in \mathrm{int}(C)$. If (a) or (b) holds, $A := \bigcup_{z \in \mathrm{int}(C)} \omega(z; g)$, $\zeta \in A$ and $g^n(\eta) = \eta$ for some $\eta \in C \setminus \{0\}$ and $n \geq 1$, then ζ dominates η and $(1-t)\zeta + t\eta \in \partial C$ for $0 \leq t \leq 1$.

REMARK 4.31. Equation (4.23) is a strong hypothesis even for linear maps in finite dimensional cones, and it yields correspondingly strong conclusions. If, in addition, C is a finite dimensional, polyhedral cone, we can use results in [2] to obtain much sharper results.

5. Specifying the location of $\omega(x; f)$

Our goal in this section is to sharpen the conclusions of Theorem 4.14 and 4.17 and to provide some further evidence for Conjectures 4.21–4.23. We begin with some simple lemmas.

LEMMA 5.1. *Let C be a closed cone in a Banach space X and let $\langle x_k \mid k \geq 1 \rangle$ and $\langle y_k \mid k \geq 1 \rangle$ be sequences in C such that x_k and y_k are comparable for all $k \geq 1$ and $d(x_k, y_k) \leq R$ for all $k \geq 1$, where d denotes Hilbert's projective metric on C. Assume that $\lim_{k \to \infty} \|x_k - \zeta\| = 0$ and $\lim_{k \to \infty} \|y_k - \eta\| = 0$, where $\zeta \neq 0$ and $\eta \neq 0$. Then ζ and η are comparable and $d(\zeta, \eta) \leq R$.*

PROOF. Since $\zeta \in C \setminus \{0\}$ and $\eta \in C \setminus \{0\}$, there exist $\varphi_1 \in C^*$ and $\varphi_2 \in C^*$ with $\varphi_1(\zeta) > 0$ and $\varphi_2(\eta) > 0$. We define $\varphi = \varphi_1 + \varphi_2 \in C^*$, so $\varphi(\zeta) > 0$ and

$\varphi(\eta) > 0$. If $x'_k = (x_k/\varphi(x_k))$ and $y'_k = (y_k/\varphi(y_k))$ for k large, $\xi' = (\xi/_{\varphi(\xi)})$ and $\eta' = (\eta/\varphi(\eta))$, $d(x'_k, y'_k) = d(x_k, y_k) \leq R$, $\|x'_k - \xi'\| \to 0$, $\|y'_k - \eta'\| \to 0$ and $d(\xi', \eta') = d(\xi, \eta)$. Thus we may as well assume from the beginning that $\varphi(x_k) = \varphi(y_k) = 1$ and $\varphi(\xi) = \varphi(\eta) = 1$. By definition there are positive reals α_k and β_k with $\alpha_k x_k \leq y_k \leq \beta_k x_k$ and $(\beta_k/\alpha_k) \leq e^R$. Applying φ gives that $\alpha_k \leq \varphi(y_k) = 1 \leq \beta_k$, so $\alpha_k \geq e^{-R}$ and $\beta_k \leq e^R$. By taking a subsequence we can assume that $\alpha_k \to \alpha \geq e^{-R}$, $\beta_k \to \beta \leq e^R$ and $\alpha\xi \leq \eta \leq \beta\xi$. It follows that ξ and η are comparable and that $d(\xi, \eta) \leq \log(\frac{\beta}{\alpha}) \leq R$. $\square$

There is a version of Lemma 5.1 for Thompson's metric $\overline{d}$. We leave the proof to the reader.

LEMMA 5.2. *Let C be a closed cone in a Banach space X and let $\langle x_k \mid k \geq 1 \rangle$ and $\langle y_k \mid k \geq 1 \rangle$ be sequences in C such that x_k and y_k are comparable for all $k \geq 1$ and $\overline{d}(x_k, y_k) \leq R$ for all $k \geq 1$, where $\overline{d}$ denotes Thompson's metric on C. Assume that $\lim_{k\to\infty} \|x_k - \zeta\| = 0$ and $\lim_{k\to\infty} \|y_k - \eta\| = 0$. Then either $\zeta = 0$ and $\eta = 0$ or ζ and η are both nonzero, ζ and η are comparable and $\overline{d}(\zeta, \eta) \leq R$.*

Our first theorem shows that information about $\omega(x; f) := \omega(x; f, \|\cdot\|)$ for any given x provides information about $\bigcup_{z \in \Sigma_q} \omega(z; f)$.

