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THE SUSPENSION ISOMORPHISM FOR COHOMOLOGY INDEX BRAIDS

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Abstract. Let X be a metric space, π be a local semiflow on X, $k \in \mathbb{N}$, E be a k-dimensional normed real vector space and $\widetilde{\pi}$ be the semiflow generated by the equation $\dot{y} = Ly$, where $L: E \to E$ is a linear map whose all eigenvalues have positive real parts. We show in this paper that for every admissible isolated π-invariant set S there is a well-defined isomorphism of degree k from the (Alexander–Spanier)-cohomology categorial Conley–Morse index of (π, S) to the cohomology categorial Conley–Morse index of $(\pi \times \widetilde{\pi}, S \times \{0\})$ such that the family of these isomorphisms commutes with cohomology index sequences. This extends previous results by Carbinatto and Rybakowski to the Alexander–Spanier-cohomology case.

1. Introduction

This paper is a sequel to the previous work [6] by M. C. Carbinatto and this author. Let H^q, $q \in \mathbb{Z}$, be the Alexander–Spanier cohomology functor with coefficients in a fixed Γ-module $\overline{G}$, where Γ is a commutative ring. Let X be a metric space, π be a local semiflow on X, $k \in \mathbb{N}$, E be a k-dimensional normed real vector space and $\widetilde{\pi}$ be the semiflow generated by the equation $\dot{y} = Ly$, where $L: E \to E$ is a linear map whose all eigenvalues have positive real parts.

Consider the *local product semiflow* $\pi \times \widetilde{\pi}$ on $X \times E$ defined by $(x, y)\pi \times \widetilde{\pi} t := (x\pi t, y\widetilde{\pi} t)$ whenever $x\pi t$ is defined.

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Whenever S is an isolated π-invariant set having a strongly π-admissible isolating neighborhood, then $S\times\{0\}$, where $0=0_E$ is the zero of E, is an isolated $\pi\times\widetilde{\pi}$-invariant set having a strongly $\pi\times\widetilde{\pi}$-admissible isolating neighborhood.

It is the main purpose of this paper to prove, in Theorem 4.1 below, that there is a well-defined isomorphism $\theta^q(\pi,\widetilde{\pi},S)$ from the cohomology categorial Conley–Morse index $H^{q-k}(\pi,S) := H^{q-k}(\mathcal{C}(\pi,S))$ of (π,S) to the cohomology categorial Conley-Morse index $H^q(\pi\times\widetilde{\pi},S\times\{0\}) := H^q(\mathcal{C}(\pi\times\widetilde{\pi},S\times\{0\}))$ of $(\pi\times\widetilde{\pi},S\times\{0\})$ such that the family of such isomorphisms commutes with cohomology index sequences. This means that, given any isolated π-invariant set S having a strongly π-admissible isolating neighborhood and given any attractor-repeller pair (A,A^*) of S relative to π, the following diagram commutes:

$$\begin{tikzcd}
\longleftarrow H^q(\pi',A') & H^q(\pi',S') \arrow[l] & H^q(\pi',A^{*\prime}) \arrow[l] & H^{q-1}(\pi',A') \arrow[l] & \arrow[l] \\
\longleftarrow H^{q-k}(\pi,A) \arrow[u, "\theta^q(\pi,\widetilde{\pi},A)"] & H^{q-k}(\pi,S) \arrow[l] \arrow[u, "\theta^q(\pi,\widetilde{\pi},S)"] & H^{q-k}(\pi,A^*) \arrow[l] \arrow[u, "\theta^q(\pi,\widetilde{\pi},A^*)"'] & H^{q-k-1}(\pi,A) \arrow[l] \arrow[u, "\theta^{q-1}(\pi,\widetilde{\pi},A)"'] & \arrow[l]
\end{tikzcd}$$

Here, the upper (resp. lower) horizontal sequence is the cohomology index sequence of $(\pi',S',A',A^{*\prime})$ (resp. (π,S,A,A^*)), where we set $\pi' := \pi\times\widetilde{\pi}$ and for $K\subset X$, $K' := K\times\{0_E\}$.

This result implies that whenever P is a finite set, $\prec$ is a strict order relation on P and $(M_i)_{i\in P}$ is a $\prec$-ordered Morse decomposition of S relative to π, then $(M_i\times\{0\})_{i\in P}$ is a $\prec$-ordered Morse decomposition of $S\times\{0\}$ relative to $\pi\times\widetilde{\pi}$ and there is a module braid isomorphism from the cohomology index braid of $(\pi,S,(M_i)_{i\in P})$ to the cohomology index braid of $(\pi\times\widetilde{\pi},S\times\{0\},(M_i\times\{0\})_{i\in P})$ 'shifted to the right by k'.

Theorem 4.1 extends [6, Theorem 3.1] to the case of Alexander–Spanier cohomology.

The construction of the suspension isomorphism for cohomology, see Theorem 5.6 below, essentially follows by a standard dualization of the proof from [6] for the homology case. In fact this construction is valid for an arbitrary cohomology theory.

On the other hand, additional ideas are required for the proof that the suspension isomorphism constructed in Theorem 5.6 commutes with cohomology index sequences. More specifically, the proof is based on the concept of weakly coexact sequences, introduced in Definition 2.1 and on some technical results about index triples and Alexander-Spanier cochain complexes established in Sections 3 and 6. We also require an anticommutativity result for 3×3-matrices of cochain maps, established in Section 2.

An application of the results of this paper and the paper [6] to some singular perturbation problems will be given in a forthcoming publication.

In this paper we use the notation and results from [5] and [6] without further explanation.

2. Weakly coexact sequences and anticommutativity of the connecting homomorphisms

We now introduce the following general concept:

Definition 2.1. A sequence

$$C_1 \xrightarrow{i} C_2 \xrightarrow{p} C_3$$

of cochain maps is called *weakly coexact* if p is epic, $p \circ i = 0$ and the map $H^q(\mu)\colon H^q(C_1) \to H^q(\ker p)$ is an isomorphism for each $q \in \mathbb{Z}$. Here, the map $\mu\colon C_1 \to \ker p$ is the (uniquely determined) cochain map with $\nu \circ \mu = i$, where $\nu\colon \ker p \to C_2$ is the inclusion map.

Remark. The concept of a *weakly coexact sequence* can be regarded as dual to Franzosa's concept of a *weakly exact sequence*, cf. [8].

Given a weakly exact sequence

$$C_1 \xrightarrow{i} C_2 \xrightarrow{p} C_3$$

and $q \in \mathbb{Z}$, define $\widehat{\partial}^q\colon H^q(C_3) \to H^{q+1}(C_1)$ by $\widehat{\partial}^q := H^{q+1}(\mu)^{-1} \circ \partial^{q*}$, where $\partial^{q*}\colon H^q(C_3) \to H^{q+1}(\ker p)$, $q \in \mathbb{Z}$, is the connecting homomorphism in the long exact cohomology sequence induced by the short exact sequence

$$0 \longrightarrow \ker p \xrightarrow{\nu} C_2 \xrightarrow{p} C_3 \longrightarrow 0.$$

Using elementary cohomology theory we obtain the following result.

Proposition 2.2. *Given a weakly coexact sequence*

$$C_1 \xrightarrow{i} C_2 \xrightarrow{p} C_3$$

of cochain maps, the corresponding cohomology sequence

$$\longrightarrow H^q(C_1) \xrightarrow{H^q(i)} H^q(C_2) \xrightarrow{H^q(p)} H^q(C_3) \xrightarrow{\widehat{\partial}^q} H^{q+1}(C_1) \longrightarrow$$

is exact. Moreover, given a commutative diagram

$$\begin{tikzcd}
C_1 \arrow[r, "i"] \arrow[d, "f_1"'] & C_2 \arrow[r, "p"] \arrow[d, "f_2"] & C_3 \arrow[d, "f_3"] \\
\widetilde{C}_1 \arrow[r, "\widetilde{i}"'] & \widetilde{C}_2 \arrow[r, "\widetilde{p}"'] & \widetilde{C}_3
\end{tikzcd}$$

of cochain maps with weakly coexact rows, the induced long cohomology ladder

$$\begin{tikzcd}
\arrow[r] & H^q(C_1) \arrow[r, "H^q(i)"] \arrow[d, "H^q(f_1)"'] & H^q(C_2) \arrow[r, "H^q(p)"] \arrow[d, "H^q(f_2)"'] & H^q(C_3) \arrow[r, "\widehat{\partial}^q"] \arrow[d, "H^q(f_3)"] & H^{q+1}(C_1) \arrow[r] \arrow[d, "H^{q+1}(f_1)"] & {} \\
\arrow[r] & H^q(\widetilde{C}_1) \arrow[r, "H^q(\widetilde{i})"'] & H^q(\widetilde{C}_2) \arrow[r, "H^q(\widetilde{p})"'] & H^q(\widetilde{C}_3) \arrow[r, "\widetilde{\widehat{\partial}^q}"'] & H^{q+1}(\widetilde{C}_1) \arrow[r] & {}
\end{tikzcd}$$

is commutative.

LEMMA 2.3. *Consider the following commutative diagram*

$$\begin{tikzcd}
A \arrow[r, "\iota"] \arrow[d, "\gamma_A"'] & B \arrow[r, "\kappa"] \arrow[d, "\gamma_B"] & C \arrow[d, "\gamma_C"] \\
\widehat{A} \arrow[r, "\widehat{\iota}"'] & \widehat{B} \arrow[r, "\widehat{\kappa}"'] & \widehat{C}
\end{tikzcd} \tag{2.1}$$

of cochain maps in which the first row is weakly coexact and the vertical arrows are isomorphisms. Then the sequence in the second row is weakly coexact.

PROOF. Since κ is epic, γ_A and γ_B are isomorphims, the commutativity of diagram (2.1) implies that $\widehat{\kappa}$ is epic. Moreover,

$$\widehat{\kappa} \circ \widehat{\iota} = (\gamma_C \circ \kappa \circ \gamma_B^{-1}) \circ (\gamma_B \circ \iota \circ \gamma_A^{-1}) = \gamma_C \circ (\kappa \circ \iota) \circ \gamma_A^{-1} = 0$$

since the sequence in the first row of diagram (2.1) is weakly coexact. The commutativity of (2.1) implies that γ_B maps $\ker \kappa$ into $\ker \widehat{\kappa}$ and so γ_B induces a cochain map ζ such that the following diagram commutes:

$$\begin{tikzcd}
A \arrow[r, "\mu"] \arrow[d, "\gamma_A"'] & \ker \kappa \arrow[r, "\nu"] \arrow[d, "\zeta"'] & B \arrow[r, "\kappa"] \arrow[d, "\gamma_B"] & C \arrow[d, "\gamma_C"] \\
\widehat{A} \arrow[r, "\widehat{\mu}"'] & \ker \widehat{\kappa} \arrow[r, "\nu"'] & \widehat{B} \arrow[r, "\widehat{\kappa}"'] & \widehat{C}
\end{tikzcd} \tag{2.2}$$

Here, μ (resp. $\widehat{\mu}$) is the uniquely determined cochain map such that $\iota = \nu \circ \mu$ (resp. $\widehat{\iota} = \widehat{\nu} \circ \widehat{\mu}$), where ν (resp. $\widehat{\nu}$) is the inclusion map. Since ζ and γ_A are isomorphisms and μ induces an isomorphism in cohomology, it follows that $\widehat{\mu}$ induces an isomorphism in cohomology. □

We require the following important result from homological algebra, which in its general and explicit form needed here is due to M. Scott Osborne:

Proposition 2.4. ([12, Proposition 9.20]) *Suppose that the diagram*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A_{1,1} \arrow[r] \arrow[d] & A_{1,2} \arrow[r] \arrow[d] & A_{1,3} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_{2,1} \arrow[r] \arrow[d] & A_{2,2} \arrow[r] \arrow[d] & A_{2,3} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_{3,1} \arrow[r] \arrow[d] & A_{3,2} \arrow[r] \arrow[d] & A_{3,3} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{2.3}$$

of cochain maps is commutative and has exact columns and rows. For every $k \in \{1,2,3\}$, let $\partial^{k,q}\colon H^q(A_{k,3}) \to H^{q+1}(A_{k,1})$, $q \in \mathbb{Z}$, be the connecting homomorphism of the long cohomology sequence associated with the k-th row of (2.3) *and $\delta^{k,q}\colon H^q(A_{3,k}) \to H^{q+1}(A_{1,k})$, $q \in \mathbb{Z}$, be the connecting homomorphism of the long cohomology sequence associated with the k-th column of* (2.3)*. Then*

$$\delta^{1,q+1} \circ \partial^{3,q} = -\partial^{1,q+1} \circ \delta^{3,q}, \qquad q \in \mathbb{Z}. \tag{2.4}$$

