Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 27, 2006, 289–298

ON A MULTIPLICITY RESULT OF J. R. WARD FOR SUPERLINEAR PLANAR SYSTEMS

Cristian Bereanu

Abstract. The purpose of this paper is to prove, under some assumptions on g, that the boundary value problem

$$u' = -g(t,u,v)v, \quad v' = g(t,u,v)u,$$
$$u(0) = 0 = u(\pi),$$

has infinitely many solutions. To prove our first main result we use a theorem of J. R. Ward and to prove the second one we use Capietto–Mawhin–Zanolin continuation theorem.

1. Introduction

Consider the following boundary value problem

$$u' = -g(t,u,v)v, \quad v' = g(t,u,v)u, \tag{1.1}$$

$$u(0) = 0 = u(\pi), \tag{1.2}$$

where g is a continuous function on $[0,\pi]\times\mathbb{R}^2$. Assume

$$g(t,u,v) \to \infty \quad \text{as } |u|+|v| \to \infty \text{ uniformly with } t \in [0,\pi], \tag{1.3}$$

$$g(t,0,0) = 0 \quad \text{for all } t \in [0,\pi], \tag{1.4}$$

$$g(t,u,v) \geq 0 \quad \text{for all } (t,u,v) \in [0,\pi]\times\mathbb{R}^2. \tag{1.5}$$

2000 *Mathematics Subject Classification.* 34B15, 34C28.
Key words and phrases. Superlinear systems, topologically distinct solutions, continuation theorem.

Under these assumptions J. R. Ward [10], among other results, proves, using essentially Rabinowitz global bifurcation theorem (see e.g. [3], [6], [8]) and the number of rotations associated to the bifurcations branches furnished by it, that boundary value problem (1.1), (1.2) has infinitely many solutions. Using the same method as in [10], we prove in Section 2 that (1.3), (1.4) are sufficient for (1.1), (1.2) to have infinitely many solutions. Remark that (1.4) is essential in the method used in [10].

Now, suppose that

$$g(0,-u,v) = g(0,u,v) \quad \text{for all } (u,v) \in \mathbb{R}^2. \tag{1.6}$$

If conditions (1.3), (1.6) hold and if the function $g(0,\cdot\,)$ is locally Lipschitzian on $\mathbb{R}^2$, we prove in Section 3 that boundary value problem (1.1), (1.2) has infinitely many solutions. We use Capietto–Mawhin–Zanolin continuation theorem [2] (see also [1], [4], [7]).

2. A first main result

Theorem 2.1. *If* $g\colon [0,\pi]\times\mathbb{R}^2 \to \mathbb{R}$ *is a continuous function satisfying* (1.3) *and* (1.4)*, then* (1.1)*,* (1.2) *has infinitely many topologically distinct solutions. Indeed, for each* $k \in \mathbb{N}$ *there is a solution* $w_k = (u_k, v_k)$ *such that the odd/even* 2π*-periodic extension* $\widetilde{w}_k$ *of* w_k *has rotation number* k*.*

Proof. Let $X = \{w = (u,v) \in C([0,\pi],\mathbb{R}^2) : u(0) = 0 = u(\pi)\}$ be a linear space equipped whit the norm $||w|| = \max_{t\in[0,\pi]} |w(t)|$ where, if $w = (u,v) \in \mathbb{R}^2$, then $|w|^2 = u^2 + v^2$. As in [10], we associate to (1.1), (1.2) the following family of boundary value problems

$$u' = -\mu v - g(t,u,v)v, \quad v' = \mu u + g(t,u,v)u, \tag{2.1}$$

$$u(0) = 0 = u(\pi). \tag{2.2}$$

Let $\mathcal{S}$ be the closure in $\mathbb{R}\times X$ of the set of all nontrivial solutions (μ, w) of (2.1), (2.2). For each $k \in \mathbb{N}$ let $C_k \subset \mathbb{R}\times X$ denote the component of $\mathcal{S}$ which meets $(k,0)$. Using [10, Theorem 3] (we can apply this theorem because g satisfies (1.4)) we have that C_k is unbounded in $\mathbb{R}\times X$ for each $k\in\mathbb{N}$. Consider $(\mu,w) \in C_k$, $w \neq 0$. Let $\widetilde{w}$ be the odd/even 2π-periodic extension of w, and let $\widetilde{g}$ be the extension of g on $[-\pi,\pi]\times\mathbb{R}^2$ defined by $\widetilde{g}(t,u,v) = g(-t,-u,v)$ for all $(t,u,v) \in [-\pi,0[\times\mathbb{R}^2$. Then, if $t \in [-\pi,\pi]\setminus\{0\}$ we have that

$$\widetilde{u}'(t) = -\mu\widetilde{v}(t) - \widetilde{g}(t,\widetilde{u}(t),\widetilde{v}(t))\widetilde{v}(t), \quad \widetilde{v}'(t) = \mu\widetilde{u}(t) + \widetilde{g}(t,\widetilde{u}(t),\widetilde{v}(t))\widetilde{u}(t).$$

