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RESONANT NONLINEAR PERIODIC PROBLEMS WITH THE SCALAR p-LAPLACIAN AND A NONSMOOTH POTENTIAL

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Abstract. We study periodic problems driven by the scalar p-Laplacian with a nonsmooth potential. Using the nonsmooth critical point theory for locally Lipschitz functions, we prove two existence theorems under conditions of resonance at infinity with respect to the first two eigenvalues of the negative scalar p-Laplacian with periodic boundary conditions.

1. Introduction

The purpose of this paper is to study nonlinear scalar periodic problems driven by the p-Laplacian differential operator with a nonsmooth potential under resonance conditions. So the problem under consideration is the following:

$$(1.1)\qquad \begin{cases} -(|x'(t)|^{p-2}x'(t))' \in \partial j(t,x(t)) & \text{a.e. on } T := [0,b], \\ x(0) = x(b), \quad x'(0) = x'(b), & 1 < p < \infty. \end{cases}$$

Here $(t,x) \mapsto j(t,x)$ is a measurable potential function, which is in general nonsmooth and locally Lipschitz in the $x \in \mathbb{R}$ variable. By $\partial j(t,x)$ we denote the generalized subdifferential of the function $x \mapsto j(t,x)$.

We prove two existence results under conditions of resonance at infinity at $\lambda_0 = 0$, the first eigenvalue of the negative scalar p-Laplacian with periodic

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boundary conditions or at $\lambda_1 > 0$, the first strictly positive eigenvalue. In fact, in the second existence result, we have a double resonance situation in the spectral interval $[0, \lambda_1]$.

Related results for semilinear problems (i.e. $p = 2$) with smooth potential (i.e. $j(t, \cdot\,) \in C^1(\mathbb{R})$), were obtained in Ahmad–Lazer [1], Mawhin [8], Iannacci–Nkashama [7], Fonda–Lupo [4], Fabry and Fonda [3] and Gossez–Omari [6]. In all these works (with the exception of Fabry–Fonda [3]) the authors allow only partial interaction with the spectrum (nonuniform nonresonance) and employ additional conditions on the right hand side nonlinearity, such as monotonicity or sign conditions. Fabry–Fonda [3] deal with the doubly resonant situation which is treated with the help of certain Landesman–Lazer type conditions.

Our approach is variational, based on the nonsmooth critical point theory for locally Lipschitz functionals. In the next section, for convenience of the reader, we recall some basic definitions and facts from this theory. Details can be found in Gasinski–Papageorgiou [5].

2. Mathematical background

Let X be a Banach space. By X^* we denote its topological dual and by $\langle\,\cdot\,,\,\cdot\,\rangle$ the duality brackets of the pair (X, X^*). The norms in X or X^* will be denoted by $\|\cdot\|$. A function $\varphi\colon X \to \mathbb{R}$ is said to be locally Lipschitz, if for every $x \in X$ we can find an open set U containing x and a constant $k_U > 0$ (depending on U) such that

$$|\varphi(y) - \varphi(z)| \le k_U \|y - z\| \quad \text{for all } y, z \in U.$$

The function $\varphi^0\colon X \times X \to \mathbb{R}$ defined by

$$\varphi^0(x; h) = \limsup_{x' \to x, \lambda \downarrow 0} \frac{\varphi(x' + \lambda h) - \varphi(x')}{\lambda}$$

is called the *generalized directional derivative* of φ. It is easy to check that $\varphi^0(x; \cdot\,)$ is sublinear and continuous. So, it is the support function of a nonempty, convex and w^*–compact set, $\partial\varphi(x)$, defined by

$$\partial\varphi(x) = \{x^* \in X^*\colon \langle x^*, h\rangle \le \varphi^0(x; h) \text{ for all } h \in X\}, \quad \text{for all } x \in X.$$

The multifunction $x \mapsto \partial\varphi(x)$ is known as the *generalized subdifferential* of φ. If $\varphi \in C^1(X)$, then φ is locally Lipschitz and $\partial\varphi(x) = \{\varphi'(x)\}$, for all $x \in X$. If φ is continuous and convex, then φ is locally Lipschitz and the generalized subdifferential coincides with the subdifferential in the sense of convex analysis, i.e.

$$\partial\varphi(x) = \{x^* \in X^*\colon \langle x^*, y - x\rangle \le \varphi(y) - \varphi(x) \text{ for all } y \in X\}.$$

If φ, $\psi: X \to \mathbb{R}$ are two locally Lipschitz functions and $\lambda \in \mathbb{R}$, then

$$\partial(\varphi + \psi)(x) \subseteq \partial\varphi(x) + \partial\psi(x) \quad \text{and} \quad \partial\varphi(\lambda x) = \lambda\partial\varphi(x) \quad \text{for all } x \in X.$$

We say that $x \in X$ is a critical point of the locally Lipschitz function $\varphi: X \to \mathbb{R}$ if $0 \in \partial\varphi(x)$. The corresponding value $c = \varphi(x)$ is called a critical value of φ. It is easy to see that if x is a local extremum point of φ (i.e. a local minimum or a local maximum), then x is a critical point of φ.

