Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 27, 2006, 333–385

TOPOLOGIES ON THE GROUP OF BOREL AUTOMORPHISMS OF A STANDARD BOREL SPACE

Sergey Bezuglyi — Anthony H. Dooley — Jan Kwiatkowski

Abstract. The paper is devoted to the study of topologies on the group $\mathrm{Aut}(X,\mathcal{B})$ of all Borel automorphisms of a standard Borel space $(X,\mathcal{B})$. Several topologies are introduced and all possible relations between them are found. One of these topologies, τ, is a direct analogue of the uniform topology widely used in ergodic theory. We consider the most natural subsets of $\mathrm{Aut}(X,\mathcal{B})$ and find their closures. In particular, we describe closures of subsets formed by odometers, periodic, aperiodic, incompressible, and smooth automorphisms with respect to the defined topologies. It is proved that the set of periodic Borel automorphisms is dense in $\mathrm{Aut}(X,\mathcal{B})$ (Rokhlin lemma) with respect to τ. It is shown that the τ-closure of odometers (and of rank 1 Borel automorphisms) coincides with the set of all aperiodic automorphisms. For every aperiodic automorphism $T \in \mathrm{Aut}(X,\mathcal{B})$, the concept of a Borel–Bratteli diagram is defined and studied. It is proved that every aperiodic Borel automorphism T is isomorphic to the Vershik transformation acting on the space of infinite paths of an ordered Borel–Bratteli diagram. Several applications of this result are given.

1. Introduction

The study of topologies on the group of transformations of an underlying space has a long history. Some of the early results in the area are the classical

2000 *Mathematics Subject Classification.* 377A04, 37B05, 54H20.
Key words and phrases. Standard Borel space, aperiodic and periodic automorphisms, odometer, Borel–Bratelli diagram.
The first named author supported in part by CRDF grant UM1-2546-KH-03.
The third named author supported in part by KBN grant 1P/03A/038/26.

results of J. Oxtoby and S. Ulam on the typical dynamical behavior of homeomorphisms which preserve a measure [26], [25]. Traditionally, this circle of problems has attracted attention in various areas of dynamical systems, notably, in measurable and topological dynamics, where it is important for many applications to understand what kind of transformations is typical for certain dynamics. Of course, this problem assumes that a topology is defined on the group of all transformations. The best known results concerning ergodic, mixing, and weakly mixing automorphisms of a measure space were obtained by P. Halmos and V. A. Rokhlin (see e.g. [17], [27]). Many results on approximation of automorphisms of a measure space can be found in the book by I. Cornfeld, S. Fomin, and Ya. Sinai [8]. The recent book by S. Alpern and V. S. Prasad [1] develops the Oxtoby–Ulam approach and contains new results on approximation of homeomorphisms of compact and non-compact manifolds.

Motivated by ideas used in measurable dynamics, we first started studying topologies in the context of Cantor minimal systems in [4] and [5], which we refer to as Cantor dynamics, but very soon it became clear that our approach could also be used for Borel automorphisms of a standard Borel space (Borel dynamics). The main goal of our two papers, the present one and [3], is to study the global properties of topologies on the group of all Borel automorphisms $\mathrm{Aut}(X,\mathcal{B})$ of a standard Borel space $(X,\mathcal{B})$ and the group of all homeomorphisms $\mathrm{Homeo}(\Omega)$ of a Cantor set Ω. Although we define and study several topologies on $\mathrm{Aut}(X,\mathcal{B})$, only two of them, the uniform (τ) and weak (p) topologies, are considered as basic since they are analogous to the topologies which are well-known in ergodic theory. Let us recall their definitions for automorphisms of a measure space.

Let $(X,\mathcal{B},\mu)$ be a standard measure space. On the group $\mathrm{Aut}(X,\mathcal{B},\mu)$ of all non-singular automorphisms of X, the uniform and weak topologies are defined by the metrics

$$d_u(S,T)=\mu(\{x\in X\mid Sx\neq Tx\})\quad\text{and}\quad d_w(S,T)=\sum_n 2^{-n}\mu(SA_n\,\Delta\,TA_n),$$

respectively, where $S,T\in\mathrm{Aut}(X,\mathcal{B},\mu)$ and (A_n) is a countable collection of Borel sets which is dense in $\mathcal{B}$. These concepts have turned out to be of crucial importance in ergodic theory. As far as we know, the first deep results on the topological properties of $\mathrm{Aut}(X,\mathcal{B},\mu)$ with respect to d_u and d_w appeared in the pioneering papers by P. Halmos [18] and V. A. Rokhlin [27] where the concept of approximation of automorphisms was introduced in abstract ergodic theory. Later, this concept was developed in many papers where the notion of approximation was considered in various areas of ergodic theory. Probably the most famous application of these ideas in measurable dynamics is the Rokhlin lemma, a statement on approximation of an aperiodic automorphism by periodic ones. However, there have been many other applications, too numerous to list here.

It seems rather curious but, to the best of our knowledge, these topologies have not so far been systematically studied in the context either of topological or Borel dynamics. However, we should mention the interesting paper [16] by E. Glasner and B. Weiss where the Rokhlin property is considered for homeomorphisms of compact metric spaces with respect to the topology of uniform convergence. For a standard Borel space, we do not have a fixed Borel measure on the underlying space (in contrast to measurable dynamics). Therefore, if we want to extend the definitions of topologies generated by metrics d_u and d_w to the group $\mathrm{Aut}(X,\mathcal{B})$, then we have to take into account the set $\mathcal{M}_1(X)$ of *all* Borel probability measures on $(X,\mathcal{B})$. Roughly speaking, we say that two Borel automorphisms S and T from $\mathrm{Aut}(X,\mathcal{B})$ are close in the uniform topology τ on $\mathrm{Aut}(X,\mathcal{B})$ if for *any* measure $\mu \in \mathcal{M}_1(X)$ the set where S and T are different is small in the measure μ. To define the topology p, which is treated as an analogue of d_w, we observe that if the symmetric difference of two Borel sets is arbitrarily small with respect to *any* $\mu \in \mathcal{M}_1(X)$, then these sets must coincide. Thus, a p-neighbourhood of T is formed by those S for which $SB = TB$ where B is a given Borel set (see Section 2 for strict definitions).

In [4], [5], we first gave a definition of the uniform topology τ analogous to d_u. Our main interest in those papers was focused on full groups and normalizers of Cantor minimal systems. In particular, we showed that the full group of a minimal homeomorphism is closed in τ. The paper [3] is devoted to the study of topologies on the group $\mathrm{Homeo}(\Omega)$ of all homeomorphisms of a Cantor set Ω. We have tried to make the content of the current paper parallel to that of [3] by considering similar topologies and similar problems. The comparison of all results from these papers would take up too much place. We mention only that on a Cantor set Ω one can study both dynamics, Borel and Cantor, and answer some questions about their interplay because $\mathrm{Homeo}(\Omega)$ is obviously a subset of $\mathrm{Aut}(\Omega,\mathcal{B})$. In particular, we can consider in both cases the topology of uniform convergence generated by the metric

$$D(S,T) = \sup_{x\in\Omega} d(Sx,Tx) + \sup_{x\in\Omega} d(S^{-1}x,T^{-1}x)$$

on the groups $\mathrm{Homeo}(\Omega)$ and $\mathrm{Aut}(\Omega,\mathcal{B})$. Notice that D coincides with the topology p defined on $\mathrm{Homeo}(\Omega)$ by clopen sets only [3]. This topology is extremely useful in the study of homeomorphisms of a Cantor set. In Borel dynamics, the topologies p and D are obviously inequivalent. Nevertheless, it seems to be interesting to study topological properties of the metric space $(\mathrm{Aut}(\Omega,\mathcal{B}),D)$ keeping in mind the parallel theory for Cantor dynamics.

The outline of the paper is as follows. We consider more systematically the definitions and properties of various natural topologies on $\mathrm{Aut}(X,\mathcal{B})$. Actually, we study simultaneously a collection of Hausdorff topologies: τ (the analogue of uniform topology), τ' (which is equivalent to τ), τ'' (which is weaker than τ), p (which is the direct analogy of the weak topology in ergodic theory and which is mostly useful in the context of Cantor dynamics), $\widetilde{p}$ (which is equivalent to p), and $\overline{p}$ (which is weaker than p). We consider also the topologies τ_0 (which is weaker than τ) and p_0 (which is equivalent to p) as natural modifications of τ and p. They all (except $\overline{p}$) make $\mathrm{Aut}(X,\mathcal{B})$ into a topological group.

The initial part of the paper is devoted to discovering all possible relations between these topologies (see Theorem 2.3 and its proof in Section 4). In Section 2, we discuss various topological properties of the group $\mathrm{Aut}(X,\mathcal{B})$ and its subsets. For example, we describe convergent sequences of automorphisms from $\mathrm{Aut}(X,\mathcal{B})$ and show that $(\mathrm{Aut}(X,\mathcal{B}),p)$ is a zero-dimensional topological space (Corollary 2.11). The group $\mathrm{Aut}(X,\mathcal{B})$ has a normal subgroup $\mathrm{Ctbl}(X)$ which consists of automorphisms with at most countable support. It turns out that $\mathrm{Ctbl}(X)$ is closed with respect to the above topologies. This fact allows us to study topologies on the quotient group $\widehat{\mathrm{Aut}}(X,\mathcal{B}) = \mathrm{Aut}(X,\mathcal{B})/\mathrm{Ctbl}(X)$. This kind of identification of Borel automorphisms is analogous to that usually used in ergodic theory.

In Sections 3 and 5, we study the following classes of Borel automorphisms: periodic, aperiodic, smooth, incompressible, and of rank 1 (the latter includes odometers). It is proved that periodic automorphisms are dense in $\mathrm{Aut}(X,\mathcal{B})$ with respect the topology τ. This allows us to prove a version of the Rokhlin lemma (Theorem 3.7). Remark that the problem of periodic approximation of an aperiodic Borel automorphism has been also studied by M. Nadkarni [24] and B. Weiss [29]. As an immediate consequence of these results, we obtain that the set $\mathcal{A}p$ of aperiodic automorphisms is nowhere dense in $(\mathrm{Aut}(X,\mathcal{B}),\tau)$. We also prove that the full group $[T]$ of a Borel automorphism $T \in \mathrm{Aut}(X,\mathcal{B})$ is closed with respect to all the topologies. We consider the set of incompressible automorphisms, $\mathcal{I}nc$, consisting of those aperiodic automorphisms which admit an invariant Borel probability measure and prove that $\mathcal{I}nc$ is a closed nowhere dense subset of $\mathcal{A}p$ with respect to p. It is shown that the τ-closure of rank 1 Borel automorphisms coincides with that of odometers (Theorem 3.19).

In the last section, we introduce the concept of Borel–Bratteli diagrams in the context of Borel dynamics. To do this, we use a remarkable result on existence of a vanishing sequence of markers [2], [24]. It is well known that Bratteli diagrams play a very important role in the study of minimal homeomorphisms of a Cantor set [19], [14]. Similarly to Cantor dynamics, we show that every aperiodic Borel automorphism is isomorphic to the Vershik transformation acting on the space of infinite paths of a Borel-Bratteli diagram. Several applications of this result are given. In particular, we prove that the set of odometers is τ-dense in aperiodic Borel automorphisms. We also discuss properties of such diagrams for automorphisms of compact and locally compact spaces. We believe that this approach to the study of Borel automorphisms will lead to further interesting developments in this area.

To conclude, we would like to make two remarks. Firstly, Borel automorphisms of a standard Borel space (or, more general, countable Borel equivalence relations) have been extensively studied in many papers during the last decade. We refer to numerous works of A. Kechris, G. Hjorth, M. Foreman, M. Nadkarni, B. Weiss and others. More comprehensive references can be found, for instance, in [2], [22], [20], [12], [24]. Secondly, it is impossible to discuss in one paper all the interesting problems related to topologies on $\mathrm{Aut}(X, \mathcal{B})$. We consider the current paper as the first step in the study of topological properties of $\mathrm{Aut}(X, \mathcal{B})$. Our primary goal is to display a wealth of new topological methods in Borel dynamics.

Throughout the paper, we use the following standard *notation*:

- $(X, \mathcal{B})$ is a standard Borel space with the σ-algebra of Borel sets $\mathcal{B} = \mathcal{B}(X)$; $\mathcal{B}_0$ is the subset of $\mathcal{B}$ consisting of uncountable Borel sets.
- $\mathrm{Aut}(X, \mathcal{B})$ is the group of all one-to-one Borel automorphisms of X with the identity map $\mathbb{I} \in \mathrm{Aut}(X, \mathcal{B})$.
- $\mathcal{A}p$ is the set of all aperiodic Borel automorphisms.
- $\mathcal{P}er$ is the set of all periodic Borel automorphisms.
- $\mathcal{M}_1(X)$ is the set of all Borel probability measures on $(X, \mathcal{B})$. Let $\mathcal{M}_1^c(X)$ denote the subset of $\mathcal{M}_1(X)$ formed by continuous (non-atomic) Borel probability measures.
- δ_x is the Dirac measure at $x \in X$.
- $E(S, T) = \{x \in X \mid Tx \neq Sx\} \cup \{x \in X \mid T^{-1}x \neq S^{-1}x\}$ where $S, T \in \mathrm{Aut}(X, \mathcal{B})$.
- $B(X)$ is the the set of Borel real-valued bounded functions, $B(X)_1 = \{f \in B(X) \mid \|f\| := \sup\{|f(x)| : x \in X\} \leq 1\}$, $\mu(f) = \int_X f\, d\mu$ where $f \in B(X)$, $\mu \in \mathcal{M}_1(X)$.
- $\mu \circ S(A) := \mu(SA)$ and $\mu \circ S(f) := \int_X f\, d(\mu \circ S) = \int_X f(S^{-1}x)\, d\mu(x)$ where $S \in \mathrm{Aut}(X, \mathcal{B})$.
- $A^c := X \setminus A$ where $A \in \mathcal{B}$.
- the term "automorphism" means a Borel automorphism of $(X, \mathcal{B})$; we deal with Borel subsets of X only.
- Ω is a Cantor set.
- $\mathrm{Homeo}(\Omega)$ is the group of all homeomorphisms of Ω.

2. Topologies on $\mathrm{Aut}(X,\mathcal{B})$

2.1. Definition of topologies on $\mathrm{Aut}(X,\mathcal{B})$. In this section, we define topologies on $\mathrm{Aut}(X,\mathcal{B})$ and establish their main properties.

Definition 2.1. The topologies τ, τ', τ'', p, $\widetilde{p}$, and $\overline{p}$ on $\mathrm{Aut}(X,\mathcal{B})$ are defined by the bases of neighbourhoods $\mathcal{U}$, $\mathcal{U}'$, $\mathcal{U}''$, $\mathcal{W}$, $\widetilde{\mathcal{W}}$, and $\overline{\mathcal{W}}$, respectively. They are: $\mathcal{U} = \{U(T;\mu_1,\dots,\mu_n;\varepsilon)\}$, $\mathcal{U}' = \{U'(T;\mu_1,\dots,\mu_n;\varepsilon)\}$, $\mathcal{U}'' = \{U''(T;\mu_1,\dots,\mu_n;\varepsilon)\}$, $\mathcal{W} = \{W(T;F_1,\dots,F_k)\}$, $\widetilde{\mathcal{W}} = \widetilde{W}(T;f_1,\dots,f_n;\varepsilon)$, and $\overline{\mathcal{W}} = \{\overline{W}(T;F_1,\dots,F_k;\mu_1,\dots,\mu_n;\varepsilon)\}$ where

$$U(T;\mu_1,\dots,\mu_n;\varepsilon) = \{S\in \mathrm{Aut}(X,\mathcal{B}) \mid \mu_i(E(S,T)) < \varepsilon,\ i=1,\dots,n\}, \tag{2.1}$$

$$U'(T;\mu_1,\dots,\mu_n;\varepsilon) = \{S\in \mathrm{Aut}(X,\mathcal{B}) \mid \sup_{F\in\mathcal{B}} \mu_i(TF\ \Delta\ SF) < \varepsilon,\ i=1,\dots,n\}, \tag{2.2}$$

$$U''(T;\mu_1,\dots,\mu_n;\varepsilon) = \{S\in \mathrm{Aut}(X,\mathcal{B}) \mid \sup_{f\in B(X)_1} |\mu_i\circ S(f) - \mu_i\circ T(f)| < \varepsilon,\ i=1,\dots,n\}, \tag{2.3}$$

$$W(T;F_1,\dots,F_k) = \{S\in \mathrm{Aut}(X,\mathcal{B}) \mid SF_i = TF_i,\ i=1,\dots,k\}, \tag{2.4}$$

$$\widetilde{W}(T;f_1,\dots,f_m;\varepsilon) = \{S\in \mathrm{Aut}(X,\mathcal{B}) \mid \|f_i\circ T^{-1} - f_i\circ S^{-1}\| < \varepsilon, i=1,\dots,m\} \tag{2.5}$$

$$\overline{W}(T;(F_i)_1^k;(\mu_j)_1^n;\varepsilon) = \{S\in \mathrm{Aut}(X,\mathcal{B}) \mid \mu_j(SF_i\Delta TF_i) + \mu_j(S^{-1}F_i\Delta T^{-1}F_i) < \varepsilon,\ i=1,\dots,k;\ j=1,\dots,n\}. \tag{2.6}$$

In all the above definitions $T\in \mathrm{Aut}(X,\mathcal{B})$, $\mu_1,\dots,\mu_n\in\mathcal{M}_1(X)$, $F_1,\dots,F_k\in\mathcal{B}$, $f_1,\dots,f_m\in B(X)$, and $\varepsilon>0$.

In the paper, we will study the topologies which are defined by their bases of neighbourhoods. With some abuse of definition, we will say that two topologies are equivalent if they are defined by equivalent bases of neighbourhoods.

If the set $E_0(S,T) = \{x\in X : Sx\neq Tx\}$ were used in (2.1) instead of $E(S,T)$, then we would obtain the topology equivalent to τ.

It is natural to define also two further topologies, which are similar to τ and p, by considering only continuous measures and uncountable Borel sets.

Definition 2.2. The topologies τ_0 and p_0 on $\mathrm{Aut}(X,\mathcal{B})$ are defined by the bases of neighbourhoods $\mathcal{U}_0 = \{U_0(T;\nu_1,\dots,\nu_n;\varepsilon)\}$ and $\mathcal{W}_0 = \{W_0(T;A_1,\dots,A_n)\}$, respectively, where $U_0(T;\nu_1,\dots,\nu_n;\varepsilon)$ and $W_0(T;A_1,\dots,A_n)$ are defined as in (2.1) and (2.4) with $\nu_i\in\mathcal{M}_1^c(X)$ and $A_i\in\mathcal{B}_0$, $i=1,\dots,n$.

Obviously, τ_0 is not stronger than τ and p_0 is not stronger than p.

Note that the topology τ was first introduced in [4] where, motivated by ergodic theory, we called it the *uniform topology*. We defined p in [5] in the context of homeomorphisms of Cantor sets. In this section, we use a number of results about these topologies which are proved lately. Namely, the following theorem is proved in Section 4.

THEOREM 2.3.

(a) *The topologies τ and τ' are equivalent.*
(b) *The topology τ ($\sim \tau'$) is strictly stronger than τ''.*
(c) *The topology τ is strictly stronger than $\overline{p}$.*
(d) *The topology τ is strictly stronger than τ_0.*
(e) *The topology p is equivalent to $\widetilde{p}$.*
(f) *The topology p ($\sim \widetilde{p}$) is equivalent to p_0.*
(g) *The topology p is strictly stronger than $\overline{p}$.*
(h) *The topology τ is not comparable with p and the topology τ'' is not comparable with $\overline{p}$ and τ_0.*

We will also introduce in Section 4 two auxiliary topologies $\widetilde{\tau}$ and $\overline{\tau}$ equivalent to τ'' which will allow us to have a more convenient description of τ'' (see Definition 4.2 and Proposition 4.4. In particular, the topology $\overline{\tau}$ is defined by neighbourhoods

$$(2.7)\quad \overline{V}(T;\mu_1,\dots,\mu_n;\varepsilon) = \Big\{ S \in \mathrm{Aut}(X,\mathcal{B}) \ \Big| \ \sup_{F\in\mathcal{B}} |\mu_j(TF) - \mu_j(SF)| < \varepsilon,\ j = 1,\dots,n \Big\}$$

where $\mu_1,\dots,\mu_n \in \mathcal{M}_1(X)$.

Given an automorphism T of $(X,\mathcal{B})$, we can associate a linear unitary operator L_T on the Banach space $B(X)$ by $(L_T f)(x) = f(T^{-1}x)$. Then the topology $\widetilde{p}$ is induced on $\mathrm{Aut}(X,\mathcal{B})$ by the strong operator topology on bounded linear operators of $B(X)$. Theorem 2.3 asserts the equivalence of p and $\widetilde{p}$. It is well known that the weak topology in ergodic theory can be also defined in a similar way using the strong topology on linear bounded operators of a Hilbert space. This observation is a justification of the name "*weak topology*" which will be used to refer to p.

PROPOSITION 2.4. *$\mathcal{U}, \mathcal{U}', \mathcal{U}'', \mathcal{U}_0$, $\mathcal{W}$, $\widetilde{\mathcal{W}}$, $\mathcal{W}_0$, and $\overline{\mathcal{W}}$ are bases of Hausdorff topologies τ, τ', τ'', τ_0, p, $\widetilde{p}$, p_0, and $\overline{p}$, respectively.* $\mathrm{Aut}(X,\mathcal{B})$ *is a topological group with respect to τ, τ', τ'', τ_0, p, $\widetilde{p}$, p_0, and* $\mathrm{Aut}(X,\mathcal{B})$ *is not a topological group with respect to $\overline{p}$.*

PROOF. The first statement is clear for τ, τ', τ_0, p, $\widetilde{p}$, p_0 and can be immediately deduced from the definition of neighbourhoods (2.1), (2.2), (2.4), (2.5).

We need to check it for τ'' and $\overline{p}$ only. Let S be a Borel automorphism taken in a given neighbourhood $U''(T;\mu_1,\ldots,\mu_n;\varepsilon)$. We will show that there exists $U_0''(S;\nu_1,\ldots,\nu_n;\delta)\subset U''(T;\mu_1,\ldots,\mu_n;\varepsilon)$. To see this, take $\nu_i=\mu_i$, $\delta=\varepsilon-c$, where

$$c=\max_{1\le i\le n}\{\sup_{f\in B(X)_1}|\mu_i\circ T(f)-\mu_i\circ S(f)|\}.$$

If $R\in U_0''$, then we get

$$\begin{aligned}\sup_{f\in B(X)_1}&|\mu_i\circ T(f)-\mu_i\circ R(f)|\\ &\le \sup_{f\in B(X)_1}|\mu_i\circ T(f)-\mu_i\circ S(f)|+\sup_{f\in B(X)_1}|\mu_i\circ S(f)-\mu_i\circ R(f)|\\ &\qquad\qquad\qquad\qquad\qquad\qquad< c+\varepsilon-c=\varepsilon,\end{aligned}$$

i.e. $R\in U''(T;\ \mu_1,\ldots,\mu_n;\ \varepsilon)$.