THEOREM 5.3. *Let notation and assumptions be as in Theorem 4.17 and let x denote a fixed element of Σ. For every $\zeta \in \bigcup_{z \in \Sigma} \omega(z; f)$, there exists $\xi \in \omega(x; f)$ with ζ comparable to ξ; and for every $\zeta \in \mathrm{co}(\bigcup_{z \in \Sigma} \omega(z; f))$ there exists $\xi \in \mathrm{co}(\omega(x; f))$ with ζ comparable to ξ. In particular, if $\mathrm{co}(\omega(x; f)) \subset \partial C$, then $\mathrm{co}(\bigcup_{z \in \Sigma} \omega(z; f)) \subset \partial C$; and if all elements of $\omega(x; f)$ are comparable, then all elements of $\bigcup_{z \in \Sigma} \omega(z; f)$ are comparable. If $\xi \in \omega(x; f)$ and $f|\gamma(x; f)$ extends in a norm continuous way to $\gamma(x; f) \cup \{\xi\}$, then ξ and $f(\xi)$ are comparable.*

PROOF. If $\zeta \in \omega(z; f)$ for some $z \in \Sigma$, there exists a sequence of integers $k_i \uparrow \infty$ with $\lim_{i\to\infty} \|f^{k_i}(z) - \zeta\| = 0$. Because $c\ell(\gamma(x; f))$ is compact, we can, by taking a further subsequence, assume that $f^{k_i}(x) \to \xi \in \omega(x; f)$, where convergence is in the norm topology. Because $d(f^{k_i}(z), f^{k_i}(x)) \leq d(z, x)$, Lemma 5.1 is applicable and implies that ζ and ξ are comparable with $d(\zeta, \xi) \leq d(z, x)$.

If $\zeta \in \mathrm{co}(\bigcup_{z \in \Sigma} \omega(z; f))$, there exist $z_j \in \Sigma$, $1 \leq j \leq n$, $\zeta_j \in \omega(z_j; f)$ and $\lambda_j > 0$ with $\sum_{j=1}^n \lambda_j = 1$ such that $\zeta = \sum_{j=1}^n \lambda_j \zeta_j$. We know that there exist $\xi_j \in \omega(x; f)$ with ξ_j comparable to ζ_j, and it follows easily that $\xi = \sum_{j=1}^n \lambda_j \xi_j \in \mathrm{co}(\omega(x; f))$ and ξ is comparable to ζ.

If $\mathrm{co}(\omega(x; f)) \subset \partial C$, no element of $\mathrm{co}(\omega(x; f))$ is comparable to an element of $\mathrm{int}(C)$ and hence no element of $\mathrm{co}(\bigcup_{z \in \Sigma} \omega(z; f))$ is comparable to an element of $\mathrm{int}(C)$. If all elements of $\omega(x; f)$ are comparable, our previous results show that all elements of $\bigcup_{z \in \Sigma} \omega(z; f)$ are comparable.

If $\zeta \in \omega(x; f)$, there exists a sequence of integers $k_i \uparrow \infty$ with $f^{k_i}(x) \to \zeta$. If $f|\gamma(x; f)$ extends continuously to $\gamma(x; f) \cup \{\zeta\}$, $f^{k_i+1}(x) \to f(\zeta)$. Since

$d(f^{k_i}(x), f^{k_i+1}(x)) \leq d(x, f(x))$, Lemma 5.1 implies that ζ and $f(\zeta)$ are comparable. $\square$

If $f \colon \Sigma_q \to \Sigma_q$ is d-nonexpansive, $\zeta = \lim_{i \to \infty} f^{k_i}(x)$, $\eta = \lim_{i \to \infty} f^{m_i}(x)$ and $\limsup_{i \to \infty} |k_i - m_i| < \infty$, one sees that ζ and η are comparable.

Theorem 5.3 does not depend on Theorem 4.13 and the special geometry of d. Thus, with the aid of Lemma 5.2, one can prove a variant of Theorem 5.3 for Thompson's metric. Details are left to the reader.