It is well-known that in the smooth critical point theory, a compactness-type condition, known as the *Palais–Smale condition* (PS-*condition* for short), plays a central role. In the present nonsmooth setting, this condition takes the following form:

Definition 2.1. A locally Lipschitz function $\varphi: X \to \mathbb{R}$ satisfies the *nonsmooth* PS-*condition,* if every sequence $\{x_n\}_{n\geq 1} \subseteq X$ such that $\varphi(x_n) \to c$ (for some $c \in \mathbb{R}$) and $m(x_n) := \inf\{\|x^*\| : x^* \in \partial\varphi(x_n)\} \to 0$ as $n \to \infty$, has a strongly convergent subsequence.

Sometimes, it is convenient to use a weaker notion, known as the *nonsmooth Cerami condition* (*nonsmooth* C-*condition* for short), which has the following form:

Definition 2.2. A locally Lipschitz function $\varphi: X \to \mathbb{R}$ satisfies the *nonsmooth* C-*condition,* if every sequence $\{x_n\}_{n\geq 1} \subseteq X$ such that $\varphi(x_n) \to c$ (for some $c \in \mathbb{R}$) and $(1 + \|x_n\|)m(x_n) \to 0$ as $n \to \infty$, has a strongly convergent subsequence.

If φ is bounded below, then the two notions are equivalent.

Our analysis will use the following basic geometric notion:

Definition 2.3. Let X be a Hausdorff topological space and let D_0, D, V be nonempty closed subsets of X such that $D_0 \subseteq D$. We say that the sets (D_0, V) link in X through D, if $D_0 \cap V = \emptyset$ and for every $\gamma \in C(D, X)$ such that $\gamma|_{D_0} = \mathrm{id}|_{D_0}$, we have $\gamma(D) \cap V \neq \emptyset$.

The following critical point theorem will be a basic tool in our study of problem (1.1) (see e.g. [9, Theorem 1]):

Theorem 2.4. *If X is a reflexive Banach space, $\varphi: X \to \mathbb{R}$ is a locally Lipschitz function which satisfies the nonsmooth* C-*condition, (D_0, V) link in X through D and $\sup_{D_0} \varphi < \inf_V \varphi$, then if*

$$c = \inf_{\gamma\in\Gamma} \sup_{v\in D} \varphi(\gamma(v)), \quad \textit{with } \Gamma = \{\gamma \in C(D, X) : \gamma|_{D_0} = \mathrm{id}|_{D_0}\},$$

one has $c \geq \inf_V \varphi$ and c is a critical value of φ, i.e. there exists a critical point $x_0 \in X$ of φ such that $c = \varphi(x_0)$.

Remark 2.5. From this theorem, by suitable choices of the linking triple (D_0, D, V), we can derive nonsmooth versions of the well-known mountain pass, saddle point and generalized mountain pass theorems (see [5]).

Finally, consider the following nonlinear eigenvalue problem

$$(2.1)_p \qquad \begin{cases} -(|x'(t)|^{p-2}x'(t))' = \lambda |x(t)|^{p-2}x(t) & \text{a.e. on } T, \\ x(0) = x(b), \quad x'(0) = x'(b), \quad 1 < p < \infty, \quad \lambda \in \mathbb{R}. \end{cases}$$

As usual, an eigenvalue is a number λ for which problem $(2.1)_p$ has a nontrivial solution, known as an eigenfunction corresponding to the eigenvalue $\lambda \in \mathbb{R}$. A simple integration argument reveals that a necessary condition for problem $(2.1)_p$ to have a nontrivial solution is that $\lambda \geq 0$. Also, $\lambda_0 = 0$ is the first eigenvalue with corresponding eigenspace $\mathbb{R}$ (the constant functions). Moreover, every nonconstant eigenfunction changes sign and has a finite number of zeros. It can be shown that the eigenvalues of $(2.1)_p$ are given by

$$\left\{\lambda_n = \left(\frac{2n\pi_p}{b}\right)^p\right\}_{n\geq 0}, \quad \text{where } \pi_p = \frac{2\pi(p-1)^{1/p}}{p\sin(\pi/p)}.$$

If $p = 2$, then $\pi_2 = \pi$ and we recover the well-known spectrum of the negative scalar Laplacian with periodic boundary conditions, namely

$$\left\{\lambda_n = \left(\frac{2n\pi}{b}\right)^2\right\}_{n\geq 0}.$$

If we replace $(2.1)_p$ by its anti-periodic counterpart, i.e.

$$(2.1)_a \qquad \begin{cases} -(|x'(t)|^{p-2}x'(t))' = \lambda |x(t)|^{p-2}x(t) & \text{a.e. on } T, \\ x(0) = -x(b), \quad x'(0) = -x'(b), \quad 1 < p < \infty, \quad \lambda \in \mathbb{R}, \end{cases}$$

one can easily check that the corresponding eigenvalues are given by

$$\left\{\lambda_n^a = \left(\frac{(2n-1)\pi_p}{b}\right)^p\right\}_{n\geq 1}.$$

In particular, when $p = 2$, this sequence reduces to the classical one, namely

$$\left\{\lambda_n^a = \left(\frac{(2n-1)\pi}{b}\right)^2\right\}_{n\geq 1}.$$

3. Resonance at $\lambda_1 > 0$

In this section we prove an existence theorem for problem (1.1) under the following hypotheses on the nonsmooth potential:

(H_j^1) $j: T \times \mathbb{R} \to \mathbb{R}$ is a function such that $j(\,\cdot\,, 0) \in L^1(T)$ and

(i) for all $x \in \mathbb{R}$, $t \to j(t, x)$ is measurable;

(ii) for almost all $t \in T$, $x \to j(t, x)$ is locally Lipschitz;

(iii) for almost all $t \in T$, all $x \in \mathbb{R}$ and all $u \in \partial j(t,x)$, we have

$$|u| \le a(t) + c|x|^{p-1} \quad \text{with } a \in L^1(T)_+,\ c > 0;$$

(iv) for almost all $t \in T$, all $x \in \mathbb{R}$ and all $u \in \partial j(t,x)$, we have

$$u \le \gamma(t) \quad \text{with } \gamma \in L^1(T)_+;$$

(v) $\limsup_{|x|\to\infty} pj(t,x)/|x|^p \le \lambda_1$ uniformly for almost all $t \in T$;

(vi) $\int_0^b j(t,c)\,dt \to \infty$ as $|c| \to \infty$.

Remark 3.1. (a) Hypothesis (H_j^1)(v) permits complete resonance at infinity with respect to the first nonzero eigenvalue $\lambda_1 > 0$.

(b) The following nonsmooth locally Lipschitz function satisfies conditions (H_j^1) (for simplicity we drop the time dependence):

$$j(x) = \begin{cases} \dfrac{\lambda_1}{p}|x|^p + \left(1 - \dfrac{\lambda_1}{p}\right) & \text{if } x < -1, \\ |x| & \text{if } -1 \le x \le -1, \\ \sqrt{x} & \text{if } x > 1. \end{cases}$$

We consider the Sobolev space

$$W^{1,p}_{\mathrm{per}}(0,b) = \{x \in W^{1,p}(0,b) : x(0) = x(b)\}.$$

Since $W^{1,p}(0,b)$ is embedded continuously (in fact compactly) in $C(T)$, the pointwise evaluations at $t = 0$ and $t = b$ make sense.

The Euler functional $\varphi\colon W^{1,p}_{\mathrm{per}}(0,b) \to \mathbb{R}$ for problem (1.1) is defined by

$$\varphi(x) = \frac{1}{p}\|x'\|_p^p - \int_0^b j(t,x(t))\,dt \quad \text{for all } x \in W^{1,p}_{\mathrm{per}}(0,b),$$

where $\|\cdot\|_p$ stands for the standard norm in $L^p(T)$. We know that φ is Lipschitz continuous on bounded sets, hence it is locally Lipschitz (see Gasinski–Papageorgiou [5, p. 59]).

Proposition 3.2. *If hypotheses* (H_j^1) *hold, then* φ *satisfies the nonsmooth* PS-*condition.*

Proof. Here and throughout the remainder of the paper we will use the same symbol $\|\cdot\|$ to denote the norms of various function spaces. Let $\{x_n\}_{n\ge1} \subseteq W^{1,p}_{\mathrm{per}}(0,b)$ be a sequence such that

$$\varphi(x_n) \to c \quad \text{for some } c \in \mathbb{R} \text{ as } n \to \infty$$

and

$$m(x_n) := \inf\{\|x^*\| : x^* \in \partial\varphi(x_n)\} \to 0 \quad \text{as } n \to \infty.$$

So, we can find $M_1 > 0$ such that $|\varphi(x_n)| \leq M_1$ for all $n \geq 1$. Also, because for fixed n, $\partial\varphi(x_n) \subseteq W^{1,p}_{\rm per}(0,b)^*$ is weakly compact, by the Weierstrass theorem, we can find $x_n^* \in \partial\varphi(x_n)$ such that $m(x_n) = \|x_n^*\|$ for all $n \geq 1$.

Let $A\colon W^{1,p}_{\rm per}(0,b) \to W^{1,p}_{\rm per}(0,b)^*$ be the nonlinear operator defined by

$$\langle A(x), y\rangle = \int_0^b |x'(t)|^{p-2}x'(t)y'(t)\,dt \quad \text{for all } x, y \in W^{1,p}_{\rm per}(0,b).$$

By $\langle\,\cdot\,,\,\cdot\,\rangle$ we denote the duality brackets for the pair $(W^{1,p}_{\rm per}(0,b), W^{1,p}_{\rm per}(0,b)^*)$. For every $n \geq 1$, we have that

$$x_n^* = A(x_n) - u_n$$

with $u_n \in L^{p'}(T)$ $(1/p + 1/p' = 1)$, $u_n(t) \in \partial j(t, x_n(t))$ a.e. on T (see Gasinski–Papageorgiou [5, p. 59]). We consider the direct sum decomposition

$$W^{1,p}_{\rm per}(0,b) = \mathbb{R} \oplus V_0$$

with $V_0 = \{v \in W^{1,p}_{\rm per}(0,b) : \int_0^b v(t)\,dt = 0\}$. We have

$$x_n = \overline{x}_n + \widehat{x}_n \quad \text{with } \overline{x}_n \in \mathbb{R},\ \widehat{x}_n \in V_0,\ n \geq 1.$$

