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FIXED POINT INDEX FOR KRASNOSEL'SKII-TYPE SET-VALUED MAPS ON COMPLETE ANRS

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Abstract. In the paper a fixed-point index for a class of the so-called Krasnoselskiĭ-type set-valued maps defined locally on arbitrary absolute neighbourhood retracts is presented. Various applications to the existence problems for constrained differential inclusions and equations are provided.

1. Introduction

Among many different generalizations of the Schauder and Banach fixed point principles the following result due to Krasnosel'skiĭ played an important role.

Theorem 1.1 (Krasnosel'skiĭ, [32]). *Let X be a nonempty closed and convex subset of a Banach space E. If $K : X \to E$ is a k-contraction (i.e. Lipschitz with constant $k \in [0,1)$), $C\colon X \to E$ is a compact map and, for all $x, y \in X$,*

$$(*)\qquad K(x) + C(y) \in X,$$

then there exists $x_0 \in X$ such that $x_0 = K(x_0) + C(x_0)$.

This result had lost some of its significance when Darbo and Sadovski introduced the concepts of a *k-set-contraction* and a *condensing* map. Namely,

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it appears that if K is a k-contraction and C is a compact map, then $K+C$ is a k-set contraction (with respect to the Kuratowski or Hausdorff measure of noncompactness). Hence, if X is additionally bounded, then the assertion of Theorem 1.1 holds true even when hypothesis $(*)$ is replaced by a weaker one: for each $x \in X$, $K(x) + C(x) \in X$ (1). However, it seems that the idea underlying the ingenious proof of Theorem 1.1 is still fruitful and has been used by different authors in order to establish some interesting generalizations of this result. These generalizations concern several directions: the authors consider less strict contraction properties (see e.g. [55]), weaker continuity properties (see e.g. [6]), more general operators (see e.g. [5], [41], [31], [11]), more general spaces (see e.g. [47], [50]) and also multivalued operators (see e.g. [43], [45], [46]) — see also references in these papers. There are also many interesting applications of Theorem 1.1 and the related results.

In the present paper we shall deal with an approach motivated to some extent by [41], [31] and that from [45], [46]. Namely it seems reasonable to consider, instead of the sum $K+C$, a composite map of the form $X \ni x \mapsto T(K(x), C(x))$, where a (usually nonlinear) operator $T\colon E\times E \to E$ replaces the sum $+\colon E\times E\to E$ and K, C are suitable set-valued maps. Our principal aim is to construct a homotopy invariant responsible for the existence of fixed points of (possibly) set-valued maps of the above or similar form that are defined locally on, no longer closed convex subsets of E, but on arbitrary complete absolute neighbourhood retracts.

The paper is organized as follows: in the rest of the first section we introduce some notation and preliminaries; in the second section we study the parameterized set-valued contractions (with non-convex values) and the properties of their fixed-point sets; in the third section we define the class of single- and set-valued Krasnosel'skiĭ-type maps and provide a construction of a homotopy invariant detecting their fixed points. The fourth section is devoted to applications, while in the additional fifth section we discuss some variants of the relevant fixed-point index theories and a general strong invariance result for constrained differential inclusions.

1.1. Preliminaries. Let X be a space (2). If $x \in X$ and $A \subset X$, then $d(x,A) := \inf_{a\in A} d(x,a)$ is the *distance* of x to A; for any $\varepsilon > 0$,

$$B(A,\varepsilon) := \{x \in X \mid d(x,A) < \varepsilon\}; \quad D(A,\varepsilon) := \{x \in Y \mid d(x,A) \le \varepsilon\}.$$

(1) In the setting of Theorem 1.1, Burton in [10] observed that condition $(*)$ may be relaxed by assuming that, for any $y \in X$, if $x = K(x) + C(y)$, then $x \in X$; the condition of Burton may still be relaxed by assuming for example that, for each $y \in X$, the map $K(\,\cdot\,) + C(y)$ is weakly inward on X.

(2) In what follows by a *space* we always mean a metric space; its distance will be denoted by d_X or, when it leads to no ambiguity, by d.

Given spaces X, Y, by a *set-valued map* Φ from X into Y (written $\Phi\colon X \multimap Y$) we mean a map which assigns to each $x \in X$ a *nonempty closed* subset $\Phi(x)$ of Y. If, for any closed (resp. open) set $U \subset Y$, the *preimage* $\Phi^{-1}(U) := \{x \in X : \Phi(x) \cap U \neq \emptyset\}$ is closed (resp. open), then we say that Φ is *upper semicontinuous* (resp. *lower semicontinuous*); Φ is *continuous* if it is upper and lower semicontinuous simultaneously. Recall that the *graph* $\mathrm{Gr}(\Phi) := \{(x,y) \in X \times Y \mid y \in \Phi(x)\}$ of an upper semicontinuous map Φ is closed; Φ is lower semicontinuous if and only if given $x \in X$, $y \in \Phi(x)$ and a sequence $x_n \to x$, there is a sequence $y_n \in \Phi(x_n)$ such that $y_n \to y$; Φ is upper semicontinuous and has compact values if and only if, for each $x \in X$ and a sequence $(x_n, y_n) \in \mathrm{Gr}(\Phi)$ such that $x_n \to x$, there exists a subsequence (y_{n_k}) such that $y_{n_k} \to y \in \Phi(x)$ (this means that the projection $p\colon \mathrm{Gr}(\Phi) \to X$ is *proper*, i.e. for each compact $K \subset X$, $p^{-1}(K)$ is compact). We say that $\Phi\colon X \multimap Y$ is *compact* if the *closure* $\mathrm{cl}\,\Phi(X)$ of the *image* $\Phi(X) := \bigcup_{x\in X}\Phi(x)$ is compact. If $X \subset Y$, then by $\mathrm{Fix}(\Phi) := \{x \in X \mid x \in \Phi(x)\}$ we denote the set of *fixed points* of Φ.

For a pair A, B of nonempty closed subsets of a space Y, the *Hausdorff distance*

$$\mathfrak{D}(A,B) := \max\{\mathfrak{d}(A,B), \mathfrak{d}(B,A)\} \leq \infty,$$

where $\mathfrak{d}(A,B) := \sup_{a\in A} d(a,B)$, is defined. It is well-known that $\mathfrak{D}$ is a metric in the (hyper)space $\mathcal{B}(Y)$ of all nonempty bounded closed subsets of Y. This metric is complete provided so is the metric in Y. We say that a map $\Phi\colon X \multimap Y$ is *H-continuous* if it is continuous with respect to the distance $\mathfrak{D}$ in Y, i.e. for each $x_0 \in X$, given $\varepsilon > 0$, there is $\delta > 0$ such that, for any $x \in X$, $\mathfrak{D}(\Phi(x), \Phi(x_0)) < \varepsilon$ provided $d(x, x_0) < \delta$. A set-valued map $F\colon X \multimap Y$ is *k-Lipschitz*, where $k \geq 0$, if for all $x, y \in X$,

$$\mathfrak{D}(F(x), F(y)) \leq kd(x,y).$$

If $k < 1$ then F is called a *set-valued k-contraction* or, simply, a *contraction*.

Let $\mathcal{C}$ be a subclass of the class of all (metric) spaces. A (nonempty) space X is an *absolute neighbourhood extensor* (resp. *absolute extensor*) *with respect to* $\mathcal{C}$ (written $X \in \mathrm{ANE}(\mathcal{C})$ (resp. $X \in \mathrm{AE}(\mathcal{C})$)) if, given a space $Z \in \mathcal{C}$ and its closed subset Z_0, any continuous map $f_0\colon Z_0 \to X$ admits a continuous extension onto a (open) neighbourhood U of Z_0 in Z (resp. onto Z), i.e. there is a map $f\colon U \to X$ (resp. $f\colon Z \to X$) such that $f(z) = f_0(z)$ for all $z \in Z_0$. Clearly $\mathrm{AE}(\mathcal{C}) \subset \mathrm{ANE}(\mathcal{C})$. If $\mathcal{C}$ is the class of *all* spaces, then $\mathrm{ANE}(\mathcal{C})$ (resp. $\mathrm{AE}(\mathcal{C})$) coincides with the class ANR of *absolute neighbourhood retracts* (resp. AR of *absolute retracts*). Observe that, for any class $\mathcal{C}$, $\mathrm{ANR} \subset \mathrm{ANE}(\mathcal{C})$ (resp. $\mathrm{AR} \subset \mathrm{AE}(\mathcal{C})$) and if $X \in \mathrm{ANE}(\mathcal{C})$ (resp. $X \in \mathrm{AE}(\mathcal{C})$) is a closed subset a space $Y \in \mathcal{C}$, then X is a neighbourhood retract (resp. retract) of Y; hence if $Y \in \mathrm{ANR}$ (resp. $Y \in \mathrm{AR}$), then $X \in \mathrm{ANR}$ (resp. $X \in \mathrm{AR}$). In particular, by the Urysohn embedding theorem, if a compact space $X \in \mathrm{ANE}(\mathcal{C}omp)$ (resp. $X \in \mathrm{AE}(\mathcal{C}omp)$), where $\mathcal{C}omp$ stands for the class of compact spaces, then $X \in \mathrm{ANR}$ (resp. $X \in \mathrm{AR}$).

2. Set-valued contractions

The well-known Covitz–Nadler theorem, generalizing the Banach fixed point principle, asserts that any set-valued contraction $F\colon X \multimap X$, defined on a complete space X, admits a fixed point. However, contrary to the single-valued case, the fixed-point set $\mathrm{Fix}(F)$ does not need to be a singleton and, hence, it is of interest to study the structure of this set (see e.g. [49]). It is clear that $\mathrm{Fix}(F)$ is always closed and compact if so are the values of F. The important result due to Ricceri [44] states that if X is a convex closed subset of a Banach space and values of F are convex, then $\mathrm{Fix}(F) \in \mathrm{AR}$. More generally (see e.g. [20, Theorem 56] and comp. [48]):

Theorem 2.1. *Let Y be a space and X be a closed convex subset of a Banach space. Suppose that $F\colon X\times Y \multimap X$ has closed convex values, for each $y \in Y$, the map $F(\,\cdot\,,y)\colon X \multimap X$ is a $k(y)$-contraction, where $k\colon Y \to [0,1)$ is continuous. If, for each $x \in X$, $F(x,\,\cdot\,)\colon Y \multimap X$ is lower semicontinuous, then:*

- (a) *given a space Z, a closed subset $Z_0 \subset Z$ and a continuous map $g\colon Z \to Y$, any continuous map $f_0\colon Z_0 \to X$ such that, for $z \in Z_0$, $f_0(z) \in \mathrm{Fix}(F(\,\cdot\,,g(z))) := \{x \in X \mid x \in F(x,g(z))\}$, admits a continuous extension $f\colon Z \to X$ such that $f(z) \in \mathrm{Fix}(F(\,\cdot\,,g(z)))$;*
- (b) *if the graph of the map $Y \ni y \mapsto\!\!\multimap \mathrm{Fix}(F(\,\cdot\,,y))$ is closed ([3]), then there is a continuous map $r\colon X \times Y \to X$ such that, for each $y \in Y$, $r(\,\cdot\,,y)$ is a retraction of X onto $\mathrm{Fix}(F(\,\cdot\,,y))$;*
- (c) *the map $Y \ni y \mapsto\!\!\multimap \mathrm{Fix}(F(\,\cdot\,,y))$ admits a continuous selection ([4]).*

In fact [20] shows the second assertion assuming that $F(x,\,\cdot\,)$ is continuous for all $x \in X$. The first assertion may be shown similarly as in [27]; the second and the third assertions follow from the first one (comp. Proposition 2.4).

If above, for all $x \in X$, $F(x,\,\cdot\,)$ is only upper semicontinuous, then the assertion of Theorem 2.1 does not hold true. To see this let $X = Y = \mathbb{R}$ and, for any $(x,y) \in \mathbb{R}^2$, let

$$F(x,y) = \begin{cases} \{0\} & \text{if } y < 0,\\ [0,1] & \text{if } y = 0,\\ \{1\} & \text{if } y > 0.\end{cases}$$

Then F satisfies the above assumptions, $F(x,\,\cdot\,)$ is upper semicontinuous, but the map $Y \ni y \mapsto\!\!\multimap \mathrm{Fix}(F(\,\cdot\,,y))$ has no continuous selections.

([3]) This holds e.g. if, for all $x \in X$, $F(x,\,\cdot\,)$ is continuous.

([4]) I.e. there is a map $t\colon Y \to X$ such that $t(y) \in \mathrm{Fix}(F(\,\cdot\,,y))$ for all $y \in Y$.

The result of Ricceri has been generalized by Górniewicz, Marano and Ślosarski in [27]. Since we shall make use of this result, let us recall some of its relevant issues. Suppose that $\mathcal{C}$ denotes a subclass of the class of all (metric) spaces and let X be a space. According to [27] a lower semicontinuous map $\Phi: X \multimap X$ is said to have the *selection property with respect to* $\mathcal{C}$ (written $\Phi \in \mathrm{SP}(X;\mathcal{C})$) whenever given a space $Z \in \mathcal{C}$, a continuous map $f: Z \to X$, a continuous function $\varepsilon: Z \to (0,\infty)$ such that

$$\Psi(z) := \mathrm{cl}\,[\Phi(f(z)) \cap B(f(z),\varepsilon(z))] \neq \emptyset$$

for all $z \in Z$, and a closed nonempty subset $Z_0 \subset Z$, every continuous selection $g_0: Z_0 \to X$ of $\Psi|_{Z_0}$ admits a continuous extension $g: Z \to X$ being a selection of Ψ. If $\mathcal{C}$ is the class of all spaces, then we write $\Phi \in \mathrm{SP}(X)$.

It appears (see [27]) that if a complete space $X \in \mathrm{AE}(\mathcal{C})$, $F \in \mathrm{SP}(X;\mathcal{C})$ is a set-valued contraction, then $\mathrm{Fix}(F) \in \mathrm{AE}(\mathcal{C})$. This result generalizes the Ricceri theorem since, as it is easy to see, if X is a convex closed subset of a Banach space, then $X \in \mathrm{AR}$ and any contraction $F: X \multimap X$ with convex values belongs to $\mathrm{SP}(X)$ in view of the Michael theorem. Moreover, in view of the Bressan, Colombo and Fryszkowski theorem (see [9], [19]), if X is a closed subset of the space $L^p(J,E)$ (of p-Bochner integrable functions defined on a finite interval J with values in a Banach space E, $1 \le p < \infty$) and a lower semicontinuous map $\Phi: X \multimap X$ has decomposable values ([5]), then $\Phi \in \mathrm{SP}(X;\mathcal{S})$ where $\mathcal{S}$ stands for the class of all separable spaces. In particular, if X is a retract in $L^p(J,E)$ and $F: X \multimap X$ is a contraction with decomposable values, then $\mathrm{Fix}(F) \in \mathrm{AR}$. Some other examples of maps having the selection property are provided in [27] (see also [24]). It is not difficult to obtain a result that generalize the above mentioned result in a way Theorem 2.1 generalizes the result of Ricceri (comp. [20, Theorem 57] for the decomposable case).

In what follows we shall also study compositions of contractions. For example suppose X_1, X_2 are spaces and let $F_1: X_1 \multimap X_2$, $F_2: X_2 \multimap X_1$ be k_1- (resp k_2-) Lipschitz maps. It is clear that the compositions $F_2 \circ F_1: X_1 \multimap X_1$ and $F_1 \circ F_2: X_2 \multimap X_2$ are k_1k_2-Lipschitz and their *mixed Cartesian product* $F_1 \otimes F_2: X_1 \times X_2 \multimap X_1 \times X_2$, defined by the formula $F_1 \otimes F_2(x_1,x_2) := F_2(x_2) \times F_1(x_1)$ for $x_1 \in X_1$, $x_2 \in X_2$, is $\max\{k_1,k_2\}$-Lipschitz ([6]). Observe that $x_1 \in \mathrm{Fix}(F_2 \circ F_1)$ if and only if there is $x_2 \in F_1(x_1)$ such that $x_1 \in F_2(x_2)$, i.e. if and only if there is $x_2 \in X_2$ such that $(x_1,x_2) \in \mathrm{Fix}(F_1 \otimes F_2)$. Hence $\mathrm{Fix}(F_2 \circ F_1) = p_1(\mathrm{Fix}(F_1 \otimes F_2))$ where $p_1: X_1 \times X_2 \to X_1$ is the projection. In particular, if X_1, X_2 are closed convex subsets of some Banach spaces, $F_1: X_1 \multimap X_2$ and $F_2: X_2 \multimap X_1$ are

([5]) Recall that a nonempty set $D \subset L^p(J,E)$ is *decomposable* if, given $u, v \in D$ and a measurable $I \subset J$, $\chi_I u + \chi_{J\setminus I} v \in D$ where χ_I denotes the characteristic function of I.

([6]) We consider a metric $d_{X_1\times X_2}((x_1,x_2),(x_1',x_2')) := \max\{d_{X_1}(x_1,x_1'), d_{X_2}(x_2,x_2')\}$ in $X_1 \times X_2$.

contractions with convex values, then so is $F_1 \otimes F_2$ and therefore, in view of the Ricceri theorem, $\mathrm{Fix}(F_1 \otimes F_2) \in \mathrm{AR}$; hence $\mathrm{Fix}(F_2 \circ F_1)$ is a continuous image of a complete AR; moreover, if $F_1: X_1 \times Y \multimap X_2$ and $F_2: X_2 \times Y \multimap X_1$, where Y is a space, satisfy conditions of Theorem 2.1, then the map $Y \ni y \mapsto\!\!\circ\ \mathrm{Fix}(F_2(F_1(\,\cdot\,, y) \times \{y\}))$ admits a selection.

However, these arguments do not help much to characterize the set $\mathrm{Fix}(F_2 \circ F_1)$ in case $k_1 k_2 < 1$ (but $k_1 \geq 1$ or $k_2 \geq 1$) and do not imply the existence of selections in case, for each $x_1 \in X_1$, $x_2 \in X_2$, either $F_1(x_1, \cdot\,)$ or $F_2(x_2, \cdot\,)$ is not lower semicontinuous.

In order to get a result in this direction we shall introduce the following definition.

Definition 2.2. Let $\mathcal{C}$ be a subclass of the class of all spaces and let X_1, X_2 be spaces. A map $\Phi: X_1 \multimap X_2$ is said to have the *selection property with respect to* $\mathcal{C}$ (written $\Phi \in \mathrm{SP}(X_1, X_2; \mathcal{C})$) whenever given a space $Z \in \mathcal{C}$, continuous maps $f_1: Z \to X_1$, $f_2: Z \to X_2$, a continuous function $\varepsilon: Z \to (0, \infty)$ such that, for all $z \in Z$,

$$\Psi(z) := \mathrm{cl}\,[\Phi(f_1(z)) \cap B(f_2(z), \varepsilon(z))] \neq \emptyset,$$

and a closed nonempty subset $Z_0 \subset Z$, every continuous selection $g_0: Z_0 \to X_2$ of $\Psi|_{Z_0}$ admits a continuous extension $g: Z \to X_2$ such that g is a selection of Ψ. If $\mathcal{C}$ is the class of all spaces, then we write $\Phi \in \mathrm{SP}(X_1, X_2)$.

Remark 2.3. (a) Again in view of the Michael theorem, if X_1 is a space, X_2 is a closed convex subset of a Banach space and $\Phi: X_1 \multimap X_2$ is lower semicontinuous with convex values, then $\Phi \in \mathrm{SP}(X_1, X_2)$. If X_2 is a closed subset of $L^p(J, E)$ and Φ is lower semicontinuous with decomposable values, then, in view of the Bressan–Colombo results (see [9]), $\Phi \in \mathrm{SP}(X_1, X_2; \mathcal{S})$ where $\mathcal{S}$ denotes the class of separable spaces.

(b) It is easy to see that if $\Phi_1 \in \mathrm{SP}(X_1, X_2; \mathcal{C})$ and $\Phi_2 \in \mathrm{SP}(X_2, X_1; \mathcal{C})$, then $\Phi_1 \otimes \Phi_2 \in \mathrm{SP}(X_1 \times X_2; \mathcal{C})$.

(c) If $\Phi \in \mathrm{SP}(X_1, X_2; \mathcal{C})$ is H-continuous, then it has the following property:

(∗) Given $Z \in \mathcal{C}$, a continuous $f: Z \to X_1$ and a closed $Z^* \subset Z$, any continuous selection $g^*: Z^* \to X_2$ of the restriction $(\Phi \circ f)|_{Z^*}$ admits a continuous extension $g: Z \to X_2$ such that $g(z) \in \Phi(f(z))$ for all $z \in Z$.

To see that take any continuous $h: Z \to X_2$ and observe that the function

$$\varepsilon'(z) := d(h(z), \Phi(f(z))) + 1, \quad z \in Z,$$

is continuous; let $\varepsilon'': Z \to [0, \infty)$ be any continuous extension of the function $Z^* \ni z \mapsto d(g^*(z), h(z)) + 1$ and let $\varepsilon(z) := \max\{\varepsilon'(z), \varepsilon''(z)\}$ for $z \in Z$. Then $\Psi(z) := \mathrm{cl}\,[\Phi(f(z)) \cap B(h(z), \varepsilon(z))] \neq \emptyset$ on Z and $g^*(z) \in \Psi(z)$ on Z^*. Hence, by Definition 2.2, the assertion follows.

Remark 2.3(c) says that Φ has a certain selection property and, being H-continuous, has the closed graph. To understand better the structure of the above defined family $\mathrm{SP}(X_1, X_2, \mathcal{C})$ let us mention the following result.

Proposition 2.4. *Let $\mathcal{C}$ be a multiplicative class of spaces* (7) *and let $X_1, X_2 \in \mathcal{C}$. If a map $\Phi: X_1 \multimap X_2$ has property* $(*)$ *and* $\mathrm{Gr}(\Phi)$ *is closed, then:*

(a) *there is retraction $R: X_1 \times X_2 \to \mathrm{Gr}(\Phi)$ such that $R(x_1, x_2) \in \{x_1\} \times \Phi(x_1)$ for all $(x_1, x_2) \in X_1 \times X_2$;*
(b) *there is a normed space E, a closed continuous embedding $i: X_2 \to E$ and a continuous map $r: X_1 \times E \to E$ such that $r(x_1, \cdot\,)$ is a retraction from E on $i(\Phi(x_1))$ for all $x_1 \in X_1$ (in particular, $\Phi(x) \in \mathrm{AR}$ for all $x \in X_1$);*
(c) *Φ is lower semicontinuous.*

On the other hand, if condition (b) *is satisfied, (or $X_2 \in \mathrm{AE}(\mathcal{C})$ and condition* (a) *holds), then Φ has property* $(*)$ *and* $\mathrm{Gr}(\Phi)$ *is closed.*

Proof. Suppose that Φ has property $(*)$ and $\mathrm{Gr}(\Phi)$ is closed. Let $Z := X_1 \times X_2$, $Z_0 := \mathrm{Gr}(\Phi)$ and let $f: X_1 \times X_2 \to X_1 \times X_2$, $g_0: Z_0 \to X_1 \times X_2$ be given by: $f(x_1, x_2) := (x_1, x_1)$ for $x_1 \in X_1$, $x_2 \in X_2$ and $g_0(x_1, x_2) = (x_1, x_2)$ for $(x_1, x_2) \in Z_0$. It is a matter of a routine calculation to show that the map $\Psi: X_1 \times X_1 \multimap X_1 \times X_2$ given by $\Psi(x_1, x_1') = \{x_1\} \times \Phi(x_1')$ has property $(*)$, i.e. there is a continuous $g: Z \to X_1 \times X_2$ such that $g|_{Z_0} = g_0$ and $g(x_1, x_2) \in (\Psi \circ f)(x_1, x_2) = \{x_1\} \times \Phi(x_1)$ for all $x_1 \in X_1$, $x_2 \in X_2$. Hence a map $R: X_1 \times X_2 \to \mathrm{Gr}(\Phi)$ such that $R(x_1, x_2) = g(x_1, x_2)$ on $X_1 \times X_2$ is a retraction.