This implies that

$$\frac{d}{dt}|\widetilde{w}(t)|^2 = 0 \quad \text{for all } t\in[-\pi,\pi]\setminus\{0\},$$

from where we deduce that $\widetilde{u}^2+\widetilde{v}^2$ is constant on $[-\pi,\pi]$. Then $t\rightarrowtail \widetilde{w}(t)/|\widetilde{w}(t)|$ may be considered as a map of the circle S^1 into itself, denoted as in [10] by $\varphi(\mu,w)$. Let $\mathrm{rot}(\varphi(\mu,w))$ be the rotating number (Brouwer degree) of $\varphi(\mu,w)$. Using Kronecker formula [9], we have that

$$\mathrm{rot}(\varphi(\mu,w))=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\widetilde{v}'\widetilde{u}-\widetilde{u}'\widetilde{v}}{\widetilde{u}^2+\widetilde{v}^2}\,dt=\mu+\frac{1}{2\pi}\int_{-\pi}^{\pi}\widetilde{g}(t,\widetilde{u},\widetilde{v})\,dt. \tag{2.3}$$

Let $k\in\mathbb{N}$, then from [10, Theorem 3] we know that $\mathrm{rot}(\varphi(\mu,w))=k$ for all $(\mu,w)\in C_k$, $w\neq 0$. We have three possible situations:

(I) the projection of C_k onto $\mathbb{R}$ is unbounded from below,
(II) the projection of C_k onto $\mathbb{R}$ is bounded,
(III) the projection of C_k onto $\mathbb{R}$ is unbounded from above.

We show that situations (II) and (III) don't hold. Using (1.3), we deduce that there exists $c\in\mathbb{R}$ such that

$$g(t,u,v)\geq c\quad\text{for all } (t,u,v)\in[0,\pi]\times\mathbb{R}^2. \tag{2.4}$$

Suppose that (II) holds. Then, because C_k is unbounded in $\mathbb{R}\times X$, there is a sequence $(\mu_n,w_n)_n$ in C_k such that $(\mu_n)_n$ is bounded in $\mathbb{R}$ and $||w_n||\to\infty$. As we have already seen, we have for all $n\in\mathbb{N}$ that

$$||w_n||^2=\widetilde{u}_n^2+\widetilde{v}_n^2\quad\text{for all } t\in[-\pi,\pi]$$

so that $|\widetilde{u}_n|+|\widetilde{v}_n|\to\infty$ uniformly in $t\in[-\pi,\pi]$. Using (1.3) we deduce that $\widetilde{g}(t,\widetilde{u}_n(t),\widetilde{v}_n(t))\to\infty$ uniformly in $t\in[-\pi,\pi]$. From this, the fact that the sequence $(\mu_n)_n$ is bounded and (2.3), we have that $\mathrm{rot}(\varphi(\mu_n,w_n))\to\infty$, a contradiction with $\mathrm{rot}(\varphi(\mu_n,w_n))=k$ for all $n\in\mathbb{N}$.

Suppose that (III) holds. Then, there is a sequence $(\mu_n,w_n)_n$ in C_k such that $\mu_n\to\infty$. Using (2.3) and (2.4) it follows that $\mathrm{rot}(\varphi(\mu_n,w_n))\to\infty$, which is again a contradiction with $\mathrm{rot}(\varphi(\mu_n,w_n))=k$ for all $n\in\mathbb{N}$.

Consequently we can only have situation (I), so, from the connectedness of C_k and $(k,0)\in C_k$ it follows that there exists $w_k\in X$ such that $(0,w_k)\in C_k$, so $w_k\neq 0$ is a solution of (1.1), (1.2). On the other hand, because $\mathrm{rot}(\varphi(0,w_k))=k$ for all $k\in\mathbb{N}$, we deduce that $w_k\neq w_j$ if $k\neq j$. □

3. A second main result

In this section $g\colon[0,\pi]\times\mathbb{R}^2\to\mathbb{R}$ is a continuous function satisfying (1.3) and (1.6). Moreover, we suppose that the function $g(0,\,\cdot\,)$ is locally Lipschitzian on $\mathbb{R}^2$. Our second main result is the following one.