Proof. The original statement of Proposition 2.4 is for chain complexes, chain maps and homology. The version stated here follows by the usual passage from cochain complexes and cochain maps to chain complexes and chain maps (and vice versa) using the index transformation $q \to -q$. □

Proposition 2.5. *Suppose that the diagram*

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
A_{1,1} \arrow[r, "\alpha_1"] \arrow[d] & A_{1,2} \arrow[r, "\beta_1"] \arrow[d, "f_1"] & A_{1,3} \arrow[d] \\
A_{2,1} \arrow[r, "\alpha_2"'] \arrow[d] & A_{2,2} \arrow[r, "\beta_2"'] \arrow[d, "f_2"] & A_{2,3} \arrow[d] \\
A_{3,1} \arrow[r, "\alpha_3"'] \arrow[d] & A_{3,2} \arrow[r, "\beta_3"'] \arrow[d] & A_{3,3} \arrow[d] \\
0 & 0 & 0
\end{tikzcd} \tag{2.5}$$

of cochain maps is commutative, has exact columns and weakly coexact last two rows. If β_1 is an epimorphism, then the first row of (2.5) *is weakly coexact. Furthermore, for every $k \in \{1,2,3\}$, let $\widehat{\partial}^{k,q}\colon H^q(A_{k,3}) \to H^{q+1}(A_{k,1})$, $q \in \mathbb{Z}$ be the connecting homomorphism of the long cohomology sequence associated with* *the k-th row of* (2.5) *and* $\delta^{k,q}\colon H^q(A_{3,k}) \to H^{q+1}(A_{1,k})$, $q \in \mathbb{Z}$, *be the connecting homomorphism of the long cohomology sequence associated with the k-th column of* (2.5). *Then*

$$\delta^{1,q+1} \circ \widehat{\partial}^{3,q} = -\widehat{\partial}^{1,q+1} \circ \delta^{3,q}, \qquad q \in \mathbb{Z}. \tag{2.6}$$

PROOF. Since $\beta_2 \circ \alpha_2 = 0$ and the cochain map from $A_{1,3}$ to $A_{2,3}$ in (2.5) is a monomorphism, it follows that $\beta_1 \circ \alpha_1 = 0$. Thus for each $k \in \{1,2,3\}$ there is a uniquely defined cochain map $\mu_k\colon A_{k,1} \to \ker\beta_k$ such that $\alpha_k = \nu_k \circ \mu_k$, where $\nu_k\colon \ker\beta_k \to A_{k,2}$ is the inclusion map. Moreover, f_1 (resp. f_2) induces the cochain map $g_1\colon \ker\beta_1 \to \ker\beta_2$ (resp. $g_2\colon \ker\beta_2 \to \ker\beta_3$). It follows that the diagram

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
A_{1,1} \arrow[r, "\mu_1"] \arrow[d] & \ker\beta_1 \arrow[r, "\nu_1"] \arrow[d, "g_1"'] & A_{1,2} \arrow[r, "\beta_1"] \arrow[d, "f_1"] & A_{1,3} \arrow[d] \\
A_{2,1} \arrow[r, "\mu_2"] \arrow[d] & \ker\beta_2 \arrow[r, "\nu_2"] \arrow[d, "g_2"'] & A_{2,2} \arrow[r, "\beta_2"] \arrow[d, "f_2"] & A_{2,3} \arrow[d] \\
A_{3,1} \arrow[r, "\mu_3"'] \arrow[d] & \ker\beta_3 \arrow[r, "\nu_3"'] \arrow[d] & A_{3,2} \arrow[r, "\beta_3"'] \arrow[d] & A_{3,3} \arrow[d] \\
0 & 0 & 0 & 0
\end{tikzcd} \tag{2.7}$$

is commutative. Now the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \ker\beta_1 \arrow[r, "\nu_1"] \arrow[d, "g_1"] & A_{1,2} \arrow[r, "\beta_1"] \arrow[d, "f_1"] & A_{1,3} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \ker\beta_2 \arrow[r, "\nu_2"] \arrow[d, "g_2"] & A_{2,2} \arrow[r, "\beta_2"] \arrow[d, "f_2"] & A_{2,3} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \ker\beta_3 \arrow[r, "\nu_3"'] \arrow[d] & A_{3,2} \arrow[r, "\beta_3"'] \arrow[d] & A_{3,3} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{2.8}$$

is commutative, has exact rows, and exact second and third column. It follows from the 3×3-Lemma that the first column (2.8) is also exact. For every $k \in \{1,2,3\}$, let $\partial^{k',q}\colon H^q(A_{k,3}) \to H^{q+1}(\ker\beta_k)$, $q \in \mathbb{Z}$, be the connecting homomorphism of the long cohomology sequence associated with the k-th row of (2.8) and $\delta^{1',q}\colon H^q(\ker\beta_3) \to H^{q+1}(\ker\beta_1)$, $q \in \mathbb{Z}$, be the connecting homomorphism of the long cohomology sequence associated with the first column of (2.8). Now, using the commutativity of diagram (2.7) we obtain the induced long commutative cohomology ladder

$$
\begin{array}{ccccccccc}
\longrightarrow & H^q(A_{1,1}) & \longrightarrow & H^q(A_{2,1}) & \longrightarrow & H^q(A_{3,1}) & \xrightarrow{\delta^{1,q}} & H^{q+1}(A_{1,1}) & \longrightarrow \\
& \Big\downarrow{\scriptstyle H^q(\mu_1)} & & \Big\downarrow{\scriptstyle H^q(\mu_2)} & & \Big\downarrow{\scriptstyle H^q(\mu_3)} & & \Big\downarrow{\scriptstyle H^{q+1}(\mu_1)} & \\
\longrightarrow & H^q(\ker\beta_1) & \longrightarrow & H^q(\ker\beta_2) & \longrightarrow & H^q(\ker\beta_3) & \xrightarrow[\delta^{1',q}]{} & H^{q+1}(\ker\beta_1) & \longrightarrow
\end{array}
\tag{2.9}
$$

Using the fact that both $H^q(\mu_2)$ and $H^q(\mu_3)$ are bijective for all $q \in \mathbb{Z}$, it follows from the Five-Lemma that $H^q(\mu_1)$ is bijective for all $q \in \mathbb{Z}$. This proves that the first row of (2.5) is weakly coexact.

Now, applying Proposition 2.4 to diagram (2.8) we obtain that

$$
\delta^{1',q+1} \circ \partial^{3',q} = -\partial^{1',q+1} \circ \delta^{3,q}, \qquad q \in \mathbb{Z}. \tag{2.10}
$$

From diagram (2.9) we obtain that

$$
(H^{q+1}(\mu_1))^{-1} \circ \delta^{1',q} = \delta^{1,q} \circ (H^q(\mu_3))^{-1}, \qquad q \in \mathbb{Z}. \tag{2.11}
$$

Now $\widehat{\partial}^{3,q} = (H^{q+1}(\mu_3))^{-1} \circ \partial^{3',q}$ and $\widehat{\partial}^{1,q+1} = (H^{q+2}(\mu_1))^{-1} \circ \partial^{1',q+1}$, $q \in \mathbb{Z}$. This together with (2.10) and (2.11) implies (2.6). □

3. Alexander–Spanier cohomology and weakly coexact sequences

Let us briefly recall the basic definitions and notations concerning Alexander–Spanier cohomology. The standard reference is [15].

Let X be a topological space. For every $q \in \mathbb{Z}$ with $q \geq 0$ let $C^q(X) = C^q(X;\overline{G})$ be the Γ-module of all functions from the cartesian product X^{q+1} to $\overline{G}$. For $q \in \mathbb{Z}$ with $q < 0$ let $C^q(X) = C^q(X,\overline{G})$ be the trivial Γ-module. The coboundary operator $\delta^q\colon C^q(X) \to C^{q+1}(X)$, $q \in \mathbb{Z}$, $q \geq 0$, is defined, for $\varphi \in C^q(X)$ by

$$
\delta^q\varphi(x_0,\ldots,x_{q+1}) = \sum_{j=0}^{q+1}(-1)^j\varphi(x_0,\ldots,\widehat{x}_j,\ldots,x_{q+1}), \quad (x_0,\ldots,x_{q+1}) \in X^{q+1}
$$

and $\delta^q = 0$ for $q \in \mathbb{Z}$, $q < 0$. It follows that $\delta^{q+1} \circ \delta^q = 0$ for $q \in \mathbb{Z}$, so $C^*(X) := (C^q(X),\delta^q)_{q\in\mathbb{Z}}$ is a cochain complex.

Given another topological space Y and an arbitrary function f, define, for $q \in \mathbb{Z}$ with $q \geq 0$,

$$
f^{\sharp q}\colon C^q(Y) \to C^q(X)
$$

by

$$
(f^\sharp\varphi)(x_0,\ldots,x_q) = \varphi(f(x_0),\ldots,f(x_q))
$$

for $\varphi \in C^q(Y)$ and $(x_0, \dots, x_q) \in X^q$. Moreover, let $f^{\sharp q}$ be the trivial map for $q \in \mathbb{Z}$ with $q < 0$. Then $f^\sharp := (f^{\sharp q})_{q\in\mathbb{Z}}$ is a cochain map.

Given $q \in \mathbb{Z}$ with $q \geq 0$, a function $\varphi \in C^q(X)$ is said to be *locally zero* if there is an open covering $\mathcal{U} = \mathcal{U}_\varphi$ such that $\varphi(x_0, \dots, x_q) = 0$ for all $(x_0, \dots, x_q) \in \bigcup_{U\in\mathcal{U}} U^{q+1}$. Let $C_0^q(X)$ be the set of all functions $\varphi \in C^q(X)$ which are locally zero. The set $C_0^q(X)$ is a Γ-submodule of $C^q(X)$. Moreover, let $C_0^q(X)$ be the trivial Γ-module for $q \in \mathbb{Z}$ with $q < 0$. Since $\delta^q(C_0^q(X)) \subset C_0^{q+1}(X)$ for $q \in \mathbb{Z}$, $C_0^*(X) := (C_0^q(X), \delta^q_{|C_0^q(X)})_{q\in\mathbb{Z}}$ is a cochain subcomplex of $C^*(X)$. The quotient cochain complex $\overline{C}^*(X) := C^*(X)/C_0^*(X)$ is called the *Alexander–Spanier cochain complex of* X. The q-th cohomology module $H^q(\overline{C}^*(X))$ is called the *q-th Alexander–Spanier cohomology module of* X and is usually denoted by $\overline{H}^q(X)$ but we will simply write $H^q(X)$, dropping the overline-sign.

If Y is another topological space and $f\colon X \to Y$ is continuous, then, for every $q \in \mathbb{Z}$, $f^{\sharp q}(C_0^q(Y)) \subset C_0^q(X)$, so $f^{\sharp q}$ induces a map $\overline{f}^{\sharp q}\colon \overline{C}^q(Y) \to \overline{C}^q(X)$. It follows that $\overline{f}^\sharp := (\overline{f}^{\sharp q})_{q\in\mathbb{Z}}$ is a cochain map, called the *cochain map induced by* f. We will again drop the overline sign and write $f^{\sharp q}$ and $f^\sharp$ for $\overline{f}^{\sharp q}$ and $\overline{f}^\sharp$.

Let A be a subspace of X and $i\colon A \to X$ be the inclusion map. Then $i^\sharp\colon \overline{C}^*(X) \to \overline{C}^*(A)$ is an epimorphism (cf. Lemma 3.1 below). The kernel of $i^\sharp$ is a cochain subcomplex of $\overline{C}^*(X)$ and is denoted by $\overline{C}^*(X, A)$. For $q \in \mathbb{Z}$ the q-th cohomology module $H^q(\overline{C}^*(X, A))$ is called the *q-th Alexander–Spanier relative cohomology module of* (X, A) and is usually denoted by $\overline{H}^q(X, A)$ but we will simply write $H^q(X, A)$ for that.