To see that $\{\overline{\mathcal{W}}\}$ is a base of neighbourhoods, we may use a slight modification of the above argument. For $S\in\overline{W}(T;F_1,\ldots,F_k;\mu_1,\ldots,\mu_n;\varepsilon)$, take

$$c=\max_{i,j}[\mu_j(SF_i\ \Delta\ TF_i)+\mu_j(S^{-1}F_i\ \Delta\ T^{-1}F_i)].$$

It is easily seen from (2.6) that

$$\overline{W}(S;F_1,\ldots,F_k;\mu_1,\ldots,\mu_n;\delta)\subset\overline{W}(T;F_1,\ldots,F_k;\mu_1,\ldots,\mu_n;\varepsilon)$$

where $\delta=\varepsilon-c$.

Now we prove that $\mathrm{Aut}(X,\mathcal{B})$ is a topological group with respect to $\tau\sim\tau'$, τ_0, $p\sim p_0\sim\widetilde{p}$, and τ''.

Consider first $(\mathrm{Aut}(X,\mathcal{B}),\tau)$. In the case of the topology τ_0 the proof is similar. Let $S,T\in\mathrm{Aut}(X,\mathcal{B})$. We need the following facts:

(i) $U(T;\mu_1,\ldots,\mu_n;\varepsilon)=U(T^{-1};\mu_1,\ldots,\mu_n;\varepsilon)^{-1}$;
(ii) $\{x\in\Omega: Sx\ne Tx\}\subset\{x\in\Omega\mid Sx\ne Rx\}\cup\{x\in\Omega\mid Rx\ne Tx\}$.

Indeed, (i) follows from the relation $E(S,T)=E(S^{-1},T^{-1})$ and (ii) is checked straightforward.

By (i), the map $T\mapsto T^{-1}$ is continuous. To prove that $(S,T)\mapsto ST$ is also continuous, we show that for any neighbourhood $U_{ST}=U(ST;\mu_1,\ldots,\mu_n;\varepsilon)$ there exist neighbourhoods $U_S=U(S;\nu_1,\ldots,\nu_k;\varepsilon_1)$, $U_T=U(T;\sigma_1,\ldots,\sigma_m;\varepsilon_2)$ such that $U_SU_T\subset U_{ST}$. Take $\varepsilon_1=\varepsilon_2=\varepsilon/4$, $k=m=2n$, $(\nu_1,\ldots,\nu_{2n})=(\mu_1,\ldots,\mu_n,\ \mu_1\circ T^{-1},\ldots,\mu_n\circ T^{-1})$, and $(\sigma_1,\ldots,\sigma_{2n})=(\mu_1,\ldots,\mu_n,\mu_1\circ S,\ldots,\mu_n\circ S)$.

Let $P \in U_S$, $Q \in U_T$. It follows from (ii) that

$$\begin{aligned} E(PQ, ST) &= \{x \mid PQx \neq STx\} \cup \{x \mid (PQ)^{-1}x \neq (ST)^{-1}x\} \\ &\subset \{x \mid PQx \neq PTx\} \cup \{x \mid PTx \neq STx\} \\ &\quad \cup \{x \mid Q^{-1}P^{-1}x \neq Q^{-1}S^{-1}x\} \cup \{x \mid Q^{-1}S^{-1}x \neq T^{-1}S^{-1}x\} \\ &= \{x \mid Qx \neq Tx\} \cup T^{-1}(\{x \mid Px \neq Sx\}) \\ &\quad \cup \{x \mid P^{-1}x \neq S^{-1}x\} \cup S(\{x \mid Q^{-1}x \neq T^{-1}x\}). \end{aligned}$$

Then, for any $i = 1, \ldots, n$,

$$\begin{aligned} \mu_i(E(PQ, ST)) \leq \mu_i(\{x \mid Qx \neq Tx\}) + \mu_i \circ T^{-1}(\{x \mid Px \neq Sx\}) \\ + \mu_i(\{x \mid P^{-1}x \neq S^{-1}x\}) + \mu_i \circ S(\{x \mid Q^{-1}x \neq T^{-1}x\}) < \varepsilon. \end{aligned}$$

Thus, $PQ \in U_{ST}$.

The proof of continuity of $(T, S) \mapsto TS$ and $T \mapsto T^{-1}$ in $(\mathrm{Aut}(X, \mathcal{B}), p)$ is straightforward.

To complete the proof of the proposition, we will show that $\mathrm{Aut}(X, \mathcal{B})$ is a topological group with respect to τ''. Again choose $S, T \in \mathrm{Aut}(X, \mathcal{B})$ and let $U''(ST) = U''(ST, \mu_1, \ldots, \mu_n; \varepsilon)$ be a given neighbourhood. Consider $U''(S) = U''(S; \mu_1, \ldots, \mu_n; \varepsilon/2)$ and $U''(T) = U''(T; \mu_1 \circ S, \ldots, \mu_n \circ S; \varepsilon/2)$. Then for $R \in U''(S), Q \in U''(T)$ we have

$$\begin{aligned} &\sup_{f \in B(X)_1} |\mu_i \circ RQ(f) - \mu_i \circ ST(f)| \\ &\quad \leq \sup_{f \in B(X)_1} |\mu_i \circ RQ(f) - \mu_i \circ SQ(f)| + \sup_{f \in B(X)_1} |\mu_i \circ SQ(f) - \mu_i \circ ST(f)| \\ &\quad = \sup_{f' = L_Q f \in B(X)_1} |\mu_i \circ R(f') - \mu_i \circ S(f')| \\ &\qquad + \sup_{f \in B(X)_1} |(\mu_i \circ S)Q(f) - (\mu_i \circ S)T(f)| < \varepsilon. \end{aligned}$$

We remark that $(\mathrm{Aut}(X, \mathcal{B}), \overline{p})$ is not a topological group as one can prove that the product $(T, S) \mapsto TS$ is not continuous in the topology $\overline{p}$. To see this, we show that there exists a $\overline{p}$-neighbourhood $\overline{W}_0$ of the identity such that in any neighbourhoods $\overline{W}_1(\mathbb{I}; (E_i); (\nu_j); \varepsilon)$ and $\overline{W}_2(\mathbb{I}; (D_i); (\lambda_j); \delta)$ of the identity one can find automorphisms R and Q, respectively, such that $RQ \notin \overline{W}_0$. Take $\overline{W}_0 = \overline{W}_0(\mathbb{I}; F; \delta_{x_0}; 1/2)$. Note that if $Sx_0 \notin F$, then $S \notin \overline{W}_0$. Find an automorphism $Q \in \overline{W}_2$ such that the point Qx_0 is of measure zero with respect to all ν_j. Finally, find some $R \in \overline{W}_1$ such that $RQx_0 \notin F$. □

Remark 2.5. (1) Let $(X, \mathcal{B})$ and $(Y, \mathcal{C})$ be two standard Borel spaces (hence, they are Borel isomorphic). It can easily be seen that $(\mathrm{Aut}(X, \mathcal{B}), \mathrm{top}(X))$ and $(\mathrm{Aut}(Y, \mathcal{C}), \mathrm{top}(Y))$ are homeomorphic where top is any of the topologies from Definitions 2.1 and 2.2.

(2) Let (X,d) be a compact metric space. In this case, we can consider the group $\mathrm{Aut}(X,\mathcal{B})$ of Borel automorphisms and its subgroup $\mathrm{Homeo}(X)$ of homeomorphisms of X. Define for $S,T \in \mathrm{Aut}(X,\mathcal{B})$ the topology of uniform convergence generated by the metric[1]

$$D(S,T) = \sup_{x\in X} d(Sx,Tx) + \sup_{x\in X} d(S^{-1}x,T^{-1}x). \tag{2.8}$$

Then $(\mathrm{Aut}(X,\mathcal{B}),D)$ is a complete metric space and $\mathrm{Homeo}(X)$ is closed in $\mathrm{Aut}(X,\mathcal{B})$. When $X=\Omega$ is a Cantor set, we studied thoroughly the Polish group $(\mathrm{Homeo}(\Omega),D)$ in [3]. It is not hard to see that in Cantor dynamics the topology on $\mathrm{Homeo}(\Omega)$ generated by D is equivalent to the topology p defined by clopen sets only. We note that, in contrast to (1), the topology generated by D on $\mathrm{Aut}(X,\mathcal{B})$ depends, in general, on the topological space (X,d). Nevertheless, we think it is worth to study the topological properties of $\mathrm{Aut}(X,\mathcal{B})$ and $\mathrm{Homeo}(X)$ for a fixed compact (or Cantor) metric space X because one can compare in this case these properties for both groups.

Let $\mathrm{Ctbl}(X)$ be defined as the subset of $\mathrm{Aut}(X,\mathcal{B})$ consisting of all automorphisms with countable support, that is $T\in \mathrm{Ctbl}(X)$ if and only if $E(S,\mathbb{I})$ is at most countable. One can show that $\mathrm{Ctbl}(X)$ is a normal subgroup, closed with respect to the topologies which we have defined (see Lemma 2.6 below). Therefore $\widehat{\mathrm{Aut}}(X,\mathcal{B}) = \mathrm{Aut}(X,\mathcal{B})/\mathrm{Ctbl}(X)$ is a Hausdorff topological group with respect to the quotient topology. Note that the quotient group $\widehat{\mathrm{Aut}}(X,\mathcal{B})$ was first considered in [28] where the simplicity of this group was proved. Considering elements from $\widehat{\mathrm{Aut}}(X,\mathcal{B})$, we identify Borel automorphisms which are different on a countable set. The class of automorphisms equivalent to a Borel automorphism T we will denote by the same symbol T and write $T\in\widehat{\mathrm{Aut}}(X,\mathcal{B})$. This corresponds to the situation in ergodic theory when two automorphisms are also identified if they are different on a set of measure 0. We studied topological properties of the group $\widehat{\mathrm{Aut}}(X,\mathcal{B})$ in [6].

Lemma 2.6. *The normal subgroup* $\mathrm{Ctbl}(X)\subset \mathrm{Aut}(X,\mathcal{B})$ *is closed with respect to the all topologies from Definitions* 2.1 *and* 2.2.

Proof. By Theorem 2.3, it is sufficient to prove the statement of the lemma for the topologies τ'', τ_0 and $\overline{p}$. Notice that we gave in [6] a direct proof of the fact that $\mathrm{Ctbl}(X)$ is closed in τ and p. The case of the topology τ_0 is similar to that of τ.

Suppose that $S\in \overline{\mathrm{Ctbl}(X)}^{\tau''}\setminus \mathrm{Ctbl}(X)$. Then there exists an uncountable Borel set F such that $SF\cap F=\emptyset$. Let μ be a continuous Borel probability

[1] In fact, the metric D can be considered on $\mathrm{Aut}(X,\mathcal{B})$ if X is a totally bounded metric space. Many of results concerning the metric D which are proved below for a compact metric space can be generalized to totally bounded spaces.

measure such that $\mu(F) = 1$ and let $f_0(x) = \chi_F(x)$. The τ''-neighbourhood $U''(S) = U''(S;\mu;1/2)$ consists of automorphisms T satisfying the condition

$$\sup_{f\in B_1(X)} |\mu\circ S(f) - \mu\circ T(f)| < 1/2.$$

By assumption, there exists $T_0 \in \mathrm{Ctbl}(X)\cap U''(S)$. Then $|\mu\circ S(f_0)-\mu\circ T_0(f_0)| < 1/2$. On the other hand, it can be easily seen that $\mu\circ T_0(f_0) = 1$ and $\mu\circ S(f_0) = 0$. This leads to a contradiction.

The fact that $\mathrm{Ctbl}(X)$ is closed in $\overline{p}$ is proved in the same way. We leave the details to the reader. □

Notice also that $\overline{\mathrm{Ctbl}(X)}^D = \mathrm{Ctbl}(X)$ when (X,d) is a compact metric space. Indeed, if $(T_n)_{n\in\mathbb{N}} \subset \mathrm{Ctbl}(X)$, then there exists a countable set $C \subset X$ such that $T_n x = x$ for all $x \in X\setminus C$. Let $D(T_n,T) \to 0$ $(n\to\infty)$. We obtain $Tx = x$ for every $x \in X\setminus C$, i.e. $T \in \mathrm{Ctbl}(X)$.

Let π be the natural projection from $\mathrm{Aut}(X,\mathcal{B})$ to $\widehat{\mathrm{Aut}}(X,\mathcal{B})$. Lemma 2.6 allows us to define the quotient topologies $\widehat{\tau} = \pi(\tau)$, $\widehat{\tau}_0 = \pi(\tau_0)$, and $\widehat{p} = \pi(p)$ on $\widehat{\mathrm{Aut}}(X,\mathcal{B})$. It turns out that $\widehat{\tau}$-neighbourhoods are defined by continuous measures and $\widehat{p}$-neighbourhoods are defined by uncountable Borel sets. In particular, this means that $\widehat{\tau}$ is equivalent to $\widehat{\tau}_0$. In [6], the following proposition was proved.

PROPOSITION 2.7. *Given a $\widehat{\tau}$-neighbourhood $\widehat{U} = \widehat{U}(T;\mu_1,\ldots,\mu_n;\varepsilon)$ and a $\widehat{p}$-neighbourhood $\widehat{W} = \widehat{W}(T;F_1,\ldots,F_m)$, there exist neighbourhoods*

$$\begin{aligned}\widehat{U}_0(T;\nu_1,\ldots,\nu_n;\varepsilon) &= U_0(T;\nu_1,\ldots,\nu_n;\varepsilon)\,\mathrm{Ctbl}(X),\\ \widehat{W}_0(T;B_1,\ldots,B_m) &= W_0(T;B_1,\ldots,B_m)\,\mathrm{Ctbl}(X)\end{aligned}$$

in $\widehat{\tau}$ and $\widehat{p}$, respectively, such that $\widehat{U}_0 \subset \widehat{U}$, $\widehat{W}' \subset \widehat{W}$ where $\nu_1,\ldots,\nu_n \in \mathcal{M}_1^c(X)$ and $B_1,\ldots,B_m \in \mathcal{B}_0$.

2.2. Properties of the topologies. We will now discuss some properties of the topologies which we introduced above. In particular, we consider convergent sequences with respect to these topologies.

REMARK 2.8. (1) We recall some facts about the uniform topology τ from the paper [4].

(a) $T_n \xrightarrow{\tau} S$ if and only if for all $x \in X$ there exists $n(x) \in \mathbb{N}$ such that for all $n > n(x)$, $T_n x = Sx$;
(b) $T_n \xrightarrow{\tau} S$ if and only if, for all $\mu \in \mathcal{M}_1(X)$, $\mu(E(T_n,S)) \to 0$ if and only if, for all $x \in X$, $\delta_x(E(T_n,S)) \to 0$;
(c) A sequence of Borel automorphisms $(T_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(\mathrm{Aut}(X,\mathcal{B}),\tau)$ if and only if $\bigcup_{n\in\mathbb{N}} X_n = X$ where $X_n = \{x \in X \mid$

$T_n x = T_{n+1} x = \dots\}$. Clearly, every Cauchy sequence converges to a Borel automorphism in τ.

(d) For a Cantor set Ω, $\mathrm{Homeo}(\Omega)$ is not closed in $\mathrm{Aut}(\Omega, \mathcal{B}(\Omega))$ with respect to τ. (In fact, it is shown in [3] that $\mathrm{Homeo}(\Omega)$ is dense in $\mathrm{Aut}(\Omega, \mathcal{B}(\Omega))$ in τ).

(2) Note that

$$T_n \xrightarrow{\tau_0} T \Leftrightarrow [\forall \mu \in \mathcal{M}_1^c(X),\ \mu(E(T_n, T)) \to 0,\ n \to \infty]. \tag{2.9}$$

It is not hard to see that (2.9) holds if the set

$$C = \bigcap_{n \in \mathbb{N}} \bigcup_{m \ge n} E(T_m, T) \tag{2.10}$$

is countable. Indeed, if $C_n = \bigcup_{m \ge n} E(T_m, T)$, then $\mu(C_n) \to 0$ for all $\mu \in \mathcal{M}_1^c(X)$. Hence $\mu(E(T_m, T)) \to 0$ as $m \to \infty$.

Observe that condition (2.10) is not necessary for τ_0-convergence. This means that even a weaker form of (1)(a) does not hold. To see this, let us consider the following example. Set $X = [0, 1)$ and denote by (ξ_n) the refining sequence of partitions of X into the intervals $A_n(i) = [i2^{-n}, (i+1)2^{-n})$, $i = 0, \dots, 2^n - 1$, $n \in \mathbb{N}$. Let μ be a continuous Borel probability measure on X. Then for all $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that $\mu(A_n(i)) < \varepsilon$ for all $n > N_\varepsilon$.

Now let $T_n(i)$ be a Borel automorphism of X such that $E(T_n(i), \mathbb{I}) = A_n(i)$. Then $\mu(E(T_n(i), \mathbb{I}) \to 0$ as $n \to \infty$. By (2.9), this sequence of automorphisms converges to the identity map in τ_0. On the other hand, for any $x \in [0, 1)$ the property $T_n(i)x \ne x$ holds for infinitely many automorphisms from the sequence $(T_n(i))$.

(3) $(\mathrm{Aut}(X, \mathcal{B}), p)$ is a complete nonseparable topological group. Note that it follows from Theorem 2.3 and (1) that $(\mathrm{Aut}(X, \mathcal{B}), \tau'')$ and $(\mathrm{Aut}(X, \mathcal{B}), \overline{p})$ are also complete spaces.

(4) Let T be a Borel automorphism and let $U(T; \mu_1, \dots, \mu_n; \varepsilon)$ be a τ-neighbourhood of T. Consider $\nu = n^{-1}(\mu_1 + \dots + \mu_n)$. Then it can be easily shown that $U(T;\ \nu;\ n^{-1}\varepsilon) \subset U(T; \mu_1, \dots, \mu_n; \varepsilon)$. This means that we can work with a single measure instead of a finite collection of measures when it is more convenient.

Remark 2.9. We notice the following three simple properties of the topology p.

(1) Without loss of generality, we may assume that p is generated by neighbourhoods $W(T; F_1, \dots, F_n)$ where $(F_1, \dots, F_n)$ is a partition of X.

(2) Let $W(T; F_1, \dots, F_n)$ be given, then for every $S \in W(T; F_1, \dots, F_n)$ one has $W(S; F_1, \dots, F_n) = W(T; F_1, \dots, F_n)$. It follows from this

observation that $W(\mathbb{I}; F_1, \ldots, F_n)$ is an open (and therefore also closed) subgroup of $(\mathrm{Aut}(X, \mathcal{B}), p)$.

(3) For any $T \in \mathrm{Aut}(X, \mathcal{B})$, we have $W(T^{-1}; TF_1, \ldots, TF_n)^{-1} = W(T; F_1, \ldots, F_n)$.

Proposition 2.10. *The sets* $W(T; F_1, \ldots, F_n)$ *are closed in* $\mathrm{Aut}(X, \mathcal{B})$ *with respect to the topologies* τ, τ', τ'', p *and* $\overline{p}$.

Proof. We begin with two simple observations. Firstly, the proposition will be proved if we can show that every set $W(T; F)$ is closed. Secondly, if the proposition holds for τ'' and $\overline{p}$, then it holds for the other topologies because they all are stronger than either τ'' or $\overline{p}$.

(1) Let us first consider $\overline{p}$. We show that $W(T; F)^c$ is open in $\overline{p}$. Suppose $S \in W(T; F)^c$, that is $SF \neq TF$. Then we have two cases:

(i) $E := TF \setminus SF \neq \emptyset$,

(ii) $TF \setminus SF = \emptyset$, that is $TF \subset SF$.

In case (i), define a neighbourhood $\overline{W}(S; F; \mu; 1/2)$ with a measure μ concentrated on E, i.e. $\mu(E) = 1, \mu(E^c) = 0$ (one can take $\mu = \delta_x$ with some $x \in E$). Let $R \in \overline{W}(S; F; \mu; 1/2)$, then $\mu(RF) = \mu(RF \,\Delta\, SF) < 1/2$ since $\mu(SF) = 0$. On the other hand, $\mu(TF \setminus RF) \geq \mu(TF) - \mu(RF) > 1/2$ since $TF \supset E$. This means that $TF \neq RF$ and $R \in W(T; F)^c$.

In case (ii), we take μ such that $\mu(SF \setminus RF) = 1$. Consider again an automorphism $R \in \overline{W}(S; F; \mu; 1/2)$. We have

$$1/2 > \mu(SF \;\Delta\; RF) = \mu(RF \setminus SF) + \mu(SF \setminus RF) = \mu((SF \setminus TF) \setminus RF)$$

and therefore

$$\mu((SF \setminus TF) \cap RF) = \mu(SF \setminus TF) - \mu((SF \setminus TF) \setminus RF) > 1/2.$$

Thus, $(SF \setminus TF) \cap RF \neq \emptyset$ and $RF \neq TF$, i.e. $R \in W(T; F)^c$.

(2) Show that $W(T; F)^c$ is open in τ''. In fact, we will consider the topology $\overline{\tau}$ equivalent to τ'' (see (2.7) and Proposition 4.4. Let $S \in W(T; F)^c$, then $SF \neq TF$ and we have the above cases (i) and (ii). In case (i), take a $\overline{\tau}$-neighbourhood $\overline{V}(S; \mu; 1/2)$ as in (2.7) where μ is concentrated on E. Then $\mu(SF) = 0$ and for $R \in \overline{V}(S; \mu; 1/2)$ we have that

$$\mu(RF) = |\mu(RF) - \mu(SF)| < 1/2.$$

Therefore, $\mu(TF \setminus RF) > 1/2$ and $R \in W(T; F)^c$.

In case (ii), we choose μ supported on $SF \setminus TF$. Then, by the method of (1), we see that for $R \in \overline{V}(S; \mu; 1/2)$ the set $(SF \setminus TF) \cap RF$ is non-empty since its measure is greater than 1/2. Therefore, $RF \neq TF$ and $R \in W(T; F)^c$. □

Observe that Proposition 2.10 does not hold for τ_0. One can easily show that $W(T;\{x\})$ is not closed with respect to τ_0.

Corollary 2.11. $(\mathrm{Aut}(X,\mathcal{B}),p)$ *is a* 0*-dimensional topological space.*

Proof. This follows from Remark 2.9(2). □

Now we consider convergent sequences in each of our topologies. Of course, the topologies are not defined by convergent sequences but it is useful for many applications to know criteria of convergence.

Remark 2.12. (1) If a sequence (T_n) of Borel automorphisms converges to S in τ'', then for any measure $\mu \in \mathcal{M}_1(X)$,

$$|\mu \circ S(f) - \mu \circ T_n(f)| \to 0$$

uniformly in $f \in B(X)_1$ as $n \to \infty$. Since τ'' is equivalent to $\overline{\tau}$ (Proposition 4.5), the above condition is equivalent to the following one: $T_n \xrightarrow{\tau''} S$ if and only if

$$|\mu(SF) - \mu(T_nF)| \to 0$$

uniformly in $F \in \mathcal{B}$.

(2) (T_n) converges to S in p if and only if for any Borel set F, $T_nF = TF$ for sufficiently large $n = n(F)$. In particular, F can be a point from X. Therefore, we see that p-convergence implies τ-convergence by Remark 2.8.

(3) It follows directly from (2.6) that $T_n \xrightarrow{\overline{p}} S$ if and only if for all $\mu \in \mathcal{M}_1(X)$ and all $F \in \mathcal{B}$

$$\mu(T_nF \,\Delta\, SF) + \mu(T_n^{-1}F \,\Delta\, S^{-1}F) \to 0. \tag{2.11}$$

In fact, one can prove the following criterion of $\overline{p}$-convergence.

