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EXISTENCE OF SOLUTIONS FOR A NONLINEAR WAVE EQUATION

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Abstract. We prove the existence of a strong solution of a periodic-Dirichlet problem for the semilinear wave equation with irrational period and with nonlinearity satisfying some general growth conditions locally around 0. We construct a new variational method, called a dual method, and using relations between critical points and critical values of the primal action and the dual action functionals we prove that the solution exists. The dual functional which we define is different from the ones known so far in that it depends on two dual variables.

1. Introduction

The aim of this paper is to prove the existence of strong solutions by introducing a so called dual variational method for a certain class of periodic-Dirichlet problems for a scalar nonlinear wave equation

$$\begin{gathered} x_{tt}(t,y) - x_{yy}(t,y) + F_x(t,y,x(t,y)) = 0, \\ x(t+T,y) = x(t,y) \quad \text{for } (t,y) \in \mathbb{R} \times \Omega, \\ x(t,0) = x(t,\pi) = 0 \quad \text{for } t \in \mathbb{R}, \end{gathered}$$

where $\Omega = (0,\pi)$, F_x denotes the derivative with respect to the third variable of the function $F: \mathbb{R} \times \Omega \times \mathbb{R} \to \mathbb{R}$ which is convex and continuously differentiable

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with respect to the third variable in some interval that will be specified later. The assumption on Ω means that we consider the so called one-dimensional or scalar wave equation. Moreover, F satisfies the Caratheodory condition, i.e. it is measurable with respect to first two variables and continuous with respect to the third one. More detailed assumptions concerning the nonlinearity will be provided later. We consider only the case when $T = 2\pi\alpha$ and α is a certain irrational number. Although problem (1.1) is fairly difficult to study, in the last decade of the former century there appeared many papers dealing with it by a variety of methods, compare [1], [3]–[7], [8], [10], [19]. The main difficulty which appears here is that the linear wave equation with the right hand side being a fixed function (instead of our nonlinear function F_x) does not possess solutions (even generalized) for all values of α (see comments in [18, pp. 242–249]). This is why most of the sources mentioned tried to find the nature of irrationality of α and relate it to solvability of (1.1) by extending KAM method, Lyapunow–Schmidt method or topological and classical variational methods. All these methods are based on the study of the spectrum of the linear equation whose structure relates strongly to the Diophantine character of α and next on developing nonlinear methods that use the properties of the spectrum of the linear d'Alembert operator $\square$. Our approach is different although to some extent it is in the spirit of the paper [5] where the variational Lyapunov–Schmidt reduction method is used. We use the same definition of α (see (T) below) but quite different and brand new variational method. These α are good enough to ensure that linear nonhomogeneous equation has a unique solution in a strong sense. Therefore we use the theorem from [18] and later develop quite new variational method by constructing new duality theory and variational relations for (1.1). In the first step using a kind of a topological idea we construct a set of functions X for which $\square^{-1}F_x(X) \subset X$. To this effect we use, in an essential way, the exact value of constant c defining the irrational α and the α itself, see condition (T) below. Such an assumption restricts locally a type of our nonlinearity F_x. Thus although we apply the variational method that bases on minimizing suitable action functionals and recovering the relationships between their critical points and critical values, we do not surpass over all subtle details connected with the wave operator and wave equation. The main difficulty in our approach is the construction of the set X and of a new duality theory related to that set X. Therefore, let us assume that the number-theoretical character of α is

(T) $T = 2\pi\alpha$, *α is any irrational number satisfying* $|\alpha - p/q| \geq cq^{-2}$ *for all* $p, q \in \mathbb{N}$ *with* $c > 0$.

On equivalent forms of this condition as well as on several properties of such numbers see e.g. [2], [3]. We only mention following [5] that for $0 < c < 1/3$, the set $W_c = \{\alpha \in \mathbb{R} : |\alpha - p/q| \geq cq^{-2}$ for all $p, q \in \mathbb{N},\ p \neq q\}$ is uncountable, has zero measure and accumulates to $\alpha = 1$ both from the left and from the right; $W_c = \emptyset$ for $c \geq 1$.

It is essential that our assumptions on the nonlinearity $F_x(t, y, \cdot\,)$ concern only some interval, i.e. we need no information about $F_x(t, y, \cdot\,)$ out of that interval. It allows us to consider superquadratic as well as subquadratic nonlinearities at infinity. We should underline as well that the existence result in our paper asserts that a solution exists in a strong sense i.e. the solution $x(t, y)$ satisfies (1.1) almost everywhere.

For the proof of Proposition 2.2 we require the following auxiliary results.

Let

$$\mathcal{G} = L^2_{\mathrm{per}}(\mathbb{R}; H^2_0(\Omega)), \quad \mathcal{G}_1 = H^1_{\mathrm{per}}(\mathbb{R}; H^1_0(\Omega))$$

and let

$$U = H^2_{\mathrm{per}}(\mathbb{R} \times \Omega) \cap H^1_{per,0}(\mathbb{R} \times \Omega), \quad U^1 = H^1_{\mathrm{per}}((0, T); W^{1,2}_0(\Omega)),$$

where H^1_{per}, H^2_{per} are usual Sobolev spaces of periodic functions with respect to the first variable with period T and

$$\begin{aligned} H^1_{\mathrm{per},0}(\mathbb{R} \times \Omega) = \{h \in H^1(\mathbb{R} \times \Omega) : h(t + T, y) = h(t, y),& \\ (t, y) \in \mathbb{R} \times \Omega,\ h(t, 0) = h(t, \pi) = 0,\ t \in \mathbb{R}\}.& \end{aligned}$$