Lemma 3.1. *If Y and Z are topological spaces and $f\colon Y \to Z$ is continuous and injective, then the induced map $f^{\sharp q}\colon \overline{C}^q(Z) \to \overline{C}^q(Y)$ is surjective for all $q \in \mathbb{Z}$.*

Proof. Let $q \in \mathbb{Z}$ and $[\varphi] \in \overline{C}^q(Y)$ be arbitrary. Define $\alpha\colon Z^{q+1} \to \overline{G}$ by

$$\alpha(z_0, \dots, z_q) = \begin{cases} \varphi(f^{-1}(z_0), \dots, f^{-1}(z_q)) & \text{if } z_i \in f(Y) \text{ for all } i \in [0\,..\,q] \\ 0 & \text{otherwise} \end{cases}$$

Since f is injective, α is well-defined. It follows that $f^{\sharp q}([\alpha]) = [\varphi]$. □

Proposition 3.2. *Let (Y, Y_1) and (Z, Z_1) be topological pairs and $f\colon Y \to Z$ be continuous and injective. Moreover, suppose that $Z_1 \cap f(Y) = f(Y_1)$, that $f(Y_1)$ is closed in Z_1 and $f\colon Y_1 \to f(Y_1)$ is a homeomorphism. Then the induced map $f^{\sharp q}\colon \overline{C}^q(Z, Z_1) \to \overline{C}^q(Y, Y_1)$ is surjective for all $q \in \mathbb{Z}$.*

Proof. Let $\eta\colon Y_1 \to Y$ and $\zeta\colon Z_1 \to Z$ be inclusion maps. Let $q \in \mathbb{Z}$ and $[\varphi] \in \overline{C}^q(Y, Y_1)$ be arbitrary. It follows that $\eta^\sharp(\varphi)$ is locally zero, i.e. there is an open covering $\mathcal{U}$ of Y_1 such that $\eta^\sharp(\varphi)|U^{q+1} = 0$ for all $U \in \mathcal{U}$. Define α as in the proof of Lemma 3.1. Then $f^\sharp([\alpha]) = [\varphi]$ and we only have to prove that $[\alpha] \in \overline{C}^q(Z, Z_1)$. Now, since $f: Y_1 \to f(Y_1)$ is a homeomorphism it follows that for each $U \in \mathcal{U}$ there is a set V_U which is open in Z_1 and such that $V_U \cap f(Y_1) = f(U)$. Since $f(Y_1)$ is closed in Z_1 we conclude that

$$\mathcal{V} := \{ V_U \mid U \in \mathcal{U} \} \cup \{Z_1 \setminus f(Y_1)\}$$

is an open covering of Z_1. Let $V \in \mathcal{V}$ and $(z_0, \dots z_q) \in V^{q+1}$ be arbitrary. Suppose first that $z_i \in f(Y)$ for all $i \in [0\,..\,q]$. Then, since $z_i \in Z_1$, it follows that $z_i \in f(Y_1)$, $i \in [0\,..\,q]$. It follows that $V \neq Z_1 \setminus f(Y_1)$ so $V = V_U$ for some $U \in \mathcal{U}$ and so $z_i \in f(U)$ for all $i \in [0\,..\,q]$. We thus obtain that $\alpha(z_0, \dots, z_q) = \varphi(y_0, \dots, y_q)$, where $y_i \in U$ with $f(y_i) = z_i$ for all $i \in [0\,..\,q]$. It follows that $\alpha(z_0, \dots, z_q) = 0$. If, on the other hand, $z_i \notin f(Y)$ for some $i \in [0\,..\,q]$, then the definition of α implies that $\alpha(z_0, \dots, z_q) = 0$ again. We thus obtain that $\zeta^\sharp(\alpha)|V^{q+1} = 0$ for all $V \in \mathcal{V}$. Hence $[\alpha] \in \overline{C}^q(Z, Z_1)$. The proposition is proved. $\square$

In this section we use the following notation: if (Y, A) and (Z, B) are topological pairs and $f: (Y, A) \to (Z, B)$ is a map then by $f_1: Y \to Z$ we denote the map f viewed as a map from Y to Z and by $f_2: A \to B$ we denote the restriction of f_1 to A.

Proposition 3.3. *Let Y be a topological space and (N_1, N_2, N_3) be a triple of subsets of Y with $N_1 \supset N_2 \supset N_3$. Let*

$$\alpha: (N_1, N_3) \to (N_1, N_2) \quad \text{and} \quad \beta: (N_2, N_3) \to (N_1, N_3)$$

be the inclusion maps. The sequence

$$0 \longrightarrow \overline{C}^*(N_1, N_2) \xrightarrow{\alpha^\sharp} \overline{C}^*(N_1, N_3) \xrightarrow{\beta^\sharp} \overline{C}^*(N_2, N_3) \longrightarrow 0 \tag{3.1}$$

is exact. In addition, for each $q \in \mathbb{Z}$, $\overline{C}^q(N_1, N_2) = \ker \beta^{\sharp q}$ and $\alpha^{\sharp q}$ is the inclusion map. The connecting homomorphism ∂^{q*}, $q \in \mathbb{Z}$, of (3.1) is just the connecting homomorphism of the triple (N_1, N_2, N_3).

Proof. For each $q \in \mathbb{Z}$ there is an inclusion induced commutative diagram

$$\begin{tikzcd}
\overline{C}^q(N_1, N_2) \arrow[r, "\alpha^{\sharp q}"] \arrow[d, "\subset"'] & \overline{C}^q(N_1, N_3) \arrow[r, "\beta^{\sharp q}"] \arrow[d, "\subset"'] & \overline{C}^q(N_2, N_3) \arrow[d, "\subset"] \\
\overline{C}^q(N_1) \arrow[r, "\alpha_1^{\sharp q}"] \arrow[d, "\beta_1^{\sharp q}"'] & \overline{C}^q(N_1) \arrow[r, "\beta_1^{\sharp q}"] \arrow[d, "f^q"'] & \overline{C}^q(N_2) \arrow[d, "g^q"] \\
\overline{C}^q(N_2) \arrow[r, "\alpha_2^{\sharp q}"'] & \overline{C}^q(N_3) \arrow[r, "\beta_2^{\sharp q}"'] & \overline{C}^q(N_3)
\end{tikzcd} \tag{3.2}$$

Since $\beta^{\sharp q}$ is a restriction of $\beta_1^{\sharp q}$, $\beta_1^{\sharp q}$ is surjective by Lemma 3.1 and $\beta_2^{\sharp q}$ is the identity function, it follows that $\beta_1^{\sharp q}$ is surjective. Similarly, $\alpha^{\sharp q}$ is a restriction of an identity map, so $\alpha^{\sharp q}$ is an inclusion map, in particular, $\overline{C}^q(N_1, N_2) \subset \overline{C}^q(N_1, N_3)$. The definition of $\overline{C}^q(N_i, N_j)$ now easily implies that $\ker \beta^{\sharp q} = \operatorname{im} \alpha^{\sharp q}$ and so (3.1) is exact. Now, by definition, the q-th component of the connecting homomorphism of the triple (N_1, N_2, N_3) is the composite

$$H^q(N_2, N_3) \xrightarrow{\gamma^q} H^q(N_2) \xrightarrow{\delta^{q*}} H^{q+1}(N_1, N_2)$$

where γ^q is induced by the inclusion $N_2 \to (N_2, N_3)$ and δ^{q*}, $q \in \mathbb{Z}$, is the connecting homomorphism of the pair (N_1, N_2), i.e. the connecting homomorphism of the short exact sequence

$$0 \longrightarrow \overline{C}^*(N_1, N_2) \longrightarrow \overline{C}^*(N_1) \longrightarrow \overline{C}^*(N_2) \longrightarrow 0.$$

The inclusion map $\zeta\colon (N_2, \emptyset) \to (N_2, N_3)$ induces, for every $n \in \mathbb{Z}$, the commutative diagram

$$\begin{tikzcd}
\overline{C}^n(N_2, N_3) \arrow[r] \arrow[d, "\zeta^{\sharp q}"'] & \overline{C}^n(N_2) \arrow[r] \arrow[d, "\mathrm{id}"] & \overline{C}^n(N_3) \arrow[d] \\
\overline{C}^n(N_2, \emptyset) \arrow[r] & \overline{C}^n(N_2) \arrow[r] & \overline{C}^n(\emptyset)
\end{tikzcd}$$

This diagram implies that $\zeta^{\sharp n}$ is the inclusion map. It follows that γ^q is induced by the family $\zeta^{\sharp n}$, $n \in \mathbb{Z}$, of inclusion maps. From (3.2) we also obtain the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \overline{C}^*(N_1, N_2) \arrow[r, "\alpha^\sharp"] \arrow[d, "\mathrm{id}"'] & \overline{C}^*(N_1, N_3) \arrow[r, "\beta^\sharp"] \arrow[d, "\subset"'] & \overline{C}^*(N_2, N_3) \arrow[r] \arrow[d, "\zeta^\sharp"] & 0 \\
0 \arrow[r] & \overline{C}^*(N_1, N_2) \arrow[r, "\subset"'] & \overline{C}^*(N_1) \arrow[r, "\beta_1^\sharp"'] & \overline{C}^*(N_2) \arrow[r] & 0
\end{tikzcd} \tag{3.3}$$

An application of the cohomology functor to (3.3) now shows that $\partial^{q*} = \delta^{q*} \circ \gamma^q$, $q \in \mathbb{Z}$, as claimed. The proof is complete. □

Proposition 3.4. *Suppose X is a metric space, π is a local semiflow on X, S is an isolated invariant set having a strongly π-admissible isolating neigborhood and (A, A^*) is an attractor-repeller pair in S, relative to π. Let (N_1, N_2, N_3) be an FM-index triple for (π, S, A, A^*) with $\mathrm{Cl}_X(N_1 \setminus N_3)$ strongly π-admissible. Let*

$$\iota\colon (N_1/N_3, \{[N_3]\}) \to (N_1/N_2, \{[N_2]\}), \quad \kappa\colon (N_2/N_3, \{[N_3]\}) \to (N_1/N_3, \{[N_3]\})$$

be inclusion induced maps. The sequence

$$\overline{C}^*(N_1/N_2,\{[N_2]\}) \xrightarrow{\iota^\sharp} \overline{C}^*(N_1/N_3,\{[N_3]\}) \xrightarrow{\kappa^\sharp} \overline{C}^*(N_2/N_3,\{[N_3]\}) \tag{3.4}$$

is weakly coexact.

Proof. For each $q\in\mathbb{Z}$ there is a commutative diagram

$$\begin{tikzcd}
\overline{C}^q(N_1/N_2,\{[N_2]\}) \arrow[r, "\iota^{\sharp q}"] \arrow[d, "\subset"'] & \overline{C}^q(N_1/N_3,\{[N_3]\}) \arrow[r, "\kappa^{\sharp q}"] \arrow[d, "\subset"'] & \overline{C}^q(N_2/N_3,\{[N_3]\}) \arrow[d, "\subset"] \\
\overline{C}^q(N_1/N_2) \arrow[r, "\iota_1^{\sharp q}"] \arrow[d] & \overline{C}^q(N_1/N_3) \arrow[r, "\kappa_1^{\sharp q}"] \arrow[d] & \overline{C}^q(N_2/N_3) \arrow[d] \\
\overline{C}^q(\{[N_2]\}) \arrow[r] & \overline{C}^q(\{[N_3]\}) \arrow[r, "\mathrm{id}"'] & \overline{C}^q(\{[N_3]\})
\end{tikzcd} \tag{3.5}$$

induced by ι and κ. Since κ_1 is injective, being an inclusion map, Lemma 3.1 implies that $\kappa_1^{\sharp q}$ is surjective. It follows that $\kappa^{\sharp q}$ is surjective. We have $\iota = \lambda\circ\xi$, where $\xi\colon(N_1/N_3,\{[N_3]\})\to(N_1/N_3,N_2/N_3)$ and $\lambda\colon(N_1/N_3,N_2/N_3)\to(N_1/N_2,\{[N_2]\})$ are inclusion maps. Now Proposition 3.3 implies that $\ker\kappa^\sharp = \overline{C}^*(N_1/N_3,N_2/N_3)$ and $\xi^\sharp$ is the inclusion map. Thus $\kappa^\sharp\circ\iota^\sharp=\kappa^\sharp\circ\xi^\sharp\circ\lambda^\sharp=0$.

We can write $\lambda=\lambda''\circ\lambda'$ where

$$\lambda'\colon(N_1/N_3,N_2/N_3)\to((N_1/N_2)/(N_2/N_3),\{[N_2/N_3]\})$$

is the projection map and

$$\lambda''\colon((N_1/N_2)/(N_2/N_3),\{[N_2/N_3]\})\to(N_1/N_2,\{[N_2]\})$$

is the canonical homeomorphism. It is well-known that N_2/N_3 is a weak deformation retract of a neighborhood of itself in N_1/N_3. It follows that $\lambda^\sharp$ induces an isomorphism in cohomology. This completes the proof. □

Remark 3.5. In the situation of Proposition 3.4 we have the following commutative diagram with weakly coexact rows

$$\begin{tikzcd}
\overline{C}^*(N_1,N_2) \arrow[r, "\alpha^\sharp"] & \overline{C}^*(N_1,N_3) \arrow[r, "\beta^\sharp"] & \overline{C}^*(N_2,N_3) \\
\overline{C}^*(N_1/N_2,\{[N_2]\}) \arrow[r, "\iota^\sharp"'] \arrow[u, "Q^{1,2,\sharp}"] & \overline{C}^*(N_1/N_3,\{[N_3]\}) \arrow[r, "\kappa^\sharp"'] \arrow[u, "Q^{1,3,\sharp}"'] & \overline{C}^*(N_2/N_3,\{[N_3]\}) \arrow[u, "Q^{2,3,\sharp}"']
\end{tikzcd} \tag{3.6}$$

where α and β are as in Proposition 3.3 and, for i, $j\in\{1,2,3\}$ with $i<j$, $Q^{i,j}\colon(N_i,N_j)\to(N_i/N_j,\{[N_j]\})$ is the canonical projection map. Applying the cohomology functor to (3.6) and using the fact that $Q^{i,j}$ induces an isomorphism in cohomology we see that the connecting homomorphism for the weakly coexact

sequence in Proposition 3.4 is identical to the map $\widehat{\partial}^q$, $q \in \mathbb{Z}$, defined in the paragraph preceding [5, Proposition 2.13].