Theorem 3.1. *If g is as above, then* (1.1), (1.2) *has infinitely many topologically distinct solutions.*

To prove the theorem above, we use Capietto–Mawhin–Zanolin continuation theorem. So, we need to make some preparations. Let X be the linear space of continuous functions $w = (u, v)$ on $[0, \pi]$ with values in $\mathbb{R}^2$ equipped with the usual norm $||w|| = \max_{t\in[0,\pi]} |w(t)|$. Consider the homotopy $\mathcal{G}\colon [0, 1] \times X \to X$ defined by $\mathcal{G}(\lambda, (u, v)) = (x, y)$, where

$$x(t) = -\int_0^t g(\lambda s, u, v)v\,ds, \qquad y(t) = v(0) - u(\pi) + \int_0^t g(\lambda s, u, v)u\,ds,$$

for all $t \in [0, \pi]$.

Lemma 3.2. *The homotopy $\mathcal{G}$ is completely continuous on* $[0, 1] \times X$.

Proof. Let $(\lambda_n, w_n)_n \subset [0, 1] \times X$ such that $\lambda_n \to \lambda_0, w_n \to w_0$. Then, if $t \in [0, \pi]$, we have

$$\begin{aligned}\bigg|\int_0^t g(\lambda_n s, u_n, v_n)v_n\,ds - \int_0^t g(\lambda_0 s, u_0, v_0)v_0\,ds\bigg| \\ \le \int_0^\pi |g(\lambda_n s, u_n, v_n)v_n - g(\lambda_0 s, u_0, v_0)v_0|\,ds =: \gamma_n, \quad (n \in \mathbb{N}).\end{aligned}$$

Using Lebesgue's dominated convergence theorem, we deduce that $\gamma_n \to 0$. Now, the continuity of $\mathcal{G}$ follows obviously. Let $(\lambda_n, w_n)_n$ be a bounded sequence in $[0, 1] \times X$. Passing if necessarily to a subsequence, we can assume that $\lambda_n \to \lambda_0$. For $n \in \mathbb{N}$, define the continuous function x_n by

$$x_n(t) = \int_0^t g(\lambda_n s, u_n, v_n)v_n\,ds, \quad (t \in [0, \pi]).$$

Let $M > 0$ such that $||w_n|| \le M$ for all $n \in \mathbb{N}$ and $M' = sup\{|g(t, u, v)v| : (t, u, v) \in [0, \pi] \times [-M, M]^2\}$. Because

$$|x_n(t)| \le \int_0^\pi |g(\lambda_n s, u_n, v_n)v_n|\,ds, \quad (t \in [0, \pi]),$$

we deduce that $\max_{t\in[0,\pi]} |x_n(t)| \le \pi M'$ for all $n \in \mathbb{N}$. Now, consider $t, t' \in [0, \pi]$ and $n \in \mathbb{N}$. We have

$$|x_n(t) - x_n(t')| \le \bigg|\int_t^{t'} |g(\lambda_n s, u_n, v_n)v_n|\,ds\bigg| \le M'|t - t'|.$$

It follows that the sequence $(x_n)_n$ is equicontinous. So, we can apply Arzela-Ascoli theorem to deduce that $(x_n)_n$ has a convergence subsequence in $C([0, \pi])$. Now, the compactness of $\mathcal{G}$ follows obviously. □

Consider the family of boundary value problems

$$u' = -g(\lambda t, u, v)v, \quad v' = g(\lambda t, u, v)u, \tag{3.1}$$

$$u(0) = 0 = u(\pi). \tag{3.2}$$

Lemma 3.3. *If* $(\lambda, w) \in [0,1] \times X$*, then* $\mathcal{G}(\lambda, w) = w$ *if and only if* w *is a solution of* (3.1)*,* (3.2)*.*

Proof. Suppose that $\mathcal{G}(\lambda, w) = w$. Then, it is clear that we have (3.1). On the other hand, it follows that $u(0) = 0$ and $v(0) = v(0) - u(\pi)$, so $u(\pi) = 0$. Conversely, suppose that w is a solution of (3.1), (3.2). Integrating on $[0,t]$ the equations in (3.1) and using the boundary condition (3.2), it follows that $\mathcal{G}(\lambda, w) = w$. □

Let $\widetilde{g}\colon [-\pi,\pi]\times\mathbb{R}^2 \to \mathbb{R}$ be a extension of g defined by $\widetilde{g}(t,u,v) = g(-t,-u,v)$ for all $(t,u,v) \in [-\pi, 0[\times\mathbb{R}^2$. Using (1.6) it follows that $\widetilde{g}$ is continuous. On the other hand, if $(u,v) \neq (0,0)$ is a solution of (3.1), (3.2), we define the $2\pi-$periodic odd/even continuous extension of (u,v) by $\widetilde{u}(t) = -u(-t)$, $\widetilde{v}(t) = v(-t)$ for all $t \in [-\pi, 0[$.