Consider a closed continuous embedding $i: X_2 \to E$, where E is a normed space, provided by the Arens-Eells theorem and a map $\Phi' := i \circ \Phi: X_1 \multimap E$. It is clear that the graph of Φ' is closed and Φ' has property $(*)$. Therefore, by (a), there is a retraction $R': X_1 \times E \to \mathrm{Gr}(\Phi')$ such that $R'(x_1, u) \in \{x_1\} \times \Phi'(x_1)$ for all $x_1 \in X_1$ and $u \in E$. Let $p: X_1 \times E \to E$ be the projection and, for $x_1 \in X_1$, $u \in E$, let

$$r(x_1, u) := p(R(x_1, u)).$$

Clearly $r(x_1, u) \in \Phi'(x_1)$ on $X_1 \times E$ and, if $u \in \Phi'(x_1)$, then $r(x_1, u) = p(R(x_1, u)) = u$.

Let $x_1 \in X_1$, $x_2 \in \Phi(x_1)$ and a sequence $x_1^n \to x_1$. In view of assertion (a), there is a sequence $x_2^n := p_2(R(x_1^n, x_2))$ such that $x_2^n \in \Phi(x_1^n)$ and $x_2^n \to x_2$, where $p_2: \mathrm{Gr}(\Phi) \to X_2$ is the projection. Therefore Φ is lower semicontinuous map.

(7) I.e. if $X_1, X_2 \in \mathcal{C}$, then $X_1 \times X_2 \in \mathcal{C}$.

Conversely suppose that condition (a) or (b) is satisfied. If condition (b) holds, then a map $R: X_1 \times X_2 \to X_1 \times X_2$ given by

$$R(x_1, x_2) := (x_1, i^{-1}(r(x_1, i(x_2)))),$$

$x_1 \in X_1$, $x_2 \in X_2$, is retraction from $X_1 \times X_2$ onto $\mathrm{Gr}(\Phi)$. Hence, in the both cases (a) and (b), $\mathrm{Gr}(\Phi)$ is closed as a retract of $X_1 \times X_2$.

Let $Z \in \mathcal{C}$, $Z_0 \subset Z$ is closed, $f: Z \to X_1$, $g_0: Z_0 \to X_2$ are continuous and $g_0(z) \in \Phi(f(z))$ on Z_0. If condition (a) is satisfied and, additionaly, $X_2 \in \mathrm{AE}(\mathcal{C})$, then we may define $g(z) := p_2(R(f(z), g_1(z)))$ for $z \in Z$, where $g_1: Z \to X_2$ is an arbitrary continuous extension of g_0. If condition (b) is satisfied, then we may define $g(z) := i^{-1}(r(f(z), g_2(z)))$ for $z \in Z$, where $g_2: Z \to E$ is an arbitrary continuous extension of $i \circ g_0$. In the both cases g is a (continuous) selection of $\Phi \circ f$ and $g|_{Z_0} = g_0$. □

Let us return to the problem of a characterization of the fixed-point set of a set-valued contraction of the form $F_2 \circ F_1$. In view of the discussion preceding Definition 2.2 and Remark 2.3(b) we get immediately the following result.

Proposition 2.5. *If complete spaces* $X_1, X_2 \in \mathrm{AE}(\mathcal{C})$ (*where a class* $\mathcal{C}$ *of spaces is arbitrary*), $F_1: X_1 \multimap X_2$, $F_2: X_2 \multimap X_1$ *are contractions such that* $F_1 \in \mathrm{SP}(X_1, X_2; \mathcal{C})$, $F_2 \in \mathrm{SP}(X_2, X_1; \mathcal{C})$, *then* $\mathrm{Fix}(F_1 \otimes F_2) \in \mathrm{AE}(\mathcal{C})$ *and* $\mathrm{Fix}(F_2 \circ F_1)$ *is a continuous image of a complete* $\mathrm{AE}(\mathcal{C})$, *namely:*

$$\mathrm{Fix}(F_2 \circ F_1) = p_1(\mathrm{Fix}(F_1 \otimes F_2))$$

where $p_1: X_1 \times X_2 \to X_1$ *is the projection onto the first factor.*

We shall address the similar question assuming instead that F_1 and F_2 are Lipschitz maps and such that the composition $F_2 \circ F_1$ is a contraction. We get a result even more general. Assume that the following general conditions are met:

Assumption 2.6.

(a) $X_0, \ldots, X_{m-1}$ are complete spaces and $X_m := X_0$;
(b) there is $j \in \{0, \ldots, m-1\}$ such that $X_j \in \mathrm{AE}(\mathcal{C})$ where $\mathcal{C}$ is a certain class of spaces;
(c) for each $i = 0, \ldots, m-1$, $F_i \in \mathrm{SP}(X_i, X_{i+1}; \mathcal{C})$ is a Lipschitz set-valued map with constant $k_i > 0$ and the product $k := k_0 \ldots k_{m-1} < 1$;
(d) $F := F_{m-1} \circ F_{m-2} \circ \ldots \circ F_0: X_0 \multimap X_m$ (according to our terminology and notation we assume silently that F has closed values).

It is convenient to observe that all spaces and maps above are indexed by the group $\mathbb{Z}_m = \{0, \ldots, m-1\}$ of integers modulo m. In what follows the *addition of indices in* $\mathbb{Z}_m$ *is performed modulo* m.

Theorem 2.7. *Under Assumption* 2.6 *the set*

$$S := \{(x_0, \ldots, x_{m-1}) \in X_0 \times \ldots \times X_{m-1} \mid x_{i+1} \in F_i(x_i),\ i \in \mathbb{Z}_m\}$$

is a nonempty absolute extensor with respect to the class $\mathcal{C}$ *and, moreover,* $\mathrm{Fix}(F) = p_0(S)$ *where* $p_0\colon \prod_{i\in\mathbb{Z}_m} X_i \to X_0$ *is the projection.*

Proof. Let us first introduce some notation. Namely let

$$\Gamma := \{(x_0, \ldots, x_{m-1}) \in X_0 \times \ldots \times X_{m-1} \mid x_{i+1} \in F_i(x_i),\ i \in \mathbb{Z}_m,\ i \neq j\}$$

and let, for $i \in \mathbb{Z}_m$, $p_i\colon \Gamma \to X_i$ be the projection. It is clear that Γ is closed and nonempty since

$$S = \{(x_0, \ldots, x_{m-1}) \in \Gamma \mid x_{j+1} \in F_j(x_j)\} \quad \text{and} \quad p_0(S) = \mathrm{Fix}(F) \neq \emptyset.$$

For technical reasons, let us introduce a new metric d on Γ given by

$$d(\gamma, \gamma') = \max\{\alpha_i d_i(p_i(\gamma), p_i(\gamma')) \mid i \in \mathbb{Z}_m\}, \quad \gamma, \gamma' \in \Gamma,$$

where, for $i \in \mathbb{Z}_m$, d_i stands for the metric in X_i and (remembering that $j \in \mathbb{Z}_m$ is fixed and such that $X_j \in \mathrm{AE}(\mathcal{C})$)

$$\alpha_{j+1} = k_{j+1}, \quad \alpha_{j+2} = 1, \quad \alpha_{j+3} = k_{j+2}^{-1},$$
$$\alpha_{j+s} = (k_{j+2} \ldots k_{j+s-1})^{-1} \quad \text{for } 4 \leq s \leq m.$$

It is clear that d is uniformly equivalent to the usual max-metric on Γ; hence the space Γ with the metric d is complete. Moreover, for any $\gamma, \gamma' \in \Gamma$ and $i \in \mathbb{Z}_m$,

$$d_i(p_i(\gamma), p_i(\gamma')) \leq \alpha_i^{-1} d(\gamma, \gamma'). \tag{2.1}$$

In order to prove that $S \in \mathrm{AE}(\mathcal{C})$, take an arbitrary space $Z \in \mathcal{C}$, a closed $Z^* \subset Z$ and a continuous map $f^*\colon Z^* \to S$. We are to show the existence of a continuous extension $f\colon Z \to S$ of f^*. To this aim let $1 < c < k^{-1}$. We shall now construct a sequence $(f_n)_{n=0}^{\infty}$, of maps $f_n\colon Z \to \Gamma$, such that:

$$f_n|_{Z^*} = f^* \qquad \text{for } n \geq 0, \tag{2.2}$$

$$p_{j+1} \circ f_n(z) \in F_j(p_j \circ f_{n-1}(z)) \qquad \text{for } n \geq 1 \text{ and } z \in Z, \tag{2.3}$$

$$d(f_n(z), f_{n-1}(z)) < k^{n-1} d(f_1(z), f_0(z)) + c^{1-n} \qquad \text{for } n \geq 1 \text{ and } z \in Z. \tag{2.4}$$

Observe that, for any $z \in Z^*$, $f^*(z) = (p_0(f^*(z)), \ldots, p_{m-1}(f^*(z)))$ and, since $X_j \in \mathrm{AE}(\mathcal{C})$, there is an extension $\widetilde{f}_0\colon Z \to X_j$ of $p_j \circ f^*\colon Z^* \to X_j$. Since, for $z \in Z^*$, $f^*(z) \in S$, we have $p_{j+1}(f^*(z)) \in F_j(p_j \circ f^*(z)) = F_j(\widetilde{f}_0(z))$. Hence, in view of Remark 2.3(c), there is a continuous extension $h_0^{j+1}\colon Z \to X_{j+1}$ of $p_{j+1} \circ f^*$ such that $h_0^{j+1}(z) \in F_j(\widetilde{f}_0(z))$ for all $z \in Z$. For $z \in Z^*$, $p_{j+2} \circ f^*(z) \in F_{j+1}(p_{j+1} \circ f^*(z)) = F_{j+1}(h_0^{j+1}(z))$ and, again by Remark 2.3(c), there is an extension $h_0^{j+2}\colon Z \to X_{j+2}$ of $p_{j+2} \circ f^*$ such that $h_0^{j+2}(z) \in F_{j+1}(h_0^{j+1}(z))$ for all $z \in Z$. Suppose that, for some $s \in \mathbb{Z}_m$, $s \geq 2$, a continuous extensions

$h_0^{j+s}\colon Z \to X_{j+s}$ of $p_{j+s} \circ f^*$ such that $h_0^{j+s}(z) \in F_{j+s-1}(h_0^{j+s-1}(z))$ for $z \in Z$ has been constructed. Since, for $z \in Z^*$, $p_{j+s+1} \circ f^*(z) \in F_{j+s}(p_{j+s} \circ f^*(z)) = F_{j+s}(h_0^{j+s}(z))$, by Remark 2.3(c), there is a continuous extension $h_0^{j+s+1}\colon Z \to X_{j+s+1}$ of $p_{j+s+1} \circ f^*$ such that, for $z \in Z$, $h_0^{j+s+1}(z) \in F_{j+s}(h_0^{j+s}(z))$ (8). In this way we have produced a family of continuous extensions $h_0^i\colon Z \to X_i$ of $p_i \circ f^*$, $i \in \mathbb{Z}_m$. For $z \in Z$, let

$$f_0(z) := (h_0^0(z), \dots , h_0^{m-1}(z)).$$

It is clear that, for $i \in \mathbb{Z}_m$, $i \neq j$, and $z \in Z$, $h_0^{i+1}(z) \in F_i(h_0^i(z))$, i.e. $f_0\colon Z \to \Gamma$ and, for $z \in Z^*$, $f_0(z) = f^*(z)$.

For $z \in Z^*$, $p_{j+1} \circ f^*(z) \in F_j(p_j \circ f^*(z)) = F_j(p_j \circ f_0(z))$. Therefore, by Remark 2.3(c) again, there is a continuous extension $h_1^{j+1}\colon Z \to X_{j+1}$ of $p_{j+1} \circ f^*$ such that $h_1^{j+1}(z) \in F_j(p_j \circ f_0(z))$. Similarly as above we see that, for $z \in Z^*$, $p_{j+2} \circ f^*(z) \in F_{j+1}(p_{j+1} \circ f^*(z)) = F_{j+1}(h_1^{j+1}(z))$, so there is a continuous extension $h_1^{j+2}\colon Z \to X_{j+2}$ of $p_{j+2} \circ f^*$ such that $h_1^{j+2}(z) \in F_{j+1}(h_1^{j+1}(z))$ for $z \in Z$. Continuing this procedure as above we produce map $h_1^i\colon Z \to X_i$, $i \in \mathbb{Z}_m$, such that if $i \neq j$, then $h_1^{i+1}(z) \in F_i(h_1^i(z))$ (and $h_1^{j+1}(z) \in F_j(p_j \circ f_0(z)))$ on Z. Therefore, the formula

$$f_1(z) := (h_1^0(z), \dots , h_1^{m-1}(z)), \quad z \in Z,$$

correctly defines a continuous map $f_1\colon Z \to \Gamma$ which, together with f_0, satisfies conditions (2.1)–(2.3).

Assume that, for some $n \geq 1$, continuous maps $f_0, \dots , f_n$ satisfying conditions (2.1)–(2.3) have been constructed. Therefore, for $z \in Z$,

$$\begin{aligned} d_{j+1}&(p_{j+1}(f_n(z)), F_j(p_j \circ f_n(z))) \\ &\leq \mathfrak{D}(F_j(p_j \circ f_{n-1}(z)), F_j(p_j \circ f_n(z))) \leq k_j d_j(p_j(f_n(z)), p_j(f_{n-1}(z))) \\ &\leq \alpha_j^{-1} k_j d(f_n(z), f_{n-1}(z)) = \frac{k}{k_{j+1}} d(f_n(z), f_{n-1}(z)) \\ &< \frac{k}{k_{j+1}} k^{n-1} d(f_1(z), f_0(z)) + \frac{k}{k_{j+1}} c^{1-n} =: \eta(z). \end{aligned}$$

Let $\eta_0, \dots , \eta_{m-1}\colon Z \to (0, \infty)$ be continuous functions such that

$$\frac{k}{k_{j+1}} d(f_n(z), f_{n-1}(z)) < \eta_0(z) < \eta_1(z) < \dots < \eta_{m-1}(z) < \eta_m(z) := \eta(z).$$

We thus get that, for $z \in Z$,

$$F_j(p_j \circ f_n(z)) \cap B_{j+1}(p_{j+1} \circ f_n(z), \eta_0(z)) \neq \emptyset \ (9).$$

(8) Remember that addition of indices is performed modulo m.

(9) Here and in what follows, $B_i(x_i, r)$, $i \in \mathbb{Z}_m$, denotes the open ball of radius r in X_i around $x_i \in X_i$.

Since $F_j \in \mathrm{SP}(X_j, X_{j+1}; \mathcal{C})$ and, for $z \in Z^*$,

$$p_{j+1} \circ f^*(z) \in \mathrm{cl}\,[F_j(p_j \circ f_n(z)) \cap B_{j+1}(p_{j+1} \circ f_n(z), \eta_0(z))],$$

we infer that there is a continuous extension $h_{n+1}^{j+1}: Z \to X_{j+1}$ of $p_{j+1} \circ f^*$ such that, for each $z \in Z$,

$$\begin{aligned} h_{n+1}^{j+1}(z) \in \mathrm{cl}\,[F_j(p_j \circ f_n(z)) \cap B_{j+1}(p_{j+1} \circ f_n(z), \eta_0(z))] \\ \subset F_j(p_j \circ f_n(z)) \cap B_{j+1}(p_{j+1} \circ f_n(z), \eta_1(z)). \end{aligned}$$

In particular, for $z \in Z$,

$$d_{j+1}(h_{n+1}^{j+1}(z), p_{j+1} \circ f_n(z)) < \eta_1(z) < \eta(z).$$

Since $f_n(z) \in \Gamma$, $p_{j+2}(f_n(z)) \in F_{j+1}(p_{j+1} \circ f_n(z))$ on Z. Hence, on Z,

$$\begin{aligned} d_{j+2}(p_{j+2}(f_n(z)), F_{j+1}(h_{n+1}^{j+1}(z)) &\le \mathfrak{D}(F_{j+1}(p_{j+1} \circ f_n(z)), F_{j+1}(h_{n+1}^{j+1}(z))) \\ &\le k_{j+1} d_{j+1}(p_{j+1}(f_n(z)), h_{n+1}^{j+1}(z)) < k_{j+1}\eta_1(z) \end{aligned}$$

and

$$F_{j+1}(h_{n+1}^{j+1}(z)) \cap B_{j+2}(p_{j+2}(f_n(z)), k_{j+1}\eta_1(z)) \neq \emptyset.$$

Since $F_{j+1} \in \mathrm{SP}(X_{j+1}, X_{j+2}; \mathcal{C})$, and, on Z^*,

$$p_{j+2} \circ f^*(z) \in F_{j+1}(p_{j+1} \circ f^*(z)) \cap B_{j+2}(p_{j+2}(f_n(z)), k_{j+1}\eta_1(z)),$$

there is a continuous extension $h_{n+1}^{j+2}: Z \to X_{j+2}$ of $p_{j+2} \circ f^*$ such that, for all $z \in Z$,

$$h_{n+1}^{j+2}(z) \in F_{j+1}(h_{n+1}^{j+1}(z)) \cap B_{j+2}(p_{j+2} \circ f_n(z), k_{j+1}\eta_2(z)).$$

Inductively we construct continuous extensions $h_{n+1}^{j+s}: Z \to X_{j+s}$ of $p_{j+s} \circ f^*$ $(3 \le s \le m)$ such that, for $z \in Z$,

$$h_{n+1}^{j+s}(z) \in F_{j+s-1}(h_{n+1}^{j+s-1}(z)) \cap B_{j+s}(p_{j+s}(f_n(z)), k_{j+1} \ldots k_{j+s-1}\eta_s(z)).$$

Let us put

$$f_{n+1}(z) := (h_{n+1}^0(z), \ldots, h_{n+1}^{m-1}(z)), \quad z \in Z.$$

It is clear that f_{n+1} is continuous, $f_{n+1}|_{Z^*} = f^*$ and $f_{n+1}: Z \to \Gamma$. Moreover, for each $z \in Z$,

$$p_{j+1} \circ f_{n+1}(z) = h_{n+1}^{j+1}(z) \in F_j(p_j \circ f_n(z))$$

and, for $s = 1, \ldots, m$, we have

$$d_{j+s}(p_{j+s}(f_{n+1}(z)), p_{j+s}(f_n(z))) < k_{j+1}\alpha_{j+s}^{-1}\eta(z),$$

i.e.

$$d(f_{n+1}(z), f_n(z)) < k^n d(f_1(z), f_0(z)) + kc^{1-n} < k^n d(f_1(z), f_0(z)) + c^{-n}.$$

This inductively completes the construction of the required sequence $(f_n)_{n=0}^{\infty}$.

Next, for any $r > 0$, let $Z_r := \{z \in Z \mid d(f_1(z), f_0(z)) < r\}$. The family $\{Z_r\}_{r>0}$ forms an open cover of Z. By (2.4) and the completeness of Γ, for each $z \in Z$, there is $f(z) := \lim_{n\to\infty} f_n(z)$; moreover, this convergence is uniform on Z_r for all $r > 0$. Hence the map $f\colon Z \to \Gamma$ is continuous. In view of properties (2.2) and (2.3), $f|_{Z^*} = f^*$ and for all $z \in Z$, $p_{j+1}(f(z)) \in F_j(p_j(f(z)))$, i.e. $f(z) \in S$. □

Corollary 2.8. *In addition to Assumption* 2.6 *suppose that, for each $i \in \mathbb{Z}_m$, the map F_i has compact values and the class $\mathcal{C}omp$ of compact spaces is contained in $\mathcal{C}$. Then the sets* $\mathrm{Fix}(F)$ *and S are compact and, moreover, $S \in$* AR.

Proof. The compactness of $\mathrm{Fix}(F)$ and, hence, of S, is clear. Therefore, by Theorem 2.7, the compact space $S \in \mathrm{AE}(\mathcal{C}) \subset \mathrm{AE}(\mathcal{C}omp)$, i.e. as we remarked in the introduction, $A \in \mathrm{AR}$. □

Suppose now that X, Y are spaces, let X be complete and consider a set-valued map $F\colon X \times Y \multimap X$. In view of the Nadler theorem, if, for each $y \in Y$, the map $F(\,\cdot\,, y)\colon X \multimap X$ is a $k(y)$-contraction ($k(y) \in [0,1)$), then the map $\mathcal{F}\colon Y \multimap X$, given by $\mathcal{F}(y) := \mathrm{Fix}(F(\,\cdot\,, y))$ for $y \in Y$, is well-defined. If $k(y)$ does not depend on y (i.e. $k(y) = k \in [0,1)$ for all $y \in Y$) and the map $F(x, \,\cdot\,)\colon Y \multimap X$ is H-upper semicontinuous uniformly with respect to $x \in X$ (i.e. for any $y_0 \in Y$ and $\varepsilon > 0$, there is $\delta > 0$ such that, for $x \in X$ and $y \in Y$, if $d_Y(y, y_0) < \delta$, then $\mathfrak{d}(F(x,y), F(x,y_0)) := \sup_{z\in F(x,y)} d(z, F(x,y_0)) < \varepsilon$), then $\mathcal{F}$ is H-upper semicontinuous. This follows by inspection of the proof of the stability result due to Lim [40] (see also [22, Lemma 15.12, Thorem 15.2]). This result is too weak for our purposes. Hence we state a stronger result.