THEOREM 5.4. *Let C be a closed, normal cone with nonempty interior in a Banach space $(X, \| \cdot \|)$. Let $f \colon \mathrm{int}(C) \to \mathrm{int}(C)$ be a map which is nonexpansive with respect to Thompson's metric $\overline{d}$. Assume the following compactness conditions on f:*

(a) *If $\langle x_k \mid k \geq 1 \rangle \subset \mathrm{int}(C)$ is any bounded sequence in the $\overline{d}$-metric such that $\lim_{k \to \infty} \overline{d}(f(x_k), x_k) = 0$, then $\langle x_k \mid k \geq 1 \rangle$ has a norm convergent subsequence.*

(b) *For every $x \in \mathrm{int}(C)$, $cl(\{f^k(x) \mid k \geq 0\})$ is compact.*

Let $\omega(z; f)$ denote the omega limit set of z under f in the norm topology. Let x denote a fixed element of $\mathrm{int}(C)$, and make the convention that 0 is comparable to 0. Then for every $\zeta \in \mathrm{co}(\bigcup_{z \in \mathrm{int}(C)} \omega(z; f))$, there exists $\xi \in \mathrm{co}(\omega(x; f))$ with ξ comparable to ζ. In particular, if $\mathrm{co}(\omega(x; f)) \subset \partial C$, then $\mathrm{co}(\bigcup_{z \in \mathrm{int}(C)} \mathrm{co}(z; f)) \subset \partial C$; and all elements of $\bigcup_{z \in \mathrm{int}(C)} \omega(z, f)$ are comparable if all elements of $\omega(x; f)$ are comparable.

If, under the hypotheses of Theorem 4.17, there exists $x \in \Sigma$ such that $\omega(x; f)$ is a finite set, then we wish to prove that all elements of $\bigcup_{z \in \Sigma} \omega(z; f)$ are comparable, so $\mathrm{co}(\bigcup_{z \in \Sigma} \omega(z; f)) \subset \partial C$. If f extends to a norm continuous map of $cl(\Sigma)$ to itself, we can apply Lemma 5.1 and Lemma 6.1 in [2] to prove this result. However, even in finite dimensions (see [9]), such a norm continuous extension may not exist.

The following technical lemma is designed to circumvent this difficulty.

LEMMA 5.5. *Let (D, ρ) be a metric space and assume that $f \colon D_0 \to D_0$ is a continuous map, where $D_0 \subset D$. For a given $x \in D_0$ assume that $cl\{f^k(x) \mid k \geq 0\} := cl(\gamma(x; f))$ is compact and let $\omega(x; f) = \bigcap_{n \geq 0} cl\{f^k(x) \mid k \geq n\}$. Assume that $\omega_* \subset \omega(x; f)$ is a nonempty set which satisfies the following properties:*

(a) *If there exists a sequence $k_i \uparrow \infty$ such that $f^{k_i}(x) \to \zeta \in \omega_*$ and $f^{k_i+1}(x) \to \eta$, then $\eta \in \omega_*$.*

(b) *ω_* is closed.*

(c) *There exists an open set $U \subset D$ with $U \cap \omega(x; f) = \omega_*$.*

Then we have that $\omega_* = \omega(x; f)$.

PROOF. We know that $\omega(x; f)$ is compact and nonempty. Let V be an open neighbourhood of ω_* in D such that $cl(V) \subset U$. We claim that there exists an integer N such that if $k \geq N$ and $f^k(x) \in V$, then $f^{k+1}(x) \in V$. If not, there exists a sequence $k_i \uparrow \infty$ with $f^{k_i}(x) \in V$ and $f^{k_i+1}(x) \notin V$. Because $cl(\gamma(x; f))$ is compact, by taking a further subsequence we can assume that $f^{k_i}(x) \to \zeta \in \overline{V} \cap \omega(x; f)$ and $f^{k_i+1}(x) \to \eta \in \omega(x; f)$. Because $\overline{V} \cap \omega(x; f) = \omega_*$, we have that $\zeta \in \omega_*$, while $\eta \notin \omega_*$, which contradicts condition (a) Thus there exists an integer N as above. However, ω_* is nonempty, so there exists $k \geq N$ with $f^k(x) \in V$, and then we have $f^j(x) \in V$ for all $j \geq k$. It follows that all elements of $\omega(x; f)$ lie in $cl(V)$ and hence in ω_*.   □

If C is a closed cone in a Banach space and $z \in C - \{0\}$, recall that C_z denotes all elements of C which are comparable to z (see (2.1)). One can also check that $cl(C_z)$ consists of all elements $x \in C$ which are dominated by z.