Lemma 3.4. *If* $(u,v) \neq (0,0)$ *is a solution of* (3.1)*,* (3.2)*, then* $(\widetilde{u},\widetilde{v}) \in C^1([-\pi,\pi],\mathbb{R}^2)$ *and* $\widetilde{u}' = -\widetilde{g}(\lambda t, \widetilde{u}, \widetilde{v})\widetilde{v}$*,* $\widetilde{v}' = \widetilde{g}(\lambda t, \widetilde{u}, \widetilde{v})\widetilde{u}$*.*

Proof. Let $t \in]0,\pi]$, then $\widetilde{u}$ is differentiable in t and

$$\widetilde{u}'(t) = u'(t) = -g(\lambda t, u(t), v(t))v(t) = -\widetilde{g}(\lambda t, \widetilde{u}(t), \widetilde{v}(t))\widetilde{v}(t).$$

Analogously, $\widetilde{v}$ is differentiable in t and

$$\widetilde{u}'(t) = \widetilde{g}(\lambda t, \widetilde{u}(t), \widetilde{v}(t))\widetilde{u}(t).$$

Now, consider $t \in [-\pi, 0[$. Then, $\widetilde{u}$ is differentiable in t and

$$\begin{aligned}\widetilde{u}'(t) = u'(-t) &= -g(-\lambda t, u(-t), v(-t))v(-t) \\ &= -g(-\lambda t, -\widetilde{u}(t), \widetilde{v}(t))\widetilde{v}(t) = -\widetilde{g}(\lambda t, \widetilde{u}(t), \widetilde{v}(t))\widetilde{v}(t).\end{aligned}$$

Note that the last equality follows by (1.6). On the other hand, $\widetilde{v}$ is differentiable in t and

$$\begin{aligned}\widetilde{v}'(t) &= -v'(-t) = -g(-\lambda t, u(-t), v(-t))u(-t) \\ &= -g(-\lambda t, -\widetilde{u}(t), \widetilde{v}(t))(-\widetilde{u}(t)) = \widetilde{g}(\lambda t, \widetilde{u}(t), \widetilde{v}(t))\widetilde{u}(t).\end{aligned}$$

We have

$$\begin{aligned}\lim_{t\searrow 0} \frac{\widetilde{u}(t) - \widetilde{u}(0)}{t} &= \lim_{t\searrow 0} \frac{u(t)}{t} = u'(0) = -g(0,0,v(0))v(0), \\ \lim_{t\nearrow 0} \frac{\widetilde{u}(t) - \widetilde{u}(0)}{t} &= \lim_{t\nearrow 0} \frac{-u(-t)}{t} = -g(0,0,v(0))v(0).\end{aligned}$$

It follows that $\widetilde{u}$ is differentiable in 0 and $\widetilde{u}'(0) = -g(0,0,v(0))v(0)$. On the other hand we have

$$\lim_{t\searrow 0} \frac{\widetilde{v}(t)-\widetilde{v}(0)}{t} = v'(0) = g(0,0,v(0))u(0) = 0, \quad \lim_{t\nearrow 0} \frac{\widetilde{v}(t)-\widetilde{v}(0)}{t} = -v'(0) = 0.$$

So, $\widetilde{v}$ is differentiable in 0 and $\widetilde{v}'(0) = 0$. Finally, $\widetilde{u}, \widetilde{v}$ are C^1 because of the continuity of $\widetilde{g}$. □

Let $w = (u,v)$ be a non-trivial solution of (3.1), (3.2). Then, using Lemma 3.4 we deduce that $(\widetilde{u}^2(t) + \widetilde{v}^2(t))' = 0$ for all $t \in [-\pi,\pi]$. It follows that $|\widetilde{w}(t)|^2 = c$ for all $t \in [-\pi,\pi]$, where $c > 0$ is a constant. Now, $\widetilde{w}(-\pi) = \widetilde{w}(\pi)$, and $\widetilde{w}(t)/|\widetilde{w}(t)| \in S^1$ for all $t \in [-\pi,\pi]$. Identifying S^1 with $[-\pi,\pi]/\{-\pi,\pi\}$ we obtain a mapping $t \rightarrowtail \widetilde{w}(t)/|\widetilde{w}(t)|$ of S^1 into itself, which we denote by $\psi(\lambda,w)$. The rotation (Brouwer degree) is defined. Using again Kronecker formula and Lemma 3.4 have that

$$\deg(\psi(\lambda,w)) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\widetilde{v}'\widetilde{u} - \widetilde{u}'\widetilde{v}}{\widetilde{u}^2+\widetilde{v}^2}\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} \widetilde{g}(\lambda t, \widetilde{u}, \widetilde{v})\,dt. \tag{3.3}$$