From the choice of the sequence $\{x_n\}_{n\geq 1} \subseteq W^{1,p}_{\rm per}(0,b)$, we have

$$|\langle x_n^*, y\rangle| \leq \varepsilon_n\|y\| \quad \text{for all } y \in W^{1,p}_{\rm per}(0,b) \text{ with } \varepsilon_n \downarrow 0.$$

First we act with the test function $y \equiv 1 \in W^{1,p}_{\rm per}(0,b)$. We obtain

$$\bigg|\int_0^b u_n(t)\,dt\bigg| \leq M_2 \quad \text{for some } M_2 > 0,\ \text{for all } n \geq 1,$$

hence

$$\bigg|\int_{\{u_n<0\}} u_n(t)\,dt\bigg| \leq \bigg|\int_0^b u_n(t)\,dt\bigg| + \int_{\{u_n>0\}} u_n(t)\,dt \leq M_2 + \|\gamma\|_1 \tag{3.1}$$

(see Hypothesis (H_j^1)(iv)). Therefore,

$$\begin{aligned}\int_0^b |u_n(t)|\,dt &= \int_{\{u_n>0\}} u_n(t)\,dt - \int_{\{u_n<0\}} u_n(t)\,dt \\ &\leq \|\gamma\|_1 + \bigg|\int_{\{u_n<0\}} u_n(t)\,dt\bigg| \leq 2\|\gamma\|_1 + M_2 =: M_3.\end{aligned} \tag{3.2}$$

Next we use as a test function $y = \widehat{x}_n \in V_0 \subseteq W^{1,p}_{\rm per}(0,b)$ and we obtain

$$\bigg|\langle A(x_n), \widehat{x}_n\rangle - \int_0^b u_n(t)\widehat{x}_n(t)\,dt\bigg| \leq \varepsilon_n\|\widehat{x}_n\|.$$

It follows that

$$\|\widehat{x}_n'\|_p^p \leq \varepsilon_n\|\widehat{x}_n\| + \int_0^b u_n(t)\widehat{x}_n(t)\,dt \leq \varepsilon_n\|\widehat{x}_n\| + M_3\|\widehat{x}_n\|_\infty$$

(see (3.2)), hence $\|\widehat{x}_n\|^p \le C_1\|\widehat{x}_n\|$ for some $C_1 > 0$ and all $n \ge 1$ (by the Poincaré–Wirtinger inequality); therefore

$$\{\widehat{x}_n\}_{n\ge 1} \subseteq W^{1,p}_{\mathrm{per}}(0,b) \text{ is bounded.} \tag{3.3}$$

Suppose that $\{x_n\}_{n\ge 1} \subseteq W^{1,p}_{\mathrm{per}}(0,b)$ is not bounded. By passing to a suitable subsequence if necessary, we may assume that $\|x_n\| \to \infty$. Then, because of (3.3), we must have $\|\overline{x}_n\| \to \infty$ as $n \to \infty$. Let

$$y_n := \frac{x_n}{\|x_n\|}, \quad n \ge 1.$$

Then, at least for a subsequence, we may assume that:

$$y_n \to y \quad \text{weakly in } W^{1,p}_{\mathrm{per}}(0,b), \quad \text{and} \quad y_n \to y \quad \text{in } C(T).$$

We have $y_n = \overline{y}_n + \widehat{y}_n$, with $\overline{y}_n = \overline{x}_n/\|x_n\| \in \mathbb{R}$, and $\widehat{y}_n = \widehat{x}_n/\|x_n\| \in V_0$, $n \ge 1$. Since $\|x_n\| \to \infty$ and (3.3) holds, we have $\widehat{y}_n \to 0$ in $W^{1,p}_{\mathrm{per}}(0,b)$; therefore $y = \overline{y} \in \mathbb{R}$.

If $y = 0$, then $y_n \to 0$ in $W^{1,p}_{\mathrm{per}}(0,b)$, a contradiction to the fact that $\|y_n\| = 1$ for all $n \ge 1$. So $y \ne 0$ and we have that $|x_n(t)| \to \infty$ for all $t \in T$; moreover, this last convergence is uniform in $t \in T$.