THEOREM 1.1 ([18]). *Let $g \in \mathcal{G}$. Then there exist $\overline{x} \in U$ being a unique solution to*

$$\begin{aligned} x_{tt}(t, y) - \Delta x(t, y) &= g(t, y), \\ x(t, 0) = x(t, \pi) &= 0, \quad t \in \mathbb{R}, \\ x(t + T, y) = x(t, y), &\quad (t, y) \in \mathbb{R} \times \Omega, \end{aligned}$$

and $\overline{x} = \Lambda g$, where $\Lambda \in \mathcal{L}(L^2_{\mathrm{per}}(\mathbb{R}; H^2_0(\Omega), U)$ with

$$\begin{aligned} \overline{x}(t, y) &= \left(\frac{2}{\pi T}\right)^{1/2} \sum_{j,k} (-j^2\alpha^{-2} + k^2)^{-1} g_{j,k} e^{ij2\pi t/T} \sin ky, \\ g_{j,k} &= \left(\frac{2}{\pi T}\right)^{1/2} \int_0^T \int_0^\pi g(t, y) e^{-ij2\pi t/T} \sin ky \, dy \, dt, \end{aligned} \tag{1.2}$$

and such that

$$\|\overline{x}\|^2_U \leq B\|g\|^2_{\mathcal{G}}, \tag{1.3}$$

$$\|\overline{x}\|^2_U \leq C\|g\|^2_{\mathcal{G}_1}, \tag{1.4}$$

with some $B \geq (1 + 16\alpha^4)\alpha^2 c^{-2}$ and $C \geq (1 + 16\alpha^4)\alpha^2 c^{-2}(\alpha/2)^{-2}$ independent on g.

Remark 1.2. From [18, pp. 239–242] we get one more estimation for the solution $\overline{x} \in U$:

$$\|\overline{x}\|^2_{U^1} \leq A\|g\|^2_{L^\infty((0,T);W^{2,2}(\Omega))}, \tag{1.5}$$

where A is independent on g. Let us observe that each $x \in U^1$ may be identified with an absolutely continuous function from $[0,T]$ to $W_0^{1,2}(\Omega)$. Thus, solution $\overline{x}$ defined by (1.2) may be estimated by

$$\|\overline{x}(t,\,\cdot\,)\|_{W_0^{1,2}(\Omega)} \leq C_1 \sup_{t\in(0,T)} \|g(t,\,\cdot\,)\|_{W^{2,2}(\Omega)}, \quad t \in [0,T],$$

where C_1 is some constant independent on g.

We assume the following hypotheses concerning the nonlinearity, where $F_x(v)$ denotes the function $(t,y) \to F_x(t,y,v(t,y))$,

(G1) there exists a function $\overline{z} \in C_0^2([0,T]\times\overline{\Omega})$ such that $(d^2/dt^2)\overline{z}-(d^2/dt^2)\overline{z} \in L^\infty((0,T);W^{2,2}(\Omega))$, $F_x(\overline{z}) \in L^\infty((0,T);W^{2,2}(\Omega))$;

(G2) F is continuously differentiable and convex with respect to the third variable in some closed neighbourhood $\widetilde{I} = [-d,d]$ of the interval

$$I = \Big[-C_1 \sup_{t\in(0,T)} \|\overline{z}(t,\,\cdot\,)\|_{W^{2,2}(\Omega)}, C_1 \sup_{t\in(0,T)} \|\overline{z}(t,\,\cdot\,)\|_{W^{2,2}(\Omega)}\Big];$$

(G3) $F_x(t+T,y,x) = F_x(t,y,x)$, $(t,y,x) \in \mathbb{R}\times\Omega\times\widetilde{I}$, $F_x(t,0,0) = F_x(t,\pi,0) = 0$, $t \in \mathbb{R}$, $F_x(t,y,0) \neq 0$, for a.e. $(t,y) \in (0,T)\times\Omega$, $(t,y) \to F(t,y,0)$ is integrable in $(0,T)\times\Omega$ and

$$\sup_{x\in I} \|F_x(t,\,\cdot\,,x)\|_{W^{2,2}(\Omega)} \leq \sup_{t\in(0,T)} \|\overline{z}(t,\,\cdot\,)\|_{W^{2,2}(\Omega)}, \quad t \in (0,T). \tag{1.6}$$

(G4) $F_{xxx}(t,\,\cdot\,,\,\cdot\,) \in L^2((\Omega\times I)$, $F_{xyy}(t,\,\cdot\,,x) \in L^2(\Omega)$ and $F_{xxy}(t,\,\cdot\,,\,\cdot\,) \in L^2(\Omega\times I))$ for a.e. $t \in (0,T)$, $F_{xt}(\,\cdot\,,y,\,\cdot\,) \in L^2((0,T)\times I))$ for a.e. $y \in \Omega$, there exists a function $l \in L^1((0,T)\times\Omega)$

$$\max_{x\in I}\{|F_x(t,y,x)|^2 + |F_{xt}(t,y,x)|^2 + |F_{xy}(t,y,x)|^2 +\} \leq l(t,y)$$

and esse $\sup_{(t,y)\in[0,T]\times\Omega} \max_{x\in I} |F_{xx}(t,y,x)|^2$ is finite.