Proposition 2.9. *Suppose again that $F\colon X \times Y \multimap X$, where X is a complete space, has compact values, for each $y \in Y$, $F(\,\cdot\,, y)$ is a $k(y)$-contraction with a continuous $k\colon Y \to [0,1)$ and, for each $x \in X$, the map $F(x, \,\cdot\,)\colon Y \multimap X$ is upper semicontinuous. Then the map $\mathcal{F}\colon Y \multimap X$, given by $\mathcal{F}(y) :=$* $\mathrm{Fix}(F(\,\cdot\,, y))$ *for $y \in Y$, is upper semicontinuous with compact values.*

Proof. Assume that a sequence $(y_n, x_n) \in \mathrm{Gr}(\mathcal{F})$, i.e. for each $n \geq 1$, $x_n \in F(x_n, y_n)$, and assume that $y_n \to y \in Y$. Let $\mathcal{K}(X)$ denote the (hyper)space of all compact subsets of X. It is clear that $(\mathcal{K}(X), \mathfrak{D})$ is a complete metric space. Consider a map $\Phi\colon \mathcal{K}(X) \to \mathcal{K}(X)$ defined by the formula $\Phi(A) = F(A \times \{y\})$ for $A \in \mathcal{K}(X)$. It is easy to see that Φ is a well-defined $k(y)$-contraction, i.e. for all $A, B \in \mathcal{K}(X)$, $\mathfrak{D}(\Phi(A), \Phi(B)) \leq k(y)\mathfrak{D}(A, B)$. In view of the Banach theorem, there is a unique fixed point of Φ, i.e. a compact $K \subset X$ such that $\Phi(K) = K$. Let $n \geq 1$ and take an arbitrary $x \in K$. Then

$$\begin{aligned} d(x_n, K) &\leq d(x_n, F(x,y)) \leq \mathfrak{d}(F(x_n,y_n), F(x,y)) \\ &\leq \mathfrak{d}(F(x_n,y_n), F(x,y_n)) + \mathfrak{d}(F(x,y_n), F(x,y)) \end{aligned}$$

$$\begin{aligned}&\leq k(y_n)d(x_n,x)+\mathfrak{d}(F(x,y_n),F(x,y))\\&\leq k(y_n)d(x_n,x)+\sup_{x\in K}\mathfrak{d}(F(x,y_n),F(x,y)).\end{aligned}$$

Since $x \in K$ was arbitrary, we infer that, for each $n \geq 1$,

$$(1-k(y_n))d(x_n,K)\leq \sup_{x\in K}\mathfrak{d}(F(x,y_n),F(x,y)).$$

For each $x \in K$, the map $F(x,\,\cdot\,)$ is upper semicontinuous with compact values. Hence it is H-upper semicontinuous and, in view of the contractivity of $F(\,\cdot\,,y)$, we gather that, for any $\varepsilon > 0$ and $x \in K$, there is $\delta_x > 0$ such that if $x' \in B(x,\delta_x)$ and $y' \in B(y,\delta_x)$, then $\mathfrak{d}(F(x',y'),F(x',y)) < \varepsilon$. The compactness of K implies that, there is $\delta > 0$ such that if $d(y',y) < \delta$, then $\mathfrak{d}(F(x,y'),F(x,y)) < \varepsilon$ for all $x \in K$. Thus, in view of the continuity of $k(\,\cdot\,)$, we see that

$$\lim_{n\to\infty} d(x_n,K)=0.$$

This implies that (x_n) has a convergent subsequence $x_{n_k} \to x$. Since clearly F is upper semicontinuous, the graph $\mathrm{Gr}(F)$ is closed and, therefore $x \in F(x,y)$, i.e. $x \in \mathcal{F}(y)$. □

Now let us return to the setting of Theorem 2.7 and Corollary 2.8. Namely suppose that

Assumption 2.10.

(a) For any $i \in \mathbb{Z}_m$, X_i, Y_i are spaces and let $X_m := X_0$ be complete; additionally let $X := X_0$;

(b) for $i \in \mathbb{Z}_m$, $F_i\colon X_i \times Y_i \multimap X_{i+1}$ has compact values; for each $y_i \in Y_i$, $F_i(\,\cdot\,,y_i)\colon X_i \multimap X_{i+1}$ is a Lipschitz map with constant $k_i(y_i) \geq 0$ where $k_i\colon Y_i \to [0,\infty)$ is continuous; the product $k_0(y_0)\ldots k_{m-1}(y_{m-1}) < 1$ on $Y_0 \times \ldots \times Y_{m-1}$ and, for each $x_i \in X_i$, the map $F_i(x_i,\,\cdot\,)\colon Y_i \multimap X_{i+1}$ is upper semicontinuous.

Proposition 2.11. *Under Assumption* 2.10, *the maps* $\mathcal{F}\colon Y := Y_0 \times \ldots \times Y_{m-1} \multimap X$ *and* $S\colon Y \multimap X_0 \times \ldots \times X_{m-1}$ *given, for* $y = (y_0,\ldots,y_{m-1}) \in Y$, *by*

$$\begin{aligned}x\in\mathcal{F}(y)\ &\Leftrightarrow\ \forall\, i\in\mathbb{Z}_m\,\exists\, x_i\in X_i\quad x=x_0,\ x_{i+1}\in F_i(x_i,y_i),\\(x_0,\ldots,x_{m-1})\in S(y)\ &\Leftrightarrow\ \forall\, i\in\mathbb{Z}_m\quad x_{i+1}\in F_i(x_i,y_i),\end{aligned}$$

are upper semicontinuous with nonempty compact values and, for $y \in Y$,

$$\mathcal{F}(y)=p_0(S(y))$$

where $p_0\colon \prod_{i\in\mathbb{Z}_m} X_i \to X = X_0$ *is the projection.*

Proof. The upper semicontinuity of $\mathcal{F}$ is a direct consequence of Proposition 2.9 since, for $y \in Y$, $\mathcal{F}(y)$ is the fixed point set of a compact-valued contraction $F(\,\cdot\,,y)$ with a continuous Lipschitz constant $k(y) := k_0(y_0)\dots k_{m-1}(y_{m-1})$ ($y = (y_0,\dots,y_{m-1}) \in Y$) and such that $F(x,\,\cdot\,)\colon Y \multimap X$ is upper semicontinuous, where for given $x \in X$ and $y = (y_0,\dots,y_{m-1}) \in Y$, $F(x,y) := G_{m-1}\circ\dots\circ G_0(x)$ with $G_i := F_i(\,\cdot\,,y_i)$ for $i \in \mathbb{Z}_m$. Moreover, for each $y \in Y$, the sets $\mathcal{F}(y)$ and $S(y)$ are compact and $\mathcal{F}(y) = p_0(S(y))$.

If $y^n \to y$ in Y and $x^n \in S(y^n)$, then $x_0^n := p_0(x^n) \in \mathcal{F}(y^n)$. By the upper semicontinuity of $\mathcal{F}$ (and passing to subsequences if necessary), $x_0^n \to x_0 \in \mathcal{F}(y) = p_0(S(y))$. If, for some $i \in \mathbb{Z}_m$, $1 \le i \le m-1$, $x_{i-1}^n := p_{i-1}(x^n) \to x_{i-1}$, then since $x_i^n := p_i(x^n) \in F_{i-1}(x_{i-1}^n, y_{i-1}^n)$, by the upper semicontinuity (and the compactness of values) of F_{i-1} and again passing to a subsequence if necessary, $x_i^n \to x_i \in F_{i-1}(x_{i-1},y_{i-1})$. Finally we have $x = (x_0,\dots,x_{m-1}) = \lim_{n\to\infty} x^n$ such that $x_i \in F_{i-1}(x_{i-1},y_{i-1})$ for all $i \in \mathbb{Z}_m$, $i \ge 1$. To show that $x \in S(y)$ observe that $x_0^n \in F_{m-1}(x_{m-1}^n, y_{m-1}^n)$; hence $x_0 \in F_{m-1}(x_{m-1},y_{m-1})$ by the closeness of $\mathrm{Gr}(F_{m-1})$. □

Remark 2.12. If above $Y_0 = \dots = Y_{m-1} =: Y$, then instead of $\mathcal{F}\colon Y^m \multimap X$ (resp. $S\colon Y^m \multimap \prod_{i\in\mathbb{Z}_m} X_i$) one may consider a map $Y \multimap X$ (resp. $Y \mapsto\!\!\circ \prod_{i\in\mathbb{Z}_m} X_i$) given by $Y \ni y \mapsto\!\!\circ\, \mathcal{F}\circ\Delta$ (resp. $Y \ni y \mapsto\!\!\circ\, S\circ\Delta$) where $\Delta\colon Y \to Y^m$ is the diagonal map, i.e. $\Delta(y) := (y,\dots,y) \in Y^m$). In what follows these new maps are also denoted by $\mathcal{F}$ (resp. S). As before these maps are upper semicontinuous with nonempty compact values and, for any $y \in Y$, $\mathcal{F}(y) = p_0(S(y))$.

In view of Theorem 2.7 we get immediately

Corollary 2.13. *Suppose that Assumption* 2.10 *is satisfied. If, for each $i \in \mathbb{Z}_m$, X_i is complete, there is $j \in \mathbb{Z}_m$ such that $X_j \in \mathrm{AE}(\mathcal{C})$, where $\mathcal{C}$ is a class of spaces containing the class $\mathcal{C}omp$ of compact spaces and, for each $i \in \mathbb{Z}_m$ and $y_i \in Y_i$, $F_i(\,\cdot\,,y_i) \in \mathrm{SP}(X_i,X_{i+1};\mathcal{C})$, then $S : Y := \prod_{i\in\mathbb{Z}_m} Y_i \to \prod_{i\in\mathbb{Z}_m} X_i$ is upper semicontinuous with compact values and $S(y) \in \mathrm{AR}$ for all $y \in Y$.*

The assertion of Corollary 2.13 holds for instance if, for each $i \in \mathbb{Z}_m$, X_i is a closed convex subset of a Banach space and the map $F_i(\,\cdot\,,y_i)$, where $y_i \in Y_i$, has compact convex values.

3. Fixed point index for Krasnosel'skiĭ-type maps

In this section we shall construct a fixed point index for the so-called Krasnosel'skiĭ-type maps. To make the setting more transparent, let us first briefly study the single-valued situation.

3.1. Single-valued Krasnosel'skiĭ-type maps. Suppose that X is a space and let $A \subset X$. Additionally let Λ be a parameter metric space.

Definition 3.1. A map $\varphi: A \times \Lambda \to X$ is called a *Krasnosel'skiĭ-type* map provided there exist:

(a) a space Y, a map $f: X \times Y \times \Lambda \to X$ such that, for each $x \in X$, $f(x, \cdot\,, \cdot): Y \times \Lambda \to X$ is continuous and, for each $y \in Y$ and $\lambda \in \Lambda$, $f(\cdot\,, y, \lambda): X \to X$ is a $k(y, \lambda)$-contraction where the function $k: Y \times \Lambda \to [0, 1)$ is continuous;

(b) a continuous compact map $c: A \times \Lambda \to Y$, such that

$$\varphi(x, \lambda) = f \diamond c(x, \lambda) := f(x, c(x, \lambda), \lambda) \quad \text{for all } x \in A \text{ and } \lambda \in \Lambda.$$

A similar class of maps has been studied in [41]. The family of all Krasnosel'skiĭ-type maps is denoted by $\mathcal{K}(A \times \Lambda, X)$. In the case when $\Lambda = \{\lambda\}$ is the singleton, we write $\mathcal{K}(A, X)$. If $\varphi = f \diamond c \in \mathcal{K}(A, X)$, i.e. when Λ is a singleton, then the dependence of λ may be skipped: the existing map $f: X \times Y \to X$ is such that $f(x, \cdot\,)$ is continuous and $f(\cdot\,, y)$ is a $k(y)$-contraction with continuous $k: Y \to [0, 1)$, while $c: A \to Y$ is continuous and compact and then

$$\varphi(x) = f \diamond c(x) := f(x, c(x)).$$

It is also clear that given $\varphi \in \mathcal{K}(A \times \Lambda, X)$ and $\lambda_0 \in \Lambda$, the map $\varphi_{\lambda_0} := \varphi(\cdot\,, \lambda_0) \in \mathcal{K}(A, X)$.

Let $\varphi = f \diamond c$ belong to $\mathcal{K}(A \times \Lambda, X)$ (resp. $\mathcal{K}(A, X)$) and suppose that the space X is complete. The map $\mathcal{F}: Y \times \Lambda \to X$ (resp. $\mathcal{F}: Y \to X$) given, for $y \in Y$ and $\lambda \in \Lambda$ by

$$\mathcal{F}(y, \lambda) := \operatorname{Fix}(f(\cdot\,, y, \lambda)) \quad (\text{resp. } \mathcal{F}(y) := \operatorname{Fix}(f(\cdot\,, y))),$$

is well-defined and, in view of Proposition 2.9, continuous. Moreover, the map $A \times \Lambda \ni (x, \lambda) \mapsto \mathcal{F}(c(x, \lambda), \lambda)$ (resp. $A \ni x \mapsto \mathcal{F}(c(x))$) is well-defined and compact. Observe that, for each $\lambda \in \Lambda$,

$$\operatorname{Fix}(\varphi(\cdot\,, \lambda)) = \operatorname{Fix}(\mathcal{F}(c(\cdot\,, \lambda), \lambda)) \quad (\text{resp. } \operatorname{Fix}(\varphi) = \operatorname{Fix}(\mathcal{F} \circ c))$$

and if A is closed then, $\operatorname{Fix}(\varphi(\cdot\,, \lambda))$ (resp. $\operatorname{Fix}(\varphi)$) is compact.

Definition 3.2. Given maps $\varphi_i \in \mathcal{K}(A, X)$, $i = 0, 1$, we say that φ_0 is *homotopic in* $\mathcal{K}$ to φ_1 (written $\varphi_0 \simeq_{\mathcal{K}} \varphi_1$) if there exists $\varphi = f \diamond c \in \mathcal{K}(A \times [0, 1], X)$ such that $\varphi_i = \varphi(\cdot\,, i)$, $i = 0, 1$ (we say that φ is a $\mathcal{K}$-*homotopy* joining φ_0 to φ_1).

In order to proceed further we assume that a complete space $X \in \mathrm{ANR}$ and $U \subset X$ is open. Assume that $\varphi = f \diamond c \in \mathcal{K}(\operatorname{cl} U, X)$ has no fixed points on the boundary $\operatorname{bd} U$ of U, i.e. $\operatorname{Fix}(\varphi) \cap \operatorname{bd} U = \emptyset$ and put

$$\operatorname{Ind}(\varphi, U) := \operatorname{ind}_G(\mathcal{F} \circ c, U) \tag{3.1}$$

where $\mathrm{ind}_G(\mathcal{F}\circ c, U)$ stands for the Granas fixed-point index of the compact map $\mathcal{F}\circ c$ (see [28]) and, as above, $\mathcal{F}(y) = \mathrm{Fix}(f(\,\cdot\,,y))$. Observe that, in general, $\mathrm{Ind}(\varphi, U)$ may strongly depend on the form of φ, i.e. the structure of φ reflected by the formula $\varphi = f\diamond c$.

Our definition constitutes a counterpart and a generalization onto the context of the fixed-point index theory of the *quasi-degree* or *quasi-rotation* of Krasnosel'skiĭ and Zabreĭko (see [33, Section 34.1] and comp. [55]).

Remark 3.3. It is not difficult to show that if $\varphi = f\diamond c \in \mathcal{K}(A,X)$, then φ is a k-ball-contraction with $k=\sup\{k(y)\mid y\in c(A)\}$, i.e. for each bounded $B\subset A$, $\beta(\varphi(B))\le k\beta(B)$, where β stands for the Hausdorff measure of noncompactness in X — see the more general Proposition 3.15 below.

Therefore if $A=\mathrm{cl}\,U$ where U is an open subset of a closed convex subset X of a Banach space, then the fixed point index of φ is available (see for instance the book [30]) and in this case our approach gives nothing new. Indeed, let $\mathrm{ind}_{\mathrm{cond}}(\varphi,U)$ denote the fixed-point index of ball-contractions. We are to show that $\mathrm{ind}_{\mathrm{cond}}(\varphi,U)=\mathrm{Ind}(\varphi,U)$. To this end we shall show that a map $h\colon \mathrm{cl}\,U\times[0,1]\to X$ given by the formula

$$h(x,\lambda)=(1-\lambda)\mathcal{F}\circ c(x)+\lambda\varphi(x)$$

(being a ball-contraction) has no fixed points on $\mathrm{bd}\,U$. To this end suppose to the contrary that $x\in\mathrm{bd}\,U$ and $x=h(x,\lambda)$ for some $\lambda\in[0,1]$. Let $y=\mathcal{F}\circ c(x)$; then $y=f(y,c(x))$ and

$$\begin{aligned}\|x-y\| &= \lambda\|y-f(x,c(x))\| = \lambda\|f(y,c(x))-f(x,c(x))\|\\ &\le \lambda k(c(x))\|x-y\| < \|x-y\|,\end{aligned}$$

a contradiction. The homotopy property of $\mathrm{ind}_{\mathrm{cond}}$ gives that

$$\mathrm{ind}_{\mathrm{cond}}(\varphi,U)=\mathrm{ind}_{\mathrm{cond}}(\mathcal{F}\circ c,U)=\mathrm{ind}_G(\mathcal{F}\circ c,U)=\mathrm{Ind}(\varphi,U).$$

However, the fixed-point index theory for *arbitrary* ball-contractions (or condensing maps) defined on arbitrary absolute neighbourhood retracts is yet unavailable. But our constructions makes it possible to deal with this construction at least for a special type of ball-contractions: namely for Krasnosel'skiĭ-type maps.

To see the immediate advantage of our approach consider the following example.

Example 3.4. Suppose that E is a Banach space, X is a bounded neighbourhood retract in E and let $\mathcal{U}=\{U(t)\}_{t\ge0}$ be a C_0-semigroup of bounded linear operators $U(t)\colon E\to E$ $(t\ge0)$ such that, for all $\|U(t)\|\le e^{\omega t}$ where $\omega<0$

(i.e. $\mathcal{U}$ is a strict contraction semigroup). Let A be the infinitesimal generator of $\mathcal{U}$ and $\mathfrak{F}\colon X \to E$ be an ℓ-Lipschitz function. Suppose that, for all $x \in \operatorname{bd} X$,

$$\liminf_{h\to 0^+} \frac{d_X(U(h)x + h\mathfrak{F}(x))}{h} = 0$$

where, for $z \in E$, $d_X(z) := \inf_{x\in X} \|x - z\|$ is the distance function. Under these assumption Bothe [8] proves that, for each $T > 0$, there exists a unique *mild solution* $x = x(x_0;\,\cdot\,)\colon [0,T] \to X$ to the problem

$$x' = Ax + \mathfrak{F}(x), \quad x(t) \in X, \quad x(0) = x_0 \in X \ (^{10}). \tag{3.2}$$

Moreover, the map $X \ni x_0 \mapsto x(x_0;\,\cdot\,) \in C([0,T],E)$, where $C([0,T],E)$ stands for the Banach space of continuous functions $[0,T] \to E$, is continuous.

We shall study the existence of a mild solution $x\colon [0,T] \to X$ to (3.2) such that $x(0) = x(T)$. This problem has been addressed in [4, Theorems 30, 37, 38] under the assumption that X is bounded and convex (and some extra assumptions concerning the semigroup $\mathcal{U}$).

It is clear that the existence of periodic (mild) solutions to (3.2) is equivalent to the existence of fixed point of the associated *Poincaré operator* $P_T\colon X \to X$ defined by the formula

$$P_T(x_0) := x(x_0;T), \quad x_0 \in X.$$

Observe that P_T is compact if the semigroup $\mathcal{U}$ is compact (and then the periodic problem may be solved via the Granas index theory); it may be also shown that P_T is a k-ball-contraction with $k = e^{(\omega+4\ell)T}$ (see [4, Theorem 25]). However it does not help much since X is no longer convex in our setting. Therefore we propose an attitude sufficient to show that P_T is a Krasnosel'skiĭ-type map.

Namely we suppose that $\mathcal{U}$ is *uniformly continuous*, i.e. $\lim_{t\to 0^+} \|U(t) - I\| = 0$ (^{11}) and $\mathfrak{F}$ is compact. Given $w \in C([0,T],E)$ and $x_0 \in E$, let $\Sigma_A(x_0,w)$ be a unique (mild) solution to the problem $x' = Ax + w$, $x(0) = x_0$. It is easy to see that, for all $t \in [0,T]$, $x_0, x_0' \in E$ and $w, w' \in C([0,T],E)$,

$$\|\Sigma_A(x_0,w)(t) - \Sigma(x_0',w')(t)\| \le e^{\omega t}\|x_0 - x_0'\| + \int_0^t e^{\omega(t-s)}\|w(s) - w'(s)\|\,ds.$$

Hence the map $f := \Sigma_A(\,\cdot\,,\,\cdot\,)(T)\colon E \times C([0,T],E) \to E$ is continuous and, for each $w \in C([0,T],E)$, $f(\,\cdot\,,w)$ is a k-contraction with $k := e^{\omega T} < 1$. Now let us consider a map $c\colon X \to C([0,T],E)$ given by

$$c(x_0)(t) = \mathfrak{F}(x(x_0;t)), \quad x_0 \in X.$$

(^{10}) I.e. $x\colon [0,T] \to E$ is a continuous function such that, for all $t \in [0,T]$,

$$x(t) = U(t)x_0 + \int_0^t U(t-s)\mathfrak{F}(x(s))\,ds.$$

(^{11}) This is, by all means, a restrictive assumption since it implies that A is defined everywhere and, for each $t \ge 0$, $U(t) = e^{At}$; moreover in this case any mild solution to (3.2) is actually a strong solution.

It is clear that c is continuous. We shall show that c is compact. To this end observe that the family $\{x(x_0;\,\cdot\,) \mid x_0 \in X\}$ is uniformly equicontinuous, that is given $\varepsilon > 0$, there is $\delta > 0$ such that, for any $t, t' \in [0,T]$, if $|t - t'| < \delta$, then $\|x(x_0;t) - x(x_0;t')\| < \varepsilon$ for all $x_0 \in X$. Indeed, assume that $0 \le t' < t \le T$; then the semigroup property yields that

$$\|x(x_0;t) - x(x_0;t')\| \le e^{\omega t'}\|U(t-t') - I\|\|x_0\| \\ + \int_0^{t'} e^{\omega(t'-s)}\|U(t-t') - I\|\|c(x_0)(s)\|\,ds + \int_{t'}^{t} e^{\omega(t-s)}\|c(x_0)(s)\|\,ds.$$

Taking the boundedness of X and $\mathfrak{F}$ and the uniform continuity of $\mathcal{U}$ into account, we infer that $\{x(x_0;\,\cdot\,)\}_{x_0\in X}$ is indeed uniformly equicontinuous. In view of the compactness of $\mathfrak{F}$, it is clear that, for each $t \in [0,T]$, the orbit $\{c(x_0)(t)\}_{x_0\in X}$ of the family $c(X)$ is relatively compact. Hence, by the Ascoli–Arzela theorem and the uniform continuity of $\mathfrak{F}$, we see that c is compact.