Let $\delta\colon \mathbb{R}^2 \to \mathbb{R}$, $\delta(u,v) = \min\{1, (u^2+v^2)^{-1}\}$. If $(u,v) \in X$, we define as before $(\widetilde{u},\widetilde{v})$ to be the odd/even extension (not necessarily continuous) of (u,v). Consider $\varphi\colon [0,1] \times X \to \mathbb{R}_+$ defined by

$$\varphi(\lambda,(u,v)) = \frac{1}{2\pi}\bigg|\int_{-\pi}^{\pi} \widetilde{g}(\lambda t, \widetilde{u}, \widetilde{v})(\widetilde{u}^2+\widetilde{v}^2)\delta(\widetilde{u},\widetilde{v})\,dt\bigg|.$$

Lemma 3.5. *The function φ defined above is continuous.*

Proof. The proof follows easily using the continuity of $\widetilde{g}, \delta$ and Lebesgue's dominated convergence theorem. □

Lemma 3.6. *There exists $R > 1$ such that $\varphi(\lambda,w) \in \mathbb{N}$ for all $(\lambda,w) \in \Sigma$ with $||w|| \geq R$, where $\Sigma = \{(\lambda,w) \in X \colon \mathcal{G}(\lambda,w) = w\}$.*

Proof. From (1.3) we have that there exists $R > 1$ such that

$$\widetilde{g}(t,u,v) > 0 \quad \text{if} \quad |(u,v)| \geq R,\ t \in [-\pi,\pi]. \tag{3.4}$$

Let $(\lambda,w) \in \Sigma$ such that $||w|| \geq R$. Using Lemmas 3.3 and 3.4 we deduce that

$$\widetilde{u}^2(t) + \widetilde{u}^2(t) = ||w||^2 \geq R^2 > 1 \quad \text{for all } t \in [-\pi,\pi]. \tag{3.5}$$

The conclusion follows from relations (3.3)–(3.5) and the definition of φ. □

Lemma 3.7. *The set $\varphi^{-1}(n) \cap \Sigma$ is bounded for each $n \in \mathbb{N}$.*

Proof. Let $n \in \mathbb{N}$ and suppose that the set $\varphi^{-1}(n) \cap \Sigma$ is unbounded. There exists a sequence $(\lambda_k, w_k) \in \Sigma$ such that $\varphi(\lambda_k,w_k) = n$ for all $k \in \mathbb{N}$ and $||w_k|| \to \infty$. Using Lemmas 3.3 and 3.4 we deduce that $\widetilde{u}_k^2 + \widetilde{v}_k^2 = ||w_k||^2$ on $[-\pi,\pi]$, which implies that $|\widetilde{u}_k(t)| + |\widetilde{v}_k(t)| \to \infty$ uniformly with $t \in [-\pi,\pi]$.

So, using (1.3), (3.3) and the definition of φ we obtain that $\varphi(\lambda_k, w_k) \to \infty$. Contradiction. $\square$

Lemma 3.8. *If $u_0, v_0 \in \mathbb{R}$, then the initial boundary value problem*

$$u' = -g(0,u,v)v, \quad v' = g(0,u,v)u, \quad u(0) = u_0, \quad v(0) = v_0 \tag{3.6}$$

has a unique solution $(u(\cdot,(u_0,v_0)), v(\,\cdot\,,(u_0,v_0)))$ which is defined on $\mathbb{R}$.

Proof. Because the function $g(0,\,\cdot\,)$ is locally Lipschitzian on $\mathbb{R}^2$, it follows that (3.6) has a unique maximal solution $(u(\,\cdot\,,(u_0,v_0)), v(\,\cdot\,,(u_0,v_0)))\colon]a,b[\to \mathbb{R}^2$. We shall prove that $]a,b[= \mathbb{R}$. Remark that (3.6) implies

$$|(u(t,(u_0,v_0)), v(t,(u_0,v_0)))| = |(u_0,v_0)|$$

for all $t \in]a,b[$. Using again (3.6) and the continuity of g, it follows that the function $(u'(\,\cdot\,,(u_0,v_0)), v'(\,\cdot\,,(u_0,v_0)))$ is bounded on $]a,b[$, which implies that $(u(\,\cdot\,,(u_0,v_0)), v(\,\cdot\,,(u_0,v_0)))$ has a continuous extension on $]a,b]$, if b is finite.