Due to the assumptions (G1)–(G4) we define a function $G\colon (0,T)\times\Omega\times\mathbb{R} \to \mathbb{R}$ by the formula

$$G(t,y,x) = \begin{cases} F(t,y,x) & \text{for } (t,y,x) \in [0,T]\times\Omega\times\widetilde{I}, \\ \infty & \text{for } (t,y,x) \in [0,T]\times\Omega\times(\mathbb{R}\backslash\widetilde{I}). \end{cases}$$

We observe that function G satisfies assumptions (G1)–(G4). Before we will proceed further we provide an example of a function which satisfies the introduced assumptions.

Example 1.3. Let

$$F(t,y,x) = -\frac{1}{12}x^4 + \frac{1}{2}xf(t,y) + \frac{1}{2}\exp(x),$$

$f, f_y, f_t \in L^\infty((0,T)\times\Omega)$, $f(t+T,y) = f(t,y)$ for a.e. $(t,y)\in(0,T)\times\Omega$ and $f(t,0) = f(t,\pi)$ for a.e. $t\in(0,T)$ and esse $\sup_{(t,y)\in(0,T)\times\Omega} f(t,y) = -1.5$. We observe that F is not convex with respect to x in the interval $(0,3)$ (for it has 2 inflection points) and it is convex for $x\in(0,1.487962065)$. We observe that inequalities

$$\frac{2}{3}d^3 + e^d - 1.5 \le d \text{ and } \frac{2}{3}(-d)^3 + e^{-d} - 1.5 \le d$$

are satisfied for $d = 1.487962065$. Let us take any function $\overline{z}\in C_0^1([0,T]\times\overline{\Omega})$ such that $(d^2/dt^2)\overline{z} - (d^2/dt^2)\overline{z} \in L^\infty((0,T)\times\Omega,\mathbb{R})$ and

$$\sup_{(t,y)\in(0,T)\times\Omega} |\overline{z}(t,y)| = 1.487962065/C_1.$$

Therefore (1.6) is satisfied. Assumptions (G4) and (G3) now obviously hold. Hence all assumptions (G1)–(G4) are satisfied.

Now we may define the following set

$$\overline{X} = \Big\{x\in U : \|x(t,\,\cdot\,)_{W_0^{1,2}(\Omega)} \in I,\ (t,y)\in(0,T),\ \|x\|_{U^1}^2 \le C_1 \sup_{t\in(0,T)} \|\overline{z}(t,\,\cdot\,)\|\Big\}.$$

Since $F = G$ on $\overline{X}$ and since we will be looking for solutions of (1.1) in $\overline{X}$, we now may rewrite (1.1) in the form

$$\begin{aligned} x_{tt}(t,y) - x_{yy}(t,y) + G_x(t,y,x(t,y)) &= 0,\\ x(t+T,y) = x(t,y) &\quad \text{for } (t,y)\in\mathbb{R}\times\Omega,\\ x(t,0) = x(t,\pi) = 0 &\quad \text{for } t\in\mathbb{R}, \end{aligned}$$

and still refer to it as (1.1). We now consider it as the Euler–Lagrange equation for the following action functional J

$$J(x) = \int_0^T\int_\Omega\left(\frac{1}{2}|x_y(t,y)|^2 - \frac{1}{2}|x_t(t,y)|^2 + G(t,y,x(t,y))\right)dy\,dt \tag{1.7}$$

where $J\colon H^1_{\mathrm{per},0}(\mathbb{R}\times\Omega)\to\mathbb{R}$.

The dual functional, which is for the first time introduced here and which will be investigated together with (1.7) now reads

$$\begin{aligned} J_D(p,q) = &-\int_0^T\int_\Omega G^*(t,y,-(p_t(t,y)-q_y(t,y)))\,dy\,dt\\ &-\frac{1}{2}\int_0^T\int_\Omega |q(t,y)|^2\,dy\,dt + \frac{1}{2}\int_0^T\int_\Omega |p(t,y)|^2\,dy\,dt \end{aligned} \tag{1.8}$$

and $J_D\colon H^1((0,T)\times\Omega)\times H^1((0,T)\times\Omega)\to\mathbb{R}$.

Let us note that by (G4) and (G3) it follows that $(t,y) \to G^*(t,y,0)$ is integrable. Therefore both action functionals are well defined, see [17].

We will look at two kinds of relationships between the functionals J and J_D on the set $\overline{X}$: the Duality Principle and the Variational Principle. The former, Theorem 4.1, relates the critical values of both functionals while the latter, Theorem 4.4, provides the necessary conditions that must be satisfied by the solution to problem (1.1). For the purpose of the necessary conditions we have to assume that a solution is approximated by a suitably convergent minimizing sequence. But in the main result we prove that such a sequence actually exists.

The introduction of the two dual variables in (1.8) is a consequence of the fact that the dual variational method from [15] requires the differential operator to be either monotone and coercive or selfadjoint and positive definite. The wave operator is actually the difference between the monotone operators $-\Delta$ and $-d^2/dt^2$. So that to tackle the problem of the existence of solutions we have had to bring up a new duality and a new variational principle in which we put two dual variables. In that consists the main difference with some results in that field, i.e. [11]–[13], [16], also obtained by a dual variational method. The variable p is connected with the operator $-d^2/dt^2$ and the variable q connected with the operator $-d^2/dy^2$. In the literature to the best knowledge of the authors, the dual functional for the wave equations connected with variational method depends only on one variable.