With the aid of Lemma 5.5, we can give examples for which $\mathrm{co}(\omega(x; f)) \subset \partial C$. In particular, under the hypotheses of Theorem 4.17, we will always have that all elements of $\omega(x; f)$ are comparable and $\mathrm{co}(\omega(x; f)) \subset \partial C$ if $\omega(x; f)$ is a finite set.

THEOREM 5.6. *Let notation and hypotheses be as in Theorem 4.17 (so, for all $x \in \Sigma$, $(\omega(x; f) \subset \partial C$). Given $x \in \Sigma$ and $z \in C \setminus \{0\}$ such that $\omega(x; f) \cap C_z \neq \emptyset$, either define*

(a) $\omega_* = \omega(x; f) \cap C_z$, *or*
(b) $\omega_* = \omega(x; f) \cap (cl(C_z))$.

If ω_* is as in case (a) and $\omega(x; f)$ is a finite set, then $\omega_* = \omega(x; f)$, $\mathrm{co}(\omega(x; f)) \subset \partial C$ and all elements of $\omega(x; f)$ are comparable. In general, if there exists an open neighbourhood U of ω_* such that $U \cap \omega(x; f) = \omega_*$ and if ω_* is closed, then $\omega_* = \omega(x; f)$.

PROOF. Suppose that ω_* is as in case (a) or case (b). If $k_i \uparrow \infty$, $f^{k_i}(x) \to \zeta \in \omega_*$ and $f^{k_i+1}(x) \to \eta$, Lemma 5.1 implies that ζ and η are comparable. In case (a), we conclude that η is comparable to z, so $\eta \in \omega(x; f) \cap C_z := \omega_*$. In case (b), η is comparable to ζ, and ζ is dominated by z, so η is dominated by z and $\eta \in \omega(x; f) \cap (cl(C_z)) := \omega_*$. If ω_* is closed (which is automatically true in case (b)) and if there exists U as in the statement of the theorem, Lemma 5.3 implies that $\omega_* = \omega(x; f)$. If $\omega(x; f)$ is finite, the existence of U and the closedness of $\omega(x; f) \cap C_z$ is obvious, so all elements of $\omega(x; f)$ are comparable to z.   □

There is, of course, an analogue of Theorem 5.5 for Thompson's metric $\overline{d}$.

THEOREM 5.7. *Let C be a closed, normal cone with nonempty interior in a Banach space $(X, \| \cdot \|)$. Assume that $f: \mathrm{int}(C) \to \mathrm{int}(C)$ is fixed point free*

and is nonexpansive with respect to Thompson's metric $\overline{d}$. Assume also that f satisfies the fixed point property on $\mathrm{int}(C)$ with respect to $\overline{d}$ and that $c\ell(\gamma(x; f))$ is compact for each $x \in \mathrm{int}(C)$. If $x \in \mathrm{int}(C)$ and $\omega(x; f, \|\cdot\|) := \omega(x; f)$ is a finite set, then either $\omega(x; f) = \{0\}$ or $\omega(x; f) \subset \partial C$ and all elements of $\omega(x; f)$ are comparable. In general, if there exists $z \in C \setminus \{0\}$ with $\omega(x; f) \cap C_z \neq \emptyset$, define

(a) $\omega_* := \omega(x; f) \cap C_z$, *or*
(b) $\omega_* = \omega(x; f) \cap (c\ell(C_z))$.

If ω_* is closed and if there exists an open set U with $\omega(x; f) \cap U = \omega_*$, then $\omega_* = \omega(x; f)$.