4. Statement of the main result

We can now state the main result of this paper.

THEOREM 4.1. *Let X be a metric space, π be a local semiflow on X, $k \in \mathbb{N}$, E be a k-dimensional normed real vector space and $\widetilde{\pi}$ be the semiflow generated by the equation $\dot{y} = Ly$, where $L: E \to E$ is a linear map with all eigenvalues having positive real parts. Then there is a family of Γ-module isomorphisms*

$$\theta^q(\pi, \widetilde{\pi}, S): H^{q-k}(\pi, S) \to H^q(\pi \times \widetilde{\pi}, S \times \{0_E\}),$$

one for each $q \in \mathbb{Z}$ and each isolated π-invariant set S having a strongly π-admissible isolating neighborhood, such that the following property is satisfied: given any isolated π-invariant set S having a strongly π-admissible isolating neighborhood and given any attractor-repeller pair (A, A^*) of S relative to π, the following diagram commutes:

$$\begin{tikzcd}
\longleftarrow H^q(\pi', A') & H^q(\pi', S') \arrow[l] & H^q(\pi', A^{*\prime}) \arrow[l] & H^{q-1}(\pi', A') \arrow[l] & \arrow[l] \\
\leftarrow H^{q-k}(\pi, A) \arrow[u, "\theta^q(\pi,\widetilde{\pi},A)"] & H^{q-k}(\pi, S) \arrow[l] \arrow[u, "\theta^q(\pi,\widetilde{\pi},S)"] & H^{q-k}(\pi, A^*) \arrow[l] \arrow[u, "\theta^q(\pi,\widetilde{\pi},A^*)"'] & H^{q-k-1}(\pi, A) \arrow[l] \arrow[u, "\theta^{q-1}(\pi,\widetilde{\pi},A)"'] & \arrow[l]
\end{tikzcd} \tag{4.1}$$

Here, the upper (resp. lower) horizontal sequence is the cohomology index sequence of $(\pi', S', A', A^{*\prime})$ (resp. (π, S, A, A^*)), where we set $\pi' := \pi \times \widetilde{\pi}$ and for $K \subset X$, $K' := K \times \{0_E\}$.

The following result will reduce the proof of Theorem 4.1 to the proof of a special case.

THEOREM 4.2. *Let X and X' be metric spaces and let π (resp. π') be a local semiflow on X (resp. on X'). Let $\gamma: X \to X'$ be a homeomorphism which conjugates π with π'.*

(a) *Let S be an isolated π-invariant set and (Y, Z) be an FM-index pair for (π, S) such that $\mathrm{Cl}_X(Y \setminus Z)$ is strongly π-admissible. Then $\gamma(S)$ is an isolated π'-invariant set and $(\gamma(Y), \gamma(Z))$ is an FM-index pair for $(\pi', \gamma(S))$ such that $\mathrm{Cl}_{X'}(\gamma(Y) \setminus \gamma(Z))$ is strongly π'-admissible. Let $\gamma_{Y,Z}: Y/Z \to \gamma(Y)/\gamma(Z)$ be the map induced by γ and, for $q \in \mathbb{Z}$, let*

$$F_q := H^q(\gamma_{Y,Z}): H^q(\gamma(Y)/\gamma(Z), \{[\gamma(Z)]\}) \to H^q(Y/Z, \{[Z]\})$$

be the induced cohomology map.

(b) *The map*

$$\langle F_q \rangle = \langle F_q \rangle_{\mathcal{C},\Phi,\mathcal{C}',\widehat{\Phi'}}: \widehat{\Phi'}(\mathcal{C}') \to \widehat{\Phi}(\mathcal{C})$$

is independent of the choice of (Y,Z). *Here,* $\mathcal{C}$ *(resp.* $\mathcal{C}'$*) is the categorial Conley–Morse index of* (π,S) *(resp.* $(\pi',\gamma(S))$*) as defined in* [5] *and* Φ *(resp.* Φ'*) is the restriction of* H^q *to* $\mathcal{C}$ *(resp.* $\mathcal{C}'$*). Define the morphism* $\kappa^q(\pi,S,\gamma)\colon H^q(\pi',\gamma(S))\to H^q(\pi,S)$ *by* $\kappa^q(\pi,S,\gamma)=\langle F_q\rangle$. $\kappa^q(\pi,S,\gamma)$ *is a* Γ*-module isomorphism.*

(c) *Given an isolated* π*-invariant set* S *having a strongly* π*-admissible isolating neighborhood and an attractor-repeller pair* (A,A^*) *of* S *relative to* π*, then* $\gamma(S)$ *is an isolated* π'*-invariant set having a strongly* π'*-admissible isolating neighborhood,* $(\gamma(A),\gamma(A^*))$ *is an attractor-repeller pair of* $\gamma(S)$ *relative to* π' *and the diagram*

$$
\begin{tikzcd}
{} & H^q(\pi,A) \arrow[l] & H^q(\pi,S) \arrow[l] & H^q(\pi,A^*) \arrow[l] & H^{q-1}(\pi,A) \arrow[l] & {} \arrow[l] \\
{} & H^q(\pi',\gamma(A)) \arrow[l] \arrow[u, "\kappa^q(\pi,A,\gamma)"] & H^q(\pi',\gamma(S)) \arrow[l] \arrow[u, "\kappa^q(\pi,S,\gamma)"] & H^q(\pi',\gamma(A^*)) \arrow[l] \arrow[u, "\kappa^q(\pi,A^*,\gamma)"'] & H^{q-1}(\pi',\gamma(A)) \arrow[l] \arrow[u, "\kappa^{q-1}(\pi,A,\gamma)"'] & {} \arrow[l]
\end{tikzcd}
\tag{4.2}
$$

commutes.

Proof. Part (a) is obvious. To prove the independence of $\langle F_q\rangle$ of the choice of (Y,Z), let $(\widehat{Y},\widehat{Z})$ be another FM-index pair for (π,S) with $\mathrm{Cl}_X(\widehat{Y}\setminus\widehat{Z})$ strongly π-admissible. By [5, Proposition 4.6, Lemma 4.8 and Proposition 2.5] we obtain sets L_1, L_2, W and $\widehat{W}$ such that $(L_1,L_2)\subset(Y\cap\widehat{Y},W\cap\widehat{W})$, $Z\subset W$, $\widehat{Z}\subset\widehat{W}$ and (L_1,L_2), (Y,W) and $(\widehat{Y},\widehat{W})$ are FM-index pairs for (π,S) such that $\mathrm{Cl}_X(L_1\setminus L_2)$, $\mathrm{Cl}_X(Y\setminus Z)$ and $\mathrm{Cl}_X(\widehat{Y}\setminus\widehat{W})$ are strongly π-admissible. We thus obtain the commutative diagram

$$
\begin{tikzcd}
H^q(Y/Z,\{[Z]\}) & H^q(\gamma(Y)/\gamma(Z),\{[\gamma(Z)]\}) \arrow[l, "H^q(\gamma_{Y,Z})"'] \\
H^q(Y/W,\{[W]\}) \arrow[u] \arrow[d] & H^q(\gamma(Y)/\gamma(W),\{[\gamma(W)]\}) \arrow[l, "H^q(\gamma_{Y,W})"'] \arrow[u] \arrow[d] \\
H^q(L_1/L_2,\{[L_2]\}) & H^q(\gamma(L_1)/\gamma(L_2),\{[\gamma(L_2)]\}) \arrow[l, "H^q(\gamma_{L_1,L_2})"'] \\
H^q(\widehat{Y}/\widehat{W},\{[\widehat{W}]\}) \arrow[u] \arrow[d] & H^q(\gamma(\widehat{Y})/\gamma(\widehat{W}),\{[\gamma(\widehat{W})]\}) \arrow[l, "H^q(\gamma_{\widehat{Y},\widehat{W}})"'] \arrow[u] \arrow[d] \\
H^q(\widehat{Y}/\widehat{Z},\{[\widehat{Z}]\}) & H^q(\gamma(\widehat{Y})/\gamma(\widehat{Z}),\{[\gamma(\widehat{Z})]\}) \arrow[l, "H^q(\gamma_{\widehat{Y},\widehat{Z}})"]
\end{tikzcd}
$$

whose vertical maps are inclusion induced. Hence, by [5, Proposition 4.5] these maps are induced by the unique morphisms in $\mathcal{C}$ (resp. in $\mathcal{C}'$) between the corresponding objects of these connected simple systems. In particular, the vertical maps are all bijective, and so we may invert the downward pointing arrows and then compose the columns to obtain the commutative diagram

$$
\begin{tikzcd}
H^q(Y/Z,\{[Z]\}) & H^q(\gamma(Y)/\gamma(Z),\{[\gamma(Z)]\}) \arrow[l, "H^q(\gamma_{Y,Z})"'] \\
H^q(\widehat{Y}/\widehat{Z},\{[\widehat{Z}]\}) \arrow[u] & H^q(\gamma(\widehat{Y})/\gamma(\widehat{Z}),\{[\gamma(\widehat{Z})]\}) \arrow[l, "H^q(\gamma_{\widehat{Y},\widehat{Z}})"] \arrow[u]
\end{tikzcd}
\tag{4.3}
$$

where the vertical maps are induced by the corresponding morphism in $\mathcal{C}$ (resp. in $\mathcal{C}'$). Now an application of [6, Proposition 2.4] to diagram (4.3) completes the proof of part (b) of the theorem. To prove part (c) let (N_1, N_2, N_3) be an FM-index triple for (π, S, A, A^*) with $\mathrm{Cl}_X(N_1 \setminus N_3)$ strongly π-admissible. It follows that

$$(N_1', N_2', N_3') := (\gamma(N_1), \gamma(N_2), \gamma(N_3))$$

is an FM-index triple for $(\pi', \gamma(S), \gamma(A), \gamma(A^*))$ such that $\mathrm{Cl}_{X'}(\gamma(N_1) \setminus \gamma(N_3))$ is strongly π'-admissible. By Proposition 3.4 we thus have the following commutative diagram of cochain maps with weakly coexact rows

$$
\begin{tikzcd}
\overline{C}^*(N_1/N_2,\{[N_2]\}) \arrow[r] & \overline{C}^*(N_1/N_3,\{[N_3]\}) \arrow[r] & \overline{C}^*(N_2/N_3,\{[N_3]\}) \\
\overline{C}^*(N_1'/N_2',\{[N_2']\}) \arrow[r] \arrow[u, "\gamma^\sharp_{N_1,N_2}"] & \overline{C}^*(N_1'/N_3',\{[N_3']\}) \arrow[r] \arrow[u, "\gamma^\sharp_{N_1,N_3}"] & \overline{C}^*(N_2'/N_3',\{[N_3']\}) \arrow[u, "\gamma^\sharp_{N_2,N_3}"]
\end{tikzcd}
\tag{4.4}
$$

Applying Proposition 2.2 to diagram (4.4) we obtain the induced long commutative ladder with exact rows. An application of the $\langle\,\cdot\,,\,\cdot\,\rangle$-operation to that ladder and using part (b) we obtain diagram (4.2). This proves part (c). $\square$

The following result is well-known (cf. [1, Lemma 3 in Section 22]).