2. The auxiliary results

Now we shall construct the aforementioned sets on which J and J_D will be considered. We observe by construction of the set $\overline{X}$ and Theorem 1.1 the following lemma may be formulated

Lemma 2.1. *For any $x \in \overline{X}$ there exists a solution $v \in U$ to the problem*

$$v_{tt}(t,y) - v_{yy}(t,y) = -G_x(t,y,x(t,y)) \quad a.e.\ on\ (0,T)\times\Omega. \tag{2.1}$$

There exists a constant $C_2 > 0$ (independent on v) such that the following estimation holds

$$\|v\|_U^2 \le C_2.$$

Proof. Fix arbitrary $x \in \overline{X}$. Since $x \in U$ and by the assumptions on G, see (G4), it follows that $G_x(\,\cdot\,,\,\cdot\,,x(\,\cdot\,,\,\cdot\,)) \in \mathcal{G}$. Hence by Theorem 1.1 there exists a unique solution $v \in U$ of the periodic-Dirichlet problem for the equation (2.1) satisfying

$$\|v\|_U^2 \le C\|G_x(x)\|_{\mathcal{G}_1}^2.$$

In order to proceed further we need some notations and calculations. Let

$$\begin{aligned} a_1 &= \int_0^T \int_\Omega \sup_{x\in\widetilde{I}} |G_{xy}(t,y,x)|^2 + \int_0^T \int_\Omega \sup_{x\in\widetilde{I}} |G_{xt}(t,y,x)|^2, \\ a_2 &= \operatorname{esse} \sup_{(t,y)\in[0,T]\times\Omega} \max_{x\in\widetilde{I}} |G_{xx}(t,y,x)|^2. \end{aligned}$$

Since for $x \in \overline{X}$ we have $\|x\|_{U^1}^2 \le C_1 \sup_{t\in(0,T)} \|\overline{z}(t,\,\cdot\,)\|_{W^{2,2}(\Omega)}$ we may estimate the norm $C\|G_x(x)\|_{\mathcal{G}_1}^2$ as follows

$$\begin{aligned} C\|G_x(x)\|_{\mathcal{G}_1}^2 &= C\int_0^T \int_\Omega |G_{xy}(t,y,x(t,y))|^2\,dy\,dt \\ &\quad + C\int_0^T \int_\Omega |G_{xt}(t,y,x(t,y))|^2 |x_t(t,y))|^2\,dy\,dt \\ &\quad + C\int_0^T \int_\Omega |G_{xt}(t,y,x(t,y))|^2\,dy\,dt \\ &\quad + C\int_0^T \int_\Omega |G_{xy}(t,y,x(t,y))|^2 |x_y(t,y))|^2\,dy\,dt \\ &\le Ca_1 + 2Ca_2C_1 \sup_{t\in(0,T)} \|\overline{z}(t,\,\cdot\,)\|_{W^{2,2}(\Omega)}. \end{aligned}$$

To conclude, we put

$$C_2 = Ca_1 + 2Ca_2C_1 \sup_{t\in(0,T)} \|\overline{z}(t,\,\cdot\,)\|_{W^{2,2}(\Omega)}. \qquad \square$$

We now define a set

$$\begin{aligned} \widetilde{X} = \Big\{ x \in U : \|x(t,\,\cdot\,)\|_{W^{1,2}(\Omega)} \in I,\ t \in (0,T)\times\Omega, \\ \|x\|_{U^1}^2 \le C_1 \sup_{t\in(0,T)} \|\overline{z}(t,\,\cdot\,)\|_{W^{2,2}(\Omega)},\ \|x\|_U^2 \le C_2 \Big\}. \end{aligned}$$

Definition of the set X. Define X as the largest subset of $\widetilde{X}$ having *property* X: for every $x \in X$ the relation

$$\widetilde{x}_{tt}(t,y) - \widetilde{x}_{yy}(t,y) = -G_x(t,y,x(t,y)) \tag{2.2}$$

implies that $\widetilde{x} \in X$.

We have to prove that there exists a nonempty set X.

PROPOSITION 2.2. *$\widetilde{X}$ has the property X i.e. $X = \widetilde{X} \neq \emptyset$.*

PROOF. It is obvious that $\widetilde{X} \neq \emptyset$. Fix arbitrary $x \in \widetilde{X}$. Since $x \in U$ and by the assumptions on G, see (G4), it follows that $G_x(\,\cdot\,,\,\cdot\,,x(\,\cdot\,,\,\cdot\,)) \in \mathcal{G}$ and also $G_x(\,\cdot\,,\,\cdot\,,x(\,\cdot\,,\,\cdot\,)) \in \mathcal{G}_1$. Hence by Theorem 1.1 there exists a unique solution $\widetilde{x} \in U$ of the periodic-Dirichlet problem for the equation

$$\widetilde{x}_{tt}(t,y) - \widetilde{x}_{yy}(t,y) = -G_x(t,y,x(t,y)) \quad \text{a.e. on } (0,T)\times\Omega. \tag{2.3}$$

Further by Remark 1.2 we obtain

$$\|\widetilde{x}\|_{U_1}^2 \le C_1 \sup_{t\in(0,T)} \|G_x(t,\,\cdot\,,x(t,\,\cdot\,))\|_{W^{2,2}(\Omega)}.$$

Thus by (1.6) it follows that

$$\|\widetilde{x}(t,\,\cdot\,)\|_{W^{2,2}(\Omega)} \le C_1 \sup_{t\in(0,T)} \|\overline{z}(t,\,\cdot\,)\|_{W^{2,2}(\Omega)}.$$

So $\|\widetilde{x}(t,\,\cdot\,)\|_{W^{2,2}(\Omega)} \in I$ for $t\in(0,T)$.