Proof. The results of Section 3 imply that $\omega(x; f) \subset \partial C$. If ω_* is defined as above in the general case, ω_* is closed and there exists an open set U as above, the same argument as in Theorem 5.6 (use Lemma 5.2 rather than Lemma 5.1) shows that $\omega_* = \omega(x; f)$. If $\omega(x; f) = \{0\}$, it is easy to see that $\lim_{k\to\infty} \|f^k(x)\| = 0$, so there will always exist $z \in C \setminus \{0\}$ with $\omega(x; f) \cap C_z \neq \emptyset$ unless $\lim_{k\to\infty} \|f^k(x)\| = 0$. For such a z, if $\omega(x; f)$ is finite, $\omega_* := \omega(x; f) \cap C_z$ is closed and there exists an open set U as in the statement of the theorem. $\square$

Recall that a closed cone in a finite dimensional Banach space X is called *polyhedral* if there exist nonzero, continuous linear functionals $\vartheta_j \in X^*$, $1 \leq j \leq N$, such that

$$(5.1) \qquad C = \{x \in X \mid \vartheta_j(x) \geq 0 \text{ for } 1 \leq j \leq N\}$$

In general, if C is a closed cone in a finite dimensional Banach space X, $\dim(C)$, the dimension of C is the dimension of the linear span of C. If $\vartheta \in C^*$, $F := \{x \in C \mid \vartheta(x) = 0\}$ is called a *face of* C, so taking $\vartheta = 0$, we see in particular that C is a face of C. If F is a face of C, F is a closed cone, and we can consider its dimension; and if $\dim(F) = \dim(C) - 1$, F is called a *facet of* C. If C is a polyhedral cone in a finite dimensional Banach space X, it is known that there exist $\vartheta_j \in X^*, 1 \leq j \leq N$, such that C is given by (5.1) and each ϑ_j defines a facet of C, see [45, Section 8.4].

Theorems 5.6 and 5.7 are most useful if one can prove that $\omega(x; f)$ is finite for some $x \in \mathrm{int}(C)$. A recent theorem in [2] provides exactly such information.

Theorem 5.8 (see [2, Theorem 6.8]). *Let C be a polyhedral cone in a finite dimensional Banach space X and assume that C is given by (5.1). If $f\colon C \to C$ is a continuous, order-preserving subhomogeneous map and $x \in C$ has a bounded orbit $\gamma(x; f) := \{f^k(x) \mid k \geq 0\}$ under f, then $\omega(x; f)$ is finite. If*

$$|\omega(x; f)| = p, \quad \lim_{k\to\infty} f^{kp}(x) = \zeta \in \omega(x; f),$$

$$\text{where } f^p(\zeta) = \zeta \text{ and } \omega(x; f) = \{f^j(\zeta) \mid 0 \leq j < p\}.$$

Furthermore, if N is as in (5.1), *then*

$$|\omega(x;f)| \leq \beta_N := \frac{N!}{[N/3]![(N+1)/3]![(N+2)/3]!},$$

where $[\delta]$ denotes the greatest integer $j \leq \delta$.

If, under the hypotheses of Theorem 5.8, $\text{int}(C) \neq \emptyset$, $f(\text{int}(C)) \subset \text{int}(C)$ and $\zeta \in \text{int}(C)$ is a periodic point of f with minimal period p (so $f^p(\zeta) = \zeta$), then earlier results of Lemmens and Scheutzow are refined in [2] and it is proved that

$$(5.2) \qquad\qquad p \leq \frac{N!}{[N/2]!} := \gamma_N.$$

With the aid of Theorems 5.7 and 5.8 we can prove a refined version of Conjecture 4.23 for which $\omega(x;f)$ is finite.