Proposition 4.3. *Let $k \in \mathbb{N}$, E be a k-dimensional normed real vector space and $\widetilde{\pi}$ be the semiflow generated by the equation $\dot{y} = Ly$, where $L: E \to E$ is a linear map with all eigenvalues having positive real parts. Let π_k be the semiflow on $\mathbb{R}^k$ generated by the ordinary differential equation $\dot{u} = u$. Then there exists a homeomorphism $\alpha_k: E \to \mathbb{R}^k$ which conjugates $\widetilde{\pi}$ with π_k.*

Theorem 4.4. *Theorem 4.1 holds whenever $k \in \mathbb{N}$, $E := \mathbb{R}^k$, and $\widetilde{\pi} := \pi_k$, where π_k is as in Proposition 4.3.*

Proof of Theorem 4.1 using Theorem 4.4. Let $\gamma: X \times E \to X \times \mathbb{R}^k$ be given by $(x, u) \mapsto (x, \alpha_k(u))$, $(x, u) \in X \times E$, where α_k is as in Proposition 4.3. Then γ is a homeomorphism which conjugates $\pi \times \widetilde{\pi}$ with $\pi \times \pi_k$. If S is an isolated π-invariant set having a strongly π-admissible isolating neighborhood then let

$$\theta^q(\pi, \pi_k, S): H^{q-k}(\pi, S) \to H^q(\pi \times \pi_k, S \times \{0_{\mathbb{R}^k}\}),$$

be the Γ-module isomorphism which exists by Theorem 4.4. Now use Theorem 4.2 with our choice of γ. It follows that $\gamma(S \times \{0_E\}) = S \times \{0_{\mathbb{R}^k}\}$. Set

$$\theta^q(\pi, \widetilde{\pi}, S) := \kappa^q(\pi, S, \gamma) \circ \theta^q(\pi, \pi_k, S).$$

By Theorem 4.2 the family of these Γ-isomorphisms clearly satisfies the conclusions of Theorem 4.1. □

The next result is the crucial step in the proof of Theorem 4.4.

Theorem 4.5. *Theorem* 4.4 *holds for* $k = 1$.

Proof of Theorem 4.4 using Theorem 4.5. The proof is by induction on $k \in \mathbb{N}$. Theorem 4.5 implies that Theorem 4.4 holds for $k = 1$. Suppose that Theorem 4.4 holds for some k. Let X be a metric space and let π be a local semiflow on X. Notice that the semiflow $\pi \times \pi_{k+1}$ is conjugated to the semiflow $(\pi \times \pi_k) \times \pi_1$ by the homeomorphism $\varphi\colon X \times \mathbb{R}^{k+1} \to (X \times \mathbb{R}^k) \times \mathbb{R}$ given by

$$\varphi(x, u_1, \dots, u_{k+1}) = ((x, u_1, \dots, u_k), u_{k+1}).$$

Let S be an isolated π-invariant set having a strongly π-admissible isolating neighborhood. We use Theorem 4.2 with $\gamma := \varphi$ to obtain the Γ-module isomorphism $\kappa^q(\pi \times \pi_{k+1}, S, \varphi)$ from $H^q((\pi \times \pi_k) \times \pi_1, (S \times \{0_{\mathbb{R}^k}\}) \times \{0_{\mathbb{R}}\})$ to $H^q(\pi \times \pi_{k+1}, S \times \{0_{\mathbb{R}^{k+1}}\})$ as in that theorem. Using Theorem 4.5 we obtain the Γ-module isomorphism $\theta^q(\pi \times \pi_k, \pi_1, S \times \{0_{\mathbb{R}^k}\})$ from $H^{q-1}(\pi \times \pi_k, S \times \{0_{\mathbb{R}^k}\})$ to $H^q((\pi \times \pi_k) \times \pi_1, (S \times \{0_{\mathbb{R}^k}\}) \times \{0_{\mathbb{R}}\})$, as in that theorem. Since the present Theorem 4.4 is valid for k, there is the Γ-module isomorphism $\theta^{q-1}(\pi, \pi_k, S)$ from $H^{(q-1)-k}(\pi, S)$ to $H^{q-1}(\pi \times \pi_k, S \times \{0_{\mathbb{R}^k}\})$, as in this theorem.

Define the Γ-module isomorphism

$$\theta^q(\pi \times \pi_{k+1}, S)\colon H^{q-k-1}(\pi, S) \to H^q(\pi \times \pi_{k+1}, S \times \{0_{\mathbb{R}^{k+1}}\})$$

by

$$\begin{aligned} &\theta^q(\pi \times \pi_{k+1}, S) \\ &\qquad := \kappa^q(\pi \times \pi_{k+1}, S, \varphi) \circ \theta^q(\pi \times \pi_k, \pi_1, S \times \{0_{\mathbb{R}^k}\}) \circ \theta^{q-1}(\pi, \pi_k, S). \end{aligned}$$

The family $\theta^q(\pi \times \pi_{k+1}, S)$ obviously satisfies the conclusions of Theorem 4.4 for $k + 1$. □

The rest of this paper is devoted to the proof of Theorem 4.5.

For the rest of this paper let X be a metric space and π be a local semiflow on X. Since Theorem 4.5 is obvious for $X = \emptyset$ we assume that $X \neq \emptyset$.

5. Construction of the suspension isomorphism

Recall the following notations from [6].

Definition 5.1. Let (N, Y, Z) be a triple of closed subsets of X with $N \neq \emptyset$ and $Z \subset Y \subset N$. Define

$$\begin{aligned} E(Y) &:= Y \times [-1,1] \cup N \times \{-1,1\}, \\ E(Z) &:= Z \times [-1,1] \cup N \times \{-1,1\}, \\ \Omega(Y,Z) &:= E(Y)/E(Z). \end{aligned}$$

Define further $I_0 := \{0\}$, $I_1 := [-1,0]$, $I_2 := [0,1]$ and

$$E_k(Y,Z) := Y \times I_k \cup E(Z), \quad k \in \{0,1,2\}.$$

Let $p_{Y,Z}: Y \times [-1,1] \cup N \times \{-1,1\} \to \Omega(Y,Z)$ be the quotient map and define

$$\begin{aligned} \Omega_1(Y,Z) &:= p_{Y,Z}(E_1(Y,Z)), \\ \Omega_2(Y,Z) &:= p_{Y,Z}(E_2(Y,Z)), \\ \Omega_0(Y,Z) &:= \Omega_1(Y,Z) \cap \Omega_2(Y,Z), \end{aligned}$$

and let $z_{Y,Z}$ be the base-point of $\Omega(Y,Z)$, i.e. $\{z_{Y,Z}\} = p_{Y,Z}(E(Z))$.

Remark 5.2. It is clear that $\Omega_0(Y,Z) = p_{Y,Z}(E_0(Y,Z))$. Moreover, for $k \in \{0,1,2\}$, $\Omega_k(Y,Z)$ and $E_k(Y,Z)/E(Z)$ are identical, both as sets and as topological spaces. In fact, since $p_{Y,Z}^{-1}(p_{Y,Z}(E_k(Y,Z))) = E_k(Y,Z)$ and $E_k(Y,Z)$ is closed in $E(Y)$, it follows that the restriction of $p_{Y,Z}$ to $E_k(Y,Z)$ is a quotient map from $E_k(Y,Z)$ to $p_{Y,Z}(E_k(Y,Z)) = \Omega_k(Y,Z)$.

For the rest of this section, let $N \neq \emptyset$ be closed in X, S be an isolated π-invariant set and (Y,Z) be an FM-index pair for (π, S) with $Y \subset N$ and $\mathrm{Cl}_X(Y \setminus Z)$ strongly π-admissible.

The following lemma is analogous to [6, Lemma 4.3], with the same proof.

Lemma 5.3. *Let $g_{Y,Z}: Y/Z \to (Y \times \{0\})/(Z \times \{0\})$ be induced by the assignment $x \mapsto (x,0)$ and $h_{Y,Z}: (Y \times \{0\})/(Z \times \{0\}) \to \Omega_0(Y,Z) = E_0(Y,Z)/E(Z)$ be inclusion induced. The map $f_{Y,Z}: Y/Z \to \Omega_0(Y,Z)$ defined by $f_{Y,Z} = h_{Y,Z} \circ g_{Y,Z}$ is a base-point preserving homeomorphism. In particular,*

$$H^q(f_{Y,Z}): H^q(\Omega_0(Y,Z), \{z_{Y,Z}\}) \to H^q(Y/Z, \{[Z]\})$$

is bijective for all $q \in \mathbb{Z}$.

We also have the following analogue of [6, Proposition 4.4].

Proposition 5.4. *Let $\ell_{Y,Z}\colon(\Omega_1(Y,Z),\Omega_0(Y,Z))\to(\Omega(Y,Z),\Omega_2(Y,Z))$ be the inclusion induced map. Then the corresponding cohomology map*

$$H^q(\ell_{Y,Z})\colon H^q(\Omega(Y,Z),\Omega_2(Y,Z))\to H^q(\Omega_1(Y,Z),\Omega_0(Y,Z))$$

is bijective for every $q\in\mathbb{Z}$.

Proof. It was proved in [6, Lemma 4.6] that $\Omega_0(Y,Z)$ is a strong deformation retract of a closed neighborhood of itself in $\Omega_1(Y,Z)$. This together with classical arguments from algebraic topology (see e.g. [10, Theorem 1.8 and its proof]) completes the proof of the proposition. □

Proposition 5.4 and standard results from algebraic topology (cf. [10, Theorem 1.4 and Remark 5.3]) imply that there exists a long exact Mayer–Vietoris cohomology sequence

$$\xleftarrow{\gamma^q} R^q \xleftarrow{\alpha^q} S^q \xleftarrow{\beta^q} T^q \xleftarrow{\gamma^{q-1}} R^{q-1} \xleftarrow{\alpha^{q-1}} S^{q-1} \xleftarrow{\beta^{q-1}} \tag{5.1}$$

where, for $q\in\mathbb{Z}$,

$$\begin{aligned} R^q &= R^q(Y,Z)=H^q(\Omega_0(Y,Z),\{z_{Y,Z}\}),\\ T^q &= T^q(Y,Z)=H^q(\Omega(Y,Z),\{z_{Y,Z}\}),\\ S^q &= S^q(Y,Z)=H^q(\Omega_1(Y,Z),\{z_{Y,Z}\})\oplus H^q(\Omega_2(Y,Z),\{z_{Y,Z}\}),\end{aligned}\tag{5.2}$$

$$\begin{aligned} &\rho_{Y,Z}:(\Omega(Y,Z),\{z_{Y,Z}\})\to(\Omega(Y,Z),\Omega_2(Y,Z)),\\ &\mu_{k,Y,Z}:(\Omega_0(Y,Z),\{z_{Y,Z}\})\to(\Omega_k(Y,Z),\{z_{Y,Z}\}),\\ &\nu_{k,Y,Z}:(\Omega_k(Y,Z),\{z_{Y,Z}\})\to(\Omega(Y,Z),\{z_{Y,Z}\}),\quad k\in\{1,2\},\end{aligned}\tag{5.3}$$

are inclusions and

$$\begin{aligned} \alpha^q &:= H^q(\mu_{1,Y,Z})-H^q(\mu_{2,Y,Z}),\\ \beta^q &:= H^q(\nu_{1,Y,Z})\oplus H^q(\nu_{2,Y,Z}),\\ \gamma^q &:= H^q(\rho_{Y,Z})\circ H^q(\ell_{Y,Z})^{-1}\circ\partial^{(q-1)*}(\Omega_1(Y,Z),\Omega_0(Y,Z),\{z_{Y,Z}\}).\end{aligned}\tag{5.4}$$

Here, $\partial^{q*}(\Omega_1(Y,Z),\Omega_0(Y,Z),\{z_{Y,Z}\})$, $q\in\mathbb{Z}$ is the connecting homomorphism of the triple $(\Omega_1(Y,Z),\Omega_0(Y,Z),\{z_{Y,Z}\})$.

The following lemma is analogous to [6, Lemma 4.7], with the same proof.

Lemma 5.5. $H^q(\Omega_1(Y,Z),\{z_{Y,Z}\})=0$ *and* $H^q(\Omega_2(Y,Z),\{z_{Y,Z}\})=0$ *for* $q\in\mathbb{Z}$.

Lemma 5.5 implies that $S^q=0$ for all $q\in\mathbb{Z}$. The exactness of diagram (5.1) therefore shows that the map γ^q is bijective for all $q\in\mathbb{Z}$. Recalling that $(\Omega(Y,Z),\{z_{Y,Z}\})=(E(Y)/E(Z),\{[E(Z)]\})$ and that, by [6, Proposition 2.2], $(E(Y),E(Z))$ is an FM-index pair for $(\pi\times\pi_1,S\times\{0\})$, and using Lemma 5.3 we thus arrive at the following result:

THEOREM 5.6.

(a) *For every $q \in \mathbb{Z}$, the map*

$$\xi^q = \xi^q_{Y,Z}\colon H^{q-1}(Y/Z,\{[Z]\}) \to H^q(E(Y)/E(Z),\{[E(Z)]\})$$

defined by

$$\xi^q = H^q(\rho_{Y,Z}) \circ H^q(\ell_{Y,Z})^{-1} \circ \widetilde{\partial}^{q-1}(Y,Z) \circ H^{q-1}(f_{Y,Z})^{-1},$$

is bijective, where we set

$$\widetilde{\partial}^q(Y,Z) := (-1)^q \partial^{q*}(\Omega_1(Y,Z),\Omega_0(Y,Z),\{z_{Y,Z}\}), \quad q \in \mathbb{Z}.$$

(b) *Whenever $(\widehat{Y},\widehat{Z})$ is another FM-index pair for (π,S) such that $\mathrm{Cl}_X(\widehat{Y}\setminus \widehat{Z})$ is strongly π-admissible, then the diagram*

$$\begin{tikzcd} \Phi(A) & \Phi'(A') \arrow[l, "F"'] \\ \Phi(B) \arrow[u, "\Phi(f)"] \arrow[r, leftrightarrow, "G"'] & \Phi'(B') \arrow[u, "\Phi'(f')"'] \end{tikzcd} \tag{5.5}$$

commutes.