Indeed, since $y_n \to \overline{y} \in \mathbb{R} \setminus \{0\}$ in $C(T)$, given $0 < \varepsilon < |\overline{y}|$, we can find $n_0 = n_0(\varepsilon) \ge 1$ such that for all $n \ge n_0$ and all $t \in T$, we have $|y_n(t) - \overline{y}| \le \varepsilon$, hence

$$|\overline{y}| - \varepsilon \le |y_n(t)| \quad \text{for all } t \in T \text{ and all } n \ge n_0.$$

Recall that $\|x_n\| \to \infty$ as $n \to \infty$. So, given $\beta > 0$, we can find $n_1 \ge n_0$ such that $\|x_n\| \ge \beta$ for all $n \ge n_1$. Hence we have

$$0 < \xi = |\overline{y}| - \varepsilon \le |y_n(t)| = \frac{|x_n(t)|}{\|x_n\|} \le \frac{|x_n(t)|}{\beta}$$

for all $t \in T$ and all $n \ge n_1$, which implies

$$\beta\xi \le |x_n(t)| \quad \text{for all } t \in T \text{ and all } n \ge n_1.$$

Since $\beta > 0$ was arbitrary, it follows that

$$\min_{t\in T} |x_n(t)| \to \infty \quad \text{as } n \to \infty. \tag{3.4}$$

We set $\theta_n := \max_{t\in T} x_n(t)$. Because of (3.4), we have that $|\theta_n| \to \infty$ as $n \to \infty$, and so, by virtue of hypothesis (H_j^1)(vi), we infer that

$$\int_0^b j(t,\theta_n)\,dt \to \infty \quad \text{as } n \to \infty. \tag{3.5}$$

From the choice of the sequence $\{x_n\}_{n\geq 1} \subseteq W^{1,p}_{\mathrm{per}}(0,b)$, we have $|\varphi(x_n)| \leq M_1$ for all $n \geq 1$, hence

$$-\frac{1}{p}\|x_n^{'}\|_p^p + \int_0^b j(t, x_n(t))\,dt \leq M_1$$

and consequently

$$-\frac{1}{p}\|x_n^{'}\|_p^p + \int_0^b j(t, \theta_n)\,dt + \int_0^b [j(t, x_n(t)) - j(t, \theta_n)]\,dt \leq M_1, \tag{3.6}$$

for all $n \geq 1$. Using the mean value theorem for locally Lipschitz functions (see for example [5, p. 53]) and a straightforward measurable selection argument, we can find $v_n : T \to \mathbb{R}$ and $\lambda_n : T \to (0,1)$, $n \geq 1$, measurable functions, such that

$$j(t, x_n(t)) - j(t, \theta_n) = v_n(t)[x_n(t) - \theta_n] \tag{3.7}$$

with $v_n(t) \in \partial j(t, \lambda_n(t)x_n(t) + (1-\lambda_n(t))\theta_n)$ a.e. on T. Using hypothesis (H_j^1)(iv), we have

$$v_n(t)[x_n(t) - \theta_n] \geq \gamma(t)[x_n(t) - \theta_n] \quad \text{a.e. on } T, \text{ for all } n \geq 1. \tag{3.8}$$

Using (3.6)–(3.8), we obtain

$$-\frac{1}{p}\|x_n^{'}\|_p^p + \int_0^b j(t, \theta_n)\,dt - \|\gamma\|_1 \|x_n - \theta_n\|_\infty \leq M_1, \quad \text{for all } n \geq 1. \tag{3.9}$$

We have

$$x_n(t) - \theta_n = \overline{x}_n + \widehat{x}_n(t) - \overline{x}_n - \max_{s\in T} \widehat{x}_n(s) = \widehat{x}_n(t) - \max_{s\in T} \widehat{x}_n(s).$$

Hence

$$\begin{aligned}\|x_n - \theta_n\|_\infty &= \max_{t\in T} |x_n(t) - \theta_n| = \max_{t\in T} |\widehat{x}_n(t) - \max_{s\in T} \widehat{x}_n(s)| \\ &= \max_{t\in T}(\widehat{\theta}_n - \widehat{x}_n(t)), \quad \text{with } \widehat{\theta}_n = \max_{s\in T} \widehat{x}_n(s),\end{aligned}$$

so that

$$\|x_n - \theta_n\|_\infty = \widehat{\theta}_n - \min_{t\in T} \widehat{x}_n(t) \leq M_4, \tag{3.10}$$

for some $M_4 > 0$, for all $n \geq 1$. Here we have used the fact that $\{\widehat{x}_n\}_{n\geq 1} \subseteq W^{1,p}_{\mathrm{per}}(0,b)$ is bounded (see (3.3)). Employing now (3.10) in (3.9), we have

$$\int_0^b j(t, \theta_n)\,dt \leq M_5 \quad \text{for some } M_5 > 0, \text{ for all } n \geq 1. \tag{3.11}$$

Here we have used the fact that $\|x_n'\|_p = \|\widehat{x}_n'\|_p$, the Poincaré–Wirtinger inequality and (3.3). Comparing (3.5) and (3.11) we reach a contradiction. This proves that $\{x_n\}_{n\geq 1} \subseteq W^{1,p}_{\rm per}(0,b)$ is bounded. So, by passing to a subsequence if necessary, we may assume that

$$x_n \to x \quad \text{weakly in } W^{1,p}_{\rm per}(0,b) \quad \text{and} \quad x_n \to x \quad \text{in } C(T).$$

Recall that we have

$$\left| \langle A(x_n), x_n - x\rangle - \int_0^b u_n(t)(x_n(t)-x(t))\,dt \right| \leq \varepsilon_n \|x_n - x\| \quad \text{with } \varepsilon_n \downarrow 0.$$