COROLLARY 5.9. *Let C be a polyhedral cone with nonempty interior in a finite dimensional Banach space X and assume that C is given by* (5.1). *Let $f: C \to C$ be a continuous, order-preserving subhomogeneous map such that $f(\text{int}(C)) \subset \text{int}(C)$, f has no fixed points in $\text{int}(C)$ and $\gamma(x;f) := \{f^k(x) \mid k \geq 0\}$ is bounded in norm for all $x \in \text{int}(C)$. Then, for every $x \in \text{int}(C)$, we have that:*

    (a) *$\omega(x;f)$ is a finite set,*
    (b) *either $\omega(x;f) = \{0\}$ or all elements of $\omega(x;f)$ are comparable, and*
    (c) *$\text{co}(\omega(x;f)) \subset \partial C$.*

Furthermore, if γ_N is defined as in (5.2), *we have that $|\omega(x;f)| \leq \gamma_{N-1}$.*

PROOF. We have already remarked that $f|\text{int}(C)$ is nonexpansive with respect to Thompson's metric $\bar{d}$. Theorem 5.8 implies that $\omega(x;f)$ is a finite set for $x \in \text{int}(C)$, so Theorem 5.7 gives conditions (b) and (c) of the corollary.

If $x \in \text{int}(C)$, $\zeta \in \omega(x;f)$ and $\zeta \neq 0$, Theorem 5.8 implies that $\omega(x;f) = \{f^j(\zeta) \mid 0 \leq j < p\}$, $p = |\omega(x;f)|$. For ϑ_j as in (5.1), let $J = \{j \mid \vartheta_j(\zeta) > 0\}$ and note that J is nonempty and $J \neq \{1, \dots, N\}$ because $\zeta \in \partial C$. Let $Y = \{y \in X \mid \vartheta_j(y) = 0 \text{ for all } j \notin J\}$ and let $K = C \cap Y$. The interior of K in $Y = \text{int}_Y(K) = \{y \in Y \mid \vartheta_j(y) > 0 \text{ for all } j \in J\}$. Since ζ and $f(\zeta)$ are comparable, $f(\text{int}_Y(K)) \subset \text{int}_Y(K)$ and $f(K) \subset K$. Since K is a polyhedral cone specified by $N' := |J| < N$ linear functionals and ζ is a periodic point of f, (5.2) implies that $p = |\omega(x;f)| \leq \gamma_{N'} \leq \gamma_{N-1}$. $\qquad\square$

Our next corollary relates to d-nonexpansive maps h.

COROLLARY 5.10. *Let C, X and f be as in Corollary 5.9. Assume, in addition, that f is homogeneous of degree one and that $r_C(f) = 1$, where $r_C(f)$ denotes the cone spectral radius of f. Let $q: C \to [0, \infty)$ be continuous, homogeneous of degree one and strictly positive on $C - \{0\}$. Define $\Sigma_q = \{x \in \text{int}(C) \mid$*

$q(x) = 1\}$ *and* $h(x) = f(x)/q(f(x))$ *for* $x \in \Sigma_q$, *so* $h\colon \Sigma_q \to \Sigma_q$. *Then for any* $y \in \mathrm{int}(C)$, *we have that* $\omega(y; f)$ *is a finite set which does not contain zero, all elements of* $\omega(y; f)$ *are comparable and* $\mathrm{co}(\omega(y; f)) \subset \partial C$. *We also have that for any* $z \in \Sigma_q$, $\omega(z; h)$ *is a finite set, all elements of* $\omega(z; h)$ *are comparable and* $\mathrm{co}(\omega(z; h)) \subset \partial C$.

PROOF. Corollary 5.10 follows directly from Theorem 5.8 and Corollary 5.9 if we note that for any $x \in \mathrm{int}(C)$ there exists $\alpha = \alpha(x) > 0$ such that $\inf\{\|f^k(x)\| \mid k \geq 0\} \geq \alpha(x)$. To see this later fact, note that because $r_C(f) = 1$ and C is finite dimensional, there exists $u \in C$, $\|u\| = 1$, with $f(u) = u$. If $x \in \mathrm{int}(C)$, there exists $\delta = \delta(x) > 0$ with $x \geq \delta u$. This implies that $f^k(x) \geq \delta f^k(u) = \delta u$ for all $k \geq 0$, and since C is a normal cone, we conclude that there $\alpha(x) > 0$ as desired. $\square$

REMARK 5.11. In general, the assumption in Corollaries 5.9 and 5.10 that $\gamma(x; f)$ is bounded for all $x \in \mathrm{int}(C)$ is crucial. In the context of Corollary 5.10, but without the assumption that $\gamma(x; f)$ is bounded for all $x \in \mathrm{int}(C)$, an example in [32] shows that $\omega(x; h)$ may be infinite. If, however, f is linear in Corollary 5.10, it is proved in [33] that $\omega(x; h)$ is finite for $x \in \mathrm{int}(C)$, even if $\gamma(x; f)$ is unbounded.