Here $A = (E(Y)/E(Z),\{[E(Z)]\})$, $B = (E(\widehat{Y})/E(\widehat{Z}),\{[E(\widehat{Z})]\})$, $A' = (Y/Z,\{[Z]\})$, $B' = (\widehat{Y}/\widehat{Z},\{[\widehat{Z}]\})$, $\mathcal{C}$ is the categorial Morse index of $(\pi\times\pi_1, S\times\{0\})$, $\mathcal{C}'$ is the categorial Morse index of (π,S), Φ is the restriction of the functor H^q to $\mathcal{C}$, Φ' is the restriction of H^{q-1} to $\mathcal{C}'$, $F = \xi^q_{Y,Z}$, $G = \xi^q_{\widehat{Y},\widehat{Z}}$ and f (resp. f') is the unique morphism in $\mathcal{C}$ (resp. in $\mathcal{C}'$) from A to B (resp. from A' to B').

(c) $\langle F\rangle_{\mathcal{C},\Phi,\mathcal{C}',\Phi'} = \langle G\rangle_{\mathcal{C},\Phi,\mathcal{C}',\Phi'}$.

REMARK. The reason for including the factor $(-1)^q$ in the definition of $\xi^q_{Y,Z}$ is crucial and will become apparent in Section 6.

PROOF. We have just proved part (a). Part (c) follows from part (b) by [6, Proposition 2.4]. To prove part (b) let us first assume that $(Y,Z)\subset(\widehat{Y},\widehat{Z})$. Given $q\in\mathbb{Z}$, consider the following diagrams in the category of Γ-modules:

$$\begin{tikzcd} H^q(\Omega(Y,Z),\{z_{Y,Z}\}) & H^q(\Omega(Y,Z),\Omega_2(Y,Z)) \arrow[l, "H^q(\rho_{Y,Z})"'] \\ H^q(\Omega(\widehat{Y},\widehat{Z}),\{z_{\widehat{Y},\widehat{Z}}\}) \arrow[u] & H^q(\Omega(\widehat{Y},\widehat{Z}),\Omega_2(\widehat{Y},\widehat{Z})) \arrow[l, "H^q(\rho_{\widehat{Y},\widehat{Z}})"] \arrow[u] \end{tikzcd}$$

$$\begin{tikzcd}
H^q(\Omega(Y,Z),\Omega_2(Y,Z)) & H^q(\Omega_1(Y,Z),\Omega_0(Y,Z)) \arrow[l, "H^q(\ell_{Y,Z})^{-1}"'] \\
H^q(\Omega(\widehat{Y},\widehat{Z}),\Omega_2(\widehat{Y},\widehat{Z})) \arrow[u] & H^q(\Omega_1(\widehat{Y},\widehat{Z}),\Omega_0(\widehat{Y},\widehat{Z})) \arrow[l, "H^q(\ell_{\widehat{Y},\widehat{Z}})^{-1}"] \arrow[u]
\end{tikzcd}$$

$$\begin{tikzcd}
H^q(\Omega_1(Y,Z),\Omega_0(Y,Z)) & H^{q-1}(\Omega_0(Y,Z),\{z_{Y,Z}\}) \arrow[l, "\widetilde{\partial}^{q-1}(Y,Z)"'] \\
H^q(\Omega_1(\widehat{Y},\widehat{Z}),\Omega_0(\widehat{Y},\widehat{Z})) \arrow[u] & H^{q-1}(\Omega_0(\widehat{Y},\widehat{Z}),\{z_{\widehat{Y},\widehat{Z}}\}) \arrow[l, "\widetilde{\partial}^{q-1}(\widehat{Y},\widehat{Z})"] \arrow[u]
\end{tikzcd}$$

$$\begin{tikzcd}
H^{q-1}(\Omega_0(Y,Z),\{z_{Y,Z}\}) & H^{q-1}(Y/Z,\{[Z]\}) \arrow[l, "H^{q-1}(f_{Y,Z})^{-1}"'] \\
H^{q-1}(\Omega_0(\widehat{Y},\widehat{Z}),\{z_{\widehat{Y},\widehat{Z}}\}) \arrow[u] & H^{q-1}(\widehat{Y}/\widehat{Z},\{[\widehat{Z}]\}) \arrow[l, "H^{q-1}(f_{\widehat{Y},\widehat{Z}})^{-1}"] \arrow[u]
\end{tikzcd}$$

Here, the vertical maps are inclusion induced. All these diagrams clearly commute (the third diagram commutes by the naturality of connecting homomorphisms of space triples). Composing these diagrams we thus obtain the commutative diagram

(5.6)
$$\begin{tikzcd}
H^q(E(Y)/E(Z),\{[E(Z)]\}) & H^{q-1}(Y/Z,\{[Z]\}) \arrow[l, "\xi^q_{Y,Z}"'] \\
H^q(E(\widehat{Y})/E(\widehat{Z}),\{[E(\widehat{Z})]\}) \arrow[u] & H^{q-1}(\widehat{Y}/\widehat{Z},\{[\widehat{Z}]\}) \arrow[l, "\xi^q_{\widehat{Y},\widehat{Z}}"] \arrow[u]
\end{tikzcd}$$

with inclusion induced vertical maps. Diagram (5.6) is just diagram (5.5) spelled out. Now an application of [5, Proposition 4.5] completes the proof of part (b) is in the special case $(Y,Z)\subset(\widehat{Y},\widehat{Z})$. In the general case we use [5, Proposition 4.6, Lemma 4.8 and Proposition 2.5] to obtain sets L_1, L_2, W and $\widehat{W}$ such that

$$(L_1,L_2)\subset(Y\cap\widehat{Y},W\cap\widehat{W}),\quad Z\subset W,\quad \widehat{Z}\subset\widehat{W}$$

and (L_1,L_2), (Y,W), $(\widehat{Y},\widehat{W})$ are FM-index pairs for (π,S) such that $\mathrm{Cl}_X(L_1\setminus L_2)$, $\mathrm{Cl}_X(Y\setminus Z)$ and $\mathrm{Cl}_X(\widehat{Y}\setminus\widehat{W})$ are strongly π-admissible. By the special case just proved we thus obtain the commutative diagram

$$\begin{tikzcd}
H^q(E(Y)/E(Z),\{[E(Z)]\}) & H^{q-1}(Y/Z,\{[Z]\}) \arrow[l, "\xi^q_{Y,Z}"'] \\
H^q(E(Y)/E(W),\{[E(W)]\}) \arrow[u] \arrow[d] & H^{q-1}(Y/W,\{[W]\}) \arrow[l, "\xi^q_{Y,W}"'] \arrow[u] \arrow[d] \\
H^q(E(L_1)/E(L_2),\{[E(L_2)]\}) & H^{q-1}(L_1/L_2,\{[L_2]\}) \arrow[l, "\xi^q_{L_1,L_2}"'] \\
H^q(E(\widehat{Y})/E(\widehat{W}),\{[E(\widehat{W})]\}) \arrow[u] \arrow[d] & H^{q-1}(\widehat{Y}/\widehat{W},\{[\widehat{W}]\}) \arrow[l, "\xi^q_{\widehat{Y},\widehat{W}}"] \arrow[u] \arrow[d] \\
H^q(E(\widehat{Y})/E(\widehat{Z}),\{[E(\widehat{Z})]\}) & H^{q-1}(\widehat{Y}/\widehat{Z},\{[\widehat{Z}]\}) \arrow[l, "\xi^q_{\widehat{Y},\widehat{Z}}"]
\end{tikzcd}$$

The vertical maps in the above diagram are all inclusion induced, thus they are induced by the unique morphisms in $\mathcal{C}(\pi\times\pi_1, S\times\{0\})$ (resp. in $\mathcal{C}(\pi,S)$) between the corresponding objects of these connected simple systems. In particular, the vertical maps are all bijective and so we may invert the downward pointing arrows then compose the columns to obtain a commutative diagram of the form (5.6) where the vertical maps are induced by the corresponding morphism in $\mathcal{C}(\pi\times\pi_1, S\times\{0\})$ (resp. in $\mathcal{C}(\pi,S)$). This completes the proof of part (b) of the theorem. □

In view of Theorem 5.6 we can now make the following definition.

DEFINITION 5.7. Given an isolated π-invariant set S having a strongly π-admissible isolating neighborhood and $q\in\mathbb{Z}$, let

$$\theta^q(\pi,\pi_1,S)\colon H^{q-1}(\pi,S)\to H^q(\pi\times\pi_1,S\times\{0_{\mathbb{R}}\})$$

be defined by $\theta^q(\pi,S):=\langle F\rangle_{\mathcal{C},\Phi,\mathcal{C}',\Phi'}$, where $\mathcal{C}$ is the categorial Morse index of $(\pi\times\pi_1,S\times\{0_{\mathbb{R}}\})$, $\mathcal{C}'$ is the categorial Morse index of (π,S), Φ is the restriction of the functor H^q to $\mathcal{C}$, Φ' is the restriction of H^{q-1} to $\mathcal{C}'$ and $F=\xi^q_{Y,Z}$ with $\xi^q_{Y,Z}$ defined in part (a) of Theorem 5.6. $\theta^q(\pi,\pi_1,S)$ is a well-defined Γ-module isomorphism, called the *suspension isomorphism from $H^{q-1}(\pi,S)$ to $H^q(\pi\times\pi_1,S\times\{0_{\mathbb{R}}\})$*.

REMARK 5.8. The proof of the existence for both the suspension isomorphism $\theta^q(\pi,\pi_1,S)$ as well as general suspension isomorphism $\theta^q(\pi,\widetilde{\pi},S)$ of Theorem 4.1 does not use any particular properties of Alexander–Spanier cohomology and so the result holds for an arbitrary (unreduced) cohomology theory with values in Γ-modules. On the other hand, the existence of long exact cohomology sequences for attractor-repeller pairs and the commutativity of the suspension isomorphism with such sequences, established in Section 6 below, does depend on special properties of Alexander–Spanier cohomology.

6. The suspension isomorphism and attractor-repeller pairs

In this section we will complete the proof of Theorem 4.5. For the rest of this section, let $N \neq \emptyset$ be closed in X, (A, A^*) be an attractor-repeller pair of S relative to π and (N_1, N_2, N_3) be an FM-index triple for (π, S, A, A^*) such that $N_1 \subset N$ and $\mathrm{Cl}_X(N_1 \setminus N_3)$ is strongly π-admissible. For $i,j \in \{1, 2, 3\}$, $i < j$, set $\Omega^{i,j} := \Omega(N_i, N_j)$, $\Omega_1^{i,j} := \Omega_1(N_i, N_j)$, $\Omega_2^{i,j} := \Omega_2(N_i, N_j)$, $\Omega_0^{i,j} := \Omega_0(N_i, N_j)$, $z^{i,j} := z_{N_i,N_j}$, $p^{i,j} := p_{N_i,N_j}$, $f^{i,j} := f_{N_i,N_j}$, $\rho^{i,j} := \rho_{N_i,N_j}$, $\ell^{i,j} := \ell_{N_i,N_j}$ and $\xi^{q,i,j} := \xi^q_{N_i,N_j}$, $q \in \mathbb{Z}$. (For the notations used here cf Definition 5.1, Lemma 5.3, Proposition 5.4, formula (5.4) and Theorem 5.6.)

Theorem 4.5 will follow from Theorem 5.6, already proved, and from the following result.

Theorem 6.1.

(a) *The inclusion induced diagram*

$$\begin{tikzcd}
\overline{C}^*(\Omega^{1,2}, \{z^{1,2}\}) \arrow[r] & \overline{C}^*(\Omega^{1,3}, \{z^{1,3}\}) \arrow[r] & \overline{C}^*(\Omega^{2,3}, \{z^{2,3}\}) \\
\overline{C}^*(\Omega^{1,2}, \Omega_2^{1,2}) \arrow[r] \arrow[u, "\rho^{1,2,\sharp}"] \arrow[d, "\ell^{1,2,\sharp}"'] & \overline{C}^*(\Omega^{1,3}, \Omega_2^{1,3}) \arrow[r] \arrow[u, "\rho^{1,3,\sharp}"] \arrow[d, "\ell^{1,3,\sharp}"'] & \overline{C}^*(\Omega^{2,3}, \Omega_2^{2,3}) \arrow[u, "\rho^{2,3,\sharp}"] \arrow[d, "\ell^{2,3,\sharp}"'] \\
\overline{C}^*(\Omega_1^{1,2}, \Omega_0^{1,2}) \arrow[r] & \overline{C}^*(\Omega_1^{1,3}, \Omega_0^{1,3}) \arrow[r] & \overline{C}^*(\Omega_1^{2,3}, \Omega_0^{1,2})
\end{tikzcd} \tag{6.1}$$

is commutative with weakly coexact rows.