Finally observe that, for all $x_0 \in X$,

$$P_T(x_0) = f \diamond c(x_0) = \Sigma(x_0; \mathfrak{F}(x(x_0;\,\cdot\,))(T).$$

Hence P_T is a Krasnosel'skiĭ-type map and its fixed points (that is periodic solutions to (3.2)) may be studied by means of the introduced index. In particular, if $X \in \mathrm{AR}$, then (3.2) has periodic solutions in view of the generalized Schauder theorem (see [28, Theorem (7.4)]).

Let us remark that the described technique allows to study the existence of *equilibria states*, i.e. points $x_0 \in X$ such that $-A(x_0) = \mathfrak{F}(x_0)$. Given $T > 0$, let $x_n\colon [0,T] \to X$ be a solution to (3.2) such that $x_n(0) = x(2^{-n}T)$. It is clear that x_n converges uniformly to a constant map $x_0 \in X$ (i.e. $x_0(t) \equiv x_0$ on $[0,T]$) such that $-A(x_0) = \mathfrak{F}(x_0)$. This result constitutes a direct (single-valued) generalization of the Deimling theorem [15, Theorem 11.5] for non-convex domains since if $A = -I$, then $U(t)x = e^{-t}x$ on E and, as it easy to see,

$$T_X^{\mathcal{U}}(x) := \left\{ u \in E \;\middle|\; \liminf_{h\to 0^+} \frac{d_X(U(h)x + hu)}{h} = 0 \right\} = x + T_X(x)$$

where $T_X(x)$ is the Bouligand tangent cone to X at $x \in X$ (see Assumption 4.2 below).

In what follows we are going to treat similar problems within the setting of set-valued Krasnosel'skiĭ-type maps.

3.2. Fixed point index for compositions of acyclic maps. In order to address the set-valued situation we shall need some more preparation. An index to be constructed will be defined via an appropriate fixed point index for set-valued maps. As it is well-known the availability of such theory is restricted to special classes of such maps (see also the Appendix below). Hence we have to recall some relevant issues.

We say that a space $A \subset X$ is *acyclic* if $\check{H}_*(A;\mathbb{Q}) = \check{H}_*(pt;\mathbb{Q})$ where pt stands for a one-point space and $\check{H}_*(\,\cdot\,;\mathbb{Q})$ is the Čech homology with compact carriers and rational coefficients (see e.g. [24]). Observe that any contractible space is acyclic; in particular if $A \in \mathrm{AR}$, then A is acyclic. Let X, Y be spaces. We say that an upper semicontinuous set-valued map $F\colon X \multimap Y$ is *acyclic* if, for each $x \in X$, $F(x)$ is compact and acyclic. The class of acyclic maps $X \multimap Y$ is denoted by $\mathcal{A}(X,Y)$.

Definition 3.5 (see [18, Definition 2.15], [51]). By an *acyclic decomposition* of a map $F\colon X \multimap Y$ we mean a sequence $D(F) = (F_0, F_1, \ldots, F_{n-1})$ representing the diagram

$$D(F)\colon X = X_0 \xrightarrow{F_0} X_1 \xrightarrow{F_1} \cdots \xrightarrow{F_{n-2}} X_{n-1} \xrightarrow{F_{n-1}} X_n = Y, \tag{3.3}$$

where X_i, $i = 0, \ldots, n$, are spaces and, for $i = 0, \ldots, n-1$, $F_i \in \mathcal{A}(X_i, X_{i+1})$, such that $F = F_{n-1} \circ \ldots \circ F_0$. We also say that the decomposition $D(F)$ determines F and that $F\colon X \multimap Y$ is *acyclic-decomposable* if there exists an acyclic decomposition determining it.

Example 3.6. (a) Clearly any acyclic, and — in particular — single-valued, map $F\colon X \multimap Y$ is acyclic-decomposable with the decomposition (F) determined by F itself.

(b) Recall Assumption 2.6 and assume that, for each $i \in \mathbb{Z}_m$, F_i has compact values. If the class $\mathcal{C}$ is multiplicative, then $F = F_{m-1} \circ \ldots \circ F_0$ is acyclic-decomposable in view of Proposition 2.4.

(c) Under assumptions of Corollary 2.13, the map $\mathcal{F}$ defined in Proposition 2.11 is acyclic-decomposable with an acyclic decomposition given by the diagram

$$D(\mathcal{F})\colon Y \xrightarrow{S} \prod_{i \in \mathbb{Z}_m} X_i \xrightarrow{p_0} X.$$

The class of acyclic decompositions of acyclic-decomposable maps $X \multimap Y$ will be denoted by $\mathcal{D}A(X,Y)$. We shall study these maps along with their acyclic decompositions. Therefore two acyclic-decomposable maps $F, F'\colon X \to Y$ are equal if and only if there are acyclic decompositions $D(F), D(F') \in \mathcal{D}A(X,Y)$ of F and F', respectively, such that $D(F) = D(F')$ ([12]). Given acyclic decompositions $D(F) \in \mathcal{D}A(X,Y)$, $D(G) \in \mathcal{D}A(Y,Z)$, where Z is a space, of

([12]) I.e. if $D(F)$ is given by (3.3) and $D(F') : X = X'_0 \xrightarrow{F'_0} X'_1 \xrightarrow{F'_1} \cdots \xrightarrow{F'_{m-2}} X'_{m-1} \xrightarrow{F'_{m-1}} X'_m = Y$, then $n = m$, $X_i = X'_i$ and $F_i = F'_i$ for all i. Clearly if $F = F'$, then $F(x) = F'(x)$ for each $x \in X$, but not conversely since the same acyclic-decomposable map may admit different decompositions.

acyclic-decomposable maps $F\colon X \multimap Y$ and $G\colon Y \multimap Z$, respectively, and $A \subset X$, then the *composition* $D(G) \circ D(F)$ and the *restriction* $D(F)|_A \in \mathcal{D}A(A, Y)$ are defined in an obvious way; evidently $D(G) \circ D(F)$ and $D(F)|_A$ are acyclic decompositions of $G \circ F$ and $F|_A$, respectively.

Two acyclic decompositions $D(F), D(G)\colon X \multimap Y$ of acyclic-decomposable maps $F, G\colon X \multimap Y$ are *homotopic*, if there exists an acyclic-decomposable map $H\colon X \times [0,1] \multimap Y$ with an acyclic decomposition $D(H) \in \mathcal{D}A(X \times [0,1], Y)$ such that

$$D(H)|_{X\times\{0\}} = D(F), \quad D(H)|_{X\times\{1\}} = F(G).$$

It is clear that the relation of homotopy is an equivalence relation.

Remark 3.7. It is easy to show (see [18, Proposition 2.14]) that if, for $i = 0, \ldots, n-1$, F_i, G_i are homotopic in $\mathcal{A}(X_i, X_{i+1})$ (i.e. there is $H_i \in \mathcal{A}(X_i \times [0,1], X_{i+1})$ such that $H_i(\,\cdot\,, 0) = F_i$ and $H_i(\,\cdot\,, 1) = G_i$ on X_i), then the decompositions $D(F) = (F_0, \ldots, F_{n-1})$ and $D(G) = (G_0, \ldots, G_{n-1})$ of acyclic-decomposable maps $G = G_{n-1} \circ \ldots \circ G_0$ and $F = F_{n-1} \circ \ldots \circ F_0$, respectively, are homotopic.

Let $X \in \mathrm{ANR}$ and let $D(F) = (F_0, \ldots, F_{n-1}) \in \mathcal{D}A(X, X)$ be an acyclic decomposition of a compact acyclic-decomposable map $F\colon X \multimap X$. In this situation Skordev and Siegberg [51], [53] (see also [18] for a more general case and [54]) define the *Lefschetz number* $L(D(F)) \in \mathbb{Q}$ of the decomposition $D(F)$. Moreover, given an open $U \subset X$ such that $\mathrm{Fix}(F) \cap \mathrm{bd}\, U = \emptyset$, they define the *fixed point index* $\mathrm{ind}_S(X, D(F), U)$ of the decomposition $D(F)$ with respect to U. These definitions rely on relatively complex algebraic constructions involving the so-called *approximation systems*. The index ind_S satisfies all the natural properties.

(a) (Existence) If $\mathrm{ind}_S(X, D(F), U) \neq 0$, then $\mathrm{Fix}(F) \cap U \neq \emptyset$.

(b) (Additivity) If $U_1, \ldots, U_k \subset U$ are open disjoint and $\mathrm{Fix}(F) \cap [\mathrm{cl}\, U \setminus \bigcup_{i=1}^k U_i] = \emptyset$, then

$$\mathrm{ind}_S(\Phi, U) = \sum_{i=1}^k \mathrm{ind}_S(\Phi, U_i).$$

(c) (Homotopy invariance) If $D(H) \in \mathcal{D}A(X \times [0,1], X)$ is an acyclic decomposition of the compact acyclic-decomposable map $H\colon X \times [0,1] \multimap X$, then $L(D(H)_{X\times\{0\}}) = L(D(H)_{X\times\{1\}})$. If, for each $t \in [0,1]$, $\mathrm{Fix}(H(\,\cdot\,, t)) \cap \mathrm{bd}\, U = \emptyset$, then

$$\mathrm{ind}_S(X, D(H)|_{X\times\{0\}}, U) = \mathrm{ind}_S(X, D(H)|_{X\times\{1\}}, U).$$

(d) (Commutativity) If $X, Y \in \mathrm{ANR}$, a decomposition $D(F) \in \mathcal{D}A(X, Y)$ determines a compact map, $G \in \mathcal{D}A(Y, X)$, $\mathrm{Fix}(G \circ F) \cap \mathrm{bd}\, U = \emptyset$,

$\mathrm{Fix}(F\circ G)\cap\mathrm{bd}\,V=\emptyset$, where $V=\{y\in Y\mid G(y)\subset U\}$ and $G[\mathrm{Fix}(F\circ G)\setminus \mathrm{cl}\,V]\cap\mathrm{Fix}(G\circ F|_U)=\emptyset$, then $\mathrm{ind}_S(X,D(G)\circ D(F),U)=\mathrm{Ind}_S(Y,D(F)\circ D(G),V)$.

(e) (Normalization) If $D(F)\in\mathcal{D}A(X,X)$, then

$$\mathrm{ind}_S(X,D(F),X)=L(D(F)).$$

(f) (Units) If $F\colon X\to F$ is acyclic-decomposable and, for each $x\in X$, $F(x)=K\subset X$, where $K\cap\mathrm{bd}\,U=\emptyset$, then, for any acyclic decomposition $D(F)$ of F,

$$\mathrm{ind}_S(X,D(F),U)=\begin{cases}1 & \text{if } K\cap U\neq\emptyset,\\ 0 & \text{otherwise.}\end{cases}$$

(g) The index ind_S is consistent with the Granas index: if $f\colon X\to X$ is a continuous compact single-valued map and $\mathrm{Fix}(f)\cap\mathrm{bd}\,U=\emptyset$, then $\mathrm{ind}_S(X,(f),U)=\mathrm{ind}_G(f,U)$ (see Example 3.6(a)).

Properties (b)–(e) were proved in [51]; property (a) follows from (b), (f) and (g) are direct consequences of the very definition.

Remark 3.8. Let $X\in\mathrm{ANR}$ and consider an acyclic decomposition (3.3) of an acyclic-decomposable map $F\colon X\multimap X$ such that $\mathrm{Fix}(F)\cap\mathrm{bd}\,U=\emptyset$.

(a) Along with (3.3) one may consider decompositions

$$(\mathrm{id},D(F))\colon X=X_0\xrightarrow{\mathrm{id}}X_0\overset{F_0}{\multimap}X_1\overset{F_1}{\multimap}\cdots X_{n-1}\overset{F_{n-1}}{\multimap}X_n=X,$$
$$(D(F),\mathrm{id})\colon X=X_0\overset{F_0}{\multimap}X_1\overset{F_1}{\multimap}\ldots X_{n-1}\overset{F_{n-1}}{\multimap}X_n\xrightarrow{\mathrm{id}}X_n=X,$$

where id stands for the identity on X_0 and X_n, respectively. Then, by the very construction of the index, it is easy to see that

$$\mathrm{ind}_S(X,(\mathrm{id},D(F)),U)=\mathrm{ind}_S(X,D(F),U)=\mathrm{ind}_S(X,(D(F),\mathrm{id}),U).$$

(b) Let $D'(F):X=X'_0\overset{F'_0}{\multimap}X'_1\overset{F'_1}{\multimap}\cdots X'_{n-1}\overset{F'_{n-1}}{\multimap}X_n=X$ be another acyclic decomposition of F. We say that the decompositions $D(F)$ and $D'(F)$ are *related* if, for each $i=0,\ldots,n$, there is a continuous map $h_i\colon X_i\to X'_i$ such that $h_0=\mathrm{id}_X=h_n$ and $h_{i+1}\circ F_i=F'_i\circ h_i$ for $i=0,\ldots,n-1$. It is clear that if decompositions $D(F)$ and $D'(F)$ are related, then

$$\mathrm{ind}_S(X,D(F),U)=\mathrm{ind}_S(X,D'(F),U).$$

(c) Suppose that in (3.3), there is $j\in\{0,\ldots,n-2\}$ such that $F_j=f_j\colon X_j\to X_{j+1}$ is a single-valued map. Then it makes sense to consider the *reduced* decomposition $D'(F)=(F_0,\ldots,F_{j-1},F_{j+1}\circ f_j,F_{j+2},\ldots,F_{n-1})$. From (b) it follows immediately that

$$\mathrm{ind}_S(X,D(F),U)=\mathrm{ind}_S(X,D'(F),U).$$

(d) If an acyclic decomposition $D(F) \in \mathcal{D}A(X,Y)$ of a compact acyclic-decomposable map $F\colon X \multimap Y$, where $X, Y \in \mathrm{ANR}$ and $Y \subset X$, and an open $U \subset X$ such that $\mathrm{Fix}(F \circ j) \cap \mathrm{bd}\, U = \emptyset$, where $j\colon Y \to X$, are given, then — in view of the commutativity property — the following *restriction property* is satisfied:

$$\mathrm{ind}_S(X, (D(F), j), U) = \mathrm{ind}_S(Y, D(F)|_Y, U \cap Y).$$

Some other indications as to the construction of the fixed point index for set-valued maps are provided Section 5.1.

3.3. Set-valued Krasnosel'skiĭ-type maps. We may now formulate the basic concepts and results of this paper.

Definition 3.9. Let X, Λ be spaces; assume that X is complete. We say that $\Phi\colon X \times \Lambda \multimap X$ is a *Krasnosel'skiĭ-type* set-valued map if:

(a) there are a space Y and a set-valued map $F\colon X \times Y \times \Lambda \multimap X$ with compact values such that, for each $x \in X$, $F(x, \cdot\,, \cdot\,)\colon Y \times \Lambda \multimap X$ is upper semicontinuous, for each $y \in Y$ and $\lambda \in \Lambda$, $F(\,\cdot\,, y, \lambda)\colon X \multimap X$ is a $k(y, \lambda)$-contraction (where the function $k\colon Y \times \Lambda \to [0, 1)$ is continuous);

(b) there is a compact upper semicontinuous map $C\colon X \times \Lambda \multimap Y$, such that

$$\Phi(x, \lambda) = F \diamond C(x, \lambda) := F(\{x\} \times C(x, \lambda) \times \{\lambda\})$$

for $x \in X$ and $\lambda \in \Lambda$.

Observe that, in view of Proposition 2.9, if Φ is a Krasnosel'skiĭ-type map, then the map $\mathcal{F}\colon Y \times \Lambda \multimap X$ given, by

$$\mathcal{F}(y, \lambda) := \mathrm{Fix}(F(\,\cdot\,, y, \lambda)) \quad \text{for } y \in Y,\ \lambda \in \Lambda,$$

is upper semicontinuous with nonempty and compact values.

Definition 3.10. We say that a Krasnosel'skiĭ-type map $\Phi = F \diamond C$ is *permissible* provided the maps C and $\mathcal{F}$ are acyclic-decomposable in the sense of Definition 3.5.

As before the family of Krasnosel'skiĭ-type maps $\Phi\colon X \times \Lambda \multimap X$ is denoted by $\mathcal{K}(X \times \Lambda, X)$ and by $\mathcal{K}(X, X)$ if Λ reduces to a point (i.e. when, in practice, Λ is not present). At the same time the class of permissible Krasnosel'skiĭ-type maps is denoted by $\mathcal{K}_p(X \times \Lambda, X)$ (resp. $\mathcal{K}_p(X, X)$). A class of mappings similar to that of permissible Krasnosel'skiĭ-type maps has been considered in [45] and [46]; the author studies there maps of the form $F \diamond C$ assuming that both maps F and C have compact convex values.

Remark 3.11. It is clear that if $\Phi = F \diamond C \in \mathcal{K}(X, X)$ and F, C are acyclic-decomposable maps, then so is Φ; however if Φ is permissible in the above sense, then — in general — Φ may be not acyclic-decomposable. For instance consider a map $F:\mathbb{R}\times\mathbb{R}\multimap\mathbb{R}$ given by $F(x,y)=\{(x+y)/2,y\}\subset\mathbb{R}$ for any $x,y\in\mathbb{R}$, and let $C:\mathbb{R}\to\mathbb{R}$ be an arbitrary compact continuous map. Then $\Phi=F\diamond C$ is not acyclic-decomposable, but for each $y\in\mathbb{R}$, $\mathcal{F}(y)=\{y\}$ is acyclic-decomposable and, hence Φ is a permissible Krasnosel'skiĭ-type map.

Example 3.6(b) justifies the definition of permissible Krasnosel'skiĭ-type maps and provides a method to construct natural examples of set-valued Krasnosel'skiĭ-type maps. Namely we have:

Example 3.12. (a) Let us assume that, for $i\in\mathbb{Z}_m$, X_i is a complete space (and there is $j\in\mathbb{Z}_m$ such that $X_j\in\mathrm{AE}(\mathcal{C})$ where $\mathcal{C}$ is a class of spaces containing the class *Comp* of compact spaces), a map $F_i:X_i\times Y\times\Lambda\multimap X_{i+1}$ (again addition is performed modulo m) is such that, for each $y\in Y$ and $\lambda\in\Lambda$, $F_i(\,\cdot\,,y,\lambda)$ is a compact-valued Lipschitz map (with constant $k_i(y,\lambda)$) belonging to $\mathrm{SP}(X_i,X_{i+1})$ and, for each $x\in X_i$, $F_i(x,\,\cdot\,,\,\cdot\,)$ is upper semicontinuous. If the product function $\prod_{i\in\mathbb{Z}_m}k_i(\,\cdot\,)$ is continuous on $Y\times\Lambda$ and takes values in $[0,1)$, and $C:X\times\Lambda\multimap Y$ is an arbitrary acyclic-decomposable map, then $\Phi:=F\diamond C$, where $F(x,y,\lambda)=G_{m-1}\circ\ldots\circ G_0(x)$ and $G_i:=F_i(\,\cdot\,,y,\lambda)$ for $x\in X$, $y\in Y$, $\lambda\in\Lambda$ and $i\in\mathbb{Z}_m$, is a permissible Krasnosel'skiĭ-type map.

(b) Suppose that $K:X\times\Lambda\multimap X_1$ has compact values, for each $\lambda\in\Lambda$, $K(\,\cdot\,,\lambda)\in\mathrm{SP}(X,X_1)$ is Lipschitz with constant $k(\lambda)$, $C:X\times\Lambda\multimap Y$ is compact acyclic-decomposable and let $T:X_1\times Y\times\Lambda\to X$ be continuous and such that, for each $y\in Y$, $\lambda\in\Lambda$, $T(\,\cdot\,,y,\lambda)$ is Lipschitz with constant $t(y,\lambda)$. If spaces X,X_1 are complete, $X_1\in\mathrm{AR}$, the function $k(\,\cdot\,)t(\,\cdot\,,\,\cdot\,)\in[0,1)$ is continuous on $Y\times\Lambda$ and $K(x,\,\cdot\,)$ is upper semicontinuous on Λ, then the map mentioned in Introduction

$$X\times\Lambda\ni(x,\lambda)\mapsto\Phi(x,\lambda):=T(K(x,\lambda)\times C(x,\lambda)\times\{\lambda\})$$

is a permissible Krasnosel'skiĭ-type mapping. Indeed, let $F:X\times Y\times\Lambda\multimap X$ be given by

$$F(x,y,\lambda):=T(K(x,\lambda)\times\{(y,\lambda)\}),\quad x\in X,\ y\in Y,\ \lambda\in\Lambda.$$

Then, for $x\in X$ and $\lambda\in\Lambda$,

$$F\diamond C(x,\lambda)=F(\{x\}\times C(x,\lambda)\times\{\lambda\})=T(K(x,\lambda)\times C(x,\lambda)\times\{\lambda\})=\Phi(x,\lambda).$$

Obviously, for each $y\in Y$, $\lambda\in\Lambda$, $F(\,\cdot\,,y,\lambda)$ is a contraction with the continuous constant $k(\lambda)t(y,\lambda)$, $F(x,\,\cdot\,,\,\cdot\,)$ is upper semicontinuous on $Y\times\Lambda$ and, in view of Corollary 2.8 and Proposition 2.11, the map $\mathcal{F}:Y\times\Lambda\to X$ is acyclic-decomposable.

Definition 3.13. Given $\Phi_i \in \mathcal{K}(X,X)$ (resp. $\Phi_i \in \mathcal{K}_p(X,X)$), $i = 0,1$, we say that Φ_0 is *homotopic in* $\mathcal{K}$ (resp. in $\mathcal{K}_p$) to Φ_1 (written $\Phi_0 \simeq_{\mathcal{K}} \Phi_1$ (resp. $\Phi_0 \simeq_{\mathcal{K}_p} \Phi_1$)) if there is $\Phi \in \mathcal{K}(X \times [0,1], X)$ (resp. $\mathcal{K}_p(X \times [0,1], X)$) such that $\Phi_i = \Phi(\,\cdot\,, i)$. We say that Φ is a $\mathcal{K}$-homotopy (resp. $\mathcal{K}_p$-homotopy) joining Φ_0 to Φ_1. Obviously relation $\simeq_{\mathcal{K}}$ (resp. $\simeq_{\mathcal{K}_p}$) is an equivalence relation.

Example 3.14. Suppose that X is a convex subset of a Banach space, for $i = 0,1$, Y_i is a space, $F_i\colon X \times Y_i \multimap X$ and $C_i\colon X \to Y_i$ are set-valued maps with compact values such that C_i is compact acyclic-decomposable, for $y \in Y_i$, $F_i(\,\cdot\,,y)$ is a $k_i(y)$-contraction with convex values, $k_i\colon Y_i \to [0,1)$ is continuous and, for each $x \in X$, $F_i(x,\,\cdot\,)$ is upper semicontinuous. Then $\Phi_i := F_i \diamond C_i \in \mathcal{K}(X,X)$, $i = 0,1$, are permissible Krasnosel'skiĭ-type maps and, moreover they are $\mathcal{K}_p$-homotopic.