Proposition 2.13. $T_n \xrightarrow{\overline{p}} S$ *if and only if for all* $F \in \mathcal{B}$,

$$SF = \limsup_{n\to\infty} T_nF, \quad S^{-1}F = \limsup_{n\to\infty} T_n^{-1}F, \tag{2.12}$$

where

$$\limsup_{n\to\infty} F_n = \bigcup_m \bigcap_{n>m} F_n.$$

Proof. We assume for simplicity that $S = \mathbb{I}$. The general case is proved similarly. To prove (2.12) we remark that for any $x \in X$ and $F \in \mathcal{B}$, the convergence $T_n \xrightarrow{\overline{p}} \mathbb{I}$ implies that

$$\delta_x(T_nF \,\Delta\, F) + \delta_x(T^{-1}F \,\Delta\, F) \to 0$$

as $n \to \infty$. This means that if $x \in F$, then there exists $n_0 = n_0(x,F)$ such that $x \in T_nF$ and $x \in T_n^{-1}F$ for all $n > n_0$. We have proved that $F \subset \bigcup_m \bigcap_{n>m} T_nF$, $F \subset \bigcup_m \bigcap_{n>m} T_n^{-1}F$. In fact, these inclusions are equalities. Indeed, if we assume that there exists $x_0 \in F^c = X \setminus F$ with $x_0 \in \bigcap_{n>m} T_nF$ for some m, then we have a contradiction to the fact that x_0 also belongs to $\bigcup_k \bigcap_{n>k} T_n F^c$. Thus, (2.12) holds.

Conversely, let $E_m = \bigcap_{n>m} T_n F$ and $\bigcup_m E_m = F$. Since $E_m \subset E_{m+1}$, we know that for any measure $\mu \in \mathcal{M}_1(X)$, $\mu E_m \to \mu F$ $(m \to \infty)$. Remark that $E_m \subset T_n F$ for all $n > m$. Therefore $E_m = E_m \cap T_n F \subset F \cap T_n F \subset F$. Thus we obtain $\mu(F \cap T_n F) \to \mu F,\ n \to \infty$. Similarly $\mu(F \cap T_n^{-1} F) \to \mu F$. By (2.11), the proof is complete. □

Proposition 2.14. *Suppose (T_n) is a sequence of Borel automorphisms and let $S \in \mathrm{Aut}(X, \mathcal{B})$. Then*

$$(T_n \xrightarrow{p} S) \Rightarrow (T_n \xrightarrow{\tau} S) \Leftrightarrow (T_n \xrightarrow{\tau''} S) \Leftrightarrow (T_n \xrightarrow{\overline{p}} S).$$

Proof. We will consider, for simplicity, the case $S = \mathbb{I}$. We note that if $T_n \xrightarrow{p} \mathbb{I}$, then for every $x \in X$ the sequence (T_n) eventually gets into $W(\mathbb{I}; \{x\})$, that is $T_n x = x$ for sufficiently large n. It follows that $T_n \xrightarrow{\tau} \mathbb{I}$ (see Remark 2.8).

By Theorem 2.3, τ is strictly stronger than τ''. To prove the second implication, we need to verify only that if a sequence of Borel automorphisms (T_n) converges in τ'', then it also converges in τ.

To see this, we assume that $T_n \xrightarrow{\tau''} \mathbb{I}$ when $n \to \infty$. Then for any measure $\mu \in \mathcal{M}_1(X)$, we have

$$\sup_{f \in B(X)_1} |\mu(f) - \mu \circ T_n(f)| \to 0, \quad n \to \infty.$$

Thus,

$$\left| \int_X (f(T_n^{-1} x) - f(x))\, d\mu \right| \to 0 \tag{2.13}$$

uniformly in f as $n \to \infty$. Take $\mu = \delta_{x_0}$ in (2.13). Then for all $\varepsilon > 0$ there exists $N = N(\varepsilon, x_0)$ such that for all $n > N$ and for all $f \in B(X)_1$, one has

$$|f(T_n^{-1} x_0) - f(x_0)| < \varepsilon. \tag{2.14}$$

To prove that $T_n \xrightarrow{\tau} \mathbb{I}$, it suffices to verify that $T_n x_0 = x_0$ for n sufficiently large (see Remark 2.8). To obtain a contradiction, we assume that for any N there exist a point x_0 and $n_0 > N$ such that $T_{n_0} x_0 \neq x_0$. Take a Borel set F of x_0 such that $T_{n_0} x_0 \notin F$. Then there exists a function $f_0 \in B(X)_1$ such that $f_0(x_0) = 1$ and $f_0(T_{n_0} x_0) = 0$. This contradicts (2.14).

As above we will show that $\overline{p}$-convergence of (T_n) implies τ-convergence. If $T_n \xrightarrow{\overline{p}} \mathbb{I}$, then for every $x \in X$, $T_n \in \overline{W}(\mathbb{I}; \{x\}; \delta_x; 1/2)$ when n is sufficiently large. In other words, $T_n x = x$. It proves that $T_n \xrightarrow{\tau} \mathbb{I}$. □

3. Approximation by periodic and aperiodic automorphisms

3.1. Periodic approximation of Borel automorphisms. Here we focus on the study of periodic and aperiodic automorphisms. We will show that for every Borel aperiodic automorphism T of $(X,\mathcal{B})$ there exists a sequence of periodic Borel automorphisms that converges to T in the uniform topology τ. In fact, this result was proved by Nadkarni in [24] although he did not consider topologies on $\mathrm{Aut}(X,\mathcal{B})$. We reproduce the main part of Nadkarni's proof here because it will be used below. We will also find the closures of some natural classes of automorphisms.

Recall some standard definitions. For $T\in \mathrm{Aut}(X,\mathcal{B})$, a point $x\in X$ is called *periodic* if there exists $n\in\mathbb{N}$ such that $T^nx=x$. The smallest such $n=n(x)$ is called the *period* of T at x. Given T, the space X can be partitioned into a disjoint union of Borel T-invariant sets $X_1,\dots,X_\infty$ where X_n is the set of points with period n, and X_∞ is the set where T is aperiodic. Such a partition related to an automorphism T will be called canonical. If $X_\infty=\emptyset$, then T is called *pointwise periodic*, $T\in\mathcal{P}er$. Denote by $\mathcal{P}er_n(x)$ the set of all automorphisms which have period n at x. By definition, $T\in\mathcal{P}er_n$ if $X_n=X$. In other words,

$$\mathcal{P}er_n=\bigcap_{x\in X}\mathcal{P}er_n(x). \tag{3.1}$$

We say that $T\in\mathcal{P}er_0$ if there exists $N\in\mathbb{N}$ such that $T^Nx=x$, $x\in X$. This means that X is a finite union of some sets $X_{n_1},\dots,X_{n_k}$. Obviously, $\mathcal{P}er_0$ is a proper subset of $\mathcal{P}er$. Finally, if $X=X_\infty$, then T is called *aperiodic*, $T\in\mathcal{A}p$.

Proposition 3.1.

(a) *For any* $n\in\mathbb{N}$, *the set* $\mathcal{P}er_n(x)$ $(x\in X)$ *is clopen with respect to all topologies from Definition* 2.1.
(b) $\overline{\mathcal{P}er_0}^{\tau}=\overline{\mathcal{P}er}^{\tau}$.
(c) $\mathcal{A}p$ *and* $\mathcal{P}er_n$ $(n\in\mathbb{N})$ *are closed with respect to all the topologies.*

Proof. (a) As mentioned above, to show that $\mathcal{P}er_n(x)$ is closed for the all topologies, it suffices to do this for τ'' and $\overline{p}$.

By Proposition 4.5, τ'' is equivalent to the topology $\overline{\tau}$ whose neighbourhoods are defined by $\overline{V}(T;\mu_1,\dots,\mu_n;\varepsilon)=\{S\in\mathrm{Aut}(X,\mathcal{B})\mid \sup_{F\in\mathcal{B}}|\mu_j(TF)-\mu_j(SF)|<\varepsilon,\ j=1,\dots,n\}$ (see Definition 4.4). Let $R\in\overline{\mathcal{P}er_n(x)}^{\overline{\tau}}$. Let $\overline{V}(R)=\overline{V}(R;\delta_{Rx},\dots,\delta_{R^nx};1/2)$. Then, there exists $P\in\overline{V}(R)\cap\mathcal{P}er_n(x)$. This means that taking one-point sets $\{x\},\dots,\{R^{n-1}x\}$, we obtain that

$$|\delta_{R^ix}(R(R^{i-1}x))-\delta_{R^ix}(P(R^{i-1}x))|<1/2,\quad i=1,\dots,n.$$

It follows that $Rx=Px$, … ,$R^nx=P^nx=x$, i.e. $R\in\mathcal{P}er_n(x)$.

Let us show that $\mathcal{P}er_n(x)$ is closed in $\overline{p}$. Take $R \in \overline{\mathcal{P}er_n(x)}^{\overline{p}}$ and consider the $\overline{p}$-neighbourhood $\overline{W}(R) = \overline{W}(R; \{x\}, \dots, \{R^{n-1}x\}; \delta_{Rx}, \dots, \delta_{R^n x}; 1/2)$. There is an automorphism $P \in \overline{W}(R) \cap \mathcal{P}er_n(x)$. The inequalities $\delta_{Rx}(Rx \Delta Px) < 1/2, \dots, \delta_{R^n x}(R(R^{n-1}x)\Delta P(R^{n-1}x)) < 1/2$ imply that $Rx = Px, \dots, R^n x = P^n x = x$.

To check that $\mathcal{P}er_n(x)$ is open with respect to the all topologies, it suffices again to show this for τ'' and $\overline{p}$ only (Theorem 2.3). This fact follows from the following observation: if $P \in \mathcal{P}er_n(x)$, then $\overline{V}(P; \delta_{Px}, \dots, \delta_{P^n x}; 1/2)$ and $\overline{W}(P; \{x\}, \dots, \{P^{n-1}x\}; \delta_{Px}, \dots, \delta_{P^n x}; 1/2)$ are subsets of $\mathcal{P}er_n(x)$.

(b) Let now $T \in \mathcal{P}er$. We can construct a sequence $(P_n) \subset \mathcal{P}er_0$ converging to T in τ. As above, consider the partition $(X_i : i = 1, 2, \dots)$ of X corresponding T, i.e. T has period i on X_i. Let $E_n = \bigcup_{i \le n} X_i$. Then $E_n \subset E_{n+1}$ and $X = \bigcup_n E_n$. Define $P_n x = Tx$ if $x \in E_n$ and $P_n x = x$ if $x \in X \setminus E_n$. Clearly, $P_n \xrightarrow{\tau} T$.

(c) The set $\mathcal{P}er_n$ is closed by (a) and (3.1). It is clear that

$$\mathcal{A}p = \mathrm{Aut}(X, \mathcal{B}) \setminus \left(\bigcup_{n \in \mathbb{N}} \bigcup_{x \in X} \mathcal{P}er_n(x) \right).$$

Hence $\mathcal{A}p$ is closed. □

Let $\mathrm{Orb}_T(x)$ denote the T-orbit of $x \in X$. Recall the definition of the *full group* $[T]$ generated by $T \in \mathrm{Aut}(X, \mathcal{B})$:

$$[T] = \{\gamma \in \mathrm{Aut}(X, \mathcal{B}) \mid \gamma x \in \mathrm{Orb}_T(x), \text{ for all } x \in X\}.$$

Then every $\gamma \in [T]$ defines a Borel function $m_\gamma : X \to \mathbb{Z}$ such that $\gamma x = T^{m_\gamma(x)} x$, $x \in X$. Thus, every $\gamma \in [T]$ defines a countable partition of X into Borel sets $A_n = \{x \in X \mid m_\gamma(x) = n\}$, $n \in \mathbb{N}$.

It is obvious that if $T \in \widehat{\mathrm{Aut}}(X, \mathcal{B})$, then one can also define the full group $[T]$ as a subgroup of $\widehat{\mathrm{Aut}}(X, \mathcal{B})$.

Proposition 3.2.

(a) *The full group* $[T]$ $(T \in \mathrm{Aut}(X, \mathcal{B}))$ *is closed in* $\mathrm{Aut}(X, \mathcal{B})$ *with respect to the topologies from Definition* 2.1.

(b) *The full group* $[T]$ $(T \in \widehat{\mathrm{Aut}}(X, \mathcal{B}))$ *is closed in* $\widehat{\mathrm{Aut}}(X, \mathcal{B})$ *with respect to* $\widehat{\tau}$.[2]

Proof. (a) It is not hard to prove this result directly for the topologies τ and p. To prove the proposition for all our topologies, it is sufficient to check that $[T]$ is closed in $\tau'' \sim \overline{\tau}$ and $\overline{p}$.

[2]We do not consider here other quotient topologies on $\widehat{\mathrm{Aut}}(X, \mathcal{B})$.

Assume that there exists $S \in \overline{[T]}^{\overline{\tau}} \setminus [T]$. Then one can find a point $y \in X$ such that $Sy \neq T^n y$ for all $n \in \mathbb{Z}$. Let μ be an atomic probability measure supported by $\{T^n y \mid n \in \mathbb{Z}\}$ such that $\mu(\{y\}) = 1/2$. Then the $\overline{\tau}$-neighbourhood $\overline{V}(S) = \overline{V}(S;\mu;1/4)$ contains an automorphism $\gamma \in [T]$. Hence for any Borel set F we have that $|\mu(SF) - \mu(\gamma F)| < 1/4$. For $F = \{\gamma^{-1}y\}$, we have a contradiction.

The proof for $\overline{p}$ is similar. We observe only that the $\overline{p}$-neighbourhood $\overline{W}(S; \{y\}; \delta_{Sy}; 1/2)$ cannot meet $[T]$ where S and y as above.

(b) Assume that $R \in \overline{[T]}^{\widehat{\tau}} \setminus [T]$ where $T \in \widehat{\mathrm{Aut}}(X,\mathcal{B})$. Then any $\widehat{\tau}$-neighbourhood $\widehat{U}(R)$ contains an element γ from $[T]$. Since R is not in $[T]$, the Borel set $A = \bigcap_{n\in\mathbb{Z}} E(R,T^n)$ is uncountable. Let μ be a continuous measure from $\mathcal{M}_1^c(X)$ such that $\mu(A) = 1$. Take an automorphism $\gamma \in \widehat{U}(R;\mu;1/2) \cap [T]$. Then there exists some n such that $\mu(\{x \in X \mid Rx = T^n x\}) > 0$. This contradicts the fact that $\mu(E(R,T^n)) = 1$ for all $n \in \mathbb{Z}$. □

Given $T \in \mathrm{Aut}(X,\mathcal{B})$, a Borel set $A \subset X$ is called a *complete section* (or simply a *T-section*) if every T-orbit meets A at least once. If there exists a complete Borel section A such that A meets every T-orbit exactly once, then T is called *smooth*. In this case, $X = \bigcup_{i\in\mathbb{Z}} T^i A$ and all the $T^i A$'s are disjoint. The set of smooth automorphisms is denoted by $\mathcal{S}m$.

A measurable set W is said to be wandering with respect to $T \in \mathrm{Aut}(X,\mathcal{B})$ if the sets $T^n W$, $n \in \mathbb{Z}$, are pairwise disjoint. The σ-ideal generated by all T-wandering sets in $\mathcal{B}$ is denoted by $\mathcal{W}(T)$. By Poincaré recurrence lemma, one can state that given $T \in \mathrm{Aut}(X,\mathcal{B})$ and $A \in \mathcal{B}$ there exists $N \in \mathcal{W}(T)$ such that for each $x \in A \setminus N$ the points $T^n x$ return to A for infinitely many positive n and also for infinitely many negative n [24]. The points from the set $A \setminus N$ are called *recurrent*.

Assume that all points from a set A are recurrent for a Borel automorphism T and A is a T-section. Then for $x \in A$, let $n(x) = n_A(x)$ be the smallest positive integer such that $T^{n(x)}x \in A$ and $T^i x \notin A$, $0 < i < n(x)$. Let $C_k = \{x \in A \mid n_A(x) = k\}$, $k \in \mathbb{N}$, then $T^k C_k \subset A$ and $\{T^i C_k \mid i = 0,\dots,k-1\}$ are pairwise disjoint. Note that some C_k's may be empty. Since $T^n x \in A$ for infinitely many positive and negative n, we obtain

$$\bigcup_{n\geq 0} T^n A = \bigcup_{n\in\mathbb{Z}} T^n A = X \quad \text{and} \quad \bigcup_{n\geq 0} T^n A = \bigcup_{k=1}^{\infty} \bigcup_{i=0}^{k-1} T^i C_k.$$

This union decomposes X into T-towers $\xi_k = \{T^i C_k \mid i = 0,\dots,k-1\}$, $k \in \mathbb{N}$, where C_k is the base and $T^{k-1}C_k$ is the top of ξ_k. In particular, the number of these towers may be finite.

The next lemma is one of the main tools in our study of Borel automorphisms.

Lemma 3.3. *Let $T \in \mathrm{Aut}(X, \mathcal{B})$ be an aperiodic Borel automorphism of a standard Borel space $(X, \mathcal{B})$. Then there exists a sequence (A_n) of Borel sets such that*

(a) $X = A_0 \supset A_1 \supset A_2 \supset \dots,$
(b) $\bigcap_n A_n = \emptyset,$
(c) *A_n and $X \setminus A_n$ are complete T-sections, $n \in \mathbb{N}$,*
(d) *for $n \in \mathbb{N}$, every point in A_n is recurrent,*
(e) *for $n \in \mathbb{N}$, $A_n \cap T^i(A_n) = \emptyset$, $i = 1, \dots, n-1$,*
(f) *for $n \in \mathbb{N}$, the base $C_k(n)$ of every non-empty T-tower is an uncountable Borel set, $k \in \mathbb{N}$.*

Proof. See [2, Lemma 4.5.3] where (a)–(c) have been proved in more general settings of countable Borel equivalence relations. It is shown in [24, Chapter 7] that one can refine the choice of (A_n) to get (d) and (e). It is clear that one can remove an at most countable set of points from each A_n to prove (f). □

Definition 3.4. A sequence of Borel sets satisfying conditions (a)–(f) of Lemma 3.3 is called a vanishing sequence of markers.

Note that usually (A_n) is called a *vanishing sequence of markers* if it satisfies (a)–(d). We have added two more conditions, (e) and (f), which we will need in the constructions in Section 5.

Remark 3.5. We will use below the following *changing-of-topology* result (see, for example, [21], [24]). Let $T \in \mathrm{Aut}(X, \mathcal{B})$ and let (ξ_n) be a sequence of at most countable partitions of X such that:

(a) ξ_{n+1} refines ξ_n;
(b) $\bigcup_n \xi_n$

generates the σ-algebra of Borel sets $\mathcal{B}$. Then we may introduce a topology ω on X such that:

(i) (X, ω) becomes a Polish 0-dimensional space,
(ii) $\mathcal{B}(\omega) = \mathcal{B}$ where $\mathcal{B}(\omega)$ is the σ-algebra generated by ω-open sets,
(iii) all elements of partitions ξ_n, $n \in \mathbb{N}$ are clopen in ω,
(iv) T is a homeomorphism of (X, ω).

In particular, by changing-of-topology, we can choose the elements of the partitions corresponding to a vanishing sequence of markers to be clopen.

Proposition 3.6. *Let $T \in \mathrm{Aut}(X, \mathcal{B})$ be an aperiodic Borel automorphism of a standard Borel space $(X, \mathcal{B})$. Then there exists a sequence of periodic automorphisms (P_n) of $(X, \mathcal{B})$ such that $P_n \xrightarrow{\tau} T$, $n \to \infty$. Moreover, the P_n can all be taken from $[T]$.*

Proof. If T is a smooth automorphism, then the proof is obvious. Let (A_n) be a vanishing sequence of markers for T. Then, as we have seen above, A_n

generates a decomposition of X into T-towers $\xi_k(n) = \{T^i C_k(n) \mid i = 0, \dots, k-1\}$ and $\bigcup_k C_k(n) = A_n$. Define

$$P_n x = \begin{cases} Tx & \text{if } x \notin B_n = \bigcup_{k=1}^{\infty} T^{k-1} C_k(n), \\ T^{-k+1} x & \text{if } x \in T^{k-1} C_k(n), \text{ for some } k. \end{cases}$$

Then P_n belongs to $[T]$ and the period of P_n on $\xi_k(n)$ is k. Note that P_n equals T everywhere on X except B_n, the union of the tops of the towers.

It follows from Lemma 3.3 that (B_n) is a decreasing sequence of Borel subsets such that $\bigcap_n B_n = \emptyset$. This means that for any $x \in X$ there exists $n(x)$ such that $x \notin B_n$, $n \geq n(x)$. Moreover, if for some $x \in X$, $P_n x = Tx$, then $P_{n+1} x = Tx$. These facts prove that for each x, all the $P_n x$ are eventually the same and equal to Tx, that is P_n converges to T in τ. □

We now give a version of the Rokhlin lemma for aperiodic Borel automorphisms. We should also remark that B. Weiss proved a measure-free version of the Rokhlin lemma [29].

Theorem 3.7 (Rokhlin lemma). *Let $m \in \mathbb{N}$ and let T be an aperiodic Borel automorphism of $(X, \mathcal{B})$. Then for any $\varepsilon > 0$ and any measures $\mu_1, \dots, \mu_p$ from $\mathcal{M}_1(X)$ there exists a Borel subset F in X such that $F, TF, \dots, T^{m-1}F$ are pairwise disjoint and*

$$\mu_i(F \cup TF \cup \dots \cup T^{m-1}F) > 1 - \varepsilon, \quad i = 1, \dots, p.$$

Proof. We will use notation from the proof of Proposition 3.6. Clearly, it suffices to consider the case of non-smooth automorphism T only. Let (A_n) be a vanishing sequence of markers. Note that for any $\mu \in \mathcal{M}_1(X)$, $\mu(A_n) \to 0$ as $n \to \infty$ because A_n decreases to the empty set. By the same reasoning, $\mu(B_n) \to 0$. For every n, the space X can be represented as a union of T-towers $\xi_k(n)$ where the height of $\xi_k(n)$ is k (see the proof of Proposition 3.6. Let

$$D_n(m) = \bigcup_{k=1}^{m-1} \xi_k(n).$$

Since $D_n(m) \subset \bigcup_{k=0}^{m-2} T^k A_n$, we see that there exists n_0 such that for $n > n_0$

$$\mu_i(D_n(m)) < \varepsilon/2, \quad i = 1, \dots, p. \tag{3.2}$$

Let $B'_n = \bigcup_{k \geq m} T^{k-1} C_k(n)$. Similarly, we can deduce that for all sufficiently large n

$$\mu_i(B'_n \cup T^{-1} B'_n \cup \dots \cup T^{-m+2} B'_n) \leq \varepsilon/2, \quad i = 1, \dots, p. \tag{3.3}$$

Let n be chosen so large that (3.2) and (3.3) hold. Define F by the following rule. In each T-tower $\xi_k(n)$, $k \geq m$, we take every m-th set beginning with $C_k(n)$, i.e.

$$F = \bigcup_{k \geq m} \bigcup_{j=0}^{[k/m]-1} T^j C_k(n).$$

Then $F \cap T^j F = \emptyset$, $j = 1, \ldots, m-1$, and

$$\mu_i(X - (F \cup TF \cup \ldots \cup T^{m-1}F)) < \varepsilon, \quad i = 1, \ldots, p,$$

in view of (3.2) and (3.3). □

It follows from Theorem 3.7 that in any τ-neighbourhood $U(T; \mu_1, \ldots, \mu_n; \varepsilon)$ of an aperiodic Borel automorphism T there exists a pointwise periodic automorphism. Thus we obtain the following corollary from the above results:

Corollary 3.8.