(b) *The diagram*

$$\begin{tikzcd}
\overline{C}^*(N_1/N_2, \{[N_2]\}) \arrow[r] & \overline{C}^*(N_1/N_3, \{[N_3]\}) \arrow[r] & \overline{C}^*(N_2/N_3, \{[N_3]\}) \\
\overline{C}^*(\Omega_0^{1,2}, \{z^{1,2}\}) \arrow[r] \arrow[u, "f^{1,2,\sharp}"] & \overline{C}^*(\Omega_0^{1,3}, \{z^{1,3}\}) \arrow[r] \arrow[u, "f^{1,3,\sharp}"] & \overline{C}^*(\Omega_0^{2,3}, \{z^{2,3}\}) \arrow[u, "f^{2,3,\sharp}"]
\end{tikzcd} \tag{6.2}$$

is commutative with weakly exact rows. Here, the horizontal maps are inclusion induced.

(c) *The following inclusion induced diagram of chain maps is commutative with exact columns and weakly exact rows:*

$$
\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
\overline{C}^*(\Omega_1^{1,2}, \Omega_0^{1,2}) \arrow[r] \arrow[d] & \overline{C}^*(\Omega_1^{1,3}, \Omega_0^{1,3}) \arrow[r] \arrow[d] & \overline{C}^*(\Omega_1^{2,3}, \Omega_0^{2,3}) \arrow[d] \\
\overline{C}^*(\Omega_1^{1,2}, \{z^{1,2}\}) \arrow[r] \arrow[d] & \overline{C}^*(\Omega_1^{1,3}, \{z^{1,3}\}) \arrow[r] \arrow[d] & \overline{C}^*(\Omega_1^{2,3}, \{z^{2,3}\}) \arrow[d] \\
\overline{C}^*(\Omega_0^{1,2}, \{z^{1,2}\}) \arrow[r] \arrow[d] & \overline{C}^*(\Omega_0^{1,3}, \{z^{1,3}\}) \arrow[r] \arrow[d] & \overline{C}^*(\Omega_0^{2,3}, \{z^{2,3}\}) \arrow[d] \\
0 & 0 & 0
\end{tikzcd}
\tag{6.3}
$$

Lemma 6.2. *For every $q \in \mathbb{Z}$ the inclusion induced chain maps*

$$\overline{C}^q(\Omega^{1,3}, \Omega_2^{1,3}) \to \overline{C}^q(\Omega^{2,3}, \Omega_2^{2,3}) \quad \textit{and} \quad \overline{C}^q(\Omega_1^{1,3}, \Omega_0^{1,3}) \to \overline{C}^q(\Omega_1^{2,3}, \Omega_0^{2,3})$$

are surjective.

Proof. Consider the following commutative diagram:

$$
\begin{tikzcd}
E(N_2) \arrow[r, "\kappa"] \arrow[d, "p^{2,3}"'] & E(N_1) \arrow[d, "p^{1,3}"] \\
\Omega^{2,3} \arrow[r, "\iota"'] & \Omega^{1,3}
\end{tikzcd}
\tag{6.4}
$$

where $\kappa: E(N_2) \to E(N_1)$ is the inclusion map. It was proved in [6] that $\iota: \Omega^{2,3} \to \Omega^{1,3}$ is continuous, injective and $\iota(z^{2,3}) = z^{1,3}$. Moreover, $\iota(\Omega_i^{2,3}) \subset \Omega_i^{1,3}$ and $\Omega_i^{1,3} \cap \iota(\Omega^{2,3}) \subset \iota(\Omega_i^{2,3})$, for $i \in \{0,1,2\}$. It follows that $\Omega_2^{1,3} \cap \iota(\Omega^{2,3}) = \iota(\Omega_2^{2,3})$ and $\Omega_0^{1,3} \cap \iota(\Omega_1^{2,3}) = \iota(\Omega_0^{2,3})$. We claim that

(6.5) $\iota(\Omega^{2,3})$ is closed in $\Omega^{1,3}$.

To prove this claim notice that, in view of the commutativity of diagram (6.4), we have that $w \in (p^{1,3})^{-1}(\iota(\Omega^{2,3}))$ if and only if there is a $w' \in E(N_2)$ such that $p^{1,3}(w) = p^{1,3}(w')$ if and only if $w \in E(N_2)$. Thus $(p^{1,3})^{-1}(\iota(\Omega^{2,3})) = E(N_2)$ and since $E(N_2)$ is closed in $E(N_1)$ it follows that $\iota(\Omega^{2,3})$ is closed in $\Omega^{1,3}$.

We also claim that

(6.6) Whenever U is open in $\Omega^{2,3}$, then $\iota(U)$ is open in $\iota(\Omega^{2,3})$.

To prove this claim, let U be an arbitrary open set in $\Omega^{2,3}$. Then $\widetilde{U} := (p^{2,3})^{-1}(U)$ is open in $E(N_2)$. Hence there is an open set $\widetilde{V}$ in $E(N_1)$ such that $\widetilde{V} \cap E(N_2) = \widetilde{U}$. We have two possible cases: either $\widetilde{U} \supset E(N_3)$ or else $\widetilde{U} \cap E(N_3) = \emptyset$. In the former case $\widetilde{V} \supset E(N_3)$. In the latter case, intersecting, if necessary, the set $\widetilde{V}$ with the set $E(N_1) \setminus E(N_3)$ (which is open in $E(N_1)$) we may assume that $\widetilde{V} \cap E(N_3) = \emptyset$. In both cases it follows that $\widetilde{V} = (p^{1,3})^{-1}(V)$ where $V := p^{1,3}(\widetilde{V})$. Thus V is open in $\Omega^{1,3}$. Now $b \in V \cap \iota(\Omega^{2,3})$ if and only if there are $w \in \widetilde{V}$ and $w' \in E(N_2)$ with $b = p^{1,3}(w) = p^{1,3}(w')$ if and only if there is a $w \in \widetilde{V} \cap E(N_2) = \widetilde{U}$ with $b = p^{1,3}(w) = \iota(p^{2,3}(w))$ if and only if $b \in \iota(p^{2,3}(\widetilde{U})) = \iota(U)$. We thus obtain $V \cap \iota(\Omega^{2,3}) = \iota(U)$ so $\iota(U)$ is open in $\iota(\Omega^{2,3})$, as claimed.

Claim (6.6) implies, in view of the continuity and injectivity of ι, that ι induces a homeomorphism of $\Omega^{2,3}$ onto $\iota(\Omega^{2,3})$ and of $\Omega^{2,3}_i$ onto $\iota(\Omega^{2,3}_i)$ for $i \in \{0,1,2\}$. Since $(p^{2,3})^{-1}(\Omega^{2,3}_i) = E_i(N_2,N_3)$ and $E_i(N_2,N_3)$ is closed in $E(N_2)$ it follows that $\Omega^{2,3}_i$ is closed in $\Omega^{2,3}$ so $\iota(\Omega^{2,3}_i)$ is closed in $\iota(\Omega^{2,3})$, which itself is closed in $\Omega^{1,3}$ by claim (6.5). Thus $\iota(\Omega^{2,3}_2)$ is closed in $\Omega^{1,3}_2$ and $\iota(\Omega^{2,3}_0)$ is closed in $\Omega^{1,3}_0$. Now Proposition 3.2 completes the proof of the lemma. $\square$

Let us make the following

Definition 6.3. A triple (M_1,M_2,M_3) is called an *admissible index triple* if there are: a metric space $\widetilde{X}$, a local semiflow $\widetilde{\pi}$ on $\widetilde{X}$, an isolated $\widetilde{\pi}$-invariant set $\widetilde{S}$ having a strongly $\widetilde{\pi}$-admissible isolating neighborhood and an attractor-repeller pair $(\widetilde{A},\widetilde{A}^*)$ of $\widetilde{S}$, rel. to $\widetilde{\pi}$, such that (M_1,M_2,M_3) is an FM-index triple for $(\widetilde{\pi},\widetilde{S},\widetilde{A},\widetilde{A}^*)$ and $\mathrm{Cl}_{\widetilde{X}}(M_1 \setminus M_3)$ is strongly $\widetilde{\pi}$-admissible.

Lemma 6.4. *The sequences*

$$\overline{C}^*(\Omega^{1,2},\{z^{1,2}\}) \longrightarrow \overline{C}^*(\Omega^{1,3},\{z^{1,3}\}) \longrightarrow \overline{C}^*(\Omega^{2,3},\{z^{2,3}\})$$

$$\overline{C}^*(N_1/N_2,\{[N_2]\}) \longrightarrow \overline{C}^*(N_1/N_3,\{[N_3]\}) \longrightarrow \overline{C}^*(N_2/N_3,\{[N_3]\})$$

are weakly coexact.

Proof. Since (N_1,N_2,N_3) is an admissible index triple, Proposition 3.4 implies that the second sequence is weakly coexact. Results in [6] imply that $(E(N_1),E(N_2),E(N_3))$ is an admissible index triple and so Proposition 3.4 implies that the first sequence is weakly coexact. $\square$

Lemma 6.5. *For $j \in \{0,1,2\}$ the sequence*

$$\overline{C}^*(\Omega^{1,2}_j,\{z^{1,2}\}) \longrightarrow \overline{C}^*(\Omega^{1,3}_j,\{z^{1,3}\}) \longrightarrow \overline{C}^*(\Omega^{2,3}_j,\{z^{2,3}\})$$

is weakly coexact.

Proof. Define $\mathbb{R}_0 := \{0\}$, $\mathbb{R}_1 :=]-\infty,0]$ and $\mathbb{R}_2 := [0,\infty[$. Fix $j \in \{0,1,2\}$. For $i \in \{1,2,3\}$ let $M_i = M^j_i := (N_i \times I_j) \cup (N \times (I_j \cap \{-1,1\}))$. By results in [6]

(M_1, M_2, M_3) is an admissible index triple and for i, $\ell \in \{0,1,2\}$ with $i < \ell$ the inclusion induced map

$$M_i/M_\ell \to E_j(N_i, N_\ell)/E(N_\ell)$$

is a base-point preserving homeomorphism. Moreover, the following inclusion induced diagram is commutative

$$\begin{array}{ccccc} M_2/M_3 & \longrightarrow & M_1/M_3 & \longrightarrow & M_1/M_2 \\ \downarrow & & \downarrow & & \downarrow \\ E_j(N_2,N_3)/E(N_3) & \longrightarrow & E_j(N_1,N_3)/E(N_3) & \longrightarrow & E_j(N_1,N_2)/E(N_2) \end{array} \tag{6.7}$$

Notice that, by Remark 5.2, $\Omega_j^{i,\ell} = E_j(N_i, N_\ell)/E(N_\ell)$, i, $\ell \in \{1,2,3\}$, $i < \ell$ and $j \in \{0,1,2\}$, both as sets and as topological spaces.