Note that

$$\int_0^b u_n(t)(x_n(t)-x(t))\,dt \to 0 \quad \text{as } n\to\infty,$$

hence

$$\lim_{n\to\infty} \langle A(x_n), x_n - x\rangle = 0.$$

But A being maximal monotone, it is generalized pseudomonotone (see [5, p. 84]). So (also remark that $A(x_n)$ being bounded in $W^{1,p}_{\rm per}(0,b)^*$, we can assume that it weakly converges in that space),

$$\langle A(x_n), x_n\rangle \to \langle A(x), x\rangle,$$

and we conclude that

$$\|x_n'\|_p \to \|x'\|_p.$$

Since $x_n' \to x'$ weakly in $L^p(T)$ and $L^p(T)$ is uniformly convex, from the Kadec–Klee property we have that $x_n' \to x'$ in $L^p(T)$. Therefore $x_n \to x$ in $W^{1,p}_{\rm per}(0,b)$, which proves that φ satisfies the nonsmooth PS-condition.

Next, let

$$V = \left\{ v \in W^{1,p}_{\rm per}(0,b) : \int_0^b |v(t)|^{p-2} v(t)\,dt = 0 \right\}.$$

PROPOSITION 3.3. *If hypotheses* (H_j^1) *hold then* $\varphi|_V$ *is coercive, i.e. if* $\|x_n\| \to \infty$ *with* $x_n \in V$*, then* $\varphi(x_n) \to \infty$ *as* $n\to\infty$.

PROOF. We argue indirectly. So suppose that the conclusion of the proposition is not true. Then we can find $\{x_n\}_{n\geq 1} \subseteq V$ such that

$$\|x_n\| \to \infty \quad \text{as } n\to\infty \quad \text{and} \quad \varphi(x_n) \leq M_6 \quad \text{for some } M_6 \geq 0, \text{ for all } n\geq 1.$$

As before, we set $y_n = x_n/\|x_n\|$, $n\geq 1$, and so we may assume that

$$y_n \to y \quad \text{weakly in } W^{1,p}_{\rm per}(0,b) \quad \text{and} \quad y_n \to y \quad \text{in } C(T).$$

Obviously, $y\in V$. We have

$$\frac{1}{p}\|y_n'\|_p^p - \int_0^b \frac{j(t,x_n(t))}{\|x_n\|^p}\,dt \leq \frac{M_6}{\|x_n\|^p}, \quad \text{for all } n\geq 1. \tag{3.12}$$

By virtue of the mean value theorem for locally Lipschitz functions and hypothesis (H_j^1)(iii), we conclude that there exist $\widehat{a} \in L^1(T)_+$ and $\widehat{c} > 0$ such that

$$|j(t,x)| \leq \widehat{a}(t) + \widehat{c}|x|^p \quad \text{a.e. on } T, \text{ for all } x \in \mathbb{R},$$

(recall that $j(\,\cdot\,,0) \in L^1(T)$). Then if

$$h_n(\,\cdot\,) := \frac{j(\,\cdot\,,x_n(\,\cdot\,))}{\|x_n\|^p}, \quad n \geq 1,$$

we have that

$$|h_n(t)| \leq \frac{\widehat{a}(t)}{\|x_n\|^p} + \widehat{c}|y_n(t)|^p \quad \text{a.e. on } T, \text{ for all } n \geq 1. \tag{3.13}$$

Because of (3.13) and the Dunford–Pettis theorem, we may assume that

$$h_n \to h \quad \text{weakly in } L^1(T), \text{ as } n \to \infty.$$

Clearly $h(t) = 0$ a.e. on $\{y = 0\}$ (see (3.13)). For given $\varepsilon > 0$ and $n \geq 1$, we introduce the set

$$C_{\varepsilon,n} = \bigg\{t \in T : x_n(t) \neq 0,\ \frac{j(t,x_n(t))}{|x_n(t)|^p} \leq \frac{1}{p}(\lambda_1 + \varepsilon)\bigg\}.$$

Note that $|x_n(t)| \to \infty$ as $n \to \infty$ for all $t \in \{y \neq 0\}$. So hypothesis (H_j^1)(v) implies that

$$\chi_{C_{\varepsilon,n}}(t) \to 1 \quad \text{a.e. on } \{y \neq 0\}, \text{ as } n \to \infty.$$

Note that

$$\|(1 - \chi_{C_{\varepsilon,n}})h_n\|_{L^1(\{y\neq 0\})} \to 0 \quad \text{as } n \to \infty,$$

hence

$$\chi_{C_{\varepsilon,n}} h_n \to h \quad \text{weakly in } L^1(\{y \neq 0\}).$$

From the definition of $C_{\varepsilon,n}$, we have

$$\chi_{C_{\varepsilon,n}}(t)h_n(t) = \chi_{C_{\varepsilon,n}}(t)\frac{j(t,x_n(t))}{|x_n(t)|^p}|y_n(t)|^p \leq \frac{1}{p}(\lambda_1 + \varepsilon)|y_n(t)|^p$$

a.e. on T. Taking weak limits in $L^1(\{y \neq 0\})$ and using Mazur's lemma we obtain

$$h(t) \leq \frac{1}{p}(\lambda_1 + \varepsilon)|y(t)|^p \quad \text{a.e. on } \{y \neq 0\}.$$