Consider (u_b, v_b) the solution of (3.6) with the initial data $(u(b,(u_0,v_0)), v(b,(u_0,v_0)))$. Let $\varepsilon > 0$ sufficiently small and define $(u,v)\colon]a, b+\varepsilon] \to \mathbb{R}^2$ by

$$(u,v) = (u(\,\cdot\,,(u_0,v_0)), v(\,\cdot\,,(u_0,v_0)))$$

on $]a,b[$, and $(u,v) = (u_b, v_b)$ on $[b, b+\varepsilon]$. It is clear that (u,v) verifies (3.6), contradiction with maximality of $(u(\,\cdot\,,(u_0,v_0)), v(\,\cdot\,,(u_0,v_0)))$. Analogously, it follows that $a = -\infty$, so $b = \infty$. $\square$

Using Lemma 3.8 we can consider the continuous function $\mathcal{U}\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\mathcal{U}(z_1,z_2) = (2z_1, z_2 + u(\pi,(z_1,z_2)))$. It is obvious that if (u,v) is a solution of (3.1), (3.2) with $\lambda = 0$, then $(0, v(0))$ is a fixed point of $\mathcal{U}$, and if (z_1,z_2) is a fixed point of $\mathcal{U}$, then $z_1 = 0$ and $(u(\,\cdot\,,(0,z_2)), v(\,\cdot\,,(0,z_2)))$ is a solution of (3.1), (3.2) with $\lambda = 0$. If $\alpha > 0$, define

$$\Omega_\alpha = \{w \in X\colon ||w|| < \alpha\}, \quad G_\alpha = \{\xi \in \mathbb{R}^2\colon |\xi| < \alpha\}.$$

Suppose that α is chosen so that there is no solution (u,v) of (3.1), (3.2) with $\lambda = 0$ such that $|v(0)| = \alpha$. The open sets Ω_α, G_α have the following properties: there are no initial values of solutions to (3.1), (3.2) with $\lambda = 0$ on ∂G_α and no solution on $\partial\Omega_\alpha$; the set of initial values in G_α of solutions to (3.1), (3.2) with $\lambda = 0$ equals the set of values at $t = 0$ of solutions in Ω_α to (3.1), (3.2) with $\lambda = 0$. If $G \subset \mathbb{R}^2, \Omega \subset X$ are two bounded open sets having the proprieties above, following Krasnosel'skii and Zabreĭko, we say that G, Ω have a common core. Following the same lines as in the proof of [5, Theorem 28.5] we have the following result.

Lemma 3.9. *If $G \subset \mathbb{R}^2, \Omega \subset X$ are two open bounded sets having a common core, then the degrees $d_B(I-\mathcal{U},G,0), d_{LS}(I-\mathcal{G}(0,\,\cdot\,),\Omega,0)$ are well defined and equal.*

In what follows we use the notation

$$(u(\,\cdot\,,\alpha),v(\,\cdot\,,\alpha)) \quad \text{for } (u(\,\cdot\,,(0,\alpha)),v(\,\cdot\,,(0,\alpha))).$$

If $R>1$ is the constant from Lemma 3.6 and $\alpha>R$, then, using (3.4) it follows that the range of $(u(\,\cdot\,,\alpha),v(\,\cdot\,,\alpha))$ is the circle of radius α. Let $\tau(\alpha)>0$ be such that $u(\tau(\alpha),\alpha)=0$ and $u(t,\alpha)\neq 0$ for all $t\in]o,\tau(\alpha)[$. On the other hand, from Lemma 3.8 and (1.6), we obtain that $u(\,\cdot\,,\alpha)$ is odd and $v(\,\cdot\,,\alpha)$ is even. So, we have that $(u(\,\cdot\,,\alpha),v(\,\cdot\,,\alpha))$ is a parametrization from $[-\tau(\alpha),\tau(\alpha)]$ to the circle of radius α.

Lemma 3.10. *$\tau(\alpha)\to 0$ as $\alpha\to\infty$.*

Proof. Because $(u(\,\cdot\,,\alpha),v(\,\cdot\,,\alpha))$ is a parametrization from $[-\tau(\alpha),\tau(\alpha)]$ to the circle of radius α, we have that

$$\int_{-\tau(\alpha)}^{\tau(\alpha)} (u'^2(t,\alpha)+v'^2(t,\alpha))^{1/2}\,dt = 2\pi\alpha$$

which implies that

$$\tau(\alpha)\inf\{(u'^2(t,\alpha)+v'^2(t,\alpha))^{1/2} : t\in[-\tau(\alpha),\tau(\alpha)]\}\le \pi\alpha. \tag{3.7}$$