Since $\widetilde{X}\subset\overline{X}$, so $\widetilde{x}$ is a solution to (2.1) corresponding to a certain $x\in\overline{X}$. Therefore by Lemma 2.1 we get $\|\widetilde{x}\|_U^2 \le C_2$.

Thus for an arbitrary $x\in\widetilde{X}$ there exists an $\widetilde{x}\in\widetilde{X}$ and satisfying (2.3). Therefore $\widetilde{X}$ has the property X. $\square$

Now we define the set on which the dual action functional will be considered. To this effect let us put

$$\begin{aligned} W_t^1 &= W_t^1((0,T)\times\Omega) = \{p\in L^2((0,T)\times\Omega) : p_t\in L^2((0,T)\times\Omega)\},\\ W_y^1 &= W_y^1((0,T)\times\Omega) = \{q\in L^2((0,T)\times\Omega) : q_y\in L^2((0,T)\times\Omega)\}. \end{aligned}$$

Definition of X^d. We say that an element $(p,q)\in W_t^1\times W_y^1$ belongs to X^d provided that there exists $x\in X$ such that, for a.e. $(t,y)\in(0,T)\times\Omega$,

$$p_t(t,y)-q_y(t,y) = -G_x(t,y,x(t,y)) \quad \text{with } q(t,y)=x_y(t,y)$$

or else

$$p(t,y)=x_t(t,y) \quad\text{and}\quad q(t,y)=x_y(t,y).$$

We will also consider sets

$$X_1^d=\{p\in W_t^1 : (p,q)\in X^d\},\quad X_2^d=\{q\in W_y^1 : (p,q)\in X^d\}.$$

We observe that both X and X^d are not subspaces. Thus even standard calculations using convexity arguments are rather difficult. What helps us is a special structure of the sets X and X^d which despite their nonlinearity makes these calculations possible.

We have said that functionals J and J_D are well defined on X and X^d. Note that the dual action functional is not necessarily bounded on X^d. Using Fenchel–Young inequality it is easily seen by (G3) nad (G4) that J_D is bounded from the above on X^d. Now we may state the main result of the paper which is the following existence theorem.

Theorem 2.3. *There exist $\overline{x}\in U$ and $(\overline{p},\overline{q})\in W_t^1\times W_y^1$ such that*

$$J_D(\overline{p},\overline{q}) = \inf_{x\in X} J(x) = J(\overline{x})$$

and the following system holds

$$\begin{aligned}\overline{x}_t(t,y) &= \overline{p}(t,y),\\ \overline{x}_y(t,y) &= \overline{q}(t,y),\\ \overline{p}_t(t,y) - \overline{q}_y(t,y) &= -G_x(t,y,\overline{x}(t,y)).\end{aligned}$$

3. Duality result

We shall construct the duality theory for the functional $J_D: X^d \to \mathbb{R}$. So that to avoid a calculation of a Fenchel–Young transform with respect to a nonlinear subset we introduce a perturbation functional defined on the whole space. Let $J_{\mathrm{pert}}: X^d \times L^2((0,T)\times\Omega;\mathbb{R}) \to \mathbb{R}$ be given by the formula

$$\begin{aligned}J_{\mathrm{pert}}(p,q,v) = {}& \int_0^T \int_\Omega G^*(t,y,-(p_t(t,y)-q_y(t,y))+v(t,y))\,dy\,dt\\ &+\frac{1}{2}\int_0^T\int_\Omega |q(t,y)|^2\,dy\,dt - \frac{1}{2}\int_0^T\int_\Omega |p(t,y)|^2\,dy\,dt.\end{aligned}$$

From [17] and by assumption (G3) it follows that J_{pert} can be defined on $L^2((0,T)\times\Omega;\mathbb{R})$. We observe that J_{pert} is convex and lower semicontinuous in the third variable for a fixed $(p,q)\in X^d$ and $J_{\mathrm{pert}}(p,q,0) = -J_D(p,q)$.

We define a kind of a Fenchel–Young transform, $J^{\#}_{\mathrm{pert}}: X^d \times X \to \mathbb{R}$, with respect to a duality pairing for the space $L_2 = L^2((0,T)\times\Omega;\mathbb{R})$, by

$$\begin{aligned}J^{\#}_{\mathrm{pert}}(p,q,x) = \sup_{v\in L_2}\bigg\{&\int_0^T\int_\Omega \langle x(t,y), v(t,y)\rangle\,dy\,dt\\ &-\int_0^T\int_\Omega G^*(t,y,-(p_t(t,y)-q_y(t,y))+v(t,y))\,dy\,dt\bigg\}\\ &-\frac{1}{2}\int_0^T\int_\Omega |q(t,y)|^2\,dy\,dt + \frac{1}{2}\int_0^T\int_\Omega |p(t,y)|^2\,dy\,dt.\end{aligned}$$

Using properties of the Fenchel–Young transform and results concerning duality of convex integral functionals [17]

$$\begin{aligned}J^{\#}_{\mathrm{pert}}(p,q,x) = {}&\int_0^T\int_\Omega G(t,y,x(t,y))\,dy\,dt\\ &-\int_0^T\int_\Omega \langle x_t(t,y), p(t,y)\rangle\,dy\,dt + \frac{1}{2}\int_0^T\int_\Omega |p(t,y)|^2\,dy\,dt\\ &+\int_0^T\int_\Omega \langle x_y(t,y), q(t,y)\rangle\,dy\,dt - \frac{1}{2}\int_0^T\int_\Omega |q(t,y)|^2\,dy\,dt.\end{aligned}$$

We provide now two lemmas which will be used in the proof of the Duality Principle.