Since $\varepsilon > 0$ was arbitrary, we let $\varepsilon \downarrow 0$, to conclude that

$$h(t) \leq \frac{\lambda_1}{p}|y(t)|^p \quad \text{a.e. on } \{y \neq 0\}.$$

Also recall that $h(t) = 0$ a.e. on $\{y = 0\}$. So finally we can say that

$$h(t) \leq \frac{\lambda_1}{p}|y(t)|^p \quad \text{a.e. on } T.$$

Therefore we have

$$\lim_{n\to\infty}\int_0^b \frac{j(t,x_n(t))}{\|x_n\|^p}\,dt = \int_0^b h(t)\,dt \le \frac{\lambda_1}{p}\|y\|_p^p. \tag{3.14}$$

Now we pass to the limit as $n\to\infty$ in (3.12), and we use (3.14) and the fact that

$$\|y'\|_p^p \le \liminf_{n\to\infty}\|y_n'\|_p^p.$$

We obtain

$$\|y'\|_p^p \le \lambda_1\|y\|_p^p,$$

therefore (cf. [2]) either $y=0$ or y is an eigenfunction corresponding to $\lambda_1>0$.

If $y=0$, then $y_n'\to 0$ in $L^p(T)$ and so $y_n\to 0$ in $W^{1,p}(0,b)$, a contradiction to the fact that $\|y_n\|=1$ for all $n\ge 1$. So y must be an eigenfunction corresponding to $\lambda_1>0$. Hence $y\in C^1(T)$ and $y(t)\ne 0$ for almost all $t\in T$.

As before, by the mean value theorem for locally Lipschitz functions and an easy measurable selection argument, we can find two measurable functions $v_n\colon T\to\mathbb{R}$ and $\lambda_n\colon T\to(0,1)$ such that, for all $n\ge 1$:

$$v_n(t)\in\partial j(t,\lambda_n(t)x_n(t)) \quad\text{and}\quad j(t,x_n(t)) = j(t,0)+v_n(t)x_n(t)$$

a.e. on T. Therefore

$$\frac{j(t,x_n(t))}{\|x_n\|^p} = \frac{j(t,0)}{\|x_n\|^p} + \frac{v_n(t)}{\|x_n\|^{p-1}}y_n(t) \tag{3.15}$$

a.e. on T, for all $n\ge 1$. Since $y_n\to y$ in $C(T)$, given $\delta>0$, we can find $n_0=n_0(\delta)\ge 1$ such that for all $n\ge n_0$, we have (see hypothesis (H_j^1)(iv))

$$\frac{v_n(t)}{\|x_n\|^{p-1}}y_n(t) \le \frac{\gamma(t)}{\|x_n\|^{p-1}}y_n(t) \tag{3.16}$$

a.e. on $\{y>\delta\}$. On the other hand, hypothesis (H_j^1)(iii) implies that

$$\frac{|v_n(t)|}{\|x_n\|^{p-1}}|y_n(t)| \le \frac{a(t)}{\|x_n\|^{p-1}}|y_n(t)| + c|y_n(t)|^p \tag{3.17}$$

a.e. on $\{y\le\delta\}$. Then

$$\int_0^b \chi_{C_{\varepsilon,n}}(t)\frac{j(t,x_n(t))}{\|x_n\|^p}\,dt = \int_{\{y>\delta\}} \chi_{C_{\varepsilon,n}}(t)\frac{j(t,x_n(t))}{\|x_n\|^p}\,dt + \int_{\{y\le\delta\}} \chi_{C_{\varepsilon,n}}(t)\frac{j(t,x_n(t))}{\|x_n\|^p}\,dt. \tag{3.18}$$

By (3.15), (3.16) and since $\chi_{C_{\varepsilon,n}}(t)\to 1$, a.e. on $\{y\ne 0\}$, we can apply Fatou's Lemma to deduce that

$$\limsup_{n\to\infty}\int_{\{y>\delta\}} \chi_{C_{\varepsilon,n}}(t)\frac{j(t,x_n(t))}{\|x_n\|^p}\,dt \le 0. \tag{3.19}$$

On the other hand, because of (3.15), (3.17) and Fatou's lemma, we have

$$\begin{aligned}\limsup_{n\to\infty}\int_{\{y\le\delta\}}&\chi_{C_{\varepsilon,n}}(t)\frac{j(t,x_n(t))}{\|x_n\|^p}\,dt\\ &\le\int_{\{y\le\delta\}}\limsup_{n\to\infty}\bigg[\chi_{C_{\varepsilon,n}}(t)\frac{j(t,x_n(t))}{\|x_n\|^p}\bigg]\,dt\\ &=\int_{\{y\le\delta\}}\limsup_{n\to\infty}\bigg[\chi_{C_{\varepsilon,n}}(t)\frac{j(t,x_n(t))}{|x_n(t)|^p}|y_n(t)|^p\bigg]\,dt\\ &\le\int_{\{y\le\delta\}}\limsup_{n\to\infty}\bigg[\chi_{C_{\varepsilon,n}}(t)\frac{(\lambda_1+\varepsilon)}{p}|y_n(t)|^p\bigg]\,dt\\ &=\int_{\{y\le\delta\}}\frac{(\lambda_1+\varepsilon)}{p}|y(t)|^p\,dt.\end{aligned}$$