On the other hand

$$u'^2(\,\cdot\,,\alpha)+v'^2(\,\cdot\,,\alpha)=\alpha^2[g(0,u(\,\cdot\,,\alpha),v(\,\cdot\,,\alpha))]^2. \tag{3.8}$$

Using (1.3) and (3.8) it follows that (3.7) holds only if $\tau(\alpha)\to 0$ as $\alpha\to\infty$. □

Consider the set $\mathcal{S}=\{\pi/n\}_n$. If $\alpha>R$, then it follows that

$$(u(\,\cdot\,+\tau(\alpha),\alpha),v(\,\cdot\,+\tau(\alpha),\alpha))=(u(\,\cdot\,,-\alpha),v(\,\cdot\,,-\alpha)),$$

which implies that if $|\alpha|>R$ then $(u(\,\cdot\,,\alpha),v(\,\cdot\,,\alpha))$ is a solution of (3.1), (3.2) with $\lambda=0$ if and only if $\tau(|\alpha|)\in\mathcal{S}$. So, if we consider the continuous function $\phi:\mathbb{R}\to\mathbb{R}, \phi(t)=u(\pi,t)$ then, if $\alpha>R$ such that $\tau(\alpha)\notin\mathcal{S}$, it follows that the degrees $d_B(I-\mathcal{U},G_\alpha,0), d_B(\phi,]-\alpha,\alpha[,0)$ are well defined. Moreover, we have the following result.

Lemma 3.11. *Let $\alpha>R$ such that $\tau(\alpha)\in\,]\pi/(n+1),\pi/n[$ for some $n\in\mathbb{N}$. Then*

$$d_B(I-\mathcal{U},G_\alpha,0)=d_B(\phi,]-\alpha,\alpha[,0)=(-1)^{n+1}.$$

Proof. Because $\mathcal{U}$ acts in $\mathbb{R}^2$, we have

$$d_B(I-\mathcal{U},G_\alpha,0)=d_B(\mathcal{U}-I,G_\alpha,0). \tag{3.9}$$

If we denote the rectangle $[-\alpha,\alpha]\times[-\alpha,\alpha]$ by $\mathcal{R}$ then, using the excision property of Brouwer degree it follows that

$$d_B(\mathcal{U}-I, G_\alpha, 0) = d_B(\mathcal{U}-I,\mathcal{R},0). \tag{3.10}$$

Now, consider the homotopy

$$h\colon [0,1]\times\overline{\mathcal{R}}\to\mathbb{R}^2,\quad h(\lambda,(z_1,z_2)) = (z_1, u(\pi,(\lambda z_1, z_2))).$$

Remark that $h(1,\cdot\,) = \mathcal{U}-I$ and $h(0,\cdot\,) = I_{\mathbb{R}}\times\phi$. Moreover, $h(\lambda,(z_1,z_2)) = 0$ if and only if $h(1,(z_1,z_2)) = 0$. It follows that $h(\lambda,(z_1,z_2))\neq 0$ for all $\lambda\in[0,1]$ and $(z_1,z_2)\in\partial\mathcal{R}$. So, we can apply the invariance by homotopy property, hence

$$d_B(\mathcal{U}-I,\mathcal{R},0) = d_B(I_{\mathbb{R}}\times\phi,\mathcal{R},0) = d_B(I_{\mathbb{R}},]-\alpha,\alpha[,0)d_B(\phi,]-\alpha,\alpha[,0). \tag{3.11}$$

Finally, because ϕ is odd it follows that $d_B(\phi,]-\alpha,\alpha[,0) = \operatorname{sgn}(\phi(\alpha))$, and so, using (3.9)–(3.11) and the definition of $\tau(\alpha)$ the conclusion of lemma follows. □

Denote, for any subset $A\subset[0,1]\times X$, the section of A at $\lambda\in[0,1]$, by $A_\lambda = \{x\in X\colon(\lambda,x)\in A\}$ Let $R>1$ be the constant from Lemma 3.6 and let k_0 be an integer such that

$$k_0 > \sup\{\varphi(\lambda,w) : (\lambda,w)\in\Sigma, ||w||\leq R\}$$

and, using Lemma 3.7, consider, for any integer $j>k_0$, the topological degree $d_{LS}(I-\mathcal{G}(0,\cdot\,),\Gamma_j,0)$, where $\Gamma_j\supset(\varphi^{-1}(j)\cap\Sigma)_0$ is an open bounded subset of X for which the Leray–Schauder degree d_{LS} is defined and such that $\Gamma_j\cap\Sigma_0 = (\varphi^{-1}(j)\cap\Sigma)_0$.

