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A NEW APPROACH TO BOUNDARY VALUE PROBLEMS ON THE HALF LINE USING WEAKLY–STRONGLY SEQUENTIALLY CONTINUOUS MAPS

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Abstract. An existence principle is established for a boundary value problem on the half line using a new theory based on weakly–strongly sequentially continuous maps.

1. Introduction

In this paper we present a new approach to establishing existence principles for boundary value problems on the half line. To illustrate our theory we will consider the problem

$$\begin{cases} y'' - m^2 y + f(t,y) = 0 & \text{a.e. on } [0,\infty), \\ y(0) = 0, \quad \lim_{t\to\infty} y(t) = 0 \quad \textit{and} \quad m > 0. \end{cases} \tag{1.1}$$

Our theory is based on a new Leray–Schauder alternative for weakly–strongly sequentially continuous maps. This alternative combines the advantages of the strong topology (i.e. the sets will be open in the strong topology) with the advantages of the weak topology (i.e. the maps will be weakly–strongly sequentially

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continuous and weakly compact) and we will see in Section 2 how easily our alternative applies to (1.1). One of the disadvantages of the standard Leray–Schauder alternative in the literature [1], [3] is that a lot of time is usually spent checking the compactness of the map. However if one uses this new approach the weak compactness of the map is immediate.

For notational purposes let $BC[0,\infty)$ denote the space of bounded continuous mappings from $[0,\infty)$ to $\mathbb{R}$. If $u \in BC[0,\infty)$ we write $|u|_\infty = \sup_{t\in[0,\infty)} |u(t)|$.

We conclude the introduction by stating the Leray–Schauder alternative from [2] (for completeness we sketch the proof).

Theorem 1.1. *Let E be a Banach space, C a closed convex subset of E, U a convex subset of C and U an open (strong topology) subset of E with $0 \in U$. Suppose $F\colon \overline{U} \to C$ is a weakly–strongly sequentially continuous map (i.e. $F\colon \overline{U} \to C$ is completely continuous i.e. if $x_n, x \in \overline{U}$ with $x_n \rightharpoonup x$ then $Fx_n \to Fx$); here $\overline{U}$ denotes the closure of U in C. In addition suppose either $\overline{U}$ is weakly compact or $F\colon \overline{U} \to C$ is weakly compact. Also assume*

$$x \neq \lambda F x \quad \text{for } x \in \partial_C U \text{ and } \lambda \in (0,1); \tag{1.2}$$

here $\partial_C U$ denotes the boundary (strong topology) of U in C. Then F has a fixed point in $\overline{U}$.

Remark 1.2. Note $\operatorname{int}_C U = U$ (interior in the strong topology) since U is open in C so as a result $\partial_C U = \partial_E U$; here $\partial_E U$ denotes the boundary of U in E.

Proof. Let μ be the Minkowski functional on $\overline{U}$ and let $r\colon E \to \overline{U}$ be given by

$$r(x) = \frac{x}{\max\{1, \mu(x)\}} \quad \text{for } x \in E.$$

Note $r\colon E \to \overline{U}$ is continuous. Also since $F\colon \overline{U} \to C$ is weakly–strongly sequentially continuous we have immediately that $rF\colon \overline{U} \to \overline{U}$ is weakly sequentially continuous. Notice also that $rF\colon \overline{U} \to \overline{U}$ is a weakly compact map if $F\colon \overline{U} \to C$ is weakly compact; note $\overline{F(\overline{U})}^w$ is weakly compact so the weak compactness of rF follows from the Krein–Šmulian theorem and

$$r(\overline{F(\overline{U})}^w) \subseteq \overline{\text{co}}\,(\{0\} \cup \overline{F(\overline{U})}^w).$$

Now standard fixed point theory for weakly sequentially continuous self maps in the literature (see [2]) guarantees that there exists $x \in \overline{U}$ with $x = rF(x)$. Thus $x = r(y)$ with $y = F(x)$ and $x \in \overline{U} = U \cup \partial U$ (note $\operatorname{int}_C U = U$ since U is also open in C). Now either $y \in \overline{U}$ or $y \notin \overline{U}$. If $y \in \overline{U}$ then $r(y) = y$ so $x = y = F(x)$, and we are finished. If $y \notin \overline{U}$ then $r(y) = y/\mu(y)$ with $\mu(y) > 1$. Then $x = \lambda y$ (i.e. $x = \lambda F(x)$) with $0 < \lambda = 1/\mu(y) < 1$; note $x \in \partial_C U$ since $\mu(x) = \mu(\lambda y) = 1$ (note $\partial_C U = \partial_E U$ since $\operatorname{int}_C U = U$). This of course contradicts (1.2). $\square$