(a) *The sets $\mathcal{P}er$ and $\mathcal{P}er_0$ are dense in* $(\mathrm{Aut}(X, \mathcal{B}), \tau)$. *Moreover, $\mathcal{P}er$ is also dense in* $\mathrm{Aut}(X, \mathcal{B})$ *with respect to topologies τ'', $\overline{p}$.*

(b) *$\mathcal{P}er \cap [T]$ is τ-dense in $[T]$ for each aperiodic T.*

(c) *The set $\mathcal{P}er$ is not dense in* $\mathrm{Aut}(X, \mathcal{B})$ *with respect to p.*

Proof. (a) and (b) are obvious. To prove (c), take an uncountable Borel set E and an aperiodic Borel automorphism T such that $TE \subsetneq E$. Then $W(T; E)$ has no periodic automorphisms. □

We observe that the following result can also be proved. The details are left to the reader.

Corollary 3.9. *Let $N \in \mathbb{N}$ and a τ-neighbourhood $U = U(T; \mu_1, \ldots, \mu_n : \varepsilon)$ be given. Define $V = U(T; (\nu_i^k)_{1 \leq i \leq n, |k| \leq N}; \delta)$ where $\nu_i^k = \mu_i \circ T^{-k}$ and $\delta = (2N)^{-1}\varepsilon$. Then $V \subset U$ and for any $S \in V$, we have that $S^j \in U(T^j; \mu_1, \ldots, \mu_n, \varepsilon)$, $j = 1, \ldots, N$.*

We will need below the following statement proved in [6].

Lemma 3.10. *The set $\mathcal{S}m$ of smooth automorphisms is dense in* $\mathrm{Aut}(X, \mathcal{B})$ *with respect to the topology p.*

Denote by $\mathcal{A}p \mod (\mathrm{Ctbl})$ the subset of $\mathrm{Aut}(X, \mathcal{B})$ consisting of automorphisms which are aperiodic everywhere except in an at most countable subset of X.

Theorem 3.11.

(a) $\overline{\mathcal{A}p}^{\tau_0} = \mathcal{A}p \mod (\mathrm{Ctbl})$.

(b) *$\mathcal{A}p$ is a nowhere dense closed subset in* $(\mathrm{Aut}(X, \mathcal{B}), \tau)$.

Proof. (a) We first show that the set $\mathcal{A}p \mod (\mathrm{Ctbl})$ is closed with respect to τ_0. Suppose $R \in \overline{(\mathcal{A}p \mod (\mathrm{Ctbl}))}^{\tau_0} \setminus (\mathcal{A}p \mod (\mathrm{Ctbl}))$. Then there exist some $m \in \mathbb{N}$ and an uncountable R-invariant Borel set B such that R has period m on B. Let μ be a continuous Borel probability measure such that $\mu(B) = 1$ and $\mu \circ R = \mu$. The τ_0-neighbourhood $U_0(R) = U_0(R; \mu; \varepsilon/m)$ contains an automorphism S from $\mathcal{A}p \mod (\mathrm{Ctbl})$. We have that

$$\mu(\{x \in B \mid Sx = Rx, \dots, S^{m-1}x = R^{m-1}x, S^m x = R^m x = x\}) > 1 - \varepsilon.$$

In other words, S is periodic on an uncountable Borel set, a contradiction. Thus, we have that $\overline{\mathcal{A}p}^{\tau_0} \subset \mathcal{A}p \mod (\mathrm{Ctbl})$.

Conversely, if $R \in \mathcal{A}p \mod (\mathrm{Ctbl})$, then we need to show that any τ_0-neighbourhood $U_0(R)$ of R contains some aperiodic automorphism. Indeed, the periodic part of R is supported by an either countable or finite set A. It is clear that if A is infinite, then one can change R on A to produce an aperiodic automorphism from $U_0(R)$. If A is finite, then we take a single aperiodic orbit $\mathrm{Orb}_R(x)$, $x \notin A$, and consider the infinite set $A \cup \mathrm{Orb}_R(x)$.

(b) It follows from Proposition 3.1(c) and Corollary 3.8 that $\mathrm{Aut}(X, \mathcal{B}) \setminus \mathcal{A}p$ is an open dense subset. Therefore, $\mathcal{A}p$ is a closed nowhere dense set in τ. $\square$

Let (X, d) be a compact metric space (in particular, X can be a Cantor set). Recall that in this case we can define the metric D on $\mathrm{Aut}(X, \mathcal{B})$ as in Remark 2.5. We proved in [3] that the set of aperiodic homeomorphisms is D-dense in $\mathrm{Homeo}(X)$ when $X = \Omega$ is a Cantor set. Here we will find the closure of $\mathcal{A}p$ in the group $\mathrm{Aut}(X, \mathcal{B})$ with respect to the metric D.

Let T be a Borel automorphism of X. Then X is decomposed into the canonical T-invariant partition $(Y_1, \dots, Y_\infty)$ where T has period n on Y_n and T is aperiodic on Y_∞. We call T *regular* if all sets Y_i, $1 \le i < \infty$, are uncountable.

Lemma 3.12. *Suppose that $T \in \mathrm{Aut}(X, \mathcal{B})$ is regular. Then for any $\varepsilon > 0$ there exists $S \in \mathcal{A}p$ such that $D(S, T) < \varepsilon$.*

Proof. Since $TY_i = Y_i$, it suffices to find an aperiodic automorphism S satisfying the condition of the lemma for each set Y_i. We can write down Y_i as $E_i \cup TE_i \cup \ldots \cup T^{i-1}E_i$ for some $E_i \in \mathcal{B}_0$. Let $(E_i(1), \dots, E_i(k_i))$ be a partition of E_i into uncountable Borel sets such that $\mathrm{diam}(E_i(j)) < \varepsilon$ for all $j = 1, \dots, k_i$. Let $R(j)$ be an aperiodic automorphism of $E_i(j)$. Set for $j = 1, \dots, k_i$

$$Sx = \begin{cases} Tx & \text{if } x \in \displaystyle\bigcup_{k=1}^{i-1} T^k E_i(j), \\ TR(j)x & \text{if } x \in E_i(j). \end{cases}$$

Clearly, $D(S, T) < \varepsilon$ on Y_i. $\square$

COROLLARY 3.13. *The set of aperiodic automorphisms from* $\widehat{\mathrm{Aut}}(X,\mathcal{B})$ *is dense with respect to* $\widehat{D}$.

To answer the question when a non-regular automorphism $T \in \mathrm{Aut}(X,\mathcal{B})$ belongs to $\overline{\mathcal{A}p}^D$, we need to introduce the following definition. We say that an automorphism T is *semicontinuous* at $x \in X$ if for any $\varepsilon > 0$ there exists $z \neq x$ such that $d(x,z) < \varepsilon$ and $d(Tx,Tz) < \varepsilon$.

THEOREM 3.14. *Let* T *be a non-regular Borel automorphism from* $\mathrm{Aut}(X,\mathcal{B})$ *and let* Y_0 *denote the set* $\bigcup_{i\in I} Y_i$ *where each* Y_i *is an at most countable set,* $i \in I \subset \mathbb{N}$. *Then* $T \in \overline{\mathcal{A}p}^D$ *if and only if for every* $x \in Y_0$ *there exists* $y \in \mathrm{Orb}_T(x)$ *such that* T *is semicontinuous at* y.

PROOF. We first suppose that for any $\varepsilon > 0$ there exists an aperiodic automorphism $S = S_\varepsilon$ such that $D(T,S) < \varepsilon$. Notice that the fact that T is pointwise periodic on Y_0 implies that $\mathrm{Orb}_T(x) \neq \mathrm{Orb}_S(x)$ for any $x \in Y_0$. Hence there exists $y \in \mathrm{Orb}_T(x)$ such that $Sy \neq Ty$. On the other hand, we have that

$$d(Ty,Sy) < \varepsilon, \quad d(T^{-1}(Sy), S^{-1}(Sy)) < \varepsilon. \tag{3.4}$$

Denoting by $z = T^{-1}Sy$, we obtain from (3.4) that $z \neq y$, $d(z,y) < \varepsilon$, and $d(Tz,Ty) < \varepsilon$, that is T is semicontinuous at y.[3]

Suppose now that for every $x \in Y_0$ there exists $y \in \mathrm{Orb}_T(x)$ such that T is semicontinuous at y. We need to show that for any $\varepsilon > 0$ there exists $S \in \mathcal{A}p$ such that $D(T,S) < \varepsilon$. It is clear that S can be taken to coincide with T on Y_∞. Therefore, we need to define S on the at most countable set Y_0. We assume here that Y_0 is infinite. It will be clear from the proof how one can deal with the case when Y_0 is finite.

Take a finite partition $(C_1,\dots,C_n)$ of X into Borel sets such that $\mathrm{diam}(C_i) < \varepsilon/2$ for all i. Denote by $A_{ij} = Y_0 \cap C_i \cap T^{-1}C_j$, $i,j = 1,\dots,n$. For each $x \in A_{ij}$, choose $y(x) \in \mathrm{Orb}_T(x)$ such that T is semicontinuous at $y(x)$. The set $Y_0' = \{y(x) : x \in Y_0\}$ is a subset of Y_0 intersecting each T-orbit of $x \in Y_0$ exactly once. Set $A'_{ij} = A_{ij} \cap Y_0'$. Let $J = \{(i_1,j_1),\dots,(i_p,j_p)\}$ be the set of those pairs (i,j) for which $A'_{ij} \neq \emptyset$. Then

$$Y_0 = \bigcup_{(i,j)\in J} \ \bigcup_{y(x)\in A'_{ij}} \mathrm{Orb}_T(y(x)). \tag{3.5}$$

Fix $(i,j) = (i_1,j_1)$. We have two possibilities:

(a) $|A'_{ij}| = \infty$,

(b) $|A'_{ij}| < \infty$.

[3] We have not used in this part of the proof the fact that T is non-regular.

If (a) holds, write down A'_{ij} as $\{\dots y_{-1}, y_0, y_1, \dots\}$. Define S on $\bigcup_{k\in\mathbb{Z}} \mathrm{Orb}_T(y_k)$. Set

$$Sz = \begin{cases} Tz & \text{if } z \in \mathrm{Orb}_T(y_k),\ z \neq y_k, \\ Ty_{k+1} & \text{if } z = y_k,\ k \in \mathbb{Z}. \end{cases}$$

In such a way, the set $\bigcup_{k\in\mathbb{Z}} \mathrm{Orb}_T(y_k)$ is included in an infinite S-orbit.

If (b) holds, then $A'_{ij} = \{z_1, \dots , z_q\}$. Let $\eta_1 = \min\{d(z_i, z_j) \mid i \neq j,\ i, j = 1, \dots , q\}$ and let $0 < \eta < \min\{\varepsilon/2, \eta_1\}$. By the hypothesis of the theorem, there exists $z \in C_i, z \neq z_1$, such that $Tz \in C_j$, $d(z, z_1) < \eta$, $d(Tz, Tz_1) < \eta$, and the T-orbit of z is infinite. To produce an S-orbit defined on $\bigcup_{k=1}^{q} \mathrm{Orb}_T(z_k)$, we can insert the T-orbits of $z_1, \dots , z_q$ into $\mathrm{Orb}_T(z)$. To do this, set $Sz = Tz_1$, $Sw = Tw, w \in \mathrm{Orb}_T(z), w \neq z$, and

$$Sw = \begin{cases} Tw & \text{if } w \in \bigcup_{k=1}^{q} \mathrm{Orb}_T(z_k),\ w \neq z_1, \dots , z_q, \\ Tz_{i+1} & \text{if } w = z_i,\ i = 1, \dots , q-1, \\ Tz & \text{if } w = z_q. \end{cases}$$

By the choice of the C_i's, we see that in both cases (a), (b)

$$d(Tx, Sx) + d(T^{-1}x, S^{-1}x) < \varepsilon \tag{3.6}$$

on the set $\bigcup_{y\in A'_{ij}} \mathrm{Orb}_T(y)$.

Take (i_2, j_2) from J. By definition of Y'_0, we notice that $\bigcup_{y\in A'_{ij}} \mathrm{Orb}_T(y)$ does not meet the set of T-orbits going through the points from A'_{i_2,j_2}. Therefore, we can apply consequently the above procedure until the automorphism S is defined everywhere on Y_0. By (3.5) and (3.6), we obtain that $D(T, S) < \varepsilon$. □

3.2. Incompressible automorphisms. Let T be an aperiodic Borel automorphism of $(X, \mathcal{B})$. Let us denote by $[T]_0$ the set of Borel bijections $\gamma : A \to B$ where A, B are Borel subsets of X and $\gamma x \in \mathrm{Orb}_T(x)$, $x \in A$. We call A and B equivalent with respect to T, $A \sim_T B$, if there exists $\gamma \in [T]_0$ such that $\gamma(A) = B$. If there exists a Borel subset A of X such that $X \sim_T A$ and $X \setminus A$ is a complete T-section, then T is called *compressible*. Otherwise, T is called *incompressible*. We denote the set of incompressible aperiodic automorphisms by $\mathcal{I}nc$. It was proved in [11] that T is compressible if and only if $[T]$ contains a smooth aperiodic automorphism. Let $M_1(T)$ denote the set of Borel probability T-invariant measures. Clearly, for some automorphisms this set may be empty. For example, if T is smooth, then $M_1(T) = \emptyset$. M. Nadkarni [24] proved that T is incompressible if and only if there exists a T-invariant Borel probability measure, i.e. $M_1(T) \neq \emptyset$.

Theorem 3.15. *The set $\mathcal{I}nc$ is a closed nowhere dense subset of $\mathcal{A}p$ with respect to the topology p.*

PROOF. We first show that $\mathcal{I}nc$ is p-closed. Let $T \in \overline{\mathcal{I}nc}^p$. Choose a sequence (ξ_n) of partitions of X such that:

(i) $\xi_n = (F_n(1), \dots, F_n(k_n))$, $F_n(i) \in \mathcal{B}_0$;
(ii) ξ_{n+1} refines ξ_n;
(iii) $\bigcup_n \xi_n$ generates the σ-algebra of Borel sets $\mathcal{B}$.

By changing-of-topology results (Remark 3.5), we may choose a topology ω on X such that:

(i) X is a Polish 0-dimensional space,
(ii) $\mathcal{B}(\omega) = \mathcal{B}$ where $\mathcal{B}(\omega)$ is the σ-algebra generated by ω-open sets,
(iii) the sets $F_n(i)$), $n \in \mathbb{N}$, $i = 0, \dots, k_n$ are clopen in ω,
(iv) T is a homeomorphism of (X, ω).

Denote by $W_n(T)$ the p-neighbourhood $W(T; F_n(1), \dots, F_n(k_n))$. Then $W_n(T)$ meets the set $\mathcal{I}nc$ for every n. Let $S_n \in \mathcal{I}nc \cap W_n(T)$ and let μ_n be an S_n-invariant probability measure. Then, for every $n \in \mathbb{N}$,

$$S_n F_n(i) = T F_n(i), \quad i = 1, \dots, k_n. \tag{3.7}$$

Notice that if $m > n$, then $F_n(i) = \bigcup_{j \in I} F_m(j)$ for each $i = 1, \dots, k_n$ where $I \subset \{1, \dots, k_m\}$. Hence by (3.7)

$$TF_n(i) = \bigcup_{j \in I} TF_m(j) = \bigcup_{j \in I} S_m F_m(i) = S_m F_n(i)$$

and

$$\mu_m(TF_n(i)) = \mu_m(F_n(i)) \quad m \geq n. \tag{3.8}$$

The set $\{\mu_n \mid n \in \mathbb{N}\}$ contains a subsequence (μ_{n_k}) which converges to a Borel probability measure μ in the weak* topology. Let us show that μ is T-invariant. For $B \in \bigcup_n \xi_n$ we obtain that $\mu_{n_k}(B) \to \mu(B)$ and $\mu_{n_k}(TB) \to \mu(TB)$ as $n_k \to \infty$, since B and TB are clopen sets (see e.g. [7]). It follows from (3.8) that for those sets B

$$\mu(TB) = \lim_{n_k \to \infty} \mu_{n_k}(TB) = \lim_{n_k \to \infty} \mu_{n_k}(B) = \mu(B).$$

Since $\bigcup_n \xi_n$ generates $\mathcal{B}$, we see that μ is T-invariant.

To finish the proof, we refer to Lemma 3.10 which provides us with the following result: $\overline{\mathcal{S}m}^p = \mathrm{Aut}(X, \mathcal{B})$. Since $\mathcal{S}m \cap \mathcal{I}nc = \emptyset$, we are done. $\square$

Observe that in [3] we proved that if X is a Cantor set, then $\overline{\mathrm{Homeo}(X)}^\tau = \mathrm{Aut}(X, \mathcal{B})$. From Theorem 3.15, we obtain that $\overline{\mathrm{Homeo}(X)}^p \subset \mathcal{I}nc$.

Let $T \in \mathrm{Aut}(X, \mathcal{B})$ be an aperiodic incompressible automorphism. Denote by

$$\mathrm{Fix}(M_1(T)) = \{R \in \mathrm{Aut}(X, \mathcal{B}) \mid \mu \circ R = \mu, \text{ for all } \mu \in M_1(T)\}$$

and

$$\mathrm{Pres}(M_1(T)) = \{R \in \mathrm{Aut}(X, \mathcal{B}) \mid R(M_1(T)) = M_1(T)\}.$$

Proposition 3.16.

(a) *Let $T \in \mathcal{I}nc$. Then* $\mathrm{Fix}(M_1(T))$ *is closed in the topologies τ, τ'', and p.*

(b) *Let T be an incompressible Borel automorphism of a compact metric space (X, d). Then* $\mathrm{Fix}(M_1(T))$ *is closed in the topology defined by the metric D.*

(c) *The set* $\mathrm{Pres}(M_1(T))$ *is closed in p.*

Proof. (a) We will first prove the statement for the topology $\overline{\tau}$ which is equivalent to τ'' by (3.7). It will follow from this result that $\mathrm{Fix}(M_1(T))$ is closed in τ. Let $R \in \overline{\mathrm{Fix}(M_1(T))}^{\overline{\tau}}$ and $\mu \in M_1(T)$. Then the $\overline{\tau}$-neighbourhood $\overline{V}(R; \mu; \varepsilon) = \{S \in \mathrm{Aut}(X, \mathcal{B}) \mid \sup_{F \in \mathcal{B}} |\mu(RF) - \mu(SF)| < \varepsilon\}$ meets $\mathrm{Fix}(M_1(T))$ for any $\varepsilon > 0$. Therefore, for any Borel set F, we have that $|\mu(RF) - \mu(F)| < \varepsilon$. Thus, $\mu \circ R = \mu$ and $R \in \mathrm{Fix}(M_1(T))$.

The fact that $\mathrm{Fix}(M_1(T))$ is closed in p can be proved similarly (see also (c)).

(b) We need to show that for every automorphism $R \in \overline{\mathrm{Fix}(M_1(T))}^D$ and every $\mu \in M_1(T)$ one has $\mu \circ R = \mu$. Take a sequence (γ_n) from $\mathrm{Fix}(M_1(T))$ such that $D(\gamma_n, R) \to 0$ as $n \to \infty$. For $A \subset X$ and $\alpha > 0$, denote by $B_\alpha(A)$ the α-neighbourhood of A. We notice that $\overline{A}^d = \bigcap_{\alpha > 0} B_\alpha(A)$ (a countable intersection). It is clear, that for any $\alpha > 0$, there exists n_α such that $\gamma_n(A) \subset B_\alpha(RA)$ for all $n \geq n_\alpha$.

Fix $\varepsilon > 0$ and $\mu \in M_1(T)$. The following statement follows easily from Luzin's theorem.

Claim. *Let $R, \mu, (\gamma_n)$ and ε be as above. Then there exists a closed subset F_ε of X such that the automorphisms R, R^{-1}, and γ_n, being restricted to F_ε, are homeomorphisms and $\mu(\widetilde{F}_\varepsilon) > 1 - \varepsilon$ where $\widetilde{F}_\varepsilon = RF_\varepsilon \cap F_\varepsilon \cap R^{-1}F_\varepsilon$.*

Clearly, one can choose the sets $\widetilde{F}_\varepsilon$ such that $\widetilde{F}_{\varepsilon_1} \subset \widetilde{F}_\varepsilon$ when $\varepsilon > \varepsilon_1$. Let C be a closed subset of $\widetilde{F}_\varepsilon$. Then for $\alpha > 0$ and sufficiently large n, $\mu(C) = \mu(\gamma_n C) \leq \mu(B_\alpha(RC))$. Hence

$$\mu(C) \leq \lim_{\alpha \to 0} \mu(B_\alpha(RC)) = \mu\Big(\bigcap_{\alpha > 0} B_\alpha(RC)\Big) = \mu(\overline{RC}^d) = \mu(RC). \tag{3.9}$$

Similarly to (3.9) we can show that $\mu(C) \leq \mu(R^{-1}C)$ and therefore $\mu(A) = \mu(RA)$ for any Borel set $A \subset \widetilde{F}_\varepsilon$.

Let now $\widetilde{F} = \bigcup_{\varepsilon > 0} \widetilde{F}_\varepsilon$ (a countable union). Clearly, $\mu(\widetilde{F}) = 1$. The above argument shows that $\mu(A) = \mu(RA)$ for any Borel set $A \subset \widetilde{F}$. It remains to check that if E is a Borel subset of $X \setminus \widetilde{F}$, then $\mu(RE) = 0$. Indeed, given $\alpha > 0$, we can find a sufficiently large $n = n(\alpha)$ such that $RE \subset B_\alpha(\gamma_n E)$. Since $\lim_{\alpha \to 0} \mu(B_\alpha(\gamma_n E)) = 0$, we have that $\mu(RE) = 0$.

(c) Let us show that rm $\operatorname{Pres}(M_1(T))$ is closed in the topology p. Indeed, if $R \in \overline{\operatorname{Pres}(M_1(T))}^p \setminus \operatorname{Pres}(M_1(T))$, then there exist $\mu_0 \in M_1(T)$ and a Borel set E such that $\mu_0(RTE) \neq \mu_0(RE)$. On the other hand, the p-neighbourhood $W(R; E, TE)$ contains some $S \in \operatorname{Pres}(M_1(T))$ and therefore $RE = SE$, $RTE = STE$. Then $\mu_0(RTE) = (\mu_0 \circ S)(TE) = \mu_0 \circ S(E) = \mu_0(RE)$, a contradiction.□

Remark 3.17. For $T \in \mathcal{I}nc$, the full group $[T]$ is a subset of $\operatorname{Fix}(M_1(T))$ and the normalizer $N[T] = \{S \in \operatorname{Aut}(X, \mathcal{B}) \mid S[T]S^{-1} = [T]\}$ is a proper subset of $\operatorname{Pres}(M_1(T))$. On the other hand, we know that in Cantor dynamics the set $\operatorname{Fix}(M_1(T))$ is the closure of $[T]$ in D [15]. In Borel dynamics the situation is different. We first recall the definition of odometers.