REFERENCES

[1] M. ABATE, *Iteration Theory of Holomorphic Maps on Taut Manifolds*, Mediterranean Press, 1989.

[2] M. AKIAN, S. GAUBERT, B. LEMMENS AND R. D. NUSSBAUM, *Iteration of order preserving subhomogeneous maps on a cone*, Math. Proc. Cambridge Philos. Soc. **140** (2006), 157–176.

[3] M. AKIAN, S. GAUBERT AND R. D. NUSSBAUM, *The Collatz–Wielandt theorem for order-preserving homogeneous maps on cones*, in preparation.

[4] A. F. BEARDON, *Iteration of contractions and analytic maps*, J. London Math. Soc. **41** (1990), 144–150.

[5] _____, *The dynamics of contractions*, Ergodic Theory Dynam. Systems **17** (1997), 1257–1266.

[6] Y. BENOIST, *Convexes hyperboliques et fonctions quasisymétriques*, Publ. Math. Inst. Hautes Études Sci. **97** (2003), 181–237.

[7] G. BIRKHOFF, *Extensions of Jentzsch's theorems*, Trans. Amer. Math. Soc. **85** (1957), 219–277.

[8] R. F. BROWN, Scott Foreman Co., Glenview, Illinois (1971).

[9] A. D. BURBANKS, R. D. NUSSBAUM AND C. T. SPARROW, *Extension of order-preserving maps on a cone*, Proc. Roy. Soc. Edinburgh **133A** (2003), 35–59.

[10] P. BUSHELL, *Hilbert's metric and positive contraction mappings in Banach space*, Arch. Rat. Mech. Anal. **52** (1973), 330–338.

[11] _____, *The Cayley–Hilbert metric and positive operators*, Linear Algebra Appl. **84** (1986), 271–280.

[12] A. Calka, *On conditions under which isometries have bounded orbits*, Colloq. Math. **48** (1984), 219–227.

[13] J. Conway, *A Course in Functional Analysis* (1990), Springer Verlag, New York.

[14] C. Dafermos and M. Slemrod, *Asymptotic behavior of non-linear contraction semi-groups*, J. Funct. Anal. **13** (1973), 97–106.

[15] G. Darbo, *Punti uniti in trasformazioni a condiminio non compatto*, Rend. Sem . Mat. Univ. Padova **24** (1955), 84–92.

[16] A. Dold, *Lectures on Algebraic Topology*, Springer Verlag, New York, 1972.

[17] N. Dunford and J. T. Schwartz, *Linear Operators*, Part I, Wiley Classics Library Edition, John Wiley and Sons, New York, 1988.

[18] M. Edelstein, *On non-expansive mappings of Banach spaces*, Proc. Cambridge Philos. Soc. **60** (1964), 439–447.

[19] S. P. Eveson and R. D. Nussbaum, *An elementary proof of the Birkhoff-Hopf theorem*, Math. Proc. Cambridge Philos. Soc. **117** (1995), 31–55.

[20] ———, *Applications of the Birkhoff–Hopf theorem to the spectral theory of positive linear operators*, Math. Proc. Cambridge Philos. Soc. **117** (1995), 491–512.

[21] D. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, Boston, 1988.

[22] D. Hilbert, *Über die gerade Linie als kürzeste Verbindungzweier Punkte*, Math. Ann. **46** (1895), 91–96.

[23] E. Hopf, *An inequality for positive linear integral operators*, J. Math. Mech. **12** (1963), 683–692.

[24] J. Kapeluszny, T. Kuczumow and S. Reich, *The Denjoy–Wolff theorem for condensing holomorphic mappings*, J. Funct. Anal. **167** (1999), 79–93.

[25] A. Karlsson, *Non-expanding maps and Busemann functions*, Ergodic Theory Dynam. Systems **21** (2001), 1447–1457.

[26] A. Karlsson, V. Metz and G. A. Noskov, *Horoballs in simplices and Minkowski spaces*, preprint.