Lemma 3.12. *There exists some integer $k>k_0$ such that*

$$d_{LS}(I-\mathcal{G}(0,\cdot\,),\Gamma_j,0)\neq 0$$

for all integers $j\geq k$.

Proof. Using the continuity of $\tau(\cdot\,)$ and Lemma 3.10 it follows that there exists some integer $k>k_0$ such that $\tau^{-1}(\pi/j)\neq\varnothing$ for all integers $j\geq k$. If $j\geq k$, let $\varepsilon>0$ such that $]\pi/j-\varepsilon,\pi/j+\varepsilon[\,\subset\,]\pi/(j+1),\pi/(j-1)[$ and

$$\Delta_j = \left\{(u(\cdot\,,\alpha),v(\cdot\,,\alpha))\colon|\alpha|>R,\ \tau(|\alpha|)\in\left]\frac{\pi}{j}-\varepsilon,\frac{\pi}{j}+\varepsilon\right[\right\}.$$

The sets Δ_j have the same properties as the sets Γ_j above. Consider

$$\alpha_j = \max\left\{\alpha>R\colon\tau(\alpha)=\frac{\pi}{j}+\varepsilon\right\},\quad \beta_j = \min\left\{\alpha>R\colon\tau(\alpha)=\frac{\pi}{j}-\varepsilon\right\}.$$

Using a continuity argument given in [4, Theorem 5.1], we have that

$$d_{LS}(I-\mathcal{G}(0,\cdot\,),\Delta_j,0) = d_{LS}(I-\mathcal{G}(0,\cdot\,),\Omega_{\beta_j}\setminus\overline{\Omega}_{\alpha_j},0).$$

But, using Lemmas 3.9, 3.11 and the additivity property of the Leray–Schauder degree, we deduce that

$$d_{LS}(I - \mathcal{G}(0, \cdot), \Omega_{\beta_j} \setminus \overline{\Omega}_{\alpha_j}, 0) = (-1)^{j+1} - (-1)^j \neq 0.$$

Now, the conclusion follows using the excision property of Leray–Schauder degree. □

Proof of Theorem 3.1. Using Lemmas 3.2, 3.5–3.7, 3.12 we can apply Capietto–Mawhin–Zanolin continuation theorem to deduce that for all $j \geq k$ there exists $w_j \in X$ such that $\varphi(1, w_j) = j$ and $\mathcal{G}(1, w_j) = w_j$. The conclusion follows using Lemma 3.3. □

Acknowledgements. I am extremely grateful to my advisor, Jean Mawhin, for proposing this subject to me, and for all his invaluable support.

References

[1] A. Capietto, M. Henrard, J. Mawhin and F. Zanolin, *A continuation approach to some forced superlinear Sturm–Liouville boundary value problems*, Topol. Methods Nonlinear Anal. **3** (1994), 81–100.

[2] A. Capietto, J. Mawhin and F. Zanolin, *Boundary value problems for forced superlinear second order ordinary differential equations*, Collége de France Seminar, vol. XII, Longman, New York, 1994, pp. 55–64.

[3] K. Gęba and P. Rabinowitz, *Topological Methods in Bifurcation Theory*, Sém. Math. Sup., vol. **91**, Presses Univ. Montréal, Montréal, 1985.

[4] M. Garcia-Huidobro, R. Manasevich and F. Zanolin, *Strong nonlinear second-order ODEs with rapidly growing terms*, J. Math. Anal. Appl. **202** (1996), 1–26.

[5] M. A. Krasnosel'skiĭ and P. P. Zabreĭko, *Geometrical Methods of Nonlinear Analysis*, Springer, Berlin, 1984.

[6] J. Mawhin, *Topological Degree Methods in Nonlinear Boundary Value Problems* (1979), Amer. Math. Soc., Providence RI.

[7] ———, *Leray–Schauder continuation theorems in the absence of a priori bounds*, Topol. Methods Nonlinear Anal. **9** (1997), 179–200.

[8] ———, *Leray–Schauder degree: A half century of extensions and applications*, Topol. Methods Nonlinear Anal. **14** (1999), 195–228.

[9] ———, *A simple approach to Brouwer degree based on differential forms*, Adv. Nonlinear Stud. **4** (2004), 535–548.

[10] J. R. Ward, *Rotation numbers and global bifurcation in systems of ordinary differential equations*, Adv. Nonlinear Stud. **5** (2005), 375–392.

Manuscript received November 10, 2005

Cristian Bereanu
Département de Mathématique
Université catholique de Louvain
B-1348 Louvain-la-Neuve, BELGIUM

E-mail address: bereanu@math.ucl.ac.be