2. Application

To illustrate how easily Theorem 1.1 can be applied in practice we consider the boundary value problem

$$\begin{cases} y'' - m^2 y + f(t,y) = 0 & \text{a.e. on } [0,\infty), \\ y(0) = 0, \quad \lim_{t\to\infty} y(t) = 0 \end{cases} \tag{2.1}$$

where $m > 0$ is a constant and $f\colon [0,\infty) \times \mathbb{R} \to \mathbb{R}$ is a L^p-Carathéodory function with $p > 1$; by this we mean

- (f_1) $t \mapsto f(t,y)$ is measurable for any $y \in \mathbb{R}$,
- (f_2) $y \mapsto f(t,y)$ is continuous for almost every $t \in [0,\infty)$,
- (f_3) for any $r > 0, \exists h_r \in L^p[0,\infty)$ such that $|f(t,y)| \le h_r(t)$ for all $|y| \le r$ and almost all $t \in [0,\infty)$ with

$$\lim_{t\to\infty} e^{-mt} \int_0^t e^{ms} h_r(s)\,ds = 0.$$

We begin by looking at the operator $H\colon L^p[0,\infty) \to BC[0,\infty)$ given by

$$Hu(t) = \int_0^\infty G(t,s)u(s)\,ds$$

where

$$G(t,s) = \begin{cases} \dfrac{e^{-mt}}{2m}[e^{ms} - e^{-ms}] & \text{for } s \le t, \\ \dfrac{e^{-ms}}{2m}[e^{mt} - e^{-mt}], & \text{for } s > t. \end{cases}$$

We must of course check that $Hu \in BC[0,\infty)$ if $u \in L^p[0,\infty)$. To see this first note for fixed $t \in [0,\infty)$ that $G(t,\,\cdot\,) \in L^q[0,\infty)$ (here $1/p + 1/q = 1$) since

$$\begin{aligned} \int_0^\infty |G(t,s)|^q\,ds = {} & \frac{e^{-mqt}}{(2m)^q} \int_0^t \left(e^{ms} - e^{-ms}\right)^q ds \\ & + \frac{1}{(2m)^q}[e^{mt} - e^{-mt}]^q \frac{1}{mq} e^{-mqt} < \infty. \end{aligned}$$

Next we show

$$\sup_{t\in[0,\infty)} \left(\int_0^\infty |G(t,s)|^q\,ds \right)^{1/q} < \infty. \tag{2.2}$$

To see this notice for fixed $t \in [0,\infty)$ that

$$\begin{aligned} \int_0^\infty |G(t,s)|^q\,ds &= \frac{e^{-mqt}}{(2m)^q} \int_0^t e^{mqs}\left(1 - e^{-2ms}\right)^q ds + \frac{1}{mq(2m)^q}[1 - e^{-2mt}]^q \\ &\le \frac{1}{mq(2m)^q}(1 - e^{-mqt}) + \frac{1}{mq(2m)^q}[1 - e^{-2mt}]^q, \end{aligned}$$

so (2.2) is immediate. Next notice if $t, x \in [0,\infty)$ and $u \in L^p[0,\infty)$ then

$$|Hu(t) - Hu(x)| \leq \|u\|_{L^p[0,\infty)} \bigg(\int_0^\infty |G(t,s) - G(x,s)|^q \, ds \bigg)^{1/q}$$

so $Hu \in BC[0,\infty)$ (see (2.2)) if we show

$$\int_0^\infty |G(t,s) - G(x,s)|^q \, ds \to 0 \quad \text{as } t \to x; \tag{2.3}$$

here $\|u\|_{L^p[0,\infty)} = (\int_0^\infty |u(s)|^p \, ds)^{1/p}$. Fix $x \in [0,\infty)$. To show (2.3) assume without loss of generality that $t < x$. Then

$$\begin{aligned}\int_0^\infty |G(t,s) - G(x,s)|^q \, ds = \frac{1}{m^q}(e^{-mt} - e^{-mx})^q \int_0^t (\sinh ms)^q \, ds \\ + \frac{1}{m^q} \int_t^x |e^{-ms} \sinh mt - e^{-mx} \sinh ms|^q \, ds \\ + \frac{1}{m^q} |\sinh mt - \sinh mx|^q \frac{1}{mq} e^{-mqx} \to 0\end{aligned}$$

as $t \to x$, so (2.3) is true. Thus $H\colon L^p[0,\infty) \to BC[0,\infty)$.