To see this observe that, by the Arens–Eells theorem (see e.g. [28]), both spaces Y_0 and Y_1 may be considered as the disjoint closed subsets of a normed space Y. Define $C\colon X \times [0,1] \multimap Y$ by the formula $C(x,\lambda) = (1-\lambda)C_0(x) + \lambda C_1(x)$ for $x \in X$. It is easy to verify that C is an acyclic-decomposable compact map. Next we show a construction of a certain extension $\widetilde{F}_i\colon X \times Y \multimap X$ of F_i, $i = 0,1$.

Let $i = 0$ and let $\{U_s, a_s\}_{s\in S}$ be the Dugundji system for $Y \setminus Y_0$ (see [7, Definition II.3.1]), i.e. S is the set of indices, $\{U_s\}_{s\in S}$ is a locally finite cover of $Y \setminus Y_0$, for all $s \in S$, $U_s \subset Y \setminus Y_0$ is open, $a_s \in Y_0$ and if $y \in U_s$, then $d(y,a_s) \le 2d(y,Y_0) := \inf_{a\in Y_0} d(y,a)$. Let $\{b_s\}_{s\in S}$ be a locally finite partition of unity subordinated to the covering $\{U_s\}_{s\in S}$. Consider a map $\widetilde{F}_0\colon X \times Y \multimap X$ given, for $x \in X$ and $y \in Y$, by the formula

$$\widetilde{F}_0(x,y) = \begin{cases} F_0(x,y) & \text{for } y \in Y_0, \\ \displaystyle\sum_{s\in S} b_s(y) F_0(x,a_s) & \text{for } y \in Y \setminus Y_0. \end{cases}$$

It is easy to show that, for any $x \in X$, $\widetilde{F}_0(x,\,\cdot\,)\colon Y \multimap X$ is upper semicontinuous (see the arguments in [7]) and has compact convex values. Let $y \in Y$; if $y \in Y_0$, then $\widetilde{F}_0(\,\cdot\,,y)$ is a $k_0(y)$-contraction. Suppose that $y \in Y \setminus Y_0$ and let $x, x' \in X$. Let $z \in \widetilde{F}_0(x,y)$. Hence $z = \sum_{s\in S} b_s(y) z_s$ where $z_s \in F_0(x,a_s)$. For each $s \in S$, there is $z_s' \in F_0(x',a_s)$ such that

$$\|z_s - z_s'\| = d(z_s, F_0(x',a_s)) \le \mathfrak{D}(F_0(x,a_s), F_0(x',a_s)) \le k(a_s)\|x - x'\|.$$

Clearly $z' := \sum_{s\in S} b_s(y) z_s' \in \widetilde{F}_0(x',y)$ and

$$\|z - z'\| \le \sum_{s\in S} b_s(y)\|z_s - z_s'\| \le \sum_{s\in S} b_s(y) k_0(a_s)\|x - x'\|.$$

Let, for $y \in Y$,

$$\widetilde{k}_0(y) = \begin{cases} k_0(y) & \text{for } y \in Y_0, \\ \sum_{s\in S} b_s(y)k_0(a_s) & \text{for } y \in Y \setminus Y_0. \end{cases}$$

Then $\widetilde{k}_0 \colon Y \to [0,1)$ is continuous and, for all $y \in Y$ and $x, x' \in X$,

$$\mathfrak{D}(\widetilde{F}_0(x,y), \widetilde{F}_0(x',y)) \le \widetilde{k}_0(y)\|x - x'\|.$$

Let $\widetilde{F}_1 \colon X \times Y \multimap X$ be an extension of F_1 constructed analogously (in particular, for each $y \in Y$, $\widetilde{F}_1(\,\cdot\,, y)$ is a $\widetilde{k}_1(y)$-contraction).

Finally define $F \colon X \times Y \times [0,1] \multimap X$ by the formula

$$F(x,y,\lambda) := (1-\lambda)\widetilde{F}_0(x,y) + \lambda\widetilde{F}_1(x,y), \quad x \in X,\ y \in Y.$$

It is then clear that, for each $x \in X$, $F(x,\,\cdot\,,\,\cdot\,) \colon Y \times [0,1] \multimap X$ is upper semicontinuous and, for all $y \in Y$ and $\lambda \in [0,1]$, $F(\,\cdot\,, y, \lambda) \colon X \multimap X$ is a $k(y,\lambda)$-contraction with $k(y,\lambda) := (1-\lambda)\widetilde{k}_0(y) + \lambda\widetilde{k}_1(y)$. Then $\Phi := F \diamond C$ is a desired $\mathcal{K}_p$-homotopy joining Φ_0 to Φ_1.

In the same manner one shows that any two Krasnosel'skiĭ-type maps of the form $F \diamond C$, where $F = F_n \circ \ldots \circ F_1$ is a composition of contractions with compact convex values, are $\mathcal{K}$-homotopic.

Our next observation situates the class of Krasnosel'skiĭ-type maps in the class of set-valued contractions with respect to the Hausdorff measure of noncompactness.

Proposition 3.15. *If $\Phi = F \diamond C \in \mathcal{K}(X,X)$, then Φ is a k-ball-contraction with $k := \sup\{k(y) \mid y \in C(X)\}$ i.e. for any bounded $B \subset X$,*

$$\beta(\Phi(B)) \le k\beta(B)$$

where β stands for the Hausdorff measure of noncompactness in X.

Proof. Let $B \subset X$ be bounded and $\mu := \beta(B)$. It is sufficient to show that, for any $\varepsilon > 0$, there is a compact set $Z \subset X$ such that $\Phi(B) \subset B(Z, k\mu + \varepsilon)$. To this end fix $\varepsilon > 0$. Then $B \subset \bigcup_{b \in I_B} B(b, \mu + k^{-1}\varepsilon)$ where the set $I_B \subset X$ is finite. Let $Z := F(I_B \times \operatorname{cl} C(B))$. The upper semicontinuity of F implies that Z is compact. Let $x \in \Phi(B)$. There is $x' \in B$, $b \in I_B$ and $y \in C(x)$, such that $x \in F(x', y)$ and $d(x', b) < \mu + k^{-1}\varepsilon$. Then

$$d(x, Z) \le d(x, F(b,y)) \le \mathfrak{D}(F(x',y), F(b,y)) \le k(y)d(x',b) < k\mu + \varepsilon. \qquad \square$$

Remark 3.16. If we assume that X is a closed convex and bounded subset of a Banach space, F and C are acyclic-decomposable maps, then Φ is an acyclic-decomposable map (see Remark 3.11) and, in view of [30] (see also [26]), Φ has fixed points. Moreover, in this case, the authors in [30] (comp. [26]) give a definition of a local index monitoring the existence of fixed points in an open subset of X (see Section 5.1; a more general approach to this problem has been presented in [21]). The attitude of [30], [26] or [21] can not be, unfortunately, applied in case X is an arbitrary ANR (see also Remark 3.3). This is understandable since even in the single-valued case the available fixed point techniques for k-set (-ball) -contractions (or condensing maps) require convex domains; some efforts to overcome this shortcoming has been undertaken in [42] where the so-called *special* ANRs and maps that remind condensing ones were studied. As we shall see below, the application of permissible Krasnosel'skiĭ-type maps allows to study their fixed points on arbitrary (complete) ANRs by means of homotopical methods. It seems therefore that (permissible) Krasnosel'skiĭ-type maps provide a compromise of sorts between compact and k-ball-contractions (or condensing maps) since they are properly chosen in order to deal with local homotopy invariants on arbitrary ANRs.

We are now in a position to define a fixed-point index for permissible Krasnosel'skiĭ-type maps. Namely, suppose that $\Phi = F \diamond C \in \mathcal{K}_p(X, X)$, a complete space $X \in$ ANR and let $U \subset X$ be open and such that $\mathrm{Fix}(\Phi) \cap \mathrm{bd}\, U = \emptyset$. Since the map $\mathcal{F} \circ C$ is acyclic-decomposable (as the composition of acyclic-decomposable maps) with the acyclic decomposition given as $D(\mathcal{F}) \circ D(C)$ (which, later on, will be identified with the pair $(\mathcal{F}, C)$) where $D(C)$ and $D(\mathcal{F})$ are the given acyclic decompositions of C and $\mathcal{F}$, respectively, and $\mathcal{F} \circ C$ is compact, we are in a position to define a *fixed-point index*

$$\mathrm{Ind}(\Phi, U) := \mathrm{ind}_S(X, (\mathcal{F}, C), U) \tag{3.4}$$

remembering that the index thus defined depends strongly on particular decompositions of $\mathcal{F}$ and C. It is clear that definition (3.4) is consistent with (3.1), i.e. if $\varphi = f \diamond c \in \mathcal{K}(X, X)$ is single-valued, then both definitions agree.

Let us collect some properties of Ind.

Theorem 3.17. *Let $\Phi = F \diamond C \in \mathcal{K}_p(X, X)$ where $X \in$ ANR and let $U \subset X$ be open.*

(a) (Existence) *If* $\mathrm{Fix}(\Phi) \cap \mathrm{bd}\, U = \emptyset$ *and* $\mathrm{Ind}(\Phi, U) \neq 0$, *then* $\mathrm{Fix}(\Phi) \cap U \neq \emptyset$.

(b) (Additivity) *If $U_1, \ldots, U_k \subset U$ are open disjoint and* $\mathrm{Fix}(\Phi) \cap \big[\mathrm{cl}\, U \setminus \bigcup_{i=1}^k U_i\big] = \emptyset$, *then*

$$\mathrm{Ind}(\Phi, U) = \sum_{i=1}^k \mathrm{Ind}(\Phi, U_i).$$

(c) (Homotopy invariance) *If $\Phi = F \diamond C \in \mathcal{K}_p(X \times [0,1], X)$ is a $\mathcal{K}_p$-homotopy joining $\Phi_0 := \Phi(\,\cdot\,, 0)$ to $\Phi_1 := \Phi(\,\cdot\,, 1)$ and* $\mathrm{Fix}(\Phi(\,\cdot\,, \lambda)) \cap \mathrm{bd}\, U = \emptyset$ *for each $\lambda \in [0,1]$, then* $\mathrm{Ind}(\Phi_0, U) = \mathrm{Ind}(\Phi_1, U)$.

Proof. (a) It is easy to see that $\mathrm{Fix}(\Phi) = \mathrm{Fix}(\mathcal{F} \circ C)$; if $\mathrm{Ind}(\Phi, U) \neq 0$, then $\mathrm{Fix}(\mathcal{F} \circ C) \neq \emptyset$.

(b) is an immediate consequence of the additivity property of ind_S.

To see (c) observe that, for $i = 0, 1$, $\Phi_i = F_i \diamond C_i$ where $F_i := F(\,\cdot\,, \,\cdot\,, i): X \times Y \multimap Y$, $C_i := C(\,\cdot\,, i): X \multimap Y$, for $y \in Y$,

$$\mathcal{F}_i(y) := \mathrm{Fix}(F_i(\,\cdot\,, y)) = \mathrm{Fix}(F(\,\cdot\,, y, i)) = \mathcal{F}(y, i).$$

Thus $\mathcal{F}_i$ is acyclic-decomposable with the acyclic decomposition being the restriction of the given acyclic decomposition of $\mathcal{F}$ to $Y \times \{i\}$. Therefore these acyclic decompositions are homotopic. Similarly, the acyclic decompositions of C_i, $i = 0, 1$, being restrictions of the given acyclic decomposition of C to $X \times \{i\}$, are homotopic. Hence the decompositions of Φ_i are homotopic in view of arguments similar to those from Remark 3.7. Thus, by the homotopy invariance of ind_S, we get the assertion. □

Remark 3.18. (a) Suppose that $\Phi = F \diamond C \in \mathcal{K}_p(X, X)$ where $X \in \mathrm{ANR}$ is complete, $F: X \times Y \multimap X$ and $C: X \multimap Y$. Suppose that $Y \in \mathrm{ANR}$ as well. Then the map $C \circ \mathcal{F}: Y \multimap Y$ is also acyclic-decomposable with the acyclic decomposition given as $D(C) \circ D(\mathcal{F})$. Given an open $V \subset Y$ such that $\mathrm{Fix}(C \circ \mathcal{F}) \cap \mathrm{bd}\, V = \emptyset$, the index $\mathrm{ind}_S(Y, D(C) \circ D(\mathcal{F}), V)$ is defined. Its nontriviality implies the existence of fixed points of Φ, too. Let, as above $U \subset X$ be open, $\mathrm{Fix}(\Phi) \cap \mathrm{bd}\, U = \emptyset$ and suppose that $V := \{y \in Y \mid \mathcal{F}(y) \subset U\}$. If $\mathrm{Fix}(C \circ \mathcal{F}) \cap \mathrm{bd}\, V = \emptyset$ and $\mathcal{F}[\mathrm{Fix}(C \circ \mathcal{F}) \setminus \mathrm{cl}\, V] \cap \mathrm{Fix}(\Phi|_U) = \emptyset$, then — in view of the commutativity property of ind_S, $\mathrm{Ind}(\Phi, U) = \mathrm{ind}_S(Y, D(C) \circ D(\mathcal{F}), V)$. In particular, if $U = X$, then $V = Y$ and $\mathrm{Ind}(\Phi, X) = L(D(C) \circ D(\mathcal{F}))$.

(b) The following slight generalization (in the spirit of the already mentioned observation of Burton in [10]) may be useful. Namely suppose that X' is a space and $\Phi = F \diamond C \in \mathcal{K}_p(X', X')$ where the maps $F: X' \times Y \multimap X'$ and $C: X' \multimap Y$ are as in Definition 3.9. Let $X \subset X'$ be a complete ANR and suppose that, for each $y \in \mathrm{cl}\, C(X)$, $\mathcal{F}(y) \subset X$. Then clearly the restriction $\mathcal{F} \circ C: X \multimap X$ and again the index $\mathrm{Ind}(\Phi, U)$ is defined provided $\mathrm{Fix}(\Phi) \cap \mathrm{bd}\, U = \emptyset$.

We shall now provide some simple statements which constitute direct extensions of the Krasnosel'skiĭ fixed point Theorem 1.1.

Proposition 3.19. *If* $\Phi = F \diamond C \in \mathcal{K}_p(X, X)$ *and* X *is a complete* AR, *then* $\mathrm{Ind}(\Phi, X) = 1$. *Similarly if* $Y \in \mathrm{AR}$ (*see Definition* 3.9), *then* $\mathrm{Ind}(\Phi, X) = 1$. *In particular, in both cases,* $\mathrm{Fix}(\Phi) \neq \emptyset$.

Proof. By the normalization property of ind_S,

$$\mathrm{Ind}(\Phi, X) := \mathrm{ind}_S(X, (\mathcal{F}, C), X) = L((\mathcal{F}, C)) = 1.$$

To see the last equality recall that, by Remark 3.8, $L((\mathcal{F},C)) = L((\mathcal{F},C,\mathrm{id}))$ where id is the identity on X. Since X is contractible, i.e. id is homotopic to the constant map $\mathrm{const}\colon X \to x_0 \in X$, we see, by Remark 3.7 and the homotopy invariance of ind_S, that $L((\mathcal{F},C)) = L(\mathrm{const}) = 1$.

The second statement follows from the commutativity property of ind_S (see Remark 3.18(a)):

$$\begin{aligned}\mathrm{Ind}(\Phi,X) &= \mathrm{ind}_S(X,(\mathcal{F},C),X)\\ &= \mathrm{ind}_S(Y,D(C)\circ D(\mathcal{F}),Y) = L(D(C)\circ D(\mathcal{F})) = 1.\end{aligned}$$

The assertions follow from the existence property of Ind. □

Proposition 3.19 may be easily generalized.

Corollary 3.20. *Suppose again that* $\Phi = F \diamond C \in \mathcal{K}_p(X,X)$*, where* X *is a complete* ANR*. If the space* Y *is contractible, then* $\mathrm{Fix}(\Phi) \neq \emptyset$*.*

Proof. Let $D(C)$ be an acyclic decomposition of $C\colon X \multimap Y$ and let $h\colon Y \times [0,1] \to Y$ be a homotopy joining the identity on Y to a point, i.e. $h(y,0) = y$ and $h(y,1) = y_0 \in Y$ for all $y \in Y$. Denote the constant map $Y \ni y \mapsto y_0$ by c. Then $\Phi' = F' \diamond C' \in \mathcal{K}_p(X \times [0,1], X)$ where $F'\colon X \times Y \times [0,1] \multimap X$ and $C'\colon X \times [0,1] \multimap Y$ are given by $F'(x,y,\lambda) = F(x,y)$ and $C'(x,\lambda) = h(C(x)\times\{\lambda\})$ for $x \in X$, $y \in Y$ and $\lambda \in [0,1]$. It is clear that Φ' provides a $\mathcal{K}_p$-homotopy joining Φ to $\Psi := F \diamond c$. Thus

$$\mathrm{Ind}(\Phi,X) = \mathrm{Ind}(\Psi,X) = L((\mathcal{F},c)).$$

But evidently $\mathcal{F}\circ c = \mathrm{Fix}(F(\,\cdot\,,y_0))$. Therefore, in view of the Units property of ind_S, we see that $\mathrm{Ind}(\Phi,X) = 1$. □

Having the general fixed-point index for Krasnosel'skiĭ-type set-valued maps on (complete) absolute neighbourhood retracts we may proceed as usual to get some natural consequences as concerns solvability of equations (inclusions) involving these maps (see for instance [28] where consequences of the Granas index are carefully studied). Leaving the most obvious results to the reader we shall study the generalized homotopy invariance which may be easily used as the source of various connectedness results (continuation, bifurcation, etc.).

Let Λ be a space. Given a set $A \subset E \times \Lambda$, where E is a Banach space, for $\lambda \in \Lambda$, we let $A(\lambda) := \{x \in E \mid (x,\lambda) \in A\}$. We say that $Z \subset E \times \Lambda$ is an ANR-*tube* if the map $\Lambda \ni \lambda \mapsto\!\!\!\circ\; Z(\lambda)$ is upper semicontinuous, there is an open neighbourhood $\Omega \subset E \times \Lambda$ of Z and a continuous map $r\colon \Omega \to E$ such that, for each $\lambda \in \Lambda$, $r(\,\cdot\,,\lambda)$ is a retraction of $\Omega(\lambda)$ onto $Z(\lambda)$. It is clear that, for any $\lambda \in \Lambda$, $Z(\lambda)$ is a complete ANR. For instance, if $X \subset E$ is an ANR, then $Z := X \times \Lambda$ is a *trivial* ANR-tube.

Let us consider the following situation. We assume that $Z \subset E \times \Lambda$ is an ANR-tube, Y is a space, $F\colon Z \times Y \multimap E$ and $C\colon Z \multimap Y$ are upper semicontinuous maps with compact values. Moreover we suppose that:

- for each $\lambda \in \Lambda$, $x \in Z(\lambda)$ and $y \in Y$, $F((x,\lambda),y) \subset Z(\lambda)$;
- C is a compact map;
- there is a continuous function $k\colon \Lambda \times Y \to [0,1)$ such that, for each $\lambda \in \Lambda$, $y \in Y$, if $x, x' \in Z(\lambda)$, then $\mathfrak{D}(F((x,\lambda),y), F((x',\lambda),y)) \le k(\lambda,y)\|x - x'\|$.

Let $\Phi(z) = F(z, C(z))$ for $z \in Z$. Then, for each $\lambda \in \Lambda$, $\Phi(\,\cdot\,,\lambda) \in \mathcal{K}(Z(\lambda), Z(\lambda))$. It is clear that if $Z = X \times \Lambda$ is a trivial ANR-tube, then $\Phi \in \mathcal{K}(X \times \Lambda, X)$.

Lemma 3.21. *Let $\mathcal{F}(\lambda, y) := \{x \in Z(\lambda) \mid x \in F((x,\lambda),y)\}$ for $\lambda \in \Lambda$ and $y \in Y$. Then the map $\mathcal{F}\colon \Lambda \times Y \multimap E$ is upper semicontinuous with nonempty compact values.*

Proof. It is clear that, for $(\lambda,y) \in \Lambda \times Y$, $\mathcal{F}(\lambda,y)$ is compact and nonempty since $Z(\lambda)$ is complete. Let $(\lambda_n, y_n, x_n) \in \mathrm{Gr}(\mathcal{F})$, i.e. $x_n \in Z(\lambda_n)$ and $x_n \in F((x_n,\lambda_n),y_n)$ for all $n \ge 1$, and let $(\lambda_n, y_n) \to (\lambda, y) \in \Lambda \times Y$. As in the proof of Proposition 2.9, there is a compact set $K \subset Z(\lambda)$ such that $K = F((K \times \{\lambda\}) \times \{y\})$. Since Ω (from the definition of an ANR-tube) is open, for large n, $n \ge N$ say, $K \subset \Omega(\lambda_n)$. Let $n \ge N$ and $x \in K$. Then

$$\begin{aligned} d(x_n, K) &\le \mathfrak{d}(F((x_n,\lambda_n),y_n), F((x,\lambda),y)) \\ &\le \mathfrak{d}(F((x_n,\lambda_n),y_n), F((r(x,\lambda_n),\lambda_n),y_n)) \\ &\quad + \mathfrak{d}(F((r(x,\lambda_n),\lambda_n),y_n), F((x,\lambda),y)) \\ &\le k(\lambda_n,y_n)(\|x_n - x\| + \|x - r(x,\lambda_n)\|) \\ &\quad + \mathfrak{d}(F((r(x,\lambda_n),\lambda_n),y_n), F((x,\lambda),y)) \end{aligned}$$

where $r\colon \Omega \to E$ is a retraction from the definition of an ANR-tube. Since $x \in K$ is arbitrary,

$$\begin{aligned} d(x_n, K) \le (1 - k(\lambda_n,y_n))^{-1} \sup_{x \in K} (k(\lambda_n,y_n)\|x - r(x,\lambda_n)\| \\ + \mathfrak{d}(F((r(x,\lambda_n),\lambda_n),y_n), F((x,\lambda),y))) \to 0 \end{aligned}$$

as $n \to \infty$. This implies that, up to a subsequence, $x_n \to x \in \mathcal{F}(\lambda, y)$ in view of the compactness of K and the upper semicontinuity of F. □

Further on let us assume additionally that:

- the maps $C\colon Z \multimap Y$ and $\mathcal{F}\colon \Lambda \times Y \multimap E$ are acyclic-decomposable.

Thus, for each $\lambda \in \Lambda$, $\Phi(\,\cdot\,,\lambda) \in \mathcal{K}_p(Z(\lambda), Z(\lambda))$ and the index $\mathrm{Ind}(\Phi(\,\cdot\,,\lambda), U(\lambda))$ is well-defined.