Lemma 3.1. *For any $x \in X$*

$$\inf_{p\in X_1^d}\sup_{q\in X_2^d} J^{\#}_{\mathrm{pert}}(p,q,x) = J(x).$$

Proof. It suffices to show that

$$\sup_{q\in X_2^d}\bigg\{\int_0^T\int_\Omega (\langle x_y(t,y),q(t,y)\rangle - |q(t,y)|^2)\,dy\,dt\bigg\} = \frac{1}{2}\int_0^T\int_\Omega |x_y(t,y)|^2\,dy\,dt$$

and

$$(3.1)\quad \inf_{p\in X_1^d}\bigg\{\int_0^T\int_\Omega (-\langle x_t(t,y),p(t,y)\rangle + |p(t,y)|^2)\,dy\,dt\bigg\} = -\frac{1}{2}\int_0^T\int_\Omega |x_t(t,y)|^2\,dy\,dt$$

using the structure of sets X_1^d and X_2^d. For this x there exists a pair $(p^x,q^x)\in X^d$ such that

$$x_t(t,y) = p^x(t,y),\quad x_y(t,y) = q^x(t,y).$$

In a consequence

$$\begin{aligned}
\frac{1}{2}&\int_0^T\int_\Omega |x_y(t,y)|^2\,dy\,dt\\
&= \int_0^T\int_\Omega \langle x_y(t,y),q^x(t,y)\rangle\,dy\,dt - \frac{1}{2}\int_0^T\int_\Omega |q^x(t,y)|^2\,dy\,dt\\
&\le \sup_{q\in X_2^d}\bigg\{\int_0^T\int_\Omega \langle x_y(t,y),q(t,y)\rangle\,dy\,dt - \frac{1}{2}\int_0^T\int_\Omega |q(t,y)|^2\,dy\,dt\bigg\}\\
&\le \frac{1}{2}\int_0^T\int_\Omega |x_y(t,y)|^2\,dy\,dt.
\end{aligned}$$

The relation (3.1) is now obvious. Thus we obtain the assertion of the lemma. □

Now by Fenchel–Young inequality it follows that

Lemma 3.2. *For any $(p,q)\in X^d$*

$$\inf_{x\in X} J^{\#}_{\mathrm{pert}}(p,q,x) \ge J_D(p,q).$$

From the definition of X^d it follows that in the above lemma we have equality for some $(p,q)\in X^d$, e.g. for

$$q(t,y) = x_y(t,y),\quad p_t(t,y) = q_y(t,y) - G_x(t,y,x(t,y))$$

4. Duality and Variational Principles

We now relate the critical values of both action functionals.

Theorem 4.1.

$$\inf_{x\in X} J(x) \geq \inf_{p\in X_1^d} \sup_{q\in X_2^d} J_D(p,q).$$

Proof. By Lemmas 3.1, 3.2 and since

$$\inf_{a\in A} \sup_{b\in B} I(a,b) \geq \sup_{b\in B} \inf_{a\in A} I(a,b)$$

we obtain

$$\begin{aligned}\inf_{x\in X} J(x) &= \inf_{x\in X} \inf_{p\in X_1^d} \sup_{q\in X_2^d} J^{\#}_{\mathrm{pert}}(p,q,x) \\ &\geq \inf_{p\in X_1^d} \sup_{q\in X_2^d} \inf_{x\in X} J^{\#}_{\mathrm{pert}}(p,q,x) \geq \inf_{p\in X_1^d} \sup_{q\in X_2^d} J_D(p,q). \qquad \square\end{aligned}$$

We observe that Theorem 4.1 is valid in case we consider minimizing sequences. Therefore we have

Corollary 4.2. *Let $\{x^j\} \subset X$ be a minimizing sequence for a functional J and let $\{p^j, q^j\} \subset X^d$ be the sequences corresponding to $\{x^j\}$ accordingly to the definition of the set X^d. Then*

$$\inf_{x^j\in X} J(x^j) \geq \inf_{p^j\in X_1^d} \sup_{q^j\in X_2^d} J_D(p^j,q^j). \tag{4.1}$$

Remark 4.3. Let us note that from the proof of the Theorem 4.1 and Lemmas 3.1 and 3.2 we can take in the right hand side of (4.1) for each p^j in "sup" only at most two values of q^j corresponding to p^j, accordingly to the definition of the set X^d. Therefore we have obvious inequalities

$$\begin{aligned}\inf_{x^j\in X} J(x^j) &\geq \inf_{p^j\in X_1^d} \sup_{q^j\in X_2^d} J_D(p^j,q^j) \\ &\geq \sup_{q^j\in X_2^d} \inf_{p^j\in X_1^d} J_D(p^j,q^j) \geq \liminf_j J_D(p^j,q^j).\end{aligned}$$

We state the necessary conditions. We observe that due to the construction of the set X and by Lemma 2.1 it follows that a minimizing sequence may be assumed, at least up to a subsequence, to be weakly convergent in U and strongly in U^1.