Hence, the commutativity of diagram (6.7) implies the commutativity of the following inclusion induced diagram

$$\begin{array}{ccccc} \overline{C}^*(M_1/M_2, \{[M_2]\}) & \longrightarrow & \overline{C}^*(M_1/M_3, \{[M_3]\}) & \longrightarrow & \overline{C}^*(M_2/M_3, \{[M_3]\}) \\ \uparrow & & \uparrow & & \uparrow \\ \overline{C}^*(\Omega_j^{1,2}, \{z^{1,2}\}) & \longrightarrow & \overline{C}^*(\Omega_j^{1,3}, \{z^{1,3}\}) & \longrightarrow & \overline{C}^*(\Omega_j^{2,3}, \{z^{2,3}\}) \end{array} \tag{6.8}$$

where the vertical arrows are isomorphims. Since the upper row of diagram (6.8) is weakly coexact by Proposition 3.4, reversing the upward pointing arrows in diagram (6.8) and using Lemma 2.3 we conclude that the lower row of diagram (6.8) is weakly coexact. □

We can finally give a

PROOF OF THEOREM 6.1. Diagram (6.2) is commutative by the definition of $f_{Y,Z}$ and it has weakly coexact rows by Lemmas 6.4 and 6.5. Consider the following inclusion induced diagram of chain maps:

$$\begin{array}{ccccc} 0 & & 0 & & 0 \\ \downarrow & & \downarrow & & \downarrow \\ \overline{C}^*(\Omega^{1,2}, \Omega_2^{1,2}) & \longrightarrow & \overline{C}^*(\Omega^{1,3}, \Omega_2^{1,3}) & \longrightarrow & \overline{C}^*(\Omega^{2,3}, \Omega_2^{2,3}) \\ \downarrow & & \downarrow & & \downarrow \\ \overline{C}^*(\Omega^{1,2}, \{z^{1,2}\}) & \longrightarrow & \overline{C}^*(\Omega^{1,3}, \{z^{1,3}\}) & \longrightarrow & \overline{C}^*(\Omega^{2,3}, \{z^{2,3}\}) \\ \downarrow & & \downarrow & & \downarrow \\ \overline{C}^*(\Omega_2^{1,2}, \{z^{1,2}\}) & \longrightarrow & \overline{C}^*(\Omega_2^{1,3}, \{z^{1,3}\}) & \longrightarrow & \overline{C}^*(\Omega_2^{2,3}, \{z^{2,3}\}) \\ \downarrow & & \downarrow & & \downarrow \\ 0 & & 0 & & 0 \end{array} \tag{6.9}$$

Both (6.9) and (6.3) are clearly commutative. Proposition 3.3 implies that the columns in each of those diagrams are exact. It follows from Lemmas 6.4 and 6.5 that the last two rows in each of those diagrams are weakly coexact. Now an application of Lemma 6.2 and Proposition 2.5 implies that the first row in each of those diagrams is weakly coexact. We have therefore proved that the rows of diagrams (6.1), (6.2) and (6.3) are weakly coexact. □

We can now complete the

Proof of Theorem 4.5. For i, $j \in \{1,2,3\}$ with $i<j$ define the following sets:

$$\begin{aligned} A_1^{i,j} &:= \overline{C}^*(\Omega^{i,j},\{z^{i,j}\}), & A_2^{i,j} &:= \overline{C}^*(\Omega^{i,j},\Omega_2^{i,j}), & A_3^{i,j} &:= \overline{C}^*(\Omega_1^{i,j},\Omega_0^{i,j}),\\ A_4^{i,j} &:= \overline{C}^*(\Omega_1^{i,j},\{z^{i,j}\}), & A_5^{i,j} &:= \overline{C}^*(\Omega_0^{i,j},\{z^{i,j}\}), & A_6^{i,j} &:= \overline{C}^*(N_i/N_j,\{[N_j]\}). \end{aligned}$$

Using Proposition 5.4 and Lemma 5.3 we see that diagrams (6.1) and (6.2) in Theorem 6.1 induce the following commutative diagrams in cohomology with long exact rows

$$\begin{array}{ccccccccc} \longleftarrow & H^q(A_1^{2,3}) & \longleftarrow & H^q(A_1^{1,3}) & \longleftarrow & H^q(A_1^{1,2}) & \longleftarrow & H^{q-1}(A_1^{2,3}) & \longleftarrow \\ & \uparrow{\scriptstyle H^q(\rho^{2,3})} & & \uparrow{\scriptstyle H^q(\rho^{1,3})} & & \uparrow{\scriptstyle H^q(\rho^{1,2})} & & \uparrow{\scriptstyle H^{q-1}(\rho^{2,3})} & \\ \longleftarrow & H^q(A_2^{2,3}) & \longleftarrow & H^q(A_2^{1,3}) & \longleftarrow & H^q(A_2^{1,2}) & \longleftarrow & H^{q-1}(A_2^{2,3}) & \longleftarrow \\ & \uparrow{\scriptstyle H^q(\ell^{2,3})^{-1}} & & \uparrow{\scriptstyle H^q(\ell^{1,3})^{-1}} & & \uparrow{\scriptstyle H^q(\ell^{1,2})^{-1}} & & \uparrow{\scriptstyle H^{q-1}(\ell^{2,3})^{-1}} & \\ \longleftarrow & H^q(A_3^{2,3}) & \longleftarrow & H^q(A_3^{1,3}) & \longleftarrow & H^q(A_3^{1,2}) & \longleftarrow & H^{q-1}(A_3^{2,3}) & \longleftarrow \end{array} \tag{6.10}$$

$$\begin{array}{ccccccccc} \longleftarrow & H^q(A_5^{2,3}) & \longleftarrow & H^q(A_5^{1,3}) & \longleftarrow & H^q(A_5^{1,2}) & \longleftarrow & H^{q-1}(A_5^{2,3}) & \longleftarrow \\ & \uparrow{\scriptstyle H^q(f^{2,3})^{-1}} & & \uparrow{\scriptstyle H^q(f^{1,3})^{-1}} & & \uparrow{\scriptstyle H^q(f^{1,2})^{-1}} & & \uparrow{\scriptstyle H^{q-1}(f^{2,3})^{-1}} & \\ \longleftarrow & H^q(A_6^{2,3}) & \longleftarrow & H^q(A_6^{1,3}) & \longleftarrow & H^q(A_6^{1,2}) & \longleftarrow & H^{q-1}(A_6^{2,3}) & \longleftarrow \end{array} \tag{6.11}$$

Note that diagram (6.3) in Theorem 6.1 can be written as

$$\begin{array}{ccccc} 0 & & 0 & & 0 \\ \downarrow & & \downarrow & & \downarrow \\ A_3^{1,2} & \longrightarrow & A_3^{1,3} & \longrightarrow & A_3^{2,3} \\ \downarrow & & \downarrow & & \downarrow \\ A_4^{1,2} & \longrightarrow & A_4^{1,3} & \longrightarrow & A_4^{2,3} \\ \downarrow & & \downarrow & & \downarrow \\ A_5^{1,2} & \longrightarrow & A_5^{1,3} & \longrightarrow & A_5^{2,3} \\ \downarrow & & \downarrow & & \downarrow \\ 0 & & 0 & & 0 \end{array} \tag{6.12}$$

Let $\widehat{\partial}^{3,q}\colon H^q(A_5^{2,3}) \to H^{q+1}(A_5^{1,2})$, (resp. $\widehat{\partial}^{1,q}\colon H^q(A_3^{2,3}) \to H^{q+1}(A_3^{1,2})$), $q \in \mathbb{Z}$, be the connecting homomorphism of the long cohomology sequence associated with the third (resp. first) row of (6.12) while $\delta^{3,q}\colon H^q(A_5^{2,3}) \to H^{q+1}(A_3^{2,3})$, (resp. $\delta^{2,q}\colon H^q(A_5^{1,3}) \to H^{q+1}(A_3^{1,3})$, resp. $\delta^{1,q}\colon H^q(A_5^{1,2}) \to H^{q+1}(A_3^{1,2})$), $q \in \mathbb{Z}$, be the connecting homomorphism of the long cohomology sequence associated with the third (resp. second, resp. first) column of (6.12). Proposition 2.5 implies that

$$\delta^{1,q+1} \circ \widehat{\partial}^{3,q} = -\widehat{\partial}^{1,q+1} \circ \delta^{3,q}, \quad q \in \mathbb{Z}. \tag{6.13}$$

Thus we obtain the following diagram

$$\begin{tikzcd}
{} & H^q(A_3^{2,3}) \arrow[l] & H^q(A_3^{1,3}) \arrow[l] & H^q(A_3^{1,2}) \arrow[l] & H^{q-1}(A_3^{2,3}) \arrow[l, "\widehat{\partial}^{1,q-1}"'] & {} \arrow[l] \\
{} & H^{q-1}(A_5^{2,3}) \arrow[l] \arrow[u, "(-1)^{q-1}\delta^{3,q-1}"'] & H^{q-1}(A_5^{1,3}) \arrow[l] \arrow[u, "(-1)^{q-1}\delta^{2,q-1}"'] & H^{q-1}(A_5^{1,2}) \arrow[l] \arrow[u, "(-1)^{q-1}\delta^{1,q-1}"'] & H^{q-2}(A_5^{2,3}) \arrow[l, "\widehat{\partial}^{3,q-2}"] \arrow[u, "(-1)^{q-2}\delta^{3,q-2}"'] & {} \arrow[l]
\end{tikzcd} \tag{6.14}$$

The diagram

$$\begin{tikzcd}
H^q(A_3^{2,3}) & H^q(A_3^{1,3}) \arrow[l] & H^q(A_3^{1,2}) \arrow[l] \\
H^{q-1}(A_5^{2,3}) \arrow[u, "(-1)^{q-1}\delta^{3,q-1}"] & H^{q-1}(A_5^{1,3}) \arrow[l] \arrow[u, "(-1)^{q-1}\delta^{2,q-1}"] & H^{q-1}(A_5^{1,2}) \arrow[l] \arrow[u, "(-1)^{q-1}\delta^{1,q-1}"']
\end{tikzcd}$$

commutes by the naturality of $(\delta_q^i)_{q\in\mathbb{Z}}$, $i \in \{1,2,3\}$ while the diagram

$$\begin{tikzcd}
H^q(A_3^{1,2}) & H^{q-1}(A_3^{2,3}) \arrow[l, "\widehat{\partial}^{1,q-1}"'] \\
H^{q-1}(A_5^{1,2}) \arrow[u, "(-1)^{q-1}\delta^{1,q-1}"] & H^{q-2}(A_5^{2,3}) \arrow[l, "\widehat{\partial}^{3,q-2}"] \arrow[u, "(-1)^{q-2}\delta^{3,q-2}"']
\end{tikzcd}$$

commutes in view of (6.13). It follows that diagram (6.14) is commutative.

Now, composing diagrams (6.10), (6.14) and (6.11) (from bottom to top) and using Theorem 5.6 we obtain the commutative diagram

$$\begin{tikzcd}
{} & H^q(A_1^{2,3}) \arrow[l] & H^q(A_1^{1,3}) \arrow[l] & H^q(A_1^{1,2}) \arrow[l] & H^{q-1}(A_1^{2,3}) \arrow[l] & {} \arrow[l] \\
{} & H^{q-1}(A_6^{2,3}) \arrow[l] \arrow[u, "\xi^{q,2,3}"'] & H^{q-1}(A_6^{1,3}) \arrow[l] \arrow[u, "\xi^{q,1,3}"'] & H^{q-1}(A_6^{1,2}) \arrow[l] \arrow[u, "\xi^{q,1,2}"'] & H^{q-2}(A_6^{2,3}) \arrow[l] \arrow[u, "\xi^{q-1,2,3}"'] & {} \arrow[l]
\end{tikzcd} \tag{6.15}$$

Applying the $\langle\,\cdot\,,\,\cdot\,\rangle$-operation to diagram (6.15) and using Definition 5.7 together with [5, Theorem 5.1] we obtain diagram (4.1). This combined with Theorem 5.6 completes the proof of Theorem 4.5. □

7. The suspension isomorphism and cohomology index braids

In this section let P be a finite set, $\prec$ be a strict partial order on P and $(M_i)_{i\in P}$ be a Morse decomposition of S relative to π. Using the notation of Theorem 4.1 we have that $(M_i')_{i\in P}$ is a $\prec$-ordered Morse decomposition of S' relative to π'. Given $(I,J)\in\mathcal{I}_2(\prec)$, $(M(I),M(J))$ is an attractor-repeller pair in $M(IJ)$ (where $IJ=I\cup J$) relative to π, so $(M(I)',M(J)')$ is an attractor-repeller pair in $M(IJ)'$ relative to π'. Setting, for $K\in\mathcal{I}(\prec)$ and $q\in\mathbb{Z}$, $H^{q'}(K):=H^q(\pi',M(K)')$, $H^q(K):=H^q(\pi,M(K))$ and $\theta^q(K):=\theta^q(\pi,\widetilde{\pi},M(K))$ and using Theorem 4.1 we thus arrive at the commutative diagram

$$\begin{tikzcd}
{} & H^{q'}(I) \arrow[l] & H^{q'}(IJ) \arrow[l] & H^{q'}(J) \arrow[l] & H^{(q-1)'}(I) \arrow[l] & {} \arrow[l] \\
{} & H^{q-k}(I) \arrow[l] \arrow[u, "\theta^q(I)"] & H^{q-k}(IJ) \arrow[l] \arrow[u, "\theta^q(IJ)"] & H^{q-k}(J) \arrow[l] \arrow[u, "\theta^q(J)"'] & H^{q-k-1}(I) \arrow[l] \arrow[u, "\theta^{q-1}(I)"'] & {} \arrow[l]
\end{tikzcd} \tag{7.1}$$

Here, the upper (resp. lower) horizontal sequence is the cohomology index sequence of $(\pi',M(IJ)',M(I)',M(J)')$ (resp. the cohomology index sequence of $(\pi,M(IJ),M(I),M(J))$ shifted to the left by k). We thus obtain the following result.

Theorem 7.1. *$(\theta^q(J))_{q\in\mathbb{Z}}$, $J\in\mathcal{I}(\prec)$, is an isomorphism from the graded module braid obtained by shifting the cohomology index braid of $(\pi,S,(M_i)_{i\in P})$ to the left by k to the cohomology index braid of $(\pi',S',(M_i')_{i\in P})$.*

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