Theorem 3.22. *Under the above assumptions, suppose that Λ is path-connected, $U\subset Z$ is open and, for each $\lambda\in\Lambda$, $\mathrm{Fix}(\Phi(\,\cdot\,,\lambda))\cap\mathrm{bd}\,U(\lambda)=\emptyset$. Then, for all $\lambda,\mu\in\Lambda$,*

$$\mathrm{Ind}(\Phi(\,\cdot\,,\lambda),U(\lambda))=\mathrm{Ind}(\Phi(\,\cdot\,,\mu),U(\mu)).$$

Proof. For $\lambda\in\Lambda$, let $x\in K(\lambda)$ if and only if $(x,\lambda)\in\mathrm{cl}\,U$ and $x\in\Phi(x,\lambda)$. Then $K(\lambda)\subset Z(\lambda)$ and the correspondence $\Lambda\ni\lambda\multimap K(\lambda)$ has compact (possibly empty) values and is upper semicontinuous (in the usual sense). To see this let $\lambda_n\to\lambda\in\Lambda$ and $x_n\in K(\lambda_n)$; then $(x_n,\lambda_n)\in\mathrm{cl}\,U$ and $x_n\in\mathcal{F}(\lambda_n,y_n)$ where $y_n\in C(x_n,\lambda_n)$, $n\geq 1$. The compactness of C and the upper semicontinuity of $\mathcal{F}$ (see Lemma 3.21) imply that, up to a subsequence, $y_n\to y\in Y$ and $x_n\to x\in\mathcal{F}(\lambda,y)$. Thus $(x,\lambda)\in\mathrm{cl}\,U$ and $x\in K(\lambda)$ since $y\in C(x,\lambda)$.

Let $C_1\colon\Omega\multimap Y$ be given by

$$C_1(x,\lambda)=C(r(x,\lambda),\lambda)$$

for $\lambda\in\Lambda$ and $x\in\Omega(\lambda)$. It is obvious that $C_1=C$ on Z, C_1 is compact and acyclic-decomposable and so is the map $\Omega\ni(x,\lambda)\multimap\Psi(x,\lambda):=\mathcal{F}(\lambda,C_1(x,\lambda))\subset E$. For any $\lambda\in\Lambda$ and $x\in\Omega(\lambda)$, $\Psi(x,\lambda)\in Z(\lambda)\subset\Omega(\lambda)$ and $\mathrm{Fix}(\Psi(\,\cdot\,,\lambda))\cap\mathrm{bd}\,U(\lambda)=\emptyset$ since $\mathrm{Fix}(\Psi(\,\cdot\,,\lambda))=\mathrm{Fix}(\Phi(\,\cdot\,,\lambda))$. Therefore

$$\mathrm{Ind}(\Phi(\,\cdot\,,\lambda),U(\lambda))=\mathrm{ind}_S(Z(\lambda),D(\Psi(\,\cdot\,,\lambda))|_{Z(\lambda)},U(\lambda))$$

where $D(\Psi(\,\cdot\,,\lambda))$ stands for the composition of the acyclic decompositions $D(\mathcal{F}(\lambda,\,\cdot\,))$ and $D(C(\,\cdot\,,\lambda))$ of $\mathcal{F}(\lambda,\,\cdot\,)$ and $C(\,\cdot\,,\lambda)$, respectively. In view of the restriction property of ind_S (see Remark 3.8(d)), for $\lambda\in\Lambda$,

$$\mathrm{ind}_S(Z(\lambda),D(\Psi(\,\cdot\,,\lambda))|_{Z(\lambda)},U(\lambda))=\mathrm{ind}_S(\Omega(\lambda),D(\Psi(\,\cdot\,,\lambda)),V(\lambda))$$

where $V\subset\Omega$ is is open subset of $E\times\Lambda$ such that $V\cap Z=U$. Finally observe that the map $\Lambda\ni\lambda\multimap X(\lambda):=\mathrm{cl}\,\Psi(\Omega(\lambda)\times\{\lambda\})=\mathrm{cl}\,\Psi(Z(\lambda)\times\{\lambda\})$ is upper semicontinuous as the closure of the composition of Ψ with the upper semicontinuous map $\lambda\multimap Z(\lambda)\times\{\lambda\}$.

Let us fix $\lambda_0\in\Lambda$. The sets $X(\lambda_0)$ and $K(\lambda_0)$ are compact, $K(\lambda_0)\subset X(\lambda_0)\subset Z(\lambda_0)\subset\Omega(\lambda_0)$ and $K(\lambda_0)\subset V(\lambda_0)$. Therefore there are open neighbourhoods Ω_0, V_0, and Λ_0 of $X(\lambda_0)$, of $K(\lambda_0)$ and of λ_0, respectively, such that $V_0\subset\Omega_0$, $\Omega_0\times\Lambda_0\subset\Omega$, $V_0\times\Lambda_0\subset V$. In view of the upper semicontinuity of $X(\,\cdot\,)$, $K(\,\cdot\,)$, without loss of generality we may assume that $X(\lambda)\subset\Omega_0$ and $K(\lambda)\subset V_0$ for any $\lambda\in\Lambda_0$. In view of the additivity and restriction properties, for each $\lambda\in\Lambda_0$,

$$\begin{aligned}\mathrm{ind}_S(\Omega(\lambda),D(\Psi(\,\cdot\,,\lambda)),V(\lambda))&=\mathrm{ind}_S(\Omega(\lambda),D(\Psi(\lambda,\,\cdot\,)),V_0)\\&=\mathrm{ind}_S(\Omega_0,D(\Psi(\,\cdot\,,\lambda))|_{\Omega_0},V_0).\end{aligned}$$

Now let $\lambda,\mu\in\Lambda$. There is a path $\sigma\colon[0,1]\to\Lambda$ such that $\lambda=\sigma(0)$ and $\mu=\sigma(1)$. If $s_0\in[0,1]$, $\lambda_0=\sigma(s_0)$, then by the above argument and the homotopy property of ind_S, $\mathrm{Ind}(\Phi(\,\cdot\,,\sigma(s)),U(\sigma(s))$ is constant for s in a neighbourhood of s_0. The connectedness of $[0,1]$ implies that it is constant for $s\in[0,1]$. Hence the assertion. □

When studying generalizations of the Krasnosel'skiĭ theorem it makes sense to consider combinations of contractions with *completely continuous maps* instead of compact ones. In this way we get a variant of the result obtained in [11].

Theorem 3.23. *Let $\Phi=F\diamond C\in\mathcal{K}_p(X,X)$, where $F\colon X\times Y\multimap X$, $C\colon X\multimap Y$ and X is a Banach space* (*see Definition* 3.9)*, but assume that C is merely completely continuous* (*i.e. C is acyclic-decomposable and, for each bounded $B\subset X$, $\mathrm{cl}\,C(B)$ is compact*)*. Assume that there is $y_0\in Y$ such that, for sufficiently large $R>0$, if $x\in X$ and $\|x\|\le R$, then*

$$\mathfrak{D}(\Phi(x),F(x,y_0))\le lR \tag{3.5}$$

where l is a constant such that $0<l<1-k(y_0)$. Then Φ has fixed points.

Proof. Let $\xi\in\mathcal{F}(y_0)$ and define $\widetilde{F}\colon X\times Y$, $\widetilde{C}\colon X\multimap Y$ by

$$\widetilde{F}(x,y):=F(x+\xi,y)-\xi,\ \widetilde{C}(x):=C(x+\xi),\quad x\in X,\ y\in Y.$$

Then, for each $x,x'\in X$ and $y\in Y$,

$$\mathfrak{D}(\widetilde{F}(x,y),\widetilde{F}(x',y))=\mathfrak{D}(F(x+\xi,y),F(x'+\xi,y))\le k(y)\|x-x'\|.$$

Hence, for each $x\in X$ and $y\in Y$, $\widetilde{F}(\,\cdot\,,y)$ is a $k(y)$-contraction and $\widetilde{F}(x,\,\cdot\,)$ is upper semicontinuous; moreover $\widetilde{C}$ and $\widetilde{\mathcal{F}}\colon Y\multimap X$ (given by $\widetilde{\mathcal{F}}(y):=\mathrm{Fix}(\widetilde{F}(\,\cdot\,,y))=\mathcal{F}(y)-\xi$ for $y\in Y$) are acyclic-decomposable. Observe that, for $x\in X$,

$$\mathfrak{D}(\widetilde{F}(x,y_0),\widetilde{F}(0,y_0))\le k(y_0)\|x\|.$$

Let $\widetilde{\Phi}(x)=\widetilde{F}(x,\widetilde{C}(x))$ for $x\in X$, $M:=\mathfrak{D}(\widetilde{F}(0,y_0),\{0\})$, take a large $r>0$ (such that, for $R=r+\|\xi\|$, condition (3.5) is satisfied) and $x\in X$ such that $\|x\|\le r$. Then $\|x+\xi\|\le R$ and

$$\begin{aligned}\mathfrak{D}(\widetilde{\Phi}(x),\{0\})&\le\mathfrak{D}(\widetilde{\Phi}(x),\widetilde{F}(x,y_0))+\mathfrak{D}(\widetilde{F}(x,y_0),\widetilde{F}(0,y_0))+M\\&\le l(r+\|\xi\|)+k(y_0)r+M\le r\end{aligned}$$

provided $r\ge(1-l-k(y_0))^{-1}(l\|\xi\|+M)$. Thus $\widetilde{\Phi}=\widetilde{F}\diamond\widetilde{C}\in\mathcal{K}_p(D(0,r),D(0,r))$ since $\widetilde{C}|_{D(0,r)}$ is compact. Obviously $D(0,r)$ is an AR and, in view of Proposition 3.19, there is $x_0\in D(0,r)$ such that $x_0\in\widetilde{\Phi}(x_0)$. Hence $x_0+\xi\in\mathrm{Fix}(\Phi)$. □

4. Applications

It appears that Krasnosel'skiĭ-type maps admit several nontrivial applications.

4.1. Coupled differential inclusions with constraints. Let E_1, E_2 be separable Banach spaces and let $J = [0,T]$ where $T > 0$. Suppose that sets $X \subset E_1$, $Y \subset E_2$ are closed bounded and let $\mathfrak{F}\colon J \times E_1 \times Y \multimap E_1$, $\mathfrak{G}\colon J \times X \times Y \multimap E_2$ be set-valued maps. Given $x_0 \in X$, $y_0 \in Y$, we look for solutions to the system of coupled differential inclusions:

$$\begin{cases} x'(t) \in \mathfrak{F}(t, x(t), y(t)), & x(0) = x_0, \\ y'(t) \in \mathfrak{G}(t, x(t), y(t)), & y(0) = y_0. \end{cases} \tag{4.1}$$

By a solution we mean a pair (x, y) of continuous functions $x\colon J \to X$, $y\colon J \to Y$ such that, for all $t \in J$, $x(t) = x_0 + \int_0^t w(s)\,ds$ and $y(t) = y_0 + \int_0^t u(s)\,ds$ where $w\colon J \to E_1$, $u\colon J \to E_2$ are (Bochner) integrable functions such that $w(s) \in \mathfrak{F}(s, x(s), y(s))$ and $u(s) \in \mathfrak{G}(s, x(s), y(s))$ for almost all $s \in J$ (in particular, functions x, y are absolutely continuous).

Let us make the following standing assumptions:

Assumption 4.1.

(a) The maps $\mathfrak{F}$, $\mathfrak{G}$ have compact convex (and nonempty) values;
(b) $\mathfrak{F}$ and $\mathfrak{G}$ are *upper-Carathéodory* maps, i.e. for all (x, y) (in $E_1 \times Y$ and $X \times Y$, respectively), $\mathfrak{F}(\,\cdot\,, x, y)$, $\mathfrak{G}(\,\cdot\,, x, y)$ are (strongly) measurable on J and, for all $t \in J$, $\mathfrak{F}(t, \,\cdot\,, \,\cdot\,)$, $\mathfrak{G}(t, \,\cdot\,, \,\cdot\,)$ are (jointly) upper semicontinuous (on $E_1 \times Y$ and $X \times Y$, respectively); moreover, $\mathfrak{F}$ is *product measurable* ([13]);
(c) $\mathfrak{F}$, $\mathfrak{G}$ are integrably bounded, i.e. there is an integrable function $c\colon J \to \mathbb{R}$ such that, for all $t \in J$, $x \in X$ and $y \in Y$,

$$\sup_{z \in \mathfrak{F}(t,x,y)} \|z\|, \quad \sup_{z \in \mathfrak{G}(t,x,y)} \|z\| \le c(t);$$

(d) for each $t \in J$ and $y \in Y$, the map $\mathfrak{F}(t, \,\cdot\,, y)\colon E_1 \multimap E_1$ is $L(t)$-*Lipschitz*, i.e. for $x_1, x_2 \in E_1$, $\mathfrak{D}(\mathfrak{F}(t, x_1, y), \mathfrak{F}(t, x_2, y)) \le L(t)\|x_1 - x_2\|$ where $L \in L^1(J, \mathbb{R})$;
(e) $\mathfrak{F}$ maps compact subsets of $J \times E_1 \times Y$ into compact ones;
(f) the map $\mathfrak{G}$ is *compact*, i.e. the set $\mathfrak{G}(J \times X \times Y)$ is relatively compact.

([13]) I.e. measurable with respect to the product σ-algebra of the Lebesgue σ-algebra in J and the σ-algebra of Borel subsets in $E_1 \times Y$. Observe also that since the considered spaces are separable, strong measurability coincides with the usual one.

As we see the assumptions are rather mild. As a consequence of (a) and (b), maps $\mathfrak{F}$, $\mathfrak{G}$ are weakly superpositionally measurable (in view of the product measurability, $\mathfrak{F}$ is even superpositionally measurable and this is what we actually need in what follows), i.e. given continuous functions $x: J \to E_1$ (resp. $x: J \to X$) and $y: J \to Y$, the map $\mathfrak{F}(\,\cdot\,, x(\,\cdot\,), y(\,\cdot\,))$ is measurable (resp. the map $\mathfrak{G}(\,\cdot\,, x(\,\cdot\,), y(\,\cdot\,))$ has a measurable selection). Therefore, by (c), the (set-valued) *Nemytskiĭ operators* $N_{\mathfrak{F}}: C(J, E_1) \times C(J, Y) \multimap L^1(J, E_1)$, $N_{\mathfrak{G}}: C(J, X) \times C(J, Y) \multimap L^1(J, E_2)$ given by

$$N_{\mathfrak{F}}(x, y) := \{w \in L^1(J, E_1) \mid w(s) \in \mathfrak{F}(s, x(s), y(s)) \text{ a.e. on } J\},$$
$$N_{\mathfrak{G}}(x, y) := \{u \in L^1(J, E_2) \mid u(s) \in \mathfrak{G}(s, x(s), y(s)) \text{ a.e. on } J\},$$

for $x \in C(J, E_1)$ (resp. $x \in C(J, X)$) and $y \in C(J, Y)$ ([14]), are well-defined. Condition (e) is satisfied if, for example, $\mathfrak{F}$ is upper semicontinuous.

Observe that the immediate approach that reduces the existence of solutions of (4.1) to the viablity issue for the Cauchy initial problem of the form

$$u'(t) = \mathfrak{H}(t, u(t)), \quad u(0) = (x_0, y_0),$$

where $u = (x, y): J \to X \times Y$, $\mathfrak{H}(t, x, y) := \mathfrak{F}(t, x, y) \times \mathfrak{G}(t, x, y)$, fails since the map $\mathfrak{H}$ does not have sufficient compactness properties in y. Therefore we shall suitably convert the solvability of (4.1) into the fixed point problem for Krasnosel'skiĭ-type maps.

First we shall try to explain our setting. Let

$$\mathcal{X} := C(J, X), \quad \mathcal{Y} := C(J, Y).$$

Suppose that, for each $x \in \mathcal{X}$, the set

$$C(x) := \left\{ y \in \mathcal{Y} \;\middle|\; y(t) = y_0 + \int_0^t u(s)\, ds \text{ for some } u \in N_{\mathfrak{G}}(x, y) \right\}$$

is nonempty and closed, i.e. $C: \mathcal{X} \multimap \mathcal{Y}$ and that $F: C(J, E_1) \times \mathcal{Y} \multimap C(J, E_1)$ is given, by

$$u \in F(x, y) \iff \exists\, w \in N_{\mathfrak{F}}(x, y) \quad u(t) = x_0 + \int_0^t w(s)\, ds$$

for $x \in C(J, E_1)$ and $y \in \mathcal{Y}$, is well-defined. Then the map $\Phi = F \diamond C: \mathcal{X} \multimap C(J, E_1)$ given by $\Phi(x) = F(x \times C(x))$ is defined and if, additionally, for each $y \in \mathcal{Y}$, the set $\mathcal{F}(y) = \operatorname{Fix}(F(\,\cdot\,, y)) \subset \mathcal{X}$, then $\mathcal{F} \circ C: \mathcal{X} \multimap \mathcal{X}$ and $x \in \operatorname{Fix}(\Phi)$ if and only if $x \in \mathcal{F} \circ C(x)$ if and only if there is $y \in C(x)$ such that $x \in F(x, y)$, i.e. the pair $(x, y) \in \mathcal{X} \times \mathcal{Y}$ is a solution to (4.1).

([14]) $C(J, E_1)$ and $C(J, X)$, $C(J, Y)$ stand for the Banach space of continuous functions $J \to E_1$ with the usual max-norm and the sets of continuous functions $J \to X$, $J \to Y$, respectively.

In order to use our theory (in particular Proposition 3.19) we shall make some additional hypotheses.

Assumption 4.2.

(a) The set X is a bounded neighbourhood retract in E_1 and, for any $y \in \mathcal{Y}$, X is strongly invariant with respect to the flow generated by the map $J \times E_1 \ni (t,x) \multimap \mathfrak{F}(t,x,y(t))$ (for the definition and the discussion of the concept of *strong invariance* of a set with respect to a flow — see Section 5.2);
(b) Y is closed convex bounded and, for all $t \in J$, $x \in X$, $y \in Y$, $\mathfrak{G}(t,x,y) \cap T_Y(y) \neq \emptyset$.

Above for $y \in Y$, $T_Y(y)$ stands for the *Bouligand contingent cone* (see e.g. [2]), i.e.

$$T_Y(y) := \left\{ v \in E_2 \;\middle|\; \liminf_{t \to 0^+} \frac{d_Y(y+tv)}{t} = 0 \right\},$$

where d_Y is the distance Y (i.e. for instance $d_Y(y) = \inf_{z \in Y} \|z-y\|$ for $y \in E_2$). Since Y is convex, for all $y \in Y$,

$$T_Y(y) = \operatorname{cl} \bigcup_{h>0} \frac{Y-y}{h}.$$

For technical reasons we introduce a new norm in $C(J,E_1)$ given, for $x \in C(J,E_1)$, by

$$\|x\|_B := \sup_{t \in J} e^{-\int_0^t L(s)\,ds} \|x(t)\|$$

for $x \in C(J,E_1)$. This formula correctly defines a (complete) norm ([15]) equivalent to the usual max-norm.

Let us now make the following statements.

Proposition 4.3. *Under Assumptions* 4.1 *and* 4.2*:*

(a) *the set $\mathcal{X}$ is a neighbourhood retract of $C(J,E_1)$ (and therefore $\mathcal{X} \in$ ANR), the set $\mathcal{Y}$ is closed convex in $C(J,E_2)$;*
(b) *the above map $C\colon \mathcal{X} \multimap \mathcal{Y}$ is well defined, compact and acyclic-decomposable;*
(c) *the map above $F\colon C(J,E_1) \times \mathcal{Y} \multimap C(J,E_1)$ is well-defined, has compact convex values, for each $x \in C(J,E_1)$, the map $F(x,\cdot\,)\colon \mathcal{Y} \multimap C(J,E_1)$ is upper semicontinuous;*
(d) *for each $y \in \mathcal{Y}$, the map $F(\,\cdot\,,y)\colon (C(J,E_1),\|\cdot\|_B) \multimap (C(J,E_1),\|\cdot\|_B)$ is a k-contraction with $k \in [0,1)$ and $\mathcal{F}(y) := \operatorname{Fix}(F(\,\cdot\,,y)) \subset \mathcal{X}$.*

([15]) Being a variant of the well-know Bielecki norm.

Proof. (a) Let U be a neighbourhood of X such that there is a retraction $r: U \to X$. Clearly $\mathcal{U} := \{x \in C(J, E_1) \mid x(J) \subset U\}$ is open in $C(J, E_1)$ (recall that we consider the compact-open topology on $C(J, E_1)$). Let $R: \mathcal{U} \to \mathcal{X}$ be given by: $R(x)(t) := r(x(t))$. It is easy to see that R is a (continuous) retraction of $\mathcal{U}$ onto $\mathcal{X}$. It is evident that $\mathcal{Y}$ is closed and convex.

(b) By [4, Theorem 20], for $x \in \mathcal{X}$, $C(x)$ is an R_δ-set in $C(J, E_2)$ (see Section 5.1 for the definition of R_δ-set); hence acyclic. We shall establish the compactness and upper semicontinuity of C. To this aim let $x_n \in \mathcal{X}$ and let $y_n \in C(x_n)$, $n \in \mathbb{N}$. It is now sufficient to show that the sequence (y_n) has a convergent subsequence. For each $n \in \mathbb{N}$, there is $u_n \in N_{\mathfrak{G}}(x_n, y_n)$ such that $y_n(t) = y_0 + \int_0^t u_n(s)\,ds$. Assumption 4.1(c) implies that the sequence (u_n) is integrably bounded (i.e. there is a function $c \in L^1(J)$ such that $\|u_n(t)\| \leq c(t)$). Hence the sequence (y_n) is uniformly equicontinuous. The compactness of $\mathfrak{G}$ implies that the fibres $\{y_n(t)\}$, $t \in J$, are relatively compact. Hence, by the Ascoli–Arzela Theorem, the sequence (y_n) has a convergent subsequence. To get the upper semicontinuity it is enough to show that if $x_n \to x \in \mathcal{X}$ and $y_n \to y$, then $y \in C(x)$. Observe that the sequence (u_n), being integrably bounded and having compact fibres, is — by the Diestel theorem (see [16, Corollary 3]) — relatively weakly compact in $L^1(J, E_2)$, i.e. (passing to a subsequence if necessary) we may assume that $u_n \rightharpoonup u \in L^1(J, E_2)$ (weak convergence in L^1). The function y_n is almost everywhere differentiable and $y_n' = u_n$ a.e. for all $n \in \mathbb{N}$. Since $y_n \to y$ in $C(J, E_2)$ and $u_n \rightharpoonup u$, by the so-called Compactness Theorem (see [1, Theorem 0.3.4]), we infer that $y(t) = y_0 + \int_0^t u(s)\,ds$, i.e. $y'(t) = u(t)$ a.e. on J. Summing up: for each $n \in \mathbb{N}$, $u_n(t) = y_n'(t) \in \mathfrak{G}(t, x_n(t), y_n(t))$ a.e. on J, $x_n \to x$ in $C(J, E_1)$, $y_n \to y$ in $C(J, E_2)$ and $u_n \rightharpoonup u$ in L^1. Thus, by the Convergence Theorem [2, Theorem 7.2.2], $y'(t) = u(t) \in \mathfrak{G}(t, x(t), y(t))$ a.e. on J. Hence $y \in C(x)$.