Theorem 4.4. *Let $\inf_{x\in X} J(x) = J(\overline{x})$, where $\overline{x} \in X$ is a limit, strong in U^1 and weak in U, of a minimizing sequence $\{x^j\} \subset X$. Then there exist $\overline{p} \in X_1^d$*

and $\overline{q} \in X_2^d$ *such that for a.e.* $(t,y) \in (0,T) \times \Omega$,

$$\overline{p}(t,y) = \overline{x}_t(t,y), \tag{4.2}$$

$$\overline{q}(t,y) = \overline{x}_y(t,y), \tag{4.3}$$

$$\overline{p}_t(t,y) - \overline{q}_y(t,y) + G_x(t,y,\overline{x}(t,y)) = 0, \tag{4.4}$$

and such that

$$J(\overline{x}) = J_D(\overline{p},\overline{q}).$$

Proof. Let $\overline{x} \in X$ be such that $J(\overline{x}) = \inf_{x^j \in X} J(x^j)$ and let $\{p^j, q^j\} \subset X^d$ denote the sequences corresponding to $\{x^j\}$ according to the definition of set X^d. We define

$$\overline{p}_t(t,y) = \overline{q}_y(t,y) - G_x(t,y,\overline{x}(t,y)),$$

for almost all $(t,y) \in (0,T) \times \Omega$ with $\overline{q}$ given by

$$\overline{q}(t,y) = \overline{x}_y(t,y).$$

It is clear that the above $(\overline{p},\overline{q})$ is a limit of a certain sequence of $\{p^j,q^j\} \in X^d$. We observe that there are two possible sequences $\{p^j,q^j\} \subset X^d$ corresponding to the sequence $\{x^j\}$ accordingly to the definition of the set X^d with $q^j = x^j_y$. Namely

$$q^j(t,y) = x^j_y(t,y), \quad p^j(t,y) = x^j_t(t,y) \tag{4.5}$$

or

$$p^j_t(t,y) = q^j_y(t,y) - G_x(t,y,x^j(t,y)), \quad q^j(t,y) = x^j_y(t,y). \tag{4.6}$$

First we investigate the convergence of both sequences. Since $\{x^j\}$ converges strongly in U^1 we get for the sequence (4.5)

$$x^j_t \to \overline{x}_t = \overline{p}, \quad x^j_y \to \overline{x}_y = \overline{q}.$$

Therefore system (4.2)–(4.4) is satisfied. Moreover, by a direct calculation, we have $J(\overline{x}) = J_D(\overline{p},\overline{q})$.

In case of sequence (4.6) we have similarly $q^j \to \overline{q} = \overline{x}_y$ and $q^j_y \rightharpoonup \overline{q}_y = \overline{x}_{yy}$ in L^2, possibly up to a subsequence. Moreover,

$$-(p^j_t(t,y) - q^j_y(t,y)) = G_x(t,y,x^j(t,y)). \tag{4.7}$$

From (4.7) we infer, that the sequence $\{p^j_t - q^j_y\}$ is bounded in L^∞ and so in L^2 and up to a subsequence it is also weakly convergent. Since x^j converges pointwisely to $\overline{x}$ therefore $\{p^j_t - q^j_y\}$ converges pointwisely to $-G_x(t,y,\overline{x}(t,y))$ also. We investigate the convergence of the sequence $\{p^j\}$. By (4.7) we get

$$p^j_t(t,y) = -G_x(t,y,x^j(t,y)) + q^j_y(t,y).$$

Hence $\{p_t^j\}$ and in a consequence $\{p^j\}$ are bounded in L^2. Therefore $\{p^j\}$ is convergent weakly to $\overline{p}$, possibly up to a subsequence. By Corollary 4.2 and Remark 4.3 we have

$$
\begin{aligned}
J(\overline{x}) &\geq \inf_{p^j\in X_1^d} \sup_{q^j\in X_2^d} J_D(p^j,q^j) \\
&\geq \sup_{q^j\in X_2^d} \inf_{p^j\in X_1^d} J_D(p^j,q^j) \geq \liminf_{j\to\infty} J_D(p^j,q^j).
\end{aligned}
\tag{4.8}
$$

We observe that

$$
\liminf_{j\to\infty} J_D(p^j,q^j) \geq J_D(\overline{p},\overline{q}).
\tag{4.9}
$$

Really, since $\{q^j\}$ is strongly convergent we have

$$
\begin{aligned}
&\liminf_{j\to\infty}\left(\frac{1}{2}\int_0^T\int_\Omega |p^j(t,y)|^2\,dy\,dt - \frac{1}{2}\int_0^T\int_\Omega |q^j(t,y)|^2\,dy\,dt\right)\\
&\quad= \liminf_{j\to\infty}\frac{1}{2}\int_0^T\int_\Omega |p^j(t,y)|^2\,dy\,dt - \lim_{j\to\infty}\frac{1}{2}\int_0^T\int_\Omega |q^j(t,y)|^2\,dy\,dt\\
&\quad\geq \frac{1}{2}\int_0^T\int_\Omega |\overline{p}(t,y)|^2\,dy\,dt - \frac{1}{2}\int_0^T\int_\Omega |\overline{q}(t,y)|^2\,dy\,dt.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\lim_{j\to\infty}\left(\int_0^T\int_\Omega G^*(t,y,-(p_t^j(t,y)-q_y^j(t,y)))\right)\\
&\qquad\qquad= \int_0^T\int_\Omega G^*(t,y,-(\overline{p}_t(t,y)-\overline{q}_y(t,y)))\,dy\,dt
\end{aligned}
\tag{4.10}
$$