Let $\{\lambda_t\}_{t=0}^{\infty}$ be a sequence of integers such that $\lambda_t \geq 2$. Denote by $p_{-1} = 1$, $p_t = \lambda_0 \dots \lambda_t$, $t = 0, 1 \dots$ Let X be the group of all p_t-adic numbers; then any element of X can be written as an infinite formal series:

$$X = \left\{ x = \sum_{i=0}^{\infty} x_i p_{i-1} \;\middle|\; x_i \in (0, \dots, \lambda_i - 1) \right\}.$$

It is well known that X is a compact metric abelian group endowed with the metric $d(x, y) = (n+1)^{-1}$ where $n = \min\{i \mid x_i \neq y_i\}$, $x = (x_i)$, $y = (y_i)$. By definition, an odometer T is the transformation acting on X as follows: $Tx = x + 1$, $x \in X$, where $1 = 1p_{-1} + 0p_0 + 0p_1 + \ldots \in X$[4]. From topological point of view, (X, T) is a strictly ergodic Cantor system and the set $M_1(T)$ is a singleton. The orbit $\operatorname{Orb}_T(0)$ is dense in X, that is every $b \in X$ can be approximated in d by integer adic numbers. If $b \in X$, then $T_b : x \mapsto x + b$ commutes with T. It is known that the topological centralizer $C(T)$ coincides with $\{T_b \mid b \in X\}$. Since $d(x + b, x + c) = d(b, c)$, $b, c \in X$, it follows from Proposition 3.16 that

$$C(T) \subset \overline{\{T^n \mid n \in \mathbb{Z}\}}^D \subset \overline{[T]}^D \subset \operatorname{Fix}(M_1(T))$$

and

$$C(T) \setminus \{T^n \mid n \in \mathbb{Z}\} \subset \operatorname{Fix}(M_1(T)) \setminus \{T^n \mid n \in \mathbb{Z}\}.$$

2.3. Borel automorphisms of rank 1. Recall the definition of *rank* 1 Borel automorphisms following [24].

Let $T \in \operatorname{Aut}(X, \mathcal{B})$ be an aperiodic non-smooth automorphism and let $\xi = (B_0, \dots, B_n)$ be a T-tower, that is all B_i's are disjoint where $B_i = T^i B_0$, $i = 1, \dots, n$. Then B_0 and B_n are called the base and and the top of ξ.

Suppose a disjoint collection $\eta = \bigcup_{j \in J} \xi(j)$ (finite or countable) of T-towers is given where $\xi(j) = (B_0(j), \dots, B_{n_j}(j))$ and $B_0(j) \in \mathcal{B}_0$ is a Borel uncountable set, $j \in J$. Then η is called a T-multitower and $Y = \bigcup_{j \in J} \bigcup_{i=0}^{n_j} B_i(j)$ is called the support of η. The cardinality of J is called the the rank of the multitower.

[4]More general, we call S an odometer if S is Borel isomorphic to T.

A multitower η' is said to refine η if every atom $B'_{i'}(j')$ of η' is a subset of some atom $B_i(j)$ of η.

Definition 3.18. We say that T has *rank at most* r if there exists a sequence (η_n) of T-multitowers of rank r or less such that η_{n+1} refines η_n and the collection of all atoms in η_n, taken over all $n \in \mathbb{N}$, generates $\mathcal{B}$. Then $Y_n \subset Y_{n+1}$ and $\bigcup_n Y_n = X$ where Y_n is the support of η_n, $n \in \mathbb{N}$. We say that T has *rank* r if T has rank at most r but does not have rank at most $r-1$. If T does not have rank r for any finite r then, by definition, T has infinite rank. Denote by $\mathcal{R}(n)$ the set of automorphisms of rank n.

A complete description of the structure of Borel automorphisms of rank 1 can be found in [24]. Here, we observe only that any $T \in \mathcal{R}(1)$ can be obtained as a τ-limit of a sequence (T_n) of partially defined Borel automorphisms. For this, we use the cutting and stacking method to produce a refining sequence (ξ_n) of towers satisfying Definition 3.18. More precisely, ξ_n is first cut into T_n-subtowers $\xi_n(k) = (C_n^0(k), \dots, C_n^{h_n}(k))$, $k = 0, \dots, p_n$, with $T_n^i(C_n^0(k)) = C_n^i(k)$, $i = 1, \dots, h_n$, and then some spacers $D_n^{h_n+1}(k), \dots, D_n^{m_n(k)}(k)$ are added to each subtower to extend $\xi_n(k)$ to $\xi'_n(k)$. One defines $T_{n+1}(C_n^{h_n}(k)) = D_n^{h_n+1}(k)$ and $T_{n+1}^i(D_n^{h_n+1}(k)) = D_n^{h_n+i+1}(k))$, $i = 1, \dots, m_n(k) - h_n - 1$. To construct ξ_{n+1} one takes successively the extended T_{n+1}-subtowers $\xi'_n(0), \dots, \xi'_n(p_n)$ and then makes from them a single T_{n+1}-tower by concatenating those subtowers and setting $T_{n+1}(D_n^{m_n(k)}(k)) = C_n^0(k+1)$, $k = 0, \dots, p_n$. Thus, the base and the top of ξ_{n+1} are $C_n^0(0)$ and $D_n^{m_n(p_n)}(p_n)$, respectively. Remark that the spacers that enlarge each $\xi_n(k)$ are taken from $X \setminus Y_n$. Finally, as $n \to \infty$, we get a Borel automorphism T of rank 1 as the limit of T_n.

If we assume that for given $T \in \mathcal{R}(1)$, from the above construction, one can choose a sequence (ξ_n) such that $Y_n = X$ for every n (no spacers can be added), then we get that T belongs to the simplest subclass of rank 1 Borel automorphism, the so called *odometers*. The exact description of odometers is given in Remark 3.17. Let us denote this subclass by $\mathcal{O}d$.

We will need the following simple fact. If $T \in \operatorname{Aut}(X, \mathcal{B})$ and $(F_1, \dots, F_n)$ is a Borel partition of X such that $TF_1 = F_2, \dots, TF_{n-1} = F_n$, $TF_n = F_1$, then the p-neighbourhood $W(T; F_1, \dots, F_n)$ contains an odometer S.

The next theorem shows that any rank 1 automorphism is a limit of odometers in τ. Later, in Section 5, this result will be strengthen.

Theorem 3.19. $\overline{\mathcal{R}(1)}^{\tau} = \overline{\mathcal{O}d}^{\tau}$.

Proof. We need to show only that given $\varepsilon > 0$, $T \in \mathcal{R}(1)$, and $\mu_1, \dots, \mu_n \in \mathcal{M}_1(X)$, there exists an odometer S such that $\mu_i(E(S,T)) < \varepsilon$ for all i.

We first assume that every measure μ_i is continuous. Let (ξ_n), $\xi_n = (C_n^0, \dots, C_n^{h_n})$, be a refining sequence of T-towers as above. Then we see that $\mu_i(Y_n) \to 0$

as $n \to \infty$. Since atoms from (ξ_n) generate $\mathcal{B}$, we have that $\mu_i(C_n^0 \cup C_n^{h_n}) \to 0$ as $n \to \infty$ for $i = 1, \dots, n$. Find $N \in \mathbb{N}$ such that $\mu_i(Y_n \cup C_n^0 \cup C_n^{h_n}) < \varepsilon$ for $n \geq N$. Clearly, we can define a new Borel automorphism S such that $Sx = Tx$ if $x \in \bigcup_{i=0}^{h_n-1} C_n^i$ and $S(C_n^{h_n}) = Y_n$, $S(Y_n) = C_n^0$. In other words, we have constructed the S-tower $(C_n^0, \dots, C_n^{h_n}, Y_n)$ which partitions X. Clearly, the definition of S can be extended to produce an odometer on X which belongs to $U(T; \mu_1, \dots, \mu_n; \varepsilon)$.

Now suppose that every measure μ_i can have points of positive measure, say $\{x_k(i)\}_{k\in\mathbb{N}}$, $i = 1, \dots, n$. Then find a finite set $Y = \{x_k(i) \mid k \in I(\mu_i) \subset \mathbb{N}\}$ where a finite subset $I(\mu_i)$ is determined by the condition

$$\sum_{k \notin I(\mu_i)} \mu_i(\{x_k(i)\}) \leq \frac{\varepsilon}{2}, \quad i = 1, \dots, m.$$

Notice that there exists a refining sequence of T-towers (ξ_n) such that, for sufficiently large n, points from Y do not belong to the base and the top of ξ_n. Indeed, it follows from the fact that (ξ_n) generates $\mathcal{B}$. The rest of the proof is the same as for continuous measures. □

4. Comparison of the topologies

In this section, our main aim is to prove Theorem 2.3, which clarifies relationships between the topologies τ, τ', τ'', τ_0, p, p_0, $\widetilde{p}$ and $\overline{p}$.

Proposition 4.1.

(a) *The topology τ is strictly stronger than τ_0.*

(b) *The topologies p and p_0 are equivalent.*

Proof. (a) We need to show only that τ_0 is not equivalent to τ. For $y \in X$, take $U = U(\mathbb{I}; \delta_y; 1/2)$. We will show that for any $U_0 = U_0(\mathbb{I}; \nu_1, \dots, \nu_n; \varepsilon)$ ($\nu_i \in \mathcal{M}_1^c(X)$) there exists a Borel automorphism S from U_0 such that $S \notin U$. To see this, take an uncountable Borel set B such that $y \in B$ and $\nu_i(B) < \varepsilon$ for all i. Let S be a freely acting automorphism on B such that $Sx = x$ for $x \in B^c$.

(b) We will prove that any p-neighbourhood $W(T; F_1, \dots, F_n)$ contains a p_0-neighbourhood $W_0(T; C_1, \dots, C_n)$ where the C_i's are uncountable. Without loss of generality, we can assume that $(F_1, \dots, F_n)$ is a partition of X. Suppose $F_1, \dots, F_k$ are uncountable sets and $F_{k+1}, \dots, F_n$ are countable (or finite) ones. Define $C_i = F_i$, $i = 1, \dots, k$, and $C_i = F_1 \cup F_i$, $i = k+1, \dots, n$. Clearly, if R is a Borel automorphism such that $RC_i = TC_i$ then $RF_i = TF_i$ for $i = 1, \dots, n$. □

Theorem 4.2. *The topologies τ and τ' are equivalent.*

Proof. It is sufficient to consider neighbourhoods of $\mathbb{I}$ only since $\mathrm{Aut}(X, \mathcal{B})$ is a topological group. Take a neighbourhood $U = U(\mathbb{I}; \mu_1, \dots, \mu_n; \varepsilon)$. We will show that $U' := U'(\mathbb{I}; \mu_1, \dots, \mu_n; \varepsilon/4) \subset U$. By definition, if $T \in U'$, then for any Borel set F we have

$$\mu_i(F \,\Delta\, T(F)) < \varepsilon/4, \quad i = 1, \dots, n. \tag{4.1}$$

We need to estimate $\mu_i(E(T, \mathbb{I}))$. Note that $E(T, \mathbb{I})$ can be partitioned as a disjoint union $X_2 \cup X_3 \cup \ldots \cup X_\infty$ where the period of T on X_k is k and T is aperiodic on X_∞. Apply Theorem 3.7 with $X = X_\infty$, $m = 2$ and $\varepsilon/4$. We obtain a Borel subset $F' \subset X_\infty$ such that $F' \cap TF' = \emptyset$ and by (4.1)

$$\mu_i(X_\infty) < \mu_i(F' \cup TF') + \frac{\varepsilon}{4} = \mu_i(F' \,\Delta\, TF') + \frac{\varepsilon}{4} < \frac{\varepsilon}{2}. \tag{4.2}$$

For every $2 \le k < \infty$, let us take Y_k such that

$$X_k = \bigcup_{j=0}^{k-1} T^j Y_k.$$

Let $F_1 = \bigcup_{2 \le k < \infty} Y_k$. Then $F_1 \,\Delta\, TF_1 = F_1 \cup TF_1$ and therefore

$$\mu_i(F_1) \le \mu_i(F_1 \,\Delta\, TF_1) < \varepsilon/4, \quad i = 1, \dots, n. \tag{4.3}$$

Denote by

$$F_2 = \bigcup_{2 \le k < \infty} \; \bigcup_{0 \le j \le [(k-1)/2]} T^{2j} Y_k.$$

Then $F_2 \cap TF_2 \subset F_1$ and $F_2 \cup TF_2 = \bigcup_{2 \le k < \infty} X_k$. Thus, we get from inequalities (4.1) and (4.3)

$$\mu_i\bigg(\bigcup_{2 \le k < \infty} X_k\bigg) = \mu_i(F_2 \,\Delta\, TF_2) + \mu_i(F_2 \cap TF_2) < \frac{\varepsilon}{2}, \quad i = 1, \dots, n$$

This result together with (4.2) shows that $T \in U$ and therefore $U' \subset U$.

Conversely, suppose $U' = U'(\mathbb{I}; \mu_1, \dots, \mu_n; \varepsilon)$ is given. We will show that $U(\mathbb{I}; \mu_1, \dots, \mu_n; \varepsilon/2) \subset U'$. Indeed, let $S \in U(\mathbb{I}; \mu_1, \dots, \mu_n; \varepsilon/2)$, then

$$\mu_i(E(S, \mathbb{I})) < \varepsilon/2, \quad i = 1, \dots, n.$$

Thus, for a Borel subset $F \subset E(S, \mathbb{I})$, we have

$$\mu_i(F \,\Delta\, S(F)) \le \mu_i(F \cup S(F)) \le \mu_i(F) + \mu_i(SF) \le 2\mu_i(E(S, \mathbb{I})) \le \varepsilon, \quad i = 1, \dots, n.$$

If $F \subset X - E(S, \mathbb{I})$, then $F \,\Delta\, SF = \emptyset$. Thus $U(\mathbb{I}; \mu_1, \dots, \mu_n; \varepsilon/2) \subset U'$. □

Proposition 4.3. *The topology* τ *(and therefore* τ'*) is strictly stronger than* $\overline{p}$.

Proof. We define another topology, denoted τ'_0, on $\mathrm{Aut}(X,\mathcal{B})$. By definition, τ'_0 is generated by the base of neighbourhoods

$$U'_0(T;\mu_1,\dots,\mu_n;\varepsilon)=\Big\{S\in\mathrm{Aut}(X,\mathcal{B})\ \Big|\ \sup_{F\in\mathcal{B}}\mu_i(TF\,\Delta\,SF)+\sup_{F\in\mathcal{B}}\mu_i(T^{-1}F\,\Delta\,S^{-1}F)<\varepsilon,\ i=1,\dots,n\Big\}.$$

Obviously, τ'_0 is stronger than τ'.

Claim 1. τ'_0 *is equivalent to* τ.

In fact, we need to show only that τ is stronger than τ'_0. This assertion can be proved in the same method which was used to establish that τ is stronger than τ' in Theorem 4.2. Using this fact, we obtain

$$\tau\succ\tau'_0\succ\tau'\sim\tau$$

and the claim is proved.

To finish this part of the proof, we note that τ'_0 is clearly stronger than $\overline{p}$.

Now we show that τ is strictly stronger that $\overline{p}$. For this, we need to find a τ-neighbourhood U of the identity such that for any $\overline{p}$-neighbourhood $\overline{W}=\overline{W}(\mathbb{I};(F_i);(\mu_j);\varepsilon)$ there exists $S\in\overline{W}$ which is not in U. Take $U=U(\mathbb{I};\delta_{x_0};1/2)$. Then $S\notin U$ if and only if $Sx_0\neq x_0$. Thus, we have to show that in every $\overline{W}$ there exists S such that $Sx_0\neq x_0$.

Claim 2. *Every* $\overline{p}$*-neighbourhood of the identity* $\overline{W}(\mathbb{I};(F_i);(\mu_j);\varepsilon)$ *contains a free automorphism* S.

Indeed, if $S\in\overline{W}$ then $\mu_j(SF_i\,\Delta\,F_i)+\mu_j(S^{-1}F_i\,\Delta\,F_i)<\varepsilon,\ i=1,\dots,n,\ j=1,\dots,m$. Given (F_i) and (μ_j), one can find a freely acting S satisfying the above condition. To see this, we can assume that $X=\mathbb{R}$ and then S can be taken as a translation $x\to x+\alpha$, $\alpha\in\mathbb{R}$. The details are left to the reader. □

Next, we will compare τ and τ''. Our goal is to prove that the uniform topology τ is strictly stronger than τ''. To do this, we will need a more convenient description of τ''.

Definition 4.4.

(a) Let $\widetilde{\tau}$ be the topology on $\mathrm{Aut}(X,\mathcal{B})$ defined by the base

$$\widetilde{V}(T;\ \mu_1,\dots,\mu_n;\varepsilon)=\Big\{S\in\mathrm{Aut}(X,\mathcal{B})\ \Big|\ \sup_{\mathcal{Q}}\Big(\sum_{i\in I}|\mu_j(TE_i)-\mu_j(SE_i)|\Big)<\varepsilon,\ j=1,\dots,n\Big\}\tag{4.4}$$

where $\mu_1, \dots, \mu_n \in \mathcal{M}_1(X)$ and supremum is taken over all finite Borel partitions $\mathcal{Q} = (E_i)_{i\in I}$ of X.

(b) Define a new topology $\overline{\tau}$ on $\mathrm{Aut}(X,\mathcal{B})$ using as neighbourhood base the sets

$$\begin{aligned} \overline{V}&(T;\mu_1,\dots,\mu_n;\varepsilon) \\ &= \{S \in \mathrm{Aut}(X,\mathcal{B}) \mid \sup_{F\in\mathcal{B}} |\mu_j(TF) - \mu_j(SF)| < \varepsilon,\ j = 1,\dots,n\} \end{aligned} \tag{4.5}$$

where $\mu_1,\dots,\mu_n \in \mathcal{M}_1(X)$.

Proposition 4.5. *The three topologies τ'', $\widetilde{\tau}$ and $\overline{\tau}$ are pairwise equivalent.*

Proof. ($\tau'' \Leftrightarrow \widetilde{\tau}$) Let $U'' = U''(\mathbb{I};\mu_1,\dots,\mu_n;\varepsilon)$ be as in (2.3). We will show that $\widetilde{V} = \widetilde{V}(\mathbb{I};\mu_1,\dots,\mu_n;\varepsilon/3) \subset U''$ where $\widetilde{V}$ is defined in (4.4). Take $S \in \widetilde{V}$. For any $f \in B(X)_1$, find $g(x) = \sum_{i\in I} a_i\chi_{E_i}$ such that

$$\|f - g\| = \sup_{x\in X} |f(x) - g(x)| < \varepsilon/3.$$

Here $|a_i| \le 1$ and $(E_i)_{i\in I}$ forms a partition of X into Borel sets. Then for $j = 1,\dots,n$,

$$\begin{aligned} \biggl|\int_X (f(S^{-1}x) - f(x))\,d\mu_j\biggr| &\le \biggl|\int_X (f(S^{-1}x) - g(S^{-1}x))\,d\mu_j\biggr| \\ &\quad + \biggl|\int_X (g(S^{-1}x) - g(x))\,d\mu_j\biggr| + \biggl|\int_X (g(x) - f(x))\,d\mu_j\biggr| \\ &\le \frac{2\varepsilon}{3} + \biggl|\int_X (g(S^{-1}x) - g(x))\,d\mu_j\biggr| = \frac{2\varepsilon}{3} + \Bigl|\sum_{i\in I} a_i(\mu_j(SE_i) - \mu_j(E_i))\Bigr| \\ &\le \frac{2\varepsilon}{3} + \sum_{i\in I} |\mu_j(SE_i) - \mu_j(E_i)| \le \varepsilon. \end{aligned}$$

This proves that $\widetilde{V} \subset U''$.

Conversely, let $\widetilde{V} = \widetilde{V}(\mathbb{I};\mu_1,\dots,\mu_n;\varepsilon)$ be given. It is sufficient to show that $U'' = U''(\mathbb{I};\mu_1,\dots,\mu_n;\varepsilon) \subset \widetilde{V}$. If $S \in U''$, then for all $f \in B(X)_1$, we have

$$\biggl|\int (f(S^{-1}x) - f(x))\,d\mu_j\biggr| < \varepsilon, \quad j = 1,\dots,n. \tag{4.6}$$

We thus need to show that for every finite Borel partition $\mathcal{Q} = (E_i)_{i\in I}$ and all $j = 1,\dots,n$,

$$\sum_{i\in I} |\mu_j(SE_i) - \mu_j(E_i)| < \varepsilon. \tag{4.7}$$

For this, we consider the $\mathcal{Q}$-measurable function

$$g_j(x) = \sum_{i\in I} a_i(j)\chi_{E_i} \in B(X)_1,$$

where for a finite set of measures (μ_j) and S we define

$$a_i(j) = \begin{cases} 1 & \text{if } \mu_j(SE_i) > \mu_j(E_i), \\ -1 & \text{if } \mu_j(SE_i) < \mu_j(E_i), \\ 0 & \text{if } \mu_j(SE_i) = \mu_j(E_i). \end{cases}$$

The definition of $g_j(x)$ and (4.6) imply that for every $j = 1, \dots, n$

$$\sum_{i\in I} |\mu_j(SE_i) - \mu_j(E_i)| = \left| \sum_{i\in I} a_i(j)(\mu_j(SE_i) - \mu_j(E_i)) \right|$$
$$= \left| \int_X \sum_{i\in I} a_i(j)(\chi_{SE_i} - \chi_{E_i})\, d\mu_j \right| = \left| \int_X (g_j(S^{-1}x) - g_j(x))\, d\mu_j \right| < \varepsilon.$$

Thus, (4.7) holds and therefore $U'' \subset \widetilde{V}$.

($\widetilde{\tau} \Leftrightarrow \overline{\tau}$) Let $\overline{V}(\mathbb{I}; \mu_1, \dots, \mu_n; \varepsilon)$ be given as in (4.5). Then, we will prove that

$$\widetilde{V}(\mathbb{I}; \mu_1, \dots, \mu_n; 2\varepsilon) \subset \overline{V}(\mathbb{I}; \mu_1, \dots, \mu_n; \varepsilon). \tag{4.8}$$

In fact, if $S \in \widetilde{V}(\mathbb{I};\ \mu_1, \dots, \mu_n;\ 2\varepsilon)$, then for every finite Borel partition $\mathcal{Q}$ into sets $(E_i)_{i\in I}$ we have

$$\sum_{i\in I} |\mu_j(SE_i) - \mu_j(E_i)| < 2\varepsilon.$$

In particular, if F is a Borel set, then for $\mathcal{Q} = \{F, X - F\}$ and $j = 1, \dots, n$,

$$2\varepsilon > |\mu_j(SF) - \mu_j(F)| + |\mu_j(S(X - F) - \mu_j(X - F)| = 2|\mu_j(SF) - \mu_j(F)|.$$

Hence $|\mu_j(SF) - \mu_j(F)| < \varepsilon$ for any Borel set F and $j = 1, \dots, n$, that is $S \in \overline{V}(\mathbb{I}; \mu_1, \dots, \mu_n; \varepsilon)$ and (4.8) is proved.