(c) It is clear that values of F are convex. In order to show that values of F are compact and that, for a fixed function $x \in C(J, E_1)$, $F(x, \cdot\,)$ is upper semicontinuous it is sufficient to show that given a sequence $(y_n, z_n) \in \operatorname{Gr}(F(x, \cdot\,))$, if $y_n \to y$ in $C(J, E_1)$, then there is a subsequence (z_{n_k}) such that $z_{n_k} \to z \in F(x, y)$. By definition, for each $n \in \mathbb{N}$ and $t \in J$, $z_n(t) = x_0 + \int_0^t w_n(s)\,ds$ where $w_n \in N_{\mathfrak{F}}(x, y_n)$. The compactness of fibres of the sequence (y_n), Assumption 4.1(c) together with the compactness of values of $\mathfrak{F}(t, x(t), \cdot\,)$, $t \in J$, implies that the sequence (w_n) is integrably bounded with relatively compact fibres; hence — again, by the already used result due to Diestel and passing to a subsequence if necessary — we may assume that $w_n \rightharpoonup w \in L^1(J, E_1)$. It is clear that, for each $t \in J$ and $n \geq 1$,

$$z_n(t) - x_0 = \int_0^t w_n(s)\,ds \in t \operatorname{cl} \operatorname{conv} F([0, t] \times x([0, t]) \times K)$$

where $K := \{y_n(s) \mid n \in \mathbb{N},\ s \in [0,t]\}$ is compact. Thus, by Assumption 4.1(e), we gather that the fibres of the sequence (z_n) are compact. Since this sequence is equicontinuous, again by the Ascoli–Arzela and Compactness theorems (passing to a subsequence) we may assume that $z_n \to z \in C(J,E_1)$ and $z(t) - x_0 = \int_0^t w(s)\,ds$. Again the use of the Convergence Theorem shows that $z \in F(x,y)$.

(d) Fix $y \in \mathcal{Y}$, let $x_1, x_2 \in C(J,E_1)$ and take an arbitrary $z_1 \in F(x_1,y)$. Then $z_1(t) = x_0 + \int_0^t w_1(s)\,ds$ where $w_1 \in N_{\mathfrak{F}}(x_1,y)$. The mentioned superpositional measurability of $\mathfrak{F}$ implies that the map $J \ni s \multimap \mathfrak{F}(s,x_2(s),y(s))$ is measurable and, hence, the function $J \ni s \mapsto d(w_1(s),\mathfrak{F}(s,x_2(s),y(s))$ is measurable, too. Therefore, for any $\varepsilon > 0$, there is an integrable selection $w_2(\,\cdot\,) \in \mathfrak{F}(\,\cdot\,,x_2(\,\cdot\,),y(\,\cdot\,))$ for a.e. $s \in J$ such that

$$\begin{aligned}\|w_1(s) - w_2(s)\| &\le d(w_1(s),\mathfrak{F}(s,x_2(s),y(s))) + \varepsilon\\ &\le \mathfrak{D}(\mathfrak{F}(s,x_1(s),y(s)),\mathfrak{F}(s,x_2(s),y(s))) + \varepsilon\\ &\le L(s)\|x_1(s) - x_2(s)\| + \varepsilon\end{aligned}$$

for almost all $s \in J$. Let $z_2(t) := x_0 + \int_0^t w_2(s)\,ds$. Then $z_2 \in F(x_2,y)$ and, for each $t \in J$,

$$\begin{aligned}e^{-\int_0^t L(s)\,ds}\|z_1(t) - z_2(t)\| &\le e^{-\int_0^t L(s)\,ds}\int_0^t \|w_1(s) - w_2(s)\|\,ds\\ &\le e^{-\int_0^t L(s)\,ds}\|x_1 - x_2\|_B \int_0^t L(s)e^{\int_0^s L(\tau)\,d\tau}\,ds + T\varepsilon\\ &= (1 - e^{-\int_0^t L(s)\,ds})\|x_1 - x_2\|_B + T\varepsilon \le k\|x_1 - x_2\|_B + T\varepsilon\end{aligned}$$

where $k := (1 - e^{-\int_0^T L(s)\,ds}) < 1$ (enlarging L if necassary we may assume that $\int_0^T L(s)\,ds > 0$). Hence, if we denote by d^B, $\mathfrak{d}^B$ and $\mathfrak{D}^B$ the distance, the Hausdorff "half"-distance and the Hausdorff distance induced by $\|\cdot\|_B$, respectively, then

$$d^B(z_1,F(x_2,y)) \le k\|x_1 - x_2\|_B.$$

Since z_1 was arbitrary, this implies that

$$\mathfrak{d}^B(F(x_1,y),F(x_2,y)) \le k\|x_1 - x_2\|$$

and, analogously

$$\mathfrak{d}^B(F(x_2,y),F(x_1,y)) \le k\|x_1 - x_2\|,$$

i.e.

$$\mathfrak{D}^B(F(x_1,y),F(x_2,y)) \le k\|x_1 - x_2\|_B.$$

The second statement of (d) follows from Assumption 4.2(a). This completes the proof. $\square$

Theorem 4.4. *Under Assumptions* 4.1 *and* 4.2*, problem* (4.1) *admits a solution.*

Proof. According to Remark 3.18(b), the index $\mathrm{Ind}(\Phi,\mathcal{X})$, where $\Phi := F\diamond C$, is defined. Remembering that $\mathcal{Y}$ is convex, i.e. $\mathcal{Y}$ is an AR, by Proposition 3.19, $\mathrm{Ind}(\Phi,\mathcal{X}) = 1$. □

Remark 4.5. (a) The separability of E_2 is not necessary if $\mathfrak{G}$ is upper semicontinuous; otherwise we need to apply the so-called Scorza-Dragoni property due to Rzeżuchowski which require separability (see [4, Theorem 1, Remark 24]). On the other hand, without separability of E_1 or E_2 the application of the Compactness Theorem from [1] is impossible and in order to proceed one has to use slightly different and more involved argument (the correct proof of the Compactness theorem makes use of the Phillips theorem asserting that $L^1(J,E_i)$ may be identified with $L^\infty(J,E_i^*)$ provided E_i is reflexive or separable).

(b) The convexity of Y may be replaced by the so-called epi-Lipschitzeanity of Y or its strict regularity in the sense of [4]. In this case however one has to assume that additionally Y is an AR. If not, then in order to have Theorem 4.4 one has to assume that $X \in \mathrm{AR}$.

(c) It would be interesting to obtain an existence result concerning the solvability of a coupled system of semilinear differential inclusions of the form

$$\begin{cases} x'(t) \in Ax(t) + \mathfrak{F}(t,x(t),y(t)), & x(0) = x_0; \\ y'(t) \in By(t) + \mathfrak{G}(t,x(t),y(t)), & y(0) = y_0, \end{cases}$$

where A, B are the (infinitesimal) generators of C_0-semigroups of linear bounded operators acting on E_1 and E_2, respectively, $\mathfrak{F}$, $\mathfrak{G}$ satisfies the above assumptions but instead of the compactness of $\mathfrak{G}$, the semigroup generated by B is compact. It seems that if the semigroup generated by A is uniformly bounded, then it is not a difficult task. The main difficulty is to get the strong invariance of the first of the above inclusions with respect to X.

4.2. Constrained compact-periodic problem. Suppose now that E is a separable Banach space, X is a neighbourhood retract in E and $G: X \multimap X$ is a compact acyclic-decomposable map with a given acyclic decomposition $D(G)$. Let $\mathfrak{F}: J \times E \multimap E$, where $J = [0,T]$, be a set-valued map with compact convex values such that:

Assumption 4.6.

(a) for each $x \in E$, $\mathfrak{F}(\,\cdot\,,x)$ is measurable, for each $t \in J$ and $x_1, x_2 \in E$,

$$\mathfrak{D}(\mathfrak{F}(t,x_1),\mathfrak{F}(t,x_2)) \le L(t)\|x_1 - x_2\|$$

where L is a positive integrable function and the function $J \ni t \mapsto \sup_{z\in\mathfrak{F}(t,0)}\|z\|$ is integrable ([16]);

(b) X is strongly invariant with respect to the flow generated by $\mathfrak{F}$.

Theorem 4.7. *Under the Assumption* 4.6*, the following compact-periodic problem*

$$u' \in \mathfrak{F}(t,u), \quad u \in X, \quad u(0) \in G(u(T))$$

admits a solution, i.e. there is an absolutely continuous function $u\colon J \to X$ such that $u'(t) \in \mathfrak{F}(t,u(t))$ a.e. on J and $u(0) \in G(u(T))$, provided the Lefschetz number $L(D(G)) \neq 0$.

Proof. Consider a map $F\colon C(J,E)\times E \multimap C(J,E)$ given by the formula

$$F(u,y) = \left\{x \in C(J,E) \;\middle|\; x(t) = y + \int_0^t w(s) \text{ for some } w \in N_{\mathfrak{F}}(u)\right\}, \qquad y \in E,\ u \in C(J,E).$$

Recall that $N_{\mathfrak{F}}(u) := \{w \in L^1(J,E) \mid w(s) \in \mathfrak{F}(s,u(s)) \text{ a.e. on } J\}$. Observe that in view of Assumption 4.6(a), $\mathfrak{F}$ has linear growth and, hence, $N_{\mathfrak{F}}\colon C(J,E) \multimap L^1(J,E)$ is well-defined. It is clear that, for each $u \in C(J,E)$, $F(u,\cdot\,)\colon E \multimap C(J,E)$ is compact convex valued and upper semicontinuous. As before we check that after the appropriate Bielecki renorming procedure, for each $y \in E$, $F(\,\cdot\,,y)\colon C(J,E) \multimap C(J,E)$ is a k-contraction where $k := 1 - e^{-\int_0^T L(s)\,ds}$.

Exactly as before we see that, in view of Assumption 4.6(b),

$$\mathcal{F}(y) = \operatorname{Fix}(F(\,\cdot\,,y)) \subset \mathcal{X} := C(J,X) \quad \text{for each } y \in X$$

(evidently $\mathcal{X} \in \mathrm{ANR}$). Let $C\colon C(J,E) \multimap E$ be given by the composition $\widetilde{G}\circ e_T$ where $e_T(u) = u(T)$ for $u \in C(J,E)$ and $\widetilde{G}\colon E \multimap E$ is an arbitrary compact acyclic-decomposable extension of G (i.e. $\widetilde{G}(y) = G(y)$ for $y \in X$). This extension may be produced as follows: let $r\colon U \to X$ be a neighbourhood retraction, let V be open and such that $X \subset V \subset \operatorname{cl} V \subset U$ and let $\phi\colon E \to [0,1]$ be continuous and $\phi|_X \equiv 1$, $\phi|_{E\setminus V} \equiv 0$. We put $\widetilde{G}(x) = \phi(x)G\circ r(x)$ for $x \in U$ and $\widetilde{G}(x) := \{0\}$ for $x \notin U$. It is clear that C is acyclic-decomposable having $(e_T, D(\widetilde{G}))$ as the acyclic decomposition.

Clearly $\Phi := F \diamond C \in \mathcal{K}(C(J,E),C(J,E))$ and, for each $y \in \operatorname{cl} C(\mathcal{X}) \subset X$, $\mathcal{F}(y) \subset \mathcal{X}$. Hence (see Remark 3.18(b)), the index $\operatorname{Ind}(\Phi,\mathcal{X})$ is defined. It is sufficient to show that $\operatorname{Ind}(\Phi,X) \neq 0$. Indeed in this case there is $u \in \mathcal{X}$ such that $u \in \mathcal{F}(C(u))$, i.e. $u(t) \in G(u(T)) + \int_0^t w(s)\,ds$, where $w \in N_{\mathfrak{F}}(u)$; thus $u(0) \in G(u(T))$ and $u'(t) = w(t) \in \mathfrak{F}(t,u(t))$ a.e. on J.

([16]) We see that, in fact $\mathfrak{F}$ is a Carathéodory map, i.e. $\mathfrak{F}(\,\cdot\,,x)$ is measurable and $\mathfrak{F}(t,\cdot\,)$ is continuous; consequently $\mathfrak{F}$ is product-measurable, hence $\mathfrak{F}(\,\cdot\,,u(\,\cdot\,))$ is measurable for any continuous function $u\colon J \to E$.

To complete the proof observe that $\mathcal{F}\circ C|_{\mathcal{X}}$ has the following acyclic decomposition

$$D(\mathcal{F}\circ C)\colon \mathcal{X} \xrightarrow{e_T} X \overset{D(G)}{\multimap} X \overset{\mathcal{F}}{\multimap} \mathcal{X}.$$

According to the commutativity property of the Skordev index, we have (comp. Remark 3.18(b))

$$\operatorname{Ind}(\Phi,\mathcal{X}) = \operatorname{ind}_S(X,D,X),$$

where

$$D\colon X \overset{\mathcal{F}}{\multimap} \mathcal{X} \xrightarrow{e_T} X \overset{D(G)}{\multimap} X.$$

Consider an acyclic-decomposable map $H\colon X\times[0,1]\multimap X$ having an acyclic decomposition

$$D(H)\colon X\times[0,1] \overset{\overline{\mathcal{F}}}{\multimap} \mathcal{X}\times[0,1] \xrightarrow{e} X \overset{D(G)}{\multimap} X$$

where $\overline{\mathcal{F}}(y,\lambda)=\mathcal{F}(y)\times\{\lambda\}$ and $e(u,\lambda):=e_{\lambda T}(u)=u(\lambda T)$ for $y\in X$, $u\in\mathcal{X}$ and $\lambda\in[0,1]$. It is clear that H provides a homotopy joining the decomposition D to

$$D'\colon X \overset{\mathcal{F}}{\multimap} \mathcal{X} \xrightarrow{e_0} X \overset{D(G)}{\multimap} X.$$

Hence $\operatorname{ind}_S(X,D,X)=\operatorname{ind}_S(X,D',X)$. On the other hand, the decomposition D' is related to the decomposition

$$D''\colon X \xrightarrow{\mathrm{id}} X \xrightarrow{\mathrm{id}} X \overset{D(G)}{\multimap} X.$$

To see this take $h_1\colon\mathcal{X}\to X$ and $h_0=h_2=h_3=\mathrm{id}\colon X\to X$ as $h_1\colon\mathcal{X}\to X$ is given by $h_1(u):=e_0(u)=u(0)\in X$ for $u\in\mathcal{X}$ and compare Remark 3.8. Therefore

$$\operatorname{ind}_S(X,D',X)=\operatorname{ind}_S(X,D'',X)=\operatorname{ind}_S(X,D(G),X)=L(D(G))\neq 0. \quad \square$$

4.3. Periodic problem for feedback controlled semilinear constrained differential equations. Suppose that E is a separable Banach space, X is a neighbourhood retract in E and let, as above, $A\colon D(A)\to E$ be the infinitesimal generator of a C_0-semigroup $\mathcal{U}=\{U(t)\}_{t\geq 0}$ of bounded linear operators $U(t)\colon E\to E$ such that $\|U(t)\|\leq e^{-\omega t}$, where $\omega>0$, for all $t\geq 0$. Suppose that Y is a compact space and let $\mathfrak{F}\colon[0,T]\times E\times Y\multimap E$ be a set-valued map.

We are going to study the existence periodic trajectories $x\colon[0,T]\to X$ of the feedback controlled semilinear differential inclusion of the form

$$x'(t)\in Ax(t)+\mathfrak{F}(t,x(t),y(t)),\quad y\in C(x), \tag{4.2}$$

where the *impulsive feedback rule* $C(x)$ defining the feedback control $y(\cdot)$ is defined as follows. Assume that a finite sequence of prescribed *switching times*, i.e. a partition $\{t_0,\ldots,t_n\}$, where $0=t_0<\ldots<t_n=T$, of the interval $J:=[0,T]$ and a *control rule*, i.e. a set-valued map $c\colon E\multimap Y$ are given. For

any function $x: J \to E$, let $C(x)$ be the set of functions $y: J \to Y$ such that $y(t) = y_i$ for $t \in [t_i, t_{i+1})$, $i = 0, \dots, n-2$, and $y(t) = y_{n-1}$ for $t \in [t_{n-1}, T]$, where $y_i \in c(x(t_i))$ for $i = 0, \dots, n-1$.

By a (mild) *solution* to our problem (4.2) we mean a continuous function $x: J \to E$ such that $x(0) = x(T)$, $x(t) \in X$ and

$$x(t) = U(t)x(0) + \int_0^t U(t-s)w(s)\,ds$$

for all $t \in J$, with $w(s) \in \mathfrak{F}(s, x(s), y(s))$, where $y \in C(x)$.

Let $\mathcal{Y}$ denote the space of piecewise constant functions on J with values in Y, i.e. $y \in \mathcal{Y}$ if and only if there are points $y_i \in Y$, $i = 0, \dots, n-1$, such that $y(t) = y_i$ for $t \in [t_i, t_{i+1})$, $i = 0, \dots, n-2$, and $y(t) = y_{n-1}$ for $t \in [t_{n-1}, T]$. We see that actually $C: C(J, E) \multimap \mathcal{Y}$.

Let us assume that:

Assumption 4.8.

(a) The map $\mathfrak{F}$ is integrably bounded, has compact convex values and maps compact subsets of $J \times E \times Y$ into compact sets;
(b) For each $x \in E$ and $y \in Y$, the map $\mathfrak{F}(\,\cdot\,, x, y): J \multimap E$ is measurable; for almost all $t \in J$ and all $x \in E$, the map $\mathfrak{F}(t, x, \,\cdot\,): Y \multimap E$ is upper semicontinuous;
(c) there is a constant $0 \le L < \omega$ such that

$$\mathfrak{D}(\mathfrak{F}(t, x_1, y), \mathfrak{F}(t, x_2, y))) \le L\|x_1 - x_2\|$$

for almost all $t \in J$, all $y \in Y$ and all $x_1, x_2 \in E$.
(d) The set-valued map $c: E \multimap Y$ is acyclic-decomposable.

As above we shall explain a setting for the problem. For any $x \in C(J, E)$ and $y \in \mathcal{Y}$, let as usual

$$N_{\mathfrak{F}}(x, y) := \{w \in L^1(J, E) \mid w(s) \in \mathfrak{F}(s, x(s), y(s)) \text{ for a.e. } s \in J\}$$

and define a map

$$F: C(J, E) \times \mathcal{Y} \multimap C(J, E)$$

by saying that, for $x \in C(J, E)$, $y \in \mathcal{Y}$, $u \in F(x, y)$ if and only if there exists $w \in N_{\mathfrak{F}}(x, y)$ such that

$$u(t) = [I - U(T)]^{-1}\bigg(\int_0^t U(t-s)w(s)\,ds + U(t)\int_t^T U(T-s)w(s)\,ds\bigg),$$

where $I: E \to E$ stands for the identity on E.

Observe that this definition is correct since $I - U(T)$ is invertible in view of the inequality $\|U(T)\| \le e^{-\omega T} < 1$. Moreover,

$$\|[I - U(T)]^{-1}\| \le (1 - e^{-\omega T})^{-1}.$$

Moreover, given $w \in L^1(J,E)$, $U(t-\,\cdot\,)w(\,\cdot\,)$ is measurable and $\|U(t-\,\cdot\,)w(\,\cdot\,)\| \in L^1([0,t],\mathbb{R})$ (similarly $U(T-\,\cdot\,)w(\,\cdot\,) \in L^1([t,T],E)$).

Observe that if, for some $y \in \mathcal{Y}$, $x \in F(x,y)$, then there is $w \in N_{\mathfrak{F}}(x,y)$ such that

$$x(t) = [I - U(T)]^{-1}\bigg(\int_0^t U(t-s)w(s)\,ds + U(t)\int_t^T U(T-s)w(s)\,ds\bigg).$$

Hence

$$x(0) = [I - U(T)]^{-1}\int_0^T U(T-s)w(s)\,ds = x(T)$$

and, since $U(t)[I-U(T)]^{-1} = [I-U(T)]^{-1}U(t)$, a simple computation shows that

$$x(t) = U(t)x(0) + \int_0^t U(t-s)w(s)\,ds,$$

i.e. x is a periodic mild solution to the problem

$$x'(t) \in Ax(t) + \mathfrak{F}(t,x(t),y(t)).$$

Similarly we show that if $x \in F(\{x\}\times C(x))$, then x is a solution to the original problem (4.2).

In addition to technical hypotheses 4.8, let us make the following structural assumption.

Assumption 4.9. We suppose that if $y \in \mathcal{Y}$ and $x\colon J \to E$ is a periodic mild solution of the inclusion $x'(t) \in Ax(t) + \mathfrak{F}(t,x(t),y(t))$, a.e. on J, then $x(t) \in X$ for all $t \in J$.

Theorem 4.10. *Suppose that Y is a compact contractible space. If Assumptions* 4.8 *and* 4.9 *are fulfilled, then there is a solution to problem* (4.2).

Proof. First we shall show that $F\colon C(J,E)\times\mathcal{Y} \multimap C(J,E)$ has compact convex values and, for each $x \in C(J,E)$, $F(x,\,\cdot\,)$ is upper semicontinuous (in $\mathcal{Y}$ we consider the natural metric $d(y,y') := \max_{i=1,\dots,n}\{d_Y(y_{i-1},y'_{i-1})\}$ where $y_{i-1} = y(t)$, $y'_{i-1} = y'(t)$ for $t \in [t_{i-1},t_i)$, $i = 1,\dots,n-1$ and $y_{n-1} = y(t)$, $y'_{n-1} = y'(t)$ for $t \in [t_{n-1},T]$). To this aim take a sequence $(y_n,z_n) \in \mathrm{Gr}(F(x,\,\cdot\,))$ such that $y_n \to y$ in $\mathcal{Y}$. Then $z_n = K(w_n)$ where $w_n \in N_{\mathfrak{F}}(x,y_n)$ and $K\colon L^1(J,E) \to C(J,E)$ is given by

$$K(w)(t) := [I - U(T)]^{-1}\bigg(\int_0^t U(t-s)w(s)\,ds + U(t)\int_t^T U(T-s)w(s)\,ds\bigg),$$

for $w \in L^1(J,E)$. Clearly K is linear and bounded.