Indeed, since

$$
\begin{gathered}
\int_0^T\int_\Omega G(t,y,x^j(t,y))\,dy\,dt \to \int_0^T\int_\Omega G(t,y,\overline{x}(t,y))\,dy\,dt,\\
\int_0^T\int_\Omega \langle p_t^j(t,y)-q_y^j(t,y),x^j(t,y)\rangle\,dy\,dt \to \int_0^T\int_\Omega \langle \overline{p}_t(t,y)-\overline{q}_y(t,y),\overline{x}(t,y)\rangle\,dy\,dt,
\end{gathered}
$$

and since

$$
\begin{aligned}
&-\int_0^T\int_\Omega G^*(t,y,-(p_t^j(t,y)-q_y^j(t,y)))\,dy\,dt\\
&\quad= \int_0^T\int_\Omega G(t,y,x^j(t,y))\,dy\,dt - \int_0^T\int_\Omega \langle p_t^j(t,y)-q_y^j(t,y),x^j(t,y)\rangle\,dy\,dt
\end{aligned}
$$

we have, using (4.4) together with properties of the Fenchel–Young, transform that (4.10) is satisfied. By (4.8), (4.9) it now follows that $J(\overline{x}) \geq J_D(\overline{p},\overline{q})$.

Moreover, by the definitions of J, J_D, relations (4.3), (4.4) and the Fenchel–Young inequality it follows that

$$\begin{aligned}
J(\overline{x}) &= \int_0^T \int_\Omega \left(\frac{1}{2}|\overline{x}_y(t,y)|^2 - \frac{1}{2}|\overline{x}_t(t,y)|^2 + G(t,y,\overline{x}(t,y))\right) dy\,dt \\
&\le -\int_0^T \int_\Omega \langle \overline{x}_t(t,y), \overline{p}(t,y)\rangle\, dy\,dt + \frac{1}{2}\int_0^T \int_\Omega |\overline{p}(t,y)|^2\, dy\,dt \\
&\quad + \int_0^T \int_\Omega \langle \overline{x}_y(t,y), \overline{q}(t,y)\rangle\, dy\,dt - \frac{1}{2}\int_0^T \int_\Omega |\overline{q}(t,y)|^2\, dy\,dt \\
&\quad + \int_0^T \int_\Omega G(t,y,\overline{x}(t,y))\, dy\,dt \\
&= -\int_0^T \int_\Omega G^*(t,y,-(\overline{p}_t(t,y) - \overline{q}_y(t,y)))\, dy\,dt \\
&\quad - \frac{1}{2}\int_0^T \int_\Omega |\overline{q}(t,y)|^2\, dy\,dt + \frac{1}{2}\int_0^T \int_\Omega |\overline{p}(t,y)|^2\, dy\,dt = J_D(\overline{p},\overline{q}).
\end{aligned}$$

Therefore we get that $J(\overline{x}) \le J_D(\overline{p},\overline{q})$. Thus $J(\overline{x}) = J_D(\overline{p},\overline{q})$, which reads

$$\begin{aligned}
\int_0^T \int_\Omega G^*(t,y,-(\overline{p}_t(t,y) - \overline{q}_y(t,y)))\, dy\,dt &+ \int_0^T \int_\Omega G(t,y,\overline{x}(t,y))\, dy\,dt \\
&+ \frac{1}{2}\int_0^T \int_\Omega |\overline{q}(t,y)|^2\, dy\,dt + \frac{1}{2}\int_0^T \int_\Omega |\overline{x}_y(t,y)|^2\, dy\,dt \\
&= \frac{1}{2}\int_0^T \int_\Omega |\overline{p}(t,y)|^2\, dy\,dt + \frac{1}{2}\int_0^T \int_\Omega |\overline{x}_t(t,y)|^2\, dy\,dt.
\end{aligned}$$

Therefore by (4.3), (4.4) and standard convexity arguments

$$\frac{1}{2}\int_0^T \int_\Omega |\overline{p}(t,y)|^2\, dy\,dt + \frac{1}{2}\int_0^T \int_\Omega |\overline{x}_t(t,y)|^2\, dy\,dt = \int_0^T \int_\Omega \langle \overline{x}_t(t,y), \overline{p}(t,y)\rangle.$$

Hence we obtain (4.2). $\square$

5. The proof of the main result

By definition of the set X we see that the functional J is bounded in X. Therefore we can choose in X a minimizing sequence for the functional J which we denote by $\{x^j\}$. Sequence $\{x^j\}$ has a subsequence denoted again by $\{x^j\}$ converging weakly in U and strongly in U^1. Therefore $\{x^j\}$ converges also strongly in $L^2((0,T)\times\Omega;\mathbb{R})$ to a certain element $\overline{x} \in U$. By construction of the set X we observe that $\overline{x} \in U$. Moreover, $\{x^j\}$ is also convergent almost everywhere and sequence $\{x_t^j\}$ is convergent strongly in $L^2((0,T)\times\Omega;\mathbb{R})$. So

$$\liminf_{j\to\infty} \inf J(x^j) \ge J(\overline{x}).$$

Hence $\inf_{x\in X} J(x^j) = J(\overline{x})$. Therefore by Theorem 4.1 we get the main result of the paper.

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