It is easy to see that if $u \in L^p[0,\infty)$ is a solution of

$$u = f(t, H(u)) \tag{2.4}$$

then $y(t) = \int_0^\infty G(t,s)u(s)\, ds$ is a solution of (2.1) (notice it is easy to check that $y'' - m^2 y = -u$ and $(f)_3$ guarantees that $\lim_{t\to\infty} y(t) = 0$). Conversely if w is a solution of (2.1) then $v = m^2 w - w''$ is a solution of (2.4).

Define an operator $F\colon L^p[0,\infty) \to L^p[0,\infty)$ by

$$Fu(t) = f(t, H(u)(t)).$$

Consequently solving (2.1) is equivalent to finding a fixed point $u \in L^p[0,\infty)$ of

$$u = Fu. \tag{2.5}$$

Theorem 2.1. *Let $f\colon [0,\infty) \times \mathbb{R} \to \mathbb{R}$ satisfy* (f_1), (f_2) *and* (f_3) *with $p > 1$ and suppose there exists a constant M_0, independent of λ, with*

$$\|y'' - m^2 y\|_{L^p[0,\infty)} \neq M_0 \tag{2.6}$$

for any solution y to the problem

$$\begin{cases} y'' - m^2 y + \lambda f(t,y) = 0 & \text{a.e. on } [0,\infty), \\ y(0) = 0, \quad \lim_{t\to\infty} y(t) = 0 \end{cases} \tag*{$(2.7)_\lambda$}$$

for any $\lambda \in (0,1)$. Then (2.1) *has at least one solution.*

Proof. We will apply Theorem 1.1 with

$$E = C = L^p[0,\infty) \quad \text{and} \quad U = \{u \in L^p[0,\infty) : \|u\|_{L^p[0,\infty)} < M_0\}.$$

Notice $\overline{U} = \{u \in L^p[0,\infty) : \|u\|_{L^p[0,\infty)} \le M_0\}$ is closed and convex, so weakly closed. Moreover, $\overline{U}$ is weakly compact (recall in a reflexive Banach space a subset is weakly compact if and only if it is closed in the weak topology and bounded in the norm topology). Also (2.6) guarantees that (1.2) holds. It remains to show $F\colon \overline{U} \to L^p[0,\infty)$ is a weakly–strongly sequentially continuous map. Let $y_n, y \in \overline{U}$ with $y_n \rightharpoonup y$ in $L^p[0,\infty)$ (i.e. $\int_0^\infty y_n g\,dt \to \int_0^\infty yg\,dt$ for all $g \in L^q[0,\infty)$ with $1/p+1/q=1$). We must show $Fy_n \to Fy$ in $L^p[0,\infty)$. Notice

$$\int_0^\infty |Fy_n(t) - Fy(t)|^p\,dt = \int_0^\infty |f(t,H(y_n)(t)) - f(t,H(y)(t))|^p\,dt.$$

If we show

$$\int_0^\infty |f(t,H(y_n)(t)) - f(t,H(y)(t))|^p\,dt \to 0 \quad \text{as } y_n \rightharpoonup y \tag{2.8}$$

then we are finished.