As before, using the integral boundedness of $\mathfrak{F}$, the upper semicontinuity of $\mathfrak{F}(t,x,\,\cdot\,)$, the Diestel compactness criterion and the Convergence theorem, we show that (passing to a subsequence if necessary) $w_n \rightharpoonup w$ (weakly in $L^1(J,E)$) and $w \in N_{\mathfrak{F}}(x,y)$. Moreover, similarly as in the proof of Proposition 4.3, we show that the sequence (z_n) is equicontinuous and has compact fibres and, hence (again passing to subsequences) $z_n \to z$ in $C(J,E)$. Since $z_n = K(w_n)$, this implies that $z = K(w)$, i.e. $z \in F(x,y)$. The convexity of the values of F is clear in view of the convexity of values of $N_{\mathfrak{F}}$ and the linearity of K.

Next we shall show that, for each $y \in \mathcal{Y}$, $F(\,\cdot\,,y)$ is a k-contraction. Let $x_1, x_2 \in C(J,E)$ and $z_1 \in F(x_1,y)$. As in the proof of the mentioned Proposition 4.3, given $w_1 \in N_{\mathfrak{F}}(x_1,y)$ such that $z_1 = K(w_1)$ and $\varepsilon > 0$, we choose $w_2 \in N_{\mathfrak{F}}(x_2,y)$ such that

$$\|w_1(s) - w_2(s)\| \le d(w_1(s), \mathfrak{F}(s, x_2(s), y(s)) + \varepsilon$$

for almost all $s \in J$ (the choice of w_2 is possible since, as it is easy to see, the map $J \times E \ni (s,x) \mapsto \mathfrak{F}(s,x,y(s))$ is product-measurable; to see this recall that this map is Carathéodory). Hence

$$\|w_1(s) - w_2(s)\| \le L\|x_1(s) - x_2(s)\| + \varepsilon.$$

Let $z_2 := K(w_2)$. After easy computations we see that, for any $t \in J$,

$$\|K(w_1)(t) - K(w_2)(t)\| \le \omega^{-1}(L \sup_{s\in J} \|x_1(s) - x_2(s)\| + \varepsilon).$$

Arguing as before, this implies that

$$\mathfrak{D}^C(F(x_1,y), F(x_2,y)) \le k \sup_{t\in J} \|x_1(t) - x_2(t)\|,$$

where $k := \omega^{-1}L < 1$ and $\mathfrak{D}^C$ denotes the Hausdorff distance in $C(J,E)$.

Let us now study the map $C: C(J,E) \multimap \mathcal{Y}$. It is clear that C may be factored in the following way $C = g \circ \widetilde{c} \circ f$, where $f: C(J,E) \to E^n$ is given by $f(x) = (x(t_0), \ldots, x(t_{n-1}))$ for $x \in C(J,E)$, $\widetilde{c}: E^n \multimap Y^n$ is given by $\widetilde{c}(z_0, \ldots, z_{n-1}) = c(z_0) \times \ldots \times c(z_{n-1})$ for $(z_0, \ldots, z_{n-1}) \in E^n$, and $g: Y^n \to \mathcal{Y}$ is given by $g(y_0, \ldots, y_{n-1})(t) = y_{i-1}$ for $t \in [t_{i-1}, t_i)$, $i = 1, \ldots, n-2$, $g(y_0, \ldots, y_{n-1})(t) = y_{n-1}$ for $t \in [y_{n-1}, T]$. It is clear that f, g are continuous, $\widetilde{c}$ is acyclic-decomposable; hence C is acyclic-decomposable. Therefore $\Phi := F \diamond C$ is a Krasnosel'skiĭ-type map.

It is clear that a map $\mathcal{F}: \mathcal{Y} \multimap C(J,E)$, defined as usual by $\mathcal{F}(y) = \operatorname{Fix}(F(\,\cdot\,,y))$ for $y \in \mathcal{Y}$, is acyclic-decomposable (already in view of the Ricceri theorem and Proposition 2.9). Thus $\Phi \in \mathcal{K}_p(C(J,E), C(J,E))$.

Observe that, in view of Assumption 4.9, for any $y \in \mathcal{Y}$, $\mathcal{F}(y) \in X$; thus $\mathcal{F} \circ C: X \multimap X$. Since, as it is easy to see $\mathcal{Y}$ is contractible, arguing as in Corollary 3.20, we show that $L((\mathcal{F}, C)) = 1$. This shows that Φ has a fixed point and completes the proof. $\square$

Let us finally remark Theorem 4.10 may be restated without Assumption 4.9 as follows.

Corollary 4.11. *If the space Y is compact contractible and Assumption* 4.8 *is satisfied, then the set*

$$S=\{x\in C(J,E)\mid x(0)=x(T),\ x'(t)\in Ax(t)+\mathfrak{F}(t,x(t),y(t))\ \text{a.e. on } J,\ y\in\mathcal{Y}\}$$

is bounded. Let $X\subset E$ be an arbitrary neighbourhood retract such that, for $x\in S$, $x(t)\in X$ for $t\in J$. Then there is a solution to problem (4.2) *such that $x(t)\in X$ for $t\in J$.*

5. Appendix

In this short last section we shall provide a discussion of the alternative approach to the fixed point index for acyclic-decomposable set-valued maps and the discussion of the mentioned strong invariance properties.

5.1. Discussion of the fixed point index. The class of acyclic-decomposable maps (see Definition 3.5) admits a different description. Suppose that an upper semicontinuous map $\Phi\colon X\multimap Y$ has compact values and let $p_\Phi\colon \mathrm{Gr}(\Phi)\to X$, $q_\Phi\colon\mathrm{Gr}(\Phi)\to Y$ be the projection of $\mathrm{Gr}(\Phi)$ onto X and into Y, respectively. Then, for each $x\in X$, $\Phi(x)=q_\Phi(p_\Phi^{-1}(x))$. Moreover, it is easy to see that p is a proper surjection, i.e. $p_\Phi^{-1}(K)$ is compact provided so is $K\subset X$. If Φ is acyclic, then, for each $x\in X$, the fiber $p_\Phi^{-1}(x)$ is acyclic. This formalism leads to the notion of an *admissible pair* due to Górniewicz (see [24]): a pair of continuous maps $X\xleftarrow{p}\Gamma\xrightarrow{q}Y$, where Γ is a space, is *admissible* if p is a proper surjection and, for each $x\in X$, the fiber $p^{-1}(x)$ is acyclic. Each admissible pair determines an acyclic-decomposable map $\Phi=\Phi_{(p,q)}\colon X\multimap Y$, $\Phi(x):=q(p^{-1}(x))$, $x\in X$, with an acyclic decomposition given by

$$D(p,q)\colon X\overset{\Phi_p}{\multimap}\Gamma\xrightarrow{q}Y,$$

where $\Phi_p(x):=p^{-1}(x)$ for $x\in X$. The class of maps determined by admissible pairs is closed under composition: given maps $\Phi_i\colon X_i\to X_{i+1}$, $i=0,1$, determined by admissible pairs, it is not difficult to show that the composition $\Phi_1\circ\Phi_0$ is determined by an admissible pair, too (see [24]). Since, as was shown above, any acyclic map is determined by an admissible pair, we see that the class of acyclic-decomposable maps (defined above after [18] or [51]) is identical with that consisting of maps determined by admissible pairs.

An adequate theory of the topological degree and the fixed point index for compact maps determined by admissible pairs was provided in [24] and based on the Vietoris theorem. If X is a normed space, $U\subset X$ is open and a compact map Φ determined by an admissible pair $\mathrm{cl}\,U\xleftarrow{p}\Gamma\xrightarrow{q}X$ is such that $\mathrm{Fix}(\Phi)\cap\mathrm{bd}\,U=\emptyset$, then Górniewicz defines (cf. [24] and comp. [34]) the *coincidence index* $\mathrm{ind}_G((p,q),U)$ detecting the existence of fixed points of Φ (or coincidences of p and q). As in the case of the Skordev index (where the index was defined for a particular decomposition of a given acyclic-decomposable map), the Górniewicz index ind_G depends on a particular admissible pair determining Φ. The index has most of the standard properties and constitutes a direct generalization of the ordinary Leray–Schauder fixed-point index. However, due to the lack of the commutativity property of ind_G, it is not known whether ind_G may be lifted to the case when $X \in \mathrm{ANR}$ ([17]). This is the reason we decided to use the Siegberg–Skordev index instead of the simpler and more intuitive approach due to Górniewicz (even though their definition requires that Φ has to be defined on the whole X).

Contrary to the fixed point index, Górniewicz [24] constructs the full Lefschetz theory of compact maps $\Phi\colon X \multimap X$, where $X \in \mathrm{ANR}$, determined by admissible pairs $X \xleftarrow{p} \Gamma \xrightarrow{q} X$, and defines the *generalized Lefschetz number* $L((p,q))$ having standard properties (see [24]). Therefore, from the view-point of applications where we mainly deal with the Lefschetz number, the use of the Górniewicz approach would be completely sufficient. It is not known whether (in case X is a normed space) approaches of Siegberg–Skordev and Górniewicz coincide.

Still a different, simple and sufficient in most of applications, construction of the fixed point index has been provided in [3]. Let $X \in \mathrm{ANR}$ and assume that a compact map $F\colon X \multimap X$ have a decomposition

$$D(F) : X = X_0 \overset{F_0}{\multimap} X_1 \overset{F_1}{\multimap} \cdots \overset{F_{n-2}}{\multimap} X_{n-1} \overset{F_{n-1}}{\multimap} X_n = X \tag{5.1}$$

(i.e. $F = F_{n-1} \circ \ldots \circ F_0$) such that $X_i \in \mathrm{ANR}$ for $i = 1, \ldots, n-1$ and, for $i = 0, \ldots, n-1$, $F_i \in R_\delta$, i.e. F_i is upper semicontinuous with R_δ-values (a compact subset K of an ANR is an *R_δ-set* if K is contractible in each of its neighbourhoods (see [29]); clearly any compact contractible or convex set is R_δ; in particular any compact AR is an R_δ-set). By [39] it is easy to see that R_δ-sets have trivial Čech cohomology (with coefficients in any abelian group); thus in view of [24, Theorem (5.1)] they are acyclic ([18]); in particular F given by (5.1) is acyclic-decomposable and $D(F) \in \mathcal{DA}(X,X)$. Let $U \subset X$ be open and $\mathrm{Fix}(F) \cap \mathrm{bd}\, U = \emptyset$. Using approximation results from [25] (comp. some refinements in [37]), Bader and Kryszewski in [3] define the fixed point index $\mathrm{ind}_{BK}(F,U)$ of F with respect to U. As in the case of the Skordev index, ind_{BK} depends strongly on a factorization $D(F)$ of F. The substantial difference between ind_{BK} and ind_S is that the latter does not require that in a decomposition $D(F) \in \mathcal{DA}(X,X)$ all spaces $X_i \in \mathrm{ANR}$, $i = 1, \ldots, n-1$. Hence, the approach of [51] is more general;

([17]) Under slightly stronger hypotheses concerning p and using a different approach, the first author in [36], [35] constructed the full coincidence index theory for compact admissible pairs $U \xleftarrow{p} \Gamma \xrightarrow{q} X$, where U is an open subset of an ANR X.

([18]) In fact [3] deals with maps having the so-called proximally ∞-connected values [17] being more general than R_δ-sets (see also [37]).

however the approach from [3] is simpler and needs no complicated algebraic apparatus.

5.2. Strong invariance. Let X be a closed subset of a Banach space E and assume that $\mathfrak{F}: J \times E \multimap E$, where $J = [0, T]$, is a set-valued map. We say that X is *strongly invariant with respect to the flow generated by* $\mathfrak{F}$ if any solution $x: J \to E$ of the Cauchy problem

$$x' \in \mathfrak{F}(t, x), \quad x(0) = x_0 \in X \tag{5.2}$$

stays in X, i.e. $x(t) \in X$ for all $t \in J$.

The following results is perhaps known. However, since we could not find a direct reference, we provide its full statement and proof.

THEOREM 5.1. *If, for all $t \in J$, the map $\mathfrak{F}(t, \cdot): E \multimap E$ is $L(t)$-Lipschitz with $L \in L^1(J, \mathbb{R})$, for all $x \in X$ and $t \in J$,*

$$\mathfrak{F}(t, x) \subset T_X(x), \tag{5.3}$$

then X is strongly invariant with respect to the flow generated by $\mathfrak{F}$.

As above

$$T_X(x) := \left\{ v \in E \;\middle|\; \liminf_{t \to 0^+} \frac{d_X(x + tv)}{t} = 0 \right\}$$

is the Bouligand contingent cone to X at $x \in X$ (d_X is the distance from X).

LEMMA 5.2. *For each $\alpha \geq 0$, all $x \in \operatorname{bd} X_\alpha$ (where $X_\alpha := \{x \in E \mid d(x) := d(x, X) \leq \alpha\}$), $t \in J$ and $u \in \mathfrak{F}(t, x)$,*

$$\limsup_{h \to 0^+} \frac{d(x + hu) - \alpha}{h} \leq L(t)\alpha.$$

PROOF. For $\alpha = 0$ this is easy: fix $x \in \operatorname{bd} X$ ($X_0 = X$), $t \in J$ and $u \in \mathfrak{F}(t, x)$. Let $x_n \in X$ and $x_n \to x$. By the lower semicontinuity of $\mathfrak{F}(t, \cdot)$, we gather that there is $u_n \in \mathfrak{F}(t, x_n)$ such that $u_n \to u$. Hence

$$u \in \operatorname*{Liminf}_{y \to x,\, y \in X} \mathfrak{F}(t, y) \subset \operatorname*{Liminf}_{y \to x,\, y \in X} T_X(y) \subset C_K(x),$$

where $C_X(x)$ stands for the Clarke tangent cone, i.e.

$$C_X(x) := \left\{ v \in E \;\middle|\; \limsup_{h \to 0^+,\ y \to x,\ y \in X} \frac{d(y + hv)}{h} = 0 \right\}.$$

Thus

$$\limsup_{h \to 0^+} \frac{d(x + hu)}{h} = 0.$$

Suppose to the contrary that there are $\alpha > 0$, $x \in \operatorname{bd} X_\alpha$, $t \in J$ and $u \in \mathfrak{F}(t, x)$ such that

$$\limsup_{h \to 0^+} \frac{d(x + hu) - \alpha}{h} > L\alpha$$

where $L := L(t)$. Let $\varepsilon \in (0,1)$ be such that

$$\limsup_{h\to 0^+} \frac{d(x+hu)-\alpha}{h} > L(\alpha+2\varepsilon).$$

Hence, for each $\eta > 0$, there exists $h \in (0,\eta)$ such that

$$d(x+hu)-\alpha > hL(\alpha+2\varepsilon).$$

Clearly we may assume that $u \neq 0$; then we put

$$\eta := \varepsilon(\|u\| + L(\alpha+1+\varepsilon))^{-1}$$

and choose an appropriate $h \in (0,\eta)$. Moreover let us take $0 < \gamma < hL\varepsilon$ and $y \in X$ such that $\|x-y\| < \alpha+\gamma$. Then

$$D(y+hu, hL(\alpha+\varepsilon)) \cap X = \emptyset.$$

If $z \in D(y+hu, hL(\alpha+\varepsilon)) \cap X$, then $\|z-y-hu\| \leq hL(\alpha+\varepsilon)$ and

$$\begin{aligned} hL(\alpha+2\varepsilon) &< d(x+hu)-\alpha \leq \|z-x-hu\| - \alpha \\ &\leq \|z-y-hu\| + \|x-y\| - \alpha \leq Lh(\alpha+\varepsilon)+\gamma, \end{aligned}$$

i.e. $\gamma > hL\varepsilon$: a contradiction.

Now we shall recall the following Drop Theorem due to Daneš [14]: *If X is a closed subset of a Banach space E, $y_0 \in E$, $d(y_0, X) > r > 0$ and $y \in X$, then there is $z_0 \in X \cap \operatorname{conv}(y, D(y_0,r))$ such that $X \cap \operatorname{conv}(z_0, D(y_0,r)) = \{z_0\}$.*

In our situation ($y_0 := y+hu$, $r = hL(\alpha+\varepsilon)$), there is $z_0 \in X \cap \operatorname{conv}(y, D(y+hu, hL(\alpha+\varepsilon)))$ such that $X \cap \operatorname{conv}(z_0, D(y+hu, hL(\alpha+\varepsilon))) = \{z_0\}$. Therefore, for some $\lambda \in [0,1]$,

$$z_0 = ty + (1-t)(y+hw) = y + (1-t)hw,$$

where $\|w-u\| \leq L(\alpha+\varepsilon)$. It is easy to see that $\lambda > 0$ for otherwise $z_0 = y+hw \in D(y+hu, hL(\alpha+\varepsilon)) \cap X$. Observe that

$$\begin{aligned} \|z_0 - x\| &\leq \|z_0-y\| + \|y-x\| \leq (1-\lambda)h\|w\| + \alpha + \gamma \\ &\leq h(\|u\| + L(\alpha+\varepsilon)) + hL\varepsilon + \alpha \\ &= h(\|u\| + L(\alpha+\varepsilon) + L\varepsilon) + \alpha < \alpha + \varepsilon. \end{aligned}$$

By Lipschitzeanity of $\mathfrak{F}$, there is $v \in \mathfrak{F}(t, z_0)$ such that

$$\|u-v\| \leq \mathfrak{D}(\mathfrak{F}(t,x), \mathfrak{F}(t,z_0)) \leq L\|x-z_0\| < L(\alpha+\varepsilon).$$

Since $v \in \mathfrak{F}(t,z_0) \subset T_X(z_0)$, there are sequences (s_n) and (v_n) such that $0 < s_n \to 0$, $v_n \to v$ and $z_0 + s_n v_n \in X$ for all n. For large n, $s_n < \lambda h$ and $\|v_n - u\| \leq L(\alpha+\varepsilon)$; hence $v_n \neq 0$ (for otherwise $\|u\| \leq L(\alpha+\varepsilon)$, i.e. $y \in D(y+hu, hL(\alpha+\varepsilon)) \cap X$).

Now observe that, for all large n,

$$z_0 + s_n v_n = \left(1 - \frac{s_n}{\lambda h}\right) z_0 + \frac{s_n}{\lambda h}(y + h(w + \lambda(v_n - w))) \\ \in \operatorname{conv}(z_0, D(y + hu, hL(\alpha + \varepsilon))) \cap X.$$

Thus $z_0 = z_0 + s_n v_n$ and $s_n v_n = 0$: a contradiction. □

Proof of Theorem 5.1. Let $x_0 \in X$ and let $x: J \to E$ be a solution to (5.2), i.e. $x(t) = x_0 + \int_0^t w(s)\,ds$, where $w \in L^1(J, E)$ and $w(s) \in \mathfrak{F}(s, x(s))$ for all $s \in J$. Let $g(t) = d(x(t))$. Functions x and g are differentiable almost everywhere. Let $t \in J$ be a point of differentiability of x and g; then $x'(t) = w(t)$ and, for small $h > 0$, $x(t+h) = x(t) + hw(t) + h\varepsilon(h)$, where $\varepsilon(h) \to 0$ if $h \to 0^+$. By the above Lemma,

$$g'(t) = \lim_{h \to 0^+} \frac{g(t+h) - g(t)}{h} = \lim_{h \to 0^+} \frac{d(x(t) + hw(t) + h\varepsilon(h)) - d(x(t))}{h} \\ \leq \limsup_{h \to 0^+} \frac{d(x(t) + hw(t)) - d(x(t))}{h} \leq L(t)g(t).$$

Hence, by the Gronwall inequality, $g \equiv 0$, i.e. $x(t) \in X$. □

Remark 5.3. (a) If E is separable and, for each $x \in E$, $\mathfrak{F}(\,\cdot\,, x)$ is measurable, then, by the parameterized version of the Michael theorem due to Kucia [38], it is easy to see that strong invariance of X implies that the tangency condition (5.3) is satisfied.

(b) It is easy to see that Theorem 5.1 holds true when $\mathfrak{F}(t, \,\cdot\,)$ is $L(t)$-Lipschitz for almost all $t \in J$ and, if E is separable, when for all $x \in \operatorname{bd} X$ and almost all $t \in J$, $\mathfrak{F}(t, x) \subset T_X(x)$.

(c) If X is a *proximinal*, i.e. for each $y \in E \setminus X$, the set $\pi_X(y) := \{x \in X \mid \|x - y\| = d(y, X)\} \neq \emptyset$, then the tangency condition (5.3) may be relaxed. Namely it is enough to assume that, for all $x \in \operatorname{bd} X$, $t \in J$ and $u \in \mathfrak{F}(t, x)$,

$$\sup_{v \in N_X^P(x)} \langle v, u \rangle_+ \leq 0 \tag{5.4}$$

where $N_X^P(x)$ stands for the *proximal normal cone to X at x* and $\langle\,\cdot\,,\,\cdot\,\rangle_+ : E \times E$ is the (positive) semi-inner product, i.e. $\langle v, x \rangle_+ := \sup_{p \in J(v)} \langle p, x \rangle$ where $J(v) := \{p \in E^* \mid \langle p, v \rangle = \|p\|^2 = \|v\|^2\}$ is the duality map. Condition (5.4) is also sufficient if X is no longer proximinal, but E is reflexive, E and E^* are smooth in the sense that their norms are Fréchet differentiable off the origin. Indications as to the proofs (in case of a Hilbert space) may be found in [13].

(d) Condition (5.3) may be replaced also by the following *external tangency condition*: there is an open neighbourhood Ω of X such that, for all $t \in J$, $x \in \Omega \setminus X$, $\mathfrak{F}(t, x) \subset \partial d_X(x)^-$, where $\partial d_X(x)^-$ denotes the (negative) polar cone to the Clarke generalized gradient $\partial d_X(x)$ of the distance function $d_X(\,\cdot\,)$, i.e. $\partial d_X(x)^- := \{u \in E \mid \text{ for all } p \in \partial d_X(x),\ \langle p, u\rangle \le 0\}$ — see [12]. Indeed, if $x(t) = x_0 + \int_0^t w(s)\,ds$, where $w(s) \in \mathfrak{F}(s, x(s))$ on J and there is $t \in [0,T]$ such that $x(t) \notin X$, then there are $t_1, t_2 \in [0,T]$ such that $x(t_1) \in X$ and $x(t) \in \Omega \setminus X$ for $t \in (t_1, t_2]$. As above the function $f := d_X \circ x$ is differentiable almost everywhere on $[t_1, t_2]$. If $f'(t)$, $t \in [t_1, t_2]$, exists, then

$$f'(t) \le \sup_{p \in \partial d_X(x(t))} \langle p, x'(t)\rangle \le 0.$$

Thus, for all $t \in (t_1, t_2)$,

$$d_X(x(t)) = d_X(x(t)) - d_X(x(t_1)) = \int_{t_1}^t f'(s)\,ds \le 0,$$

a contradiction.

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