Conversely, let $\widetilde{V} = \widetilde{V}(\mathbb{I}; \mu_1, \dots, \mu_n; \varepsilon)$ be given as in (4.4). We will prove that $\overline{V}(\mathbb{I}; \mu_1, \dots, \mu_n; \varepsilon/2) \subset \widetilde{V}$. Take $S \in \overline{V}(\mathbb{I}; \mu_1, \dots, \mu_n; \varepsilon/2)$. This means that for every $F \in \mathcal{B}(X)$ and all $j = 1, \dots, n$,

$$|\mu_j(SF) - \mu_j(F)| < \varepsilon/2. \tag{4.9}$$

Let $\mathcal{Q} = (E_i)_{i\in I}$ be a finite partition into Borel sets and denote by $F_+(j) = \bigcup_{i\in I_+(j)} E_i$ where $I_+(j) = \{i \in I \mid \mu_j(SE_i) \geq \mu_j(E_i)\}$. Then $F_-(j) := X - F_+(j) = \bigcup_{i\in I_-(j)} E_i$, where $I_-(j) = \{i \in I \mid \mu_j(SE_i) < \mu_j(E_i)\}$. We know, by (4.9), that for $j = 1, \dots, n$,

$$|\mu_j(SF_+(j)) - \mu_j(F_+(j))| < \varepsilon/2,$$
$$|\mu_j(SF_-(j)) - \mu_j(F_-(j))| < \varepsilon/2.$$

In other words,

$$\frac{\varepsilon}{2} > |\mu_j(SF_+(j)) - \mu_j(F_+(j))|$$
$$= \left| \sum_{i\in I_+(j)} (\mu_j(SE_i) - \mu_j(E_i)) \right| = \sum_{i\in I_+(j)} |\mu_j(SE_i) - \mu_j(E_i)|.$$

Similarly,

$$\frac{\varepsilon}{2} > |\mu_j(SF_-(j)) - \mu_j(F_-(j))| = \sum_{i\in I_-(j)} |\mu_j(SE_i) - \mu_j(E_i)|.$$

Therefore

$$\sum_{i\in I} |\mu_j(SE_i) - \mu_j(E_i)| < \varepsilon,$$

i.e. $S \in \widetilde{V}(\mathbb{I}; \mu_1, \dots \mu_n; \varepsilon)$. □

Theorem 4.6. *The topology τ (and therefore τ') is strictly stronger than τ''.*

Proof. The theorem will be proved in two steps. We first show that τ is stronger than τ'' (we give two proofs of this fact). Then we prove that τ'' cannot be equivalent to τ.

Step 1. ($\tau \succ \tau''$) First proof. As mentioned above, it suffices to consider neighbourhoods of $\mathbb{I}$ only. Take $U'' = U''(\mathbb{I}; \mu_1, \dots, \mu_n; \varepsilon)$ as in (2.3). We will show that $U = U(\mathbb{I}; \mu_1, \dots, \mu_n; \varepsilon/2) \subset U''$.

Indeed, for $T \in U$, one has $\mu_i(E(T, \mathbb{I})) < \varepsilon/2$, $i = 1, \dots, n$. Then, for $f \in B(X)_1$,

$$\begin{aligned}\sup_{\|f\|\le 1} |\mu_i \circ T(f) - \mu_i(f)| &= \sup_{\|f\|\le 1} \left| \int_X (f(T^{-1}x) - f(x))\, d\mu_i \right| \\ &= \sup_{\|f\|\le 1} \left| \int_{E(T,\mathbb{I})} (f(T^{-1}x) - f(x))\, d\mu_i \right| \le \sup_{\|f\|\le 1} 2\|f\| \mu_i(E(T, \mathbb{I})) < \varepsilon.\end{aligned}$$

Thus, $U \subset U''$.

($\tau \succ \tau''$) Second proof. By Theorem 4.2 and Proposition 4.5, the statement will be proved if we show that τ' is stronger than $\overline{\tau} \sim \tau''$. To this end, we note that for given $\mu_1, \dots, \mu_n \in \mathcal{M}_1(X)$ and $\varepsilon > 0$, one has

$$\overline{V}(\mathbb{I}; \mu_1, \dots, \mu_n; \varepsilon) \supset U'(\mathbb{I}; \mu_1, \dots, \mu_n; \varepsilon). \tag{4.10}$$

Indeed, (4.10) and (4.5) in view of the following simple observation. If $S \in U'(\mathbb{I}; \mu_1, \dots, \mu_n; \varepsilon)$, then

$$\sup_{F\in\mathcal{B}} \mu_i(F \,\Delta\, SF) < \varepsilon, \quad i = 1, \dots, n.$$

Since $|\mu_i(F) - \mu_i(SF)| \le \mu_i(F \,\Delta\, SF)$, we get that

$$\sup_{F\in\mathcal{B}} |\mu_i(F) - \mu_i(SF)| < \varepsilon,$$

i.e. $S \in \overline{V}(\mathbb{I};\ \mu_1, \dots, \mu_n;\ \varepsilon)$.

Step 2. ($\tau \nsim \tau''$) The theorem would be proved if we could show that the following assertion holds. To see that $\tau'' \sim \overline{\tau}$ cannot be equivalent to τ, we should exhibit a τ-neighbourhood, say $U(\mathbb{I}; \mu; \varepsilon)$, that does not contain a $\overline{\tau}$-neighbourhood. This means that we need to prove the following claim.

CLAIM 1. *There exists a τ-neighbourhood $U(\mathbb{I};\mu;\varepsilon)$ such that for every $\overline{\tau}$-neighbourhood $\overline{V}(\mathbb{I};\nu_1,\dots,\nu_n;\delta)=\overline{V}$ one can find a Borel automorphism S that belongs to $\overline{V}$ but does not belong to $U(\mathbb{I};\mu;\varepsilon)$.*

Step 2(a). We first discuss the case where μ is a purely atomic measure. The next claim shows that it is impossible to distinguish the topologies τ and $\tau''\sim\overline{\tau}$ by using atomic measures only.

CLAIM 2. *Let $U=U(\mathbb{I};\mu_1,\dots,\mu_n;\varepsilon)$ be such that all of the measures μ_i, $i=1,\dots,n$, are purely atomic. Then there exists $\overline{V}=\overline{V}(\mathbb{I};\nu_1,\dots,\nu_n;\delta)$ such that $\overline{V}\subset U$.*

To prove this claim, fix a τ-neighbourhood $U(\mathbb{I};\mu_1,\dots,\mu_n;\varepsilon)$ with atomic measures μ_i. Then there exists a set (at most countable) $\{x_j^i \mid j\in\mathbb{N};\ i=1,\dots,n\}$ of points in X such that $\mu_i(\{x_j^i\})>0$ and $\sum_j\mu_i(\{x_j^i\})=1$. Choose n_0 such that for all $i=1,\dots,n$,

$$\sum_{j>n_0}\mu_i(\{x_j^i\})<\varepsilon.$$

For $\{x_j^i \mid 1\le j\le n_0,\ 1\le i\le n\}$, define atomic measures ν_i, $i=1,\dots,n$:

$$\nu_i(\{x_j^i\})=b_j^i>0$$

where $\sum_{j=1}^{n_0}b_j^i=1$ and $b_j^i\ne b_{j_1}^i$ for all i and $j\ne j_1$. Let

$$\delta<\min_{1\le i\le n}\Big[\min_{j\ne j_1,1\le j,j_1\le n_0}(|b_j^i-b_{j_1}^i|,b_j^i)\Big]. \tag{4.11}$$

Then we have

$$\overline{V}(\mathbb{I};\nu_1,\dots,\nu_n;\delta)\subset U(\mathbb{I};\mu_1,\dots,\mu_n;\varepsilon).$$

Indeed, if $S\in\overline{V}(\mathbb{I};\nu_1,\dots,\nu_n;\delta)$, then for each point $\{x_j^i\}$ ($1\le j\le n_0$, $i=1,\dots,n$) we get

$$|\nu_i(\{x_j^i\})-\nu_i(\{Sx_j^i\})|<\delta. \tag{4.12}$$

Then (4.11) and (4.12) imply that $Sx_j^i=x_j^i$ for all i and $1\le j\le n_0$. In other words, ν_i is S-invariant. Then $x_j^i\notin E(S,\mathbb{I})$ for all $i=1,\dots,n$, $j=1,\dots,n_0$. Therefore

$$\mu_i(E(S,\mathbb{I}))\le\sum_{j>n_0}\mu_i(\{x_j^i\})<\varepsilon,$$

that is $S\in U(\mathbb{I};\mu_1,\dots,\mu_n;\varepsilon)$. The claim is proved.

Step 2(b). In view of Claim 2, we must consider continuous measures from $\mathcal{M}_1(X)$. Since we want to find an example of $U(\mathbb{I};\mu;\varepsilon)$ satisfying Claim 1, we may assume, without loss of generality, that $\mu(\{x\})=0$, for all $x\in X$.

CLAIM 3. *Let $U(\mathbb{I};\mu;\varepsilon)$ be given where μ is a purely continuous measure on X. Let also $\overline{V}(\mathbb{I};\ \nu_1,\dots,\nu_n;\delta)$ be a $\overline{\tau}$-neighbourhood such that μ and all ν_i's satisfy the condition $\mu\ll\nu_1\sim\dots\sim\nu_n$ (i.e. μ is absolutely continuous with* *respect to all ν_i's which are, in turn, pairwise equivalent). Then there exists $S \in \overline{V}(\mathbb{I}; \nu_1, \ldots, \nu_n; \delta)$ but $S \notin U(\mathbb{I}; \mu; \varepsilon)$.*

Let $f_1(x) = 1$ and

$$f_i(x) = \frac{d\nu_i}{d\nu_1(x)}, \quad i = 2, \ldots, n.$$

Given $\delta > 0$, choose simple functions $g_i(x)$ where $g_1 = 1$,

$$g_i = \sum_{j \in I_i} a_{ij} \chi_{E_{ij}}, \quad i = 2, \ldots, n,$$

and such that

$$\int_X |f_i - g_i| \, d\nu_1 < \delta/2, \quad i = 1, \ldots, n.$$

Define a new measure ν_i' on X by $d\nu_i'(x) = g_i(x)\, d\nu_i(x)$, $i = 1, \ldots, n$. Let $\mathcal{Q}_i$ be the partition of X defined by $(E_{ij})_{j \in I_i}$. The intersection of all $\mathcal{Q}_i$'s is a new partition $\mathcal{Q} = (F_{\overline{k}})_{\overline{k} \in \Lambda}$ of X into Borel sets. Every $F_{\overline{k}}$ is of the form $\bigcap_{i=1}^n E_{ij_i}$ where $j_i \in I_i$ and $\overline{k}$ is a multiindex $(j_1, \ldots, j_n)$ taken from a subset $\Lambda \subset I_1 \times \ldots \times I_n$. Let $S(\overline{k}) \colon F_{\overline{k}} \to F_{\overline{k}}$ be a free Borel map of $F_{\overline{k}}$ onto itself preserving ν_1. Then for any $F' \subset F_{\overline{k}}$ and $i = 1, \ldots, n$, we get

$$|\nu_i'(F') - \nu_i'(S(\overline{k})F')| = |a_{ij_i}\nu_1(F') - a_{ij_i}\nu_1(S(\overline{k})F')| = 0, \quad \overline{k} \in \Lambda.$$

It may be that some of the $F_{\overline{k}}$ have zero ν_1-measure. Thus, $S(\overline{k})$ preserves the measures $\nu_1'(= \nu_1), \nu_2', \ldots, \nu_n'$, for all $\overline{k} \in \Lambda$. Define a free Borel automorphism S of X by $Sx = S(\overline{k})x$ when $x \in F_{\overline{k}}$, $\overline{k} \in \Lambda$. Then $S \in \operatorname{Aut}(X, \mathcal{B})$ and $Sx \neq x$, $x \in X$, which means that $E(S, \mathbb{I}) = X$ and $\mu(E(S, \mathbb{I})) = 1$. Moreover, S preserves the measures $\nu_1' = \nu_1, \nu_2', \ldots, \nu_n'$, since for any Borel F,

$$\begin{aligned}
\nu_i'(SF) &= \nu_i'\bigg(\bigcup_{\overline{k} \in \Lambda} (SF \cap F_{\overline{k}}) \bigg) = \sum_{\overline{k} \in \Lambda} \nu_i'(S(F \cap F_{\overline{k}})) \\
&= \sum_{\overline{k} \in \Lambda} \nu_i'(S(\overline{k})(F \cap F_{\overline{k}})) = \sum_{\overline{k} \in \Lambda} \nu_i'(F \cap F_{\overline{k}})) = \nu_i'(F).
\end{aligned}$$

On the other hand, if $F \in \mathcal{B}(X)$, then

$$\begin{aligned}
|\nu_i(F) - \nu_i(SF)| &= \bigg| \int_F f_i \, d\nu_1 - \int_{SF} f_i \, d\nu_1 \bigg| \\
&= \bigg| \int_F (f_i - g_i) \, d\nu_1 + \int_F g_i \, d\nu_1 - \int_{SF} (f_i - g_i) \, d\nu_1 - \int_{SF} g_i \, d\nu_1 \bigg| \\
&\leq \int_F |f_i - g_i| \, d\nu_1 + \int_{SF} |f_i - g_i| \, d\nu_1 + \bigg| \int_F g_i \, d\nu_1 - \int_{SF} g_i \, d\nu_1 \bigg| \\
&< \delta + |\nu_i'(F) - \nu_i'(SF)| = \delta.
\end{aligned}$$

Hence, we have shown that $S \in \overline{V}(\mathbb{I}; \nu_1, \ldots, \nu_n; \delta)$ but $S \notin U(\mathbb{I}; \mu; \varepsilon)$ if $\varepsilon < 1$, and therefore Claim 3 is proved.

Claim 4. *Suppose $X_1 \subset X$ is a Borel set such that $\mu(X_1) > 0$ but $\nu_1(X_1) = \ldots = \nu_n(X_1) = 0$. Then there exists $S \in \mathrm{Aut}(X, \mathcal{B})$ such that for any $\delta > 0$, $S \in \overline{V}(\mathbb{I}; \nu_1, \ldots, \nu_n; \delta)$ but $S \notin U(\mathbb{I}; \mu; \varepsilon)$ if $\varepsilon < \mu(X_1)$.*

To see that this claim holds, it is sufficient to take S as a free Borel automorphism on X_1 and put $Sx = x$ on $X - X_1$. Then $\nu_i \circ S = \nu_i$, $i = 1, \ldots, n$, and therefore $S \in \overline{V}(\mathbb{I}; \nu_1, \ldots, \nu_n; \delta)$ for any $\delta > 0$. On the other hand, $E(S, \mathbb{I}) = X_1$ and then S cannot be in $U(\mathbb{I}; \mu; \varepsilon)$ if $\varepsilon < \mu(X_1)$.

Step 2(c). Using Claims 3 and 4, we can prove the theorem in the general case. To do this, it suffices to prove Claim 1. Let $U(\mathbb{I}; \mu; \varepsilon)$ be given where μ is a continuous measure on X. Let $\nu_1, \ldots, \nu_n$ be measures from $\mathcal{M}_1(X)$. Consider all possible relations between $\nu_1, \ldots, \nu_n$. For ν_1 and ν_2 there exists a partition of X into three Borel sets A, B, and C such that ν_1 and ν_2 are equivalent on C, ν_1 is zero and ν_2 is positive on B, and ν_2 is zero and ν_1 is positive on A. Then ν_1 is supported on $A \cup C$ and ν_2 is supported on $B \cup C$. Considering the three measures ν_1, ν_2, and ν_3, this fact can be applied to each of the sets A, B, C. We thus obtain a new partition $(X(1), X(2), X(3), X(1,2), X(1,3), X(2,3), X(1,2,3))$ of X such that $\nu_i > 0$, $\nu_j = 0$, $j \neq i$ on $X(i)$, $\nu_i \sim \nu_k$, $\nu_j = 0$, $j \neq ik$, on $X(i,k)$, and $\nu_1 \sim \nu_2 \sim \nu_3$ on $X(1,2,3)$. It is clear that a similar statement holds for $\nu_1, \ldots, \nu_n$. Namely, for every nonempty subset $K \subset \{1, \ldots, n\}$ there exists a subset $X(K)$ in X such that all the ν_i's are equivalent on $X(K)$ if $i \in K$, and $\nu_j(X(K)) = 0$ if $j \notin K$. Moreover, the sets $X(K)$ define a partition of X as K runs over all subsets in $\{1, \ldots, n\}$.

Now consider the measure μ together with $\nu_1, \ldots, \nu_n$. Without loss of generality, we can assume that $\mu(X(K)) > 0$ for all K. Every $X(K)$ can be decomposed into two sets $X(K)'$ and $X(K)''$ (some of these sets might be of zero μ-measure) such that $\mu \ll \nu_i$ on $X(K)'$ and μ is singular with respect to ν_i on $X(K)''$ where $i \in K$. In fact, the latter condition holds for all $\nu_1, \ldots, \nu_n$ by definition of $X(K)$. This means that $\mu(X(K)'') > 0$ whereas $\nu_i(X(K)'') = 0$, $i = 1, \ldots, n$. Denote $X' = \bigcup_K X(K)'$, $X'' = \bigcup_K X(K)''$. By Claim 4, we may define a free Borel automorphism S on X'' which preserves each $X(K)''$. To define S on X', we use the proof of Claim 3. Given $\nu_1, \ldots, \nu_n$ and δ, take $\delta_1 < 2^{-n}\delta$. The proof of Claim 3 allows us to find a Borel one-to-one map $S_K \colon X'_K \to X'_K$ such that $S_K x \neq x$, $x \in X(K)'$ and $|\nu_i(F) - \nu_I(S_K F)| < \delta_1$, $F \subset X(K)'$. Then $Sx = S_K x$, $x \in X(K)'$, defines a one-to-one Borel map on X'. Therefore, $S \in \mathrm{Aut}(X, \mathcal{B})$ and $S \in \overline{V}(\mathbb{I}; \nu_1, \ldots, \nu_n; \delta)$ and since S is free, $S \notin U(\mathbb{I}; \mu; \varepsilon)$ if $\varepsilon < 1$. The proof of Theorem 4.6 is complete. □

Theorem 4.7. *The topologies p and $\widetilde{p}$ are equivalent.*

Proof. Let $T \in \mathrm{Aut}(X, \mathcal{B})$ and let $W(T) = W(T; F_1, \ldots, F_n)$ be a p-neighbourhood of T. Then for every $S \in W(T)$, one has $SF_i = TF_i$, for all i.

Let $f_i = \chi_{F_i}$. Then for any $0 < \varepsilon < 1$ we see that $\widetilde{W}(T; f_1, \dots, f_n; \varepsilon) \subset W(T)$. Indeed, if

$$\sup_{x\in X} |f_i(S^{-1}x) - f_i(T^{-1}x)| < \varepsilon,$$

then $|\chi_{SF_i}(x) - \chi_{TF_i}(x)| < \varepsilon$ for all $x \in X$. This implies that $SF_i = TF_i$.

To prove the converse statement, we take a $\widetilde{p}$-neighbourhood $\widetilde{W}(T) = \widetilde{W}(T; f_1, \dots, f_n; \varepsilon)$. We need to show that $\widetilde{W}(T)$ contains a p-neighbourhood $W(T) = W(T; F_1, \dots, F_m)$. Given $\varepsilon > 0$, find for each f_i a Borel function $g_i(x)$ such that $g_i(x) = \sum_{j\in I(i)} a_j(i)\chi_{E_j(i)(x)}$ and

$$\sup_{x\in X} |f_i(x) - g_i(x)| < \varepsilon/2, \quad i = 1, \dots, n.$$

Note that $|I(i)| < \infty$ since f_i is bounded. Take the p-neighbourhood $W(T) = W(T; (E_j(i) : j \in I(i),\ i = 1, \dots, n))$. If $S \in W(T)$, then

$$\begin{aligned}\sup_{x\in X} |f_i(T^{-1}x) \ - f_i(S^{-1}x)| &\le \sup_{x\in X} |f_i(T^{-1}x) - g_i(T^{-1}x)| \\ &\quad + \sup_{x\in X} |g_i(T^{-1}x) - g_i(S^{-1}x)| + \sup_{x\in X} |g_i(S^{-1}x) - f_i(S^{-1}x)| \\ &< \varepsilon + \sup_{x\in X} \bigg| \sum_{j\in I(i)} a_j(i)(\chi_{TE_j(i)} - \chi_{SE_j(i)} \bigg| = \varepsilon.\end{aligned}$$

Thus, $S \in \widetilde{W}(T)$. □

Proposition 4.8.

(a) *The topologies τ and p are not comparable.*
(b) *The topologies τ'' and $\overline{p}$ are not comparable.*
(c) *The topologies τ_0 and τ'' are not comparable.*

Proof. We prove only (c) here. We will show that for the τ''-neighbourhood $U'' = U''(\mathbb{I}; \delta_y; \varepsilon_0)$, $e_0 < 1$, and any τ_0-neighbourhood $U_0 = U_0(\mathbb{I}; \nu_1, \dots, \nu_n; \varepsilon)$ there exists $T \in U_0$ such that $T \notin U''$. It suffices to take T such that $\nu_i(E(T, \mathbb{I})) < \varepsilon$ for all i and $Ty \neq y$. Clearly such a T can always be found. Then there is a Borel function $f_0 \in B(X)_1$ such that $|f_0(T^{-1}y) - f_0(y)| = 1$. Hence $T \notin U''$. The fact that τ'' cannot be stronger than τ_0 is proved as in Theorem 4.6.

The proofs of (a) and (b) use the method similar to that applied in Claim 1 and in the proof of Theorem 4.6. We leave the details to the reader. □

Remark 4.9. Note that when the underlying space X in Definition 2.1 is a Polish space, one can consider various modifications of the definition of τ, τ', τ'' and p, $\overline{p}$. In particular, we can show that replacing Borel functions from $B(X)_1$ by continuous functions from $C(X)_1$ does not affect the topology τ'' (see (2.3)). To see this, we may use the following statement in the proof of Proposition 4.5 which establishes the equivalence of τ, $\widetilde{\tau}$, and $\overline{\tau}$:

Let $g(x) \in B(X)_1$ and $S \in \mathrm{Aut}(X,\mathcal{B})$. Then for any $\delta > 0$ and any $\mu \in \mathcal{M}_1(X)$, there exists a continuous function $f \in C(X)_1$ such that

$$\left| \int_X (f(x) - g(x))\, d\mu \right| < \delta, \qquad \left| \int_X (f(S^{-1}x) - g(S^{-1}x))\, d\mu \right| < \delta.$$

In the case when X is a Cantor set, observe that in the definition of τ' (3.2) it suffices to take the supremum over only the countable family of clopen sets. This follows easily from regularity of Borel measures on Cantor sets.

5. Bratteli diagram for a Borel automorphism

5.1. Borel–Bratteli diagrams. In this section, we show that every aperiodic Borel automorphism of $(X,\mathcal{B})$ can be represented as a Borel transformation acting on the space of infinite paths of a Bratteli diagram. More precisely, we define a modification of the concept of Bratteli diagram that is suitable for Borel automorphisms.

Definition 5.1. A Borel–Bratteli diagram is an infinite graph $B = (V,E)$ such that the vertex set V and the edge set E can be partitioned into sets $V = \bigcup_{i\ge 0} V_i$ and $E = \bigcup_{i\ge 1} E_i$ having the following properties:

(a) $V_0 = \{v_0\}$ is a single point, every V_i and E_i are at most countable sets;

(b) there exist a range map r and a source map s from E to V so that $r(E_i) \subset V_i$, $s(E_i) \subset V_{i-1}$, $s^{-1}(v) \neq \emptyset$ for all $v \in V$, and $r^{-1}(v) \neq \emptyset$ for all $v \in V \setminus V_0$;

(c) for every $v \in V \setminus V_0$, the set $r^{-1}(v)$ is finite.