First we show for each $t \in [0,\infty)$ that

$$y_n \rightharpoonup y \quad \text{implies} \quad H(y_n(t)) \to H(y(t)). \tag{2.9}$$

Fix $t \in [0,\infty)$. Then

$$|H(y_n(t)) - H(y(t))| = \left| \int_0^\infty G(t,s)[y_n(s) - y(s)]\,ds \right| \to 0$$

as $y_n \rightharpoonup y$ since $G(t,\cdot\,) \in L^q[0,\infty)$. Now (2.9) together with the fact that f is a L^p-Carathéodory function (see (f_2)) gives

$$y_n \rightharpoonup y \Rightarrow f(t,H(y_n)(t)) \to f(t,H(y)(t)) \quad \text{a.e. on } [0,\infty). \tag{2.10}$$

Also for $u \in \overline{U}$ and $t \in [0,\infty)$ we have

$$\begin{aligned} |H(u(t))| &= \left| \int_0^\infty G(t,s)u(s)\,ds \right| \le \|u\|_{L^p[0,\infty)} \sup_{t\in[0,\infty)} \left(\int_0^\infty |G(t,s)|^q\,ds \right)^{1/q} \\ &\le M_0 \sup_{t\in[0,\infty)} \left(\int_0^\infty |G(t,s)|^q\,ds \right)^{1/q}. \end{aligned}$$

Thus there exists a $r > 0$ with

$$|H(u(t))| \le r \quad \text{for all } t \in [0,\infty) \text{ and } u \in \overline{U}. \tag{2.11}$$

Now (2.8) follows immediately from (2.10), (2.11), (f_3) and the Lebesgue dominated convergence theorem.

We may now apply Theorem 1.1 to deduce that F has a fixed point in $\overline{U}$. □

Example 2.2. Suppose $f\colon [0,\infty)\times\mathbb{R}\to\mathbb{R}$ is continuous with $(f)_3$ (with $p>1$) holding and suppose

$$\exists M>0 \text{ such that } |y|>M \text{ implies } yf(t,y)\le 0 \text{ for all } t\in[0,\infty). \tag{2.12}$$

Then (2.1) has at least one solution.

To see this we will use Theorem 2.1. Let y be a solution of $(2.7)_\lambda$ for some $\lambda\in(0,1)$. We *claim* $|y(t)|\le M$ for $t\in[0,\infty)$. If not, there exists a $t\in(0,\infty)$ with $|y(t)|>M$ and so $\sup_{[0,\infty)}|y(t)|=|y(t_0)|>M$ with $t_0\in(0,\infty)$. Consequently (2.12) implies

$$y(t_0)y''(t_0)=m^2[y(t_0)]^2-\lambda y(t_0)f(t_0,y(t_0))>0,$$

which contradicts the maximality of $|y(t_0)|$. Hence $|y(t)|\le M$ for $t\in[0,\infty)$. Let h_M be given as in (f_3) (with $r=M$) and notice

$$\|y''-m^2y\|_{L^p[0,\infty)}=\|-\lambda f(t,y)\|_{L^p[0,\infty)}\le\|h_M\|_{L^p[0,\infty)}$$

for any solution y to $(2.7)_\lambda$ for any $\lambda\in(0,1)$. If we take

$$M_0=\|h_M\|_{L^p[0,\infty)}+1$$

then (2.6) is satisfied so Theorem 2.1 guarantees that (2.1) has a solution.

Remark 2.4. It is easy to see that $f\colon[0,\infty)\times\mathbb{R}\to\mathbb{R}$ continuous could be replaced by (f_1) and (f_2) and $yf(t,y)\le 0$ for all $t\in[0,\infty)$ in (2.12) could be replaced by $yf(t,y)\le 0$ for a.e. $t\in[0,\infty)$.

The argument in Theorem 2.1 establishes the following existence principle for the operator equation

$$u=Tu \tag{2.13}$$

where $T\colon L^p[0,\infty)\to L^p[0,\infty)$ with $p>1$.

Theorem 2.4. *Suppose there is a constant M_0, independent of λ, with*

$$\|y\|_{L^p[0,\infty)}\ne M_0 \tag{2.14}$$

for any solution y to the problem

$$y=\lambda Ty\quad\textit{for any } \lambda\in(0,1).$$

In addition assume $T\colon\overline{U}\to L^p[0,\infty)$ is a weakly–strongly sequentially continuous map where $\overline{U}=\{u\in L^p[0,\infty):\|u\|_{L^p[0,\infty)}\le M_0\}$. Then (2.13) *has at least one solution in $\overline{U}$.*

Remark 2.5. There is an analogue of Theorem 2.2 for the operator equation (2.13) where $T\colon E\to E$ with E a reflexive Banach space.

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