[27] A. Karlsson and G. Noskov, *The Hilbert metric and Gromov hyperbolicity*, Enseign. Math. **48** (2002), 73–89.

[28] E. Kohlberg and A. Neyman, *Asymptotic behaviour of nonexpansive mappings in normed linear spaces*, Israel J. Math. **38** (1981), 269–275.

[29] M. A. Krasnosel'skiĭ, J. A. Lipshitz and A. V. Sobolev, *Positive Linear Systems: the Method of Positive Operators*, Sigma Series in Applied Math., vol. 5, Heldermann–Verlag, 1989.

[30] M. A. Krasnosel'skiĭ and A. V. Sobolev, *Spectral clearance of a focusing operator*, Funct. Anal. Appl. **17** (1983), 58–59.

[31] C. Kuratowski, *Sur les espaces complets*, Fund. Math. **15** (1930), 301–309.

[32] B. Lins, *A Denjoy–Wolff theorem for Hilbert metric nonexpansive maps on a polyhedral domain*, preprint; Math. Proc. Cambridge Philos. Society (2007) (to appear).

[33] B. Lins and R. D. Nussbaum, *Iterated linear maps on a cone and Denjoy–Wolff theorems*, Linear Algebra Appl. **416** (2006), 615–626.

[34] J. Mallet-Paret and R. D. Nussbaum, *Eigenvalues for a class of homogeneous cone maps arising from max-plus operators*, Discrete Contin. Dynam. Systems **8** (2002), 519–562.

[35] R. D. Nussbaum, *Eigenvalues of nonlinear operators and the linear Krein–Rutman theorem*, Springer Lecture Notes in Math. **886**, 309–331.

[36] ———, *Convexity and log convexity for the spectral radius*, Linear Algebra Appl. **73** (1986), 59–122.

[37] _____, *Iterated nonlinear maps and Hilbert's projective metric*, Mem. Amer. Math. Soc. **391** (1988).

[38] _____, *Iterated nonlinear maps and Hilbert's projective metric,* II, Mem. Amer. Math. Soc. **401** (1989).

[39] _____, *Finsler structures for the part metric and Hilbert's projective metric and applications to ordinary differential equations*, Differential Integral Equations **7** (1994), 1649–1707.

[40] _____, *Periodic points of linear operators and Perron–Frobenius operators*, Integral Equations Operator Theory **39** (2001), 41–97.

[41] _____, *Entropy minimization, Hilbert's projective metric and scaling integral kernels*, J. Funct. Anal. **115** (1993), 45–99.

[42] _____, *The fixed point index for local condensing maps*, Ann. Mat. Pura Appl. **89** (1971), 217–258.

[43] _____, *Some asymptotic fixed point theorems*, Trans. Amer. Math. Soc. **171** (1972), 349–375.

[44] _____, *Generalizing the fixed point index*, Math. Ann. **228** (1977), 259–278.

[45] S. Roehrig and R. Sine, *The structure of ω-limit sets of non-expansive maps*, Proc. Amer. Math. Soc. **81** (1981), 398–400.

[46] B. N. Sadovskiĭ, *A fixed-point principle*, Funct. Anal. Appl. **4** (1967), 74–76.

[47] H. Samelson, *On the Perron–Frobenius Theorem*, Michigan Math. J. **4** (1957), 57–59.

[48] H. Schaefer, *Topololgical Vector Spaces*, Springer Verlag, New York, 1971.

[49] A. Schrijver, *Theory of Linear and Integer Programming*, John Wiley, Chichester, 1986.

[50] E. Socié-Méthou, *Behaviour of distance functions in Hilbert–Finsler geometry*, Differential Geom. Appl. **20** (2004), 1–10.

[51] A. C. Thompson, *On certain contraction mappings in a partially ordered vector space*, Proc. Amer. Math. Soc. **14** (1963), 438–443.

[52] P. P. Zabreĭko, M. A. Krasnosel'skiĭ and Yu. V. Pokornyĭ, *On a class of positive linear operators*, Funct. Anal. Appl. **5** (1972), 272–279.

Roger D. Nussbaum
Mathematics Department
Rutgers University
Piscataway, New Jersey 08854-8019, USA

E-mail address: nussbaum@math.rutgers.edu