The pair (V_i, E_i) will be called the i-th level. We write $e(v,v')$ to denote an edge e such that $s(e) = v$, $r(e) = v'$.

A finite or infinite sequence of edges, $(e_i : e_i \in E_i)$ such that $s(e_i) = r(e_{i-1})$ is called a finite or infinite path, respectively. It follows from the definition that every vertex $v \in V_i$ is connected to v_0 by a finite path and the set of all such paths $E(v_0, v)$ is finite. Given a Borel–Bratteli diagram $B = (V,E)$, we denote the set of infinite paths by Y_B.

Let $B = (V,E,\ge)$ be a Borel–Bratteli diagram (V,E) equipped with a partial order $\ge$ defined on each E_i, $i = 1,2,\ldots$, such that edges e, e' are comparable if and only if $r(e) = r(e')$; in other words, a linear order $\ge$ is defined on each (finite) set $r^{-1}(v)$, $v \in V \setminus V_0$. For a Borel–Bratteli diagram (V,E) equipped with such a partial order $\ge$ on E, one can also define a partial lexicographic order on the set $E_{k+1} \circ \ldots \circ E_l$ of all paths from V_k to V_l: $(e_{k+1},\ldots,e_l) > (f_{k+1},\ldots,f_l)$ if and only if for some i with $k+1 \le i \le l$, $e_j = f_j$ for $i < j \le l$ and $e_i > f_i$. Then we see that any two paths from $E(v_0,v)$, the (finite) set of all paths connecting v_0 and v, are comparable with respect to the introduced lexicographic order.

We call a path $e = (e_1, \dots, e_i, \dots)$ maximal (minimal) if every e_i has a maximal (minimal) number amongst all elements from $r^{-1}(r(e_i))$. Notice that there are unique minimal and maximal paths in $E(v_0, v)$ for each $v \in V_i$, $i \geq 0$.

Definition 5.2. A Borel–Bratteli diagram $B = (V, E)$ together with a partial order $\geq$ on E is called an ordered Borel–Bratteli diagram $B = (V, E, \geq)$ if the space Y_B has no cofinal minimal and maximal paths. This means that Y_B does not contain paths $e = (e_1, \dots, e_i, \dots)$ such that for all sufficiently large i the edges e_i have maximal (minimal) number in the set $r^{-1}(r(e_i))$.

It follows from the definitions that Y_B is a 0-dimensional Polish space in the natural topology defined by clopen sets.

To every Borel–Bratteli diagram $B = (V, E)$, we can associate a sequence of infinite matrices. To do this, consider the set E_n of all edges between the levels V_{n-1} and V_n. Let us enumerate the vertices of V_n as $(v_1(n), \dots, v_k(n), \dots)$, $n \in \mathbb{N}$. Define the matrix $M_n = (a_{ik})_{i,k=1}^{\infty}$ where $a_{ik} = |E(v_k(n-1), v_i(n))|$. We notice that only finitely many entries of each row in M_n are non-zero because of the relation

$$\sum_{k=1}^{\infty} a_{ik} = |r^{-1}(v_i^n)| < \infty.$$

Moreover, in each column of M_n there exists at least one non-zero entry. Denote by $\mathcal{M}$ the set of infinite matrices having the above two properties of their rows and columns. It is easy to see that $\mathcal{M}$ is closed with respect to matrix multiplication.

Since the notion of ordered Bratteli diagram has been discussed in many recent papers (see, e.g. [9], [10]), we recall only the principal definitions. We refer to [14], [19] for more detailed expositions of this material. We will also use in our proofs the notions of *telescoping and splitting* of a Bratteli diagram defined in [14]. By definition, the telescoping $B' = (V', E', \geq)$ of $B = (V, E, \geq)$ with respect to a sequence $0 = m_0 < m_1 < m_2 < \dots$ is obtained if we set $V'_n = V_{m_n}$ and $E'_n = E_{m_{n-1}+1} \circ \dots \circ E_{m_n}$. The set E'_n has the induced lexicographic order defined above. The operation converse to telescoping a Bratteli diagram is splitting. This means that between to consecutive levels, say V_{n-1} and V_n, we add a new level V' which is a disjoint union of finite sets $V'(v)$ such that the number of vertices in $V'(v)$ equals the number of edges in $r^{-1}(v)$, $v \in V_n$. It is easy to see how to introduce an order on the edge set of the new diagram so that by telescoping one gets the original ordered diagram back.

For each ordered Borel–Bratteli diagram $B = (V, E, \geq)$, define a Borel transformation φ (also called the Vershik automorphism) acting on Y_B as follows. Given $y = (e_1, e_2, \dots) \in Y_B$, let k be the smallest number such that e_k is not a maximal edge. Let f_k be the successor of e_k in $r^{-1}(r(e_k))$. Then we define $\varphi(x) = (f_1, \dots, f_{k-1}, f_k, e_{k+1}, \dots)$ where $(f_1, \dots, f_{k-1})$ is the minimal path in

$E(v_0, r(f_{k-1}))$. Obviously, φ is a one-to-one mapping of Y_B onto itself. Moreover, φ is a homeomorphism of Y_B.

It is sometimes convenient to use another description of infinite paths in an ordered Borel–Bratteli diagram $B = (V, E, \geq)$. Take $v \in V_n$ and consider the finite set $E(v_0, v)$. The lexicographic order on $E(v_0, v)$ gives us an enumeration of its elements from 0 to $h(n, v) - 1$ where 0 is assigned to the minimal path and $h(n, v) - 1$ is assigned to the maximal path in $E(v_0, v)$. Note that $h(1, v) = |r^{-1}(v)|$, $v \in V_1$ and by induction

$$h(n, v) = \sum_{w \in s(r^{-1}(v))} |E(w, v)| h(n-1, w), \quad v \in V_n. \tag{5.1}$$

Let $y = (e_1, e_2, \dots)$ be an infinite path from Y_B. Consider a sequence (P_n) of enlarging finite paths defined by y: $P_n = (e_1, \dots, e_n) \in E(v_0, r(e_n))$, $n \in \mathbb{N}$. Then every P_n can be identified with a pair (i_n, v_n) where $v_n = r(e_n)$ and $i_n \in [0, h(n, v_n) - 1]$ is the number assigned to P_n in $E(v_0, v_n)$. Thus, every $y = (e_n) \in Y_B$ can be uniquely represented as an infinite sequence (i_n, v_n) with $v_n = r(e_n)$ and $0 \leq i_n \leq h(n, v_n) - 1$. We also observe that $i_n \to \infty$, $n \to \infty$, since the Borel–Bratteli diagram B has no infinite cofinal minimal paths and $(h(n, v_n) - i_n) \to \infty$, $n \to \infty$, since there is no infinite cofinal maximal paths in B.

Thus, given an ordered Borel–Bratteli diagram $B = (V, E, \geq)$, we have defined a dynamical system (Y_B, φ). Our goal is to show that every Borel automorphism can be realized as a Vershik transformation acting on the space of infinite paths of an ordered Borel–Bratteli diagram.

5.2. Construction of a Borel–Bratteli diagram by a Borel automorphism and a vanishing sequence of markers. Let T be an aperiodic automorphism of $(X, \mathcal{B})$. Take a vanishing sequence of markers (A_n) with $X = A_0 \supset A_1 \supset A_2 \supset \dots$ (see Definition 3.4)). We give a construction of an ordered Borel–Bratteli diagram coming from (A_n) and T.

Since A_1 is a recurrent complete T-section of X, there exists a partition $\xi_1 = \bigcup_{v \in V_1} \xi_1(v)$ of X formed by at most countable collection of disjoint T-towers $\xi_1(v) = (A_1(v), T(A_1(v)), \dots, T^{h(1,v)-1}(A_1(v)))$ where $\bigcup_{v \in V_1} A_1(v) = A_1$ and V_1 is a subset of $\mathbb{N}$. Here $h(1, v)$ is the height of T-tower $\xi_1(v)$.

Because A_2 is a subset of A_1, we can assume (refining, if necessary, the partition ξ_1) that A_2 is a union of some sets $A_1(v)$. Then we again define ξ_2 as a disjoint collection of T-towers $\xi_2(v) = (A_2(v), T(A_2(v)), \dots, T^{h(2,v)-1}(A_2(v)))$, $v \in V_2 \subset \mathbb{N}$, with $\bigcup_{v \in V_2} A_2(v) = A_2$. We apply this construction for every A_n and find the corresponding partition ξ_n consisting of T-towers $\xi_n(v)$ of finite height $h(n, v)$, $v \in V_n \subset \mathbb{N}$. Note that at each step, A_{n+1} is a ξ_n-set and hence ξ_{n+1} refines ξ_n. Moreover, for any partition ξ from the sequence (ξ_n) and any

Borel set D in X we can refine ξ such that D becomes a ξ-set. This means that we can assume that the collection of atoms of (ξ_n) separates points in X.

Now define an ordered Borel–Bratteli diagram B using the sets A_n and generated by them partitions ξ_n of X, $n \in \mathbb{N}$. Let $V_0 = \{v_0\}$ be a singleton (relating to $A_0 = X$). The set V_1 gives vertices at the first level in B. To define E_1, we take $v \in V_1$ and draw $h(1, v)$ edges connecting v_0 and v. Enumerate these edges from 1 to $h(1, v)$ in an arbitrary order. Set $s(e) = v_0$, $r(e) = v$ for e connecting v_0 and v. Thus, the set $r^{-1}(v)$ becomes linearly ordered for every $v \in V_1$.

To define the diagram B for the next level, we take V_2, obtained from the partition ξ_2, as the set of vertices. Fix a vertex $v \in V_2$ and consider the T-tower $\xi_2(v)$. It can be easily seen that $\xi_2(v)$ intersects a finite number of T-towers from the partition ξ_1, say $\xi_1(v_1), \dots, \xi_1(v_s)$, $v_1, \dots, v_s \in V_1$. Notice that these towers are not necessarily different and that some of the v_i's may be met several times. We see that $h(2, v) = h(1, v_1) + \dots + h(1, v_s)$. Take the vertices $v_1, \dots, v_s$ from V_1 and draw the edges connecting each of them with $v \in V_2$. We get the sets $E(v_i, v)$, $i = 1, \dots, s$. The number of edges in $E(v_i, v)$ equals the multiplicity of the vertex v_i in the set $v_1, \dots, v_s$. Define $s(e) = v_i$, $r(e) = v$ where $e \in E(v_i, v)$. To introduce a linear order on $r^{-1}(v)$, $v \in V_2$ we consider again the T-towers $\xi_1(v_1), \dots, \xi_1(v_s)$ and enumerate them from 1 to s according to the natural order in which they appear in $\xi_2(v)$ when we go along $\xi_2(v)$ from the base to the top. Enumerate the corresponding edges $e \in r^{-1}(v)$ in the same order from 1 to s. This procedure is applied to every vertex from V_2 to define the entire set E_2 together with a partial order on E_2.

Repeating this method for every n, we construct the n-th level (V_n, E_n) and establish a partial order on E_n. We see that conditions (a)–(c) of Definition 5.1 hold for the infinite graph $B = (V, E)$ where $V = \bigcup_{n \geq 0} V_n$ and $E = \bigcup_{n \geq 1} E_n$ are defined as above. The partial order which we have described on E determines an ordered Borel–Bratteli diagram $B = (V, E, \geq)$ according to Definition 5.2. Indeed, it is easy to see that B has no cofinal maximal and minimal paths. It follows from the fact that $\bigcap_n T^k(A_n) = \emptyset$ for any $k \in \mathbb{Z}$. We also observe that every infinite path $y \in Y_B$ is completely determined by the infinite sequence $\{(i_n, v_n)\}_n$, $v \in V_n$, $0 \leq i \leq h(n, v_n) - 1$ such that $T^{i_{n+1}}(A_{n+1}(v_{n+1})) \subset T^{i_n}(A_n(v_n))$, $n \in \mathbb{N}$. The height $h(n, v)$ can be found by (5.1).

Notice that if one takes a subsequence (A_{n_m}) in (A_n) and constructs a new ordered Borel–Bratteli diagram B' by (A_{m_n}) and T, then B' turns out to be a telescoping of B. If a set $A_n(v)$ is partitioned into a finite number of uncountable Borel sets $A_n^v(w)$ and respectively $\xi_n(v)$ is cut into a finite number of new T-towers $\xi_n^v(w)$ with base $A_n^v(w)$, then the above construction gives an ordered Borel–Bratteli diagram B'' which is a splitting of B.

The next theorem shows that any aperiodic Borel automorphism T can be realized as a Vershik transformation acting on the space of infinite paths of an ordered Borel–Bratteli diagram.

Theorem 5.3. *Let T be an aperiodic Borel automorphism acting on a standard Borel space $(X, \mathcal{B})$. Then there exists an ordered Borel–Bratteli diagram $B = (V, E, \geq)$ and a Vershik automorphism $\varphi: Y_B \to Y_B$ such that (X, T) is isomorphic to (Y_B, φ).*

Proof. Let (A_n) be a vanishing sequence of markers for T and let $\xi_n = (\xi_n(v) : v \in V_n)$ be a collection of disjoint T-towers where

$$\xi_n(v) = (A_n(v), T(A_n(v)), \ldots, T^{h(n,v)-1}(A_n(v))), \quad n \in \mathbb{N}.$$

As mentioned above, we can assume that the atoms of $(\xi_n,\ n \in \mathbb{N})$ generate the Borel structure on X. By changing-of-topology results (see Remark 3.5), we may choose a topology ω on X such that:

(i) X is a Polish 0-dimensional space,
(ii) $\mathcal{B}(\omega) = \mathcal{B}$ where $\mathcal{B}(\omega)$ is the σ-algebra generated by ω-open sets,
(iii) all sets $T^j(A_n(v))$, $v \in V_n$, $j = 0, \ldots, h(n, v) - 1$, $n \geq 1$, are clopen in ω,
(iv) T is a homeomorphism of (X, ω).

Next, we observe that for fixed $n \in \mathbb{N}$ and $\varepsilon > 0$ one can cut each T-tower $\xi_n(v)$, $v \in V_n$, into disjoint clopen towers of the same height such that the diameter of every element of the new towers is less than ε. Therefore, without loss of generality, we may assume that

$$\sup_{0 \leq j < h(n,v), v \in V_n} [\operatorname{diam} T^j(A_n(v))] \to 0, \quad n \to \infty. \tag{5.2}$$

Now applying the above construction to (A_n) and T (see Subsection 5.2), we can find an ordered Borel–Bratteli diagram $B = (V, E, \geq)$ together with a Vershik transformation acting on the space of infinite paths Y_B. Define a map $F: X \to Y_B$. Given $x \in X$, choose the unique sequence $\{(i_n, v_n)\}_n$, $0 \leq i \leq h(n, v_n) - 1$, $v_n \in V_n$, such that

$$T^{i_{n+1}}(A_{n+1}(v_{n+1})) \subset T^{i_n}(A_n(v_n)) \tag{5.3}$$

and $\{x\} = \bigcap_n T^{i_n}(A_n(v_n))$. As noticed in Section 5.1, such a sequence defines a unique infinite path $y \in Y_B$. We set $F(x) = y$. It is clear that F is a continuous injection of X into Y_B. Moreover, it easily follows from the construction of B and definition of Vershik transformation φ, acting on Y_B, that $F(Tx) = \varphi(Fx)$, $x \in X$. To finish the proof, we need to show only that $F(X) = Y_B$. Take a path $y \in Y_B$. Then y defines an infinite sequence $\{(i_n, v_n)\}_n$, $v \in V_n$, $0 \leq i \leq$

$h(n, v_n) - 1$ as in Section 5.1. By (5.2) and (5.3) we get a single point x such that $F(x) = y$. □

Remark 5.4. The idea of the proof of Theorem 5.3 was shown to us by B. Miller [23]. He has pointed out that if (5.2) does not hold, then F may be a map onto a proper subset of Y_B. However, one can show that in any case $F(X)$ contains a dense G_δ-set. Indeed, define

$$D = \bigcap_{n=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcup_{k \geq n}^{\infty} \bigcup_{v \in I(m,k)} A_k(v)$$

where $I(m,k) = \{v \in V_k \mid \operatorname{diam}(A_k(v)) < 1/m\}$. Then $F(D)$ is a dense G_δ-subset of Y_B.

We observe that given aperiodic $T \in \operatorname{Aut}(X, \mathcal{B})$ one can find a vanishing sequence of markers such that the corresponding ordered Borel–Bratteli diagram has a finite number of vertices at each level. To do this, we follow a suggestion of B. Miller [23] using maximal sets.

By definition, an uncountable Borel set A is called *k-maximal* for an aperiodic automorphism $T \in \operatorname{Aut}(X, \mathcal{B})$ if $A \cap T^i A = \emptyset$, $i = 1, \ldots, k-1$, and A cannot be extended to a larger set having this property. It is easy to show that A is k-maximal if and only if

$$X = \bigcup_{|i|<k} T^i A \quad \text{and} \quad A \cap T^i A = \emptyset, \quad i = 1, \ldots, k-1. \tag{5.4}$$

Hence, a maximal set A is a complete section for T such that every point from A is recurrent. The existence of maximal sets can be easily deduced, for instance, from the tower construction used in Section 2. Indeed, let $\xi = (\xi(v) : v \in V)$, where $\xi(v) = (B(v), TB(v), \ldots, T^{h(v)-1}B)$, be a partition of X into T-towers. Define

$$A = \bigcup_{v \in V} \bigcup_{i=0}^{h(v)k-1} T^{ik} B(v). \tag{5.5}$$

It follows from (5.4) that A is a k-maximal set.

Proposition 5.5. *Given aperiodic $T \in \operatorname{Aut}(X, \mathcal{B})$, there exists a vanishing sequence of markers (A_i) such that the corresponding ordered Borel–Bratteli diagram has a finite number of vertices at each level.*

Proof. Let $A_0 = X$ and let A_1 be a k-maximal set for T. Since every point $x \in A_1$ returns to A_1, one can define the induced aperiodic Borel automorphisms T_{A_1} acting on A_1:

$$T_{A_1} x = T^{m(x)} x, \quad \text{where } m(x) = \min\{m > 0 \mid T^m x \in A_1\}.$$

Then T_{A_1} is again an aperiodic automorphism of A_1 and we can find an k-maximal set $A_2 \subset A_1$ for T_{A_1}. Let now $A_{i+1} \subset A_i$ be a k-maximal set for T_{A_i}, $i \in \mathbb{N}$. It follows from (5.4) that the construction used in Subsection 5.2 gives a finite number of T-towers over A_1 which cover X. By the same reasoning, we see that A_1 is covered by a finite number of disjoint T_{A_1}-towers constructed over A_2. Therefore, X is covered by a finite number of disjoint T-towers constructed over the base A_2. It is easy to see that we will have this property at every stage of the construction.

The sequence $(A_i)_{i\in\mathbb{N}}$ of decreasing k-maximal sets which we have defined, may have a non-empty intersection, $A_\infty = \bigcap_{i\in\mathbb{N}} A_i$. Obviously, A_∞ is a wandering set with respect to T. We set $A_i' = (A_i \setminus A_\infty) \cup (\bigcup_{|j|>i} T^j A_\infty)$. Then $A_i' \supset A_{i+1}'$, $\bigcap_i A_i' = \emptyset$, and therefore (A_i') is a desired vanishing sequence of markers. □

The concept of Borel–Bratteli diagram can be used to obtain another proof of the following result (see [24, Theorem 8.9]).

Corollary 5.6. *If T is an aperiodic homeomorphism of a Polish space X, then there exists a compact metric space Y and a homeomorphism S of Y such that T is homeomorphic to the restriction of S to an S-invariant dense G_δ-subset of Y.*

Proof. We will use the method of proof of Theorem 5.3 and Proposition 5.5 to construct a vanishing sequence of markers (A_n) such that the corresponding ordered Borel–Bratteli diagram B has a finite number of vertices at each level and such that every sequence satisfying (5.3) has a non-empty intersection.

We start with a vanishing sequence of markers (D_n) satisfying (5.2). Let ξ_1 be a partition of X into T-towers constructed over D_1. Define the k-maximal set A_1 as in (5.5). Clearly, $A_1 \supset D_2$. Take the induced automorphism T_{A_1} and construct the partition ξ_2 of A_1 into T_{A_1}-towers over D_2. Let A_2 be a k-maximal subset of A_1 defined again as in (5.5). Continuing this process we define a decreasing sequence of Borel sets (A_n). Let (η_n) be the partitions of X into T-towers constructed over the A_n's. By Proposition 5.5, the corresponding ordered Borel–Bratteli diagram B has a finite number of vertices at each level. Moreover, we can assume that the atoms of the partitions η_n, $n \in \mathbb{N}$, separate points of X (see Subsection 5.2). If $\{T^{i_n}(A_n(v_n))\}$ is a sequence of atoms satisfying (5.3), then the intersection of these atoms contains at most one point. But, in fact, this intersection is non-empty because it contains the intersection of a similar sequence of atoms of partitions defined by (D_n).

As in the proof of Theorem 5.3, take the topology ω satisfying conditions (i)–(iv). Clearly, the ordered Borel–Bratteli diagram B has maximal and minimal paths (not necessarily unique). To define a homeomorphism S of a compact space

Y, we use Forrest's construction of path-sequence dynamical system generated by a Bratteli diagram [13]. By definition, the set $Y = Y_1 \cup Y_2$ where Y_1 is the set of all infinite paths from Y_B different from orbits of cofinal maximal and minimal paths and Y_2 is the set of all equivalent pairs in the sense of Forrest (x, y) with x maximal and y minimal paths in Y_B. Let S be the homeomorphism defined in [13] on Y. Then (Y, S) is a Cantor dynamical system and the set Y_1 is an S-invariant dense G_δ-subset of Y. There is a homeomorphism between the action of S on Y_1 and the action of T on X. □

Remark 5.7. Suppose that a Borel–Bratteli diagram $B = (V, E, \geq)$ has a finite number of vertices at each level, i.e. $|V_n| < \infty$, $n \in \mathbb{N}$. Then either

(a) $\limsup_n |V_n| = K < \infty$, or
(b) $\limsup_n |V_n| = \infty$.

If (a) holds, then there exists a subsequence $(n(k))$ such that $|V_{n(k)}| = K$ for all k. By telescoping, we can produce a new Borel–Bratteli diagram B' isomorphic to B such that the number of vertices at each level of B' is exactly K. If (b) holds, then there exists a subsequence $(n(k))$ such that $|V_{n(k)}| < |V_{n(k+1)}|$ for all k. This means that for the Borel–Bratteli diagram obtained by telescoping with respect to $(n(k))$, the number of vertices at each level is a strictly monotonically increasing sequence.

Another application of Proposition 5.5 consists of a description of the closures of odometers. We recall that in the case of Cantor dynamics the closure of odometers in D coincides with the set of moving homeomorphisms (by definition, S is moving if for any proper clopen set E the sets $E \setminus SE$ and $SE \setminus E$ are not empty). In Borel dynamics the notion of moving automorphisms has no sense.

Theorem 5.8.

(a) $\overline{\mathcal{O}d}^{\tau} = \mathcal{A}p$ *and* $\overline{\mathcal{O}d}^{\tau_0} = \mathcal{A}p \mod (\mathrm{Ctbl})$.
(b) $\overline{\mathcal{O}d}^{p} \subset \mathcal{I}nc$.
(3) $\overline{\mathcal{S}m}^{D} \supset \overline{\mathcal{O}d}^{D} \supset \mathcal{A}p$ *assuming that* (X, d) *is a compact metric space.*

Proof. (a) Given $T \in \mathcal{A}p$ and $\varepsilon > 0$, choose a natural number $n > 1/\varepsilon$. By Proposition 5.5, we can find a partition ξ of X into a finite number of disjoint T-towers $\xi(v) = (A(v), TA(v), \ldots, T^{h(v)-1}A(v))$, $v \in V$, $|V| < \infty$, such that $\min(h(v) : v \in V) \geq 2n$. We call the ξ-set

$$L(i) = \bigcup_{v \in V} T^{h(v)-i-1}A(v), \quad i = 0, \ldots, n-1,$$

the i-th level in the partition ξ. Let μ be a Borel probability measure on X. Then there is a pair $(L(i_0), L(i_0+1))$ such that $\mu(L(i_0) \cup L(i_0+1)) < \varepsilon$, $i_0 = 0, \ldots, n-1$. Take now the set $L(i_0)$ and construct a new partition ξ' of X into disjoint T-towers $\xi'(j)$, $j = 1, \ldots, k$, with the base $L(i_0)$ and the top $L(i_0 + 1)$. Define the automorphism S of X to coincide with T everywhere except on the top level $L(i_0+1)$. We define S on $L(i_0+1)$ as a Borel one-to-one map from the top of the tower $\xi'(j)$ onto the base of $\xi'(j+1)$, $j-1,\dots,k-1$, and the top of $\xi'(k)$ is sent by S onto the base of $\xi'(1)$. In such a way, the space X is represented as an S-tower. It is easily seen that the definition of S can be refined to produce an odometer S_1 which agrees with S everywhere except the top of the S-tower. By construction, $\mu(E(S_1,T))<\varepsilon$. The fact that the set $\mathcal{O}d$ is dense in $\mathcal{A}p$ with respect to τ follows now from the latter inequality and Remark 2.8(d).

To prove the other statement of (a), we note first that by Theorem 3.11

$$\overline{\mathcal{O}d}^{\tau_0}\subset\mathcal{A}p \mod (\mathrm{Ctbl})$$

since $\mathcal{O}d\subset\mathcal{A}p$ and $\overline{\mathcal{A}p}^{\tau_0}=\mathcal{A}p \mod (\mathrm{Ctbl})$. On the other hand, the topology τ is stronger than τ_0 and therefore

$$\overline{\mathcal{O}d}^{\tau_0}\supset\overline{\mathcal{O}d}^{\tau}=\mathcal{A}p$$

Hence $\mathcal{A}p \mod (\mathrm{Ctbl})\subset\overline{\mathcal{O}d}^{\tau_0}$ and we are done.

(b) This follows from the fact that $\mathcal{I}nc$ is closed in p (Theorem 3.15).

(c) We note that once we have a finite partition of X into T-towers as in Proposition 5.5, then every tower can be additionally cut into finitely (or countably) many subtowers such that the diameter of every atom is sufficiently small. To construct either an odometer or a smooth automorphism, we can use the same method as in the proof of (a). $\square$

5.3. Special Borel–Bratteli diagrams. In this subsection we first define the notion of special Borel–Bratteli diagrams and then indicate a class of automorphisms which are completely described by these diagrams.

By definition, an ordered Borel–Bratteli diagram $B=(V,E,\geq)$ is called *special* if it satisfies the following conditions:

(i) $V=\bigcup_{n\geq 0}V_n$ and for $n\geq 1$, $V_n=\bigcup_{j\geq n}V_{nj}$, where $V_{nj}\cap V_{nj'}=\emptyset$, $j\neq j'$, and $2\leq |V_{nj}|<\infty$. The set V_{nn} is a union of two disjoint sets $V_{nn}(0)$ and $V_{nn}(1)$ with $|V_{nn}(0)|\geq 2$.

(ii) $E=\bigcup_{n\geq 1}E_n$ where each E_n is a union of disjoint finite subsets E_{nj} for $j\geq n$, such that for $j>n$ and $e\in E_{nj}$, one has $s(e)\in V_{n-1,j}$, $r(e)\in V_{nj}$ and $s(E_{nj})=V_{n-1,j}$. The set E_{nn} is a union of two disjoint subsets $E_{nn}(0)$ and $E_{nn}(1)$ such that $s(e)\in V_{n-1,n-1}$, $r(e)\in V_{nn}(0)$ if $e\in E_{nn}(0)$ and $s(e)\in V_{n-1,n}$, $r(e)\in V_{nn}(1)$ if $e\in E_{nn}(1)$. Moreover, $|r^{-1}(v)|=1$ if v is either in V_{nj}, $j>n$, or $v\in V_{nn}(1)$. If $v\in V_{nn}(0)$, then $|r^{-1}(v)|\geq 4$. The edges $e_1<\dots<e_m$ from $|r^{-1}(v)|$, $v\in V_{nn}(0)$ are ordered such that $s(e_1)\in V_{n-1,n}$, $s(e_i)\in V_{n-1,n-1}$, $i=2,\dots,m-1$ and $s(e_m)$ is either in $V_{n-1,n}(1)$ or in $V_{n-1,j}$, $j>n-1$. By definition, $s(E_{nj})=V_{n-1,j}$.

Figure 1 is an example of a special Borel–Bratteli diagram which illustrates the above definition (we do not indicate a partial order on edges of this diagram). It follows from the definition (see also Figure 1) that the set of infinite paths does not have cofinal minimal and maximal paths.

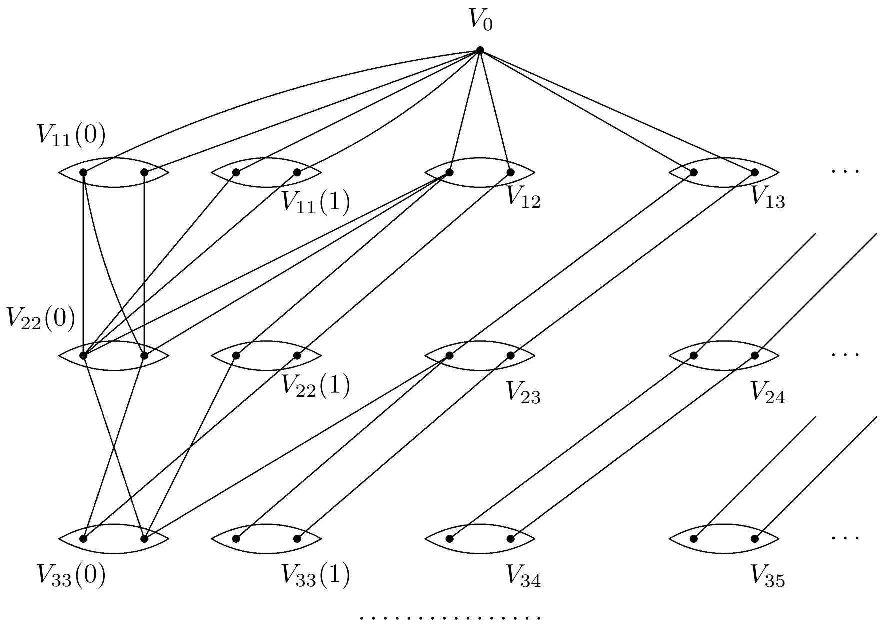

Figure 1

In the proof of Theorem 5.3 we used the fact that a given Borel dynamical system (X, T) together with a vanishing sequence of markers can be topologized by a topology ω (without changing the Borel structure) such that X becomes a 0-dimensional Polish space, all atoms of partitions defined by the sequence of markers are clopen, and T becomes a homeomorphism. The next theorem shows that if additionally the space X is *locally compact* in ω, then applying the construction given in Subsection 5.3 we get a special Borel–Bratteli diagram.

Theorem 5.9. *Let X be a locally compact 0-dimensional Polish space and let T be a homeomorphism of X. Then T is homeomorphic to a Vershik transformation acting on the space of infinite paths of a special Borel–Bratteli diagram.*

Proof. Let (A_n) be a vanishing sequence of markers for T and let ξ_n, $n \in \mathbb{N}$, be a partition of X into disjoint T-towers

$$\xi_n(v) = (T^j(A_n(v) : j = 0, \dots , h(n, v) - 1), \quad v \in V_n$$

constructed as in Subsection 5.2. Our assumptions imply that X can be taken to be a locally compact 0-dimensional Polish space, T is a homeomorphism, and each atom $T^j(A_n(v))$ of ξ_n, $n \in \mathbb{N}$ is a clopen set.

CLAIM 1. *Let $X = \bigcup_{i\geq 0} X_i$ be a partition into compact clopen disjoint sets X_i, then $B_n := \bigcup_{i\geq n} X_i$, $n = 0, 1, \dots,$ is a vanishing sequence of markers for T.*

PROOF. To see this, we note that A_n, $n \in N$, is clopen because the complement

$$A_n^c = X \setminus A_n = \bigcup_{v\in V_n} \bigcup_{j=1}^{h(n,v)-1} T^j(A_n(v))$$

is open. Then for any compact set Y, we have that $|\{n \mid A_n \cap Y \neq \emptyset\}| < \infty$, that is $A_n \subset Y^c$ for sufficiently large n (recall that the $A_n \supset A_{n+1}$ and $\bigcap_n A_n = \emptyset$). Given i, choose $n_i \in \mathbb{N}$ such that $(\bigcup_{j=1}^{i} X_j) \cap A_n = \emptyset$ for $n \geq n_i$. Then $A_{n_i} \subset X_{i+1} \cup X_{i+2} \cup \dots$, hence B_n is a complete T-section. Thus, (B_n) is a vanishing sequence of markers. It is easy to see that the partition ξ_n (we do not change our notation here), defined by B_n and the homeomorphism T, also consists of clopen sets for all n. The claim is proved.

CLAIM 2. *There exists a decomposition of X into clopen compact sets X_0, X_1, $X_2 \dots$ such that partitions ξ_n, $n \in \mathbb{N}$, constructed by the vanishing sequence of markers $A_n := X_n \cup X_{n+1} \cup \dots$, $n = 0, 1, \dots,$ have the following properties:*

$$h(n,v) = 1 \quad \text{if } A_n(v) \subset \bigcup_{i>n} X_i, \tag{5.6}$$

$$X_n = \bigcup_{v\in V_{nn}} A_n(v) \quad \text{where } V_{nn} \subset V_n,\ |V_{nn}| < \infty, \tag{5.7}$$

$$\bigcup_{i=1}^{n-1} X_i = \bigcup_{v\in V_{nn}(0)} \bigcup_{j\geq 1}^{h(n,v)-1} T^j(A_n(v)) \tag{5.8}$$

where $V_{nn}(0) = \{v \in V_{nn} \mid h(n,v) > 1\}$.

PROOF. Take a partition of X into compact clopen sets $(Y_0, Y_1, \dots)$. Denote $B_n := \bigcup_{i\geq n} Y_i$, $n \in \mathbb{N}$, and let $T^j(B_n(v))$, $j = 0, \dots, h(n,v) - 1$, $v \in V_n$, be elements of the partition η_n of X into clopen T-towers constructed by T and B_n. Without loss of generality, we can assume that the T-towers $(T^j(B_n(v)) : j = 0, \dots, h(n,v) - 1)$ are chosen such that

$$\lim_{n\to\infty} \Big[\sup_{v\in V_n} (\operatorname{diam} T^j(B_n(v))) \Big] = 0. \tag{5.9}$$

Consider these T-towers over B_n, $n \geq 1$. We can also assume that every Y_i is an η_n-set. Since $Y_0 \cup \dots \cup Y_{n-1}$ is compact, we can chose minimal $m = m(n) > n$ such that $h(n,v) = 1$ if $B_n(v) \subset Y_i$, $i > m(n)$. Let V_{nn} be a finite subset of V_n such that the disjoint T-towers $\eta_n(v)$, $v \in V_{nn}$, cover $Y_0 \cup \dots \cup Y_{m(n)}$. Some of those towers may be of height 1. Note that the set $Y_0 \cup \dots \cup Y_{n-1}$ is covered by a finite subset of T-towers $\eta_n(v)$ of heights greater than 1. Denote these towers by $V_{nn}(0)$, that is

$$Y_0 \cup \ldots \cup Y_{n-1} = \bigcup_{v \in V_{nn}(0)} \bigcup_{j=1}^{h(n,v)-1} T^j(B_n(v)).$$

Let $V_{nn}(1) = V_{nn} \setminus V_{nn}(0)$. Then $h(v, n) = 1$ when $v \in V_{nn}(1)$.

Define a sequence of clopen compact disjoint sets (X_n) by induction. Let $X_0 = Y_0$ and set

$$\begin{aligned} X_1 &= Y_1 \cup \ldots \cup Y_{m_1}, & m_1 &= m(1),\\ X_2 &= Y_{m_1+1} \cup \ldots \cup Y_{m_2}, & m_2 &= m(m_1),\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ X_{k+1} &= Y_{m_k+1} \cup \ldots \cup Y_{m_{k+1}}, & m_{k+1} &= m(m_k),\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

Let $A_n := X_n \cup X_{n+1} \cup \ldots$, $n \in \mathbb{N}$. Using (A_n) and T, construct refining partitions (ξ_n), satisfying (5.9). It follows from the construction that (ξ_n) is a telescoping of (η_n). Then X is partitioned by ξ_n into disjoint T-towers $\xi_n(v) = (T^j(A_n(v)) : j = 0, \ldots, h(n,v) - 1)$, $v \in V_n$, such that conditions (5.6)–(5.8) hold. The decomposition of $X = X_0 \cup X_1 \cup \ldots$ and towers $\xi_n(v)$ are shown in Figure 2. The claim is proved.

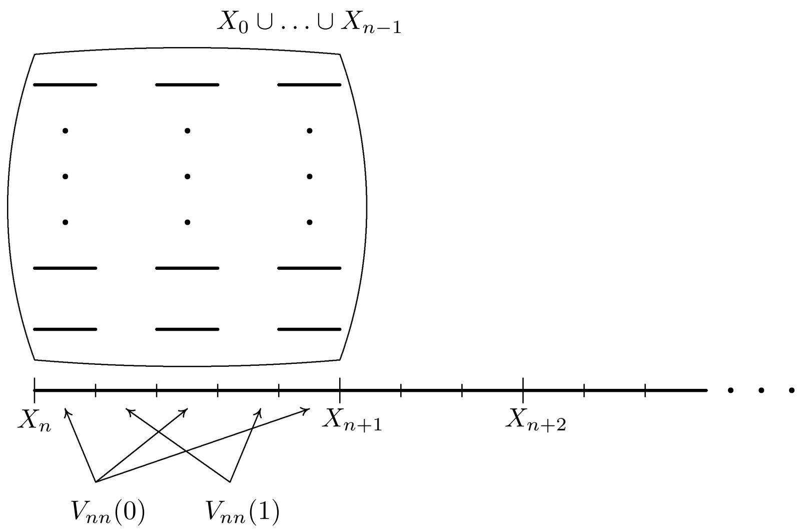

Figure 2

To complete the proof of the theorem, we need to show that the ordered Borel–Bratteli diagram $B = (V, E, \geq)$ constructed by the vanishing sequence of markers (A_n) defined in the proof of Claim 2 satisfies properties (i) and (ii) of the definition of a special Borel–Bratteli diagram.

Denote by $V_{nj} = \{v \in V_n \mid A_n(v) \subset X_j\}$, $j \geq n$, $n \in \mathbb{N}$. Let E_{nj} be the set of edges between V_{nj} and $V_{n-1,j}$. Then V_n is partitioned into non-empty finite sets $V_{nj},\ j \geq n$. Clearly, we can assume that $|V_{nj}| \geq 2$ for all n, j refining, in case of need, the partition ξ_n. It is obvious that for every $A_n(v)$, $v \in V_{nj}$, $j > n$, there exists a unique set $A_{n-1}(v')$, $v' \in V_{n-1,j}$ such that $A_n(v) \subset A_{n-1}(v')$. This means that the set $E(v', v)$ of edges connecting v' and v has exactly one element.

The set V_{nn} is divided into two sets $V_{nn}(0) = \{v \in V_{nn} \mid h(n,v) > 1\}$ and $V_{nn}(1) = \{v \in V_{nn} \mid h(n,v) = 1\}$, $n \in \mathbb{N}$. Then

$$X_0 \cup \ldots \cup X_{n-1} = \bigcup_{v \in V_{nn}(0)} \bigcup_{j=1}^{h(n,v)-1} T^j(A_n(v)), \tag{5.10}$$

$$X_n \cap T^{-1}(X_n \cup X_{n+1} \cup \ldots) = \bigcup_{v \in V_{nn}(1)} A_n(v). \tag{5.11}$$

Moreover,

$$\bigcup_{v \in V_{nn}(0)} A_n(v) = X_n \cap T^{-1}(X_0 \cup X_1 \cup \ldots \cup X_{n-1}) = X_n \cap T^{-1}(X_{n-1}). \tag{5.12}$$

Relations (5.10)–(5.12) show that $r^{-1}(v)$ consists of a single edge connecting $v \in V_{nn}(1)$ and a vertex $w \in V_{n-1,n}$. Without loss of generality, we can assume that $|V_{nn}(0)| \geq 2$. If $v \in V_{nn}(0)$, then the set $r^{-1}(v)$ has a unique edge $e(w,v)$ connecting v with some $w \in V_{n-1,n}$ and a number of edges which connect v with vertices from $V_{n-1,n-1}$. It follows from the construction of refining sequence (ξ_n) that $s(E_{nj}) = V_{n-1,j}$ for $j > n$. We get also from (5.12) that $s(E_{nn}(0)) = V_{n-1,n-1}$ where $E_{nn}(0)$ is the set of edges arriving to vertices from $V_{nn}(0)$. A non-trivial linear order on $r^{-1}(v)$ should be defined for $v \in V_{nn}(0)$. We assign the least value 1 to the edge $e(w,v)$. The maximal value of the defined order on $r^{-1}(v)$ is assigned to an edge e such that $s(e)$ is either in $V_{n-1,n}$ or in $V_{n-1,n-1}(1)$. It is obvious that the set Y_B of infinite paths is uncountable and has no cofinal maximal and minimal points. The fact that T is isomorphic to the Vershik automorphism acting on Y_B is proved as in Theorem 5.3 and Proposition 5.5.□

Acknowledgements. We are grateful to many people for discussing our results. We would like to express our special thanks to A. Kechris, G. Hjorth, and B. Miller for numerous talks which helped us to improve the first version of the paper. The first-named author thanks the University of New South Wales for the warm hospitality and the Australian Research Council for its support.

S. B. and J. K. acknowledge also the support of the Torun University and the Kharkov Institute for Low Temperature Physics during exchange visits.

References

[1] S. Alpern and V. S. Prasad, *Typical Dynamics of Volume Preserving Homeomorphisms*, Cambridge University Press, 2000.

[2] H. Becker and A. Kechris, *The Descriptive Set Theory of Polish Group Actions*, London Math. Soc., Lecture Notes Series, vol. 232, Cambridge University Press, 1996, pp. xii + 136.

[3] S. Bezuglyi, A. H. Dooley and J. Kwiatkowski, *Topologies on the group of homeomorphisms of a Cantor set*, Topol. Methods Nonlinear Anal. **27** (2006), 299–331.

[4] S. Bezuglyi and J. Kwiatkowski, *The topological full group of a Cantor minimal system is dense in the full group*, Topol. Methods in Nonlinear Anal. **16** (2000), 371–397.

[5] ______, *Topologies on full groups and normalizers of Cantor minimal systems*, Math. Physics, Analysis and Geometry **9** (2002), no. 3, 1–10.

[6] S. Bezuglyi and K. Medynets, *Smooth automorphisms and path-connectedness in Borel dynamics*, Indag. Math. **15** (2004), 453–468.

[7] P. Billingsley, *Convergence of Probability Measures*, John Willey & Sons, 1969.

[8] I. Cornfeld, S. Fomin and Ya. Sinai, *Ergodic Theory, Grundlehren der mathematischen Wissenschaften*, vol. 245, Springer–Verlag, 1982.

[9] A. H. Dooley and T. Hamachi, *Markov odometer actions not of product type*, Ergodic Theory Dynam. Systems **23** (2003), 813–82.

[10] ______, *Nonsingular dynamical systems, Bratteli diagrams and Markov odometers*, Israel J. Math. **138** (2003), 93–123.

[11] R. Dougherty, S. Jackson and A. Kechris, *The structure of hyperfinite Borel equivalence relations*, Trans. Amer. Math. Soc. **341** (1994), 193–225.

[12] M. Foreman, A. Kechris, A. Louveau and B. Weiss, London Math. Soc. Lect. Note Series, vol. 277, Cambridge University Press, 2000.

[13] A. Forrest, *K-groups associated with substitution minimal systems*, Israel J. Math. **98** (1997), 101–139.

[14] T. Giordano, I. Putnam and C. Skau, *Topological orbit equivalence and C^*-crossed products*, J. Reine Angew. Math. **469** (1995), 51–111.

[15] ______, *Full groups of Cantor minimal systems*, Israel J. Math. **111** (1999), 285–320.

[16] E. Glasner and B. Weiss, *The topological Rohlin property and topological entropy*, Amer. J. Math. **123** (2001), 1055–1070.

[17] *Lectures on Ergodic Theory*, The Mathematical Society of Japan, Publications of the Mathematical Society of Japan, 1956, pp. vii+99.

[18] P. Halmos, *Approximation theories for measure preserving transformations*, Trans. Amer. Math. Soc. **55** (1944), 1–18.

[19] R. H. Herman, I. Putnam and C. Skau, *Ordered Bratteli diagrams, dimension groups, and topological dynamics*, Intern. J. Math. **3** (1992), 827–864.

[20] G. Hjorth, *Classification and Orbit Equivalence Relations*, Math. Surveys and Monographs, vol. 75, Amer. Math. Soc., 2000.

[21] A. Kechris, *Classical Descriptive Set Theory*, Springer, 1995.

[22] ______, *Countable Borel equivalence relations*, J. Math. Logic **2** (2002), 1–80.

[23] B. Miller, private communication.

[24] M. Nadkarni, *Basic Ergodic Theory*, 2nd Edition, Birkhäuser, 1998.

[25] J. Oxtoby, *Measure and category.*, A survey of the analogies between topological and measure spaces. Second edition, Graduate Texts in Mathematics, 2, Springer–Verlag, New York–Berlin, 1980, pp. x+106.

[26] J. Oxtoby, S. M. Ulam, *Measure-preserving homeomorphisms and metrical transitivity*, Ann. of Math. **42** (1941), 874–920.

[27] V. A. Rokhlin, *Selected topics from the metric theory of dynamical systems*, Uspekhi Matem. Nauk (N.S.) **4** (1949), no. 2, 57–128. (Russian)

[28] R. M. Shortt, *Normal subgroups of measurable automorphisms*, Fund. Math. **135** (1990), 177–187.

[29] B. Weiss, *Measurable dynamics*, Contemp. Math. **26** (1984), 395–421.

Manuscript received November 10, 2004

Sergey Bezuglyi
Institute for Low Temperature Physics
Kharkov, UKRAINE

E-mail address: bezglui@ilt.kharkov.ua

Anthony H. Dooley
University of New South Wales
Sydney, AUSTRALIA

E-mail address: tony@maths.unsw.edu.au

Jan Kwiatkowski
College of Economics and Computer Science
Olsztyn, POLAND

E-mail address: jkwiat@mat.uni.torun.pl