Again, let $\varepsilon\downarrow 0$ to obtain

$$\limsup_{n\to\infty}\int_{\{y\le\delta\}}\chi_{C_{\varepsilon,n}}(t)\frac{j(t,x_n(t))}{\|x_n\|^p}dt\le\frac{\lambda_1}{p}\int_{\{y\le\delta\}}|y(t)|^p\,dt. \tag{3.20}$$

Returning to (3.18), passing to the limit as $n\to\infty$ and using (3.19) and (3.20), we have

$$\limsup_{n\to\infty}\int_0^b\chi_{C_{\varepsilon,n}}(t)\frac{j(t,x_n(t))}{\|x_n\|^p}\,dt\le\frac{\lambda_1}{p}\int_{\{y\le\delta\}}|y(t)|^p\,dt.$$

But recall that

$$\limsup_{n\to\infty}\int_0^b\chi_{C_{\varepsilon,n}}(t)\frac{j(t,x_n(t))}{\|x_n\|^p}\,dt=\limsup_{n\to\infty}\int_0^b\frac{j(t,x_n(t))}{\|x_n\|^p}\,dt.$$

Therefore, it follows that

$$\limsup_{n\to\infty}\int_0^b\frac{j(t,x_n(t))}{\|x_n\|^p}\,dt\le\frac{\lambda_1}{p}\int_{\{y\le\delta\}}|y(t)|^p\,dt.$$

Because $\delta>0$ was arbitrary, we let $\delta\downarrow 0$, and so

$$\limsup_{n\to\infty}\int_0^b\frac{j(t,x_n(t))}{\|x_n\|^p}\,dt\le\frac{\lambda_1}{p}\int_{\{y\le 0\}}|y(t)|^p\,dt.$$

Since $y\in V\setminus\{0\}$, we have

$$\frac{\lambda_1}{p}\int_{\{y\le 0\}}|y(t)|^p\,dt<\frac{\lambda_1}{p}\|y\|_p^p.$$

Thus finally we can say that

$$\limsup_{n\to\infty}\int_0^b\frac{j(t,x_n(t))}{\|x_n\|^p}\,dt<\frac{\lambda_1}{p}\|y\|_p^p. \tag{3.21}$$

Now we return all the way back to (3.12), pass to the limit as $n\to\infty$ and use (3.21). So, we obtain

$$\|y'\|_p^p<\lambda_1\|y\|_p^p,$$

a contradiction to the fact that $y \in V\setminus\{0\}$ (see Drabek–Manasevich [2]). This proves the coercivity of $\varphi|_V$. $\square$

Now we are ready to prove the first existence theorem.

Theorem 3.4. *If hypotheses* (H_j^1) *hold, then problem* (1.1) *has a solution* $x \in C^1(T)$.

Proof. Because of Proposition 3.3, we can find $\beta > 0$ such that $-\beta \leq \inf_V \varphi$. Also by virtue of hypothesis (H_j^1)(vi), we can find $c > 0$ large enough, such that

$$\varphi(\pm c) = -\int_0^b j(t, \pm c)\,dt < -\beta \leq \inf_V \varphi. \tag{3.22}$$

We introduce the sets

$$D_0 := \{-c, c\}, \quad D := \{x \in W^{1,p}_{\mathrm{per}}(0,b) : -c \leq x(t) \leq c \text{ for all } t \in T\}.$$

We claim that the sets D_0 and V link in $W^{1,p}_{\mathrm{per}}(0,b)$, through D. To this end let $\gamma \in C(D, W^{1,p}_{\mathrm{per}}(0,b))$ be such that $\gamma|_{D_0} = \mathrm{id}|_{D_0}$. Consider the continuous map $\eta\colon W^{1,p}_{\mathrm{per}}(0,b) \to \mathbb{R}$ defined by

$$\eta(x) = \int_0^b |x(t)|^{p-2}x(t)\,dt.$$

Note that $\eta(-c) < 0 < \eta(c)$. Hence $(\eta\circ\gamma)(-c) < 0 < (\eta\circ\gamma)(c)$. Also $\eta\circ\gamma \in C(D)$. Then, by the Intermediate Value Theorem, we can find $x \in D$ such that $(\eta\circ\gamma)(x) = \eta(\gamma(x)) = 0$. From the definition of η, it follows that $\gamma(x) \in V$, and so we have $\gamma(D) \cap V \neq \emptyset$. Therefore the claim is true.

Because of (3.22) and Proposition 3.2, we can apply Theorem 2.4 and obtain $x \in W^{1,p}_{\mathrm{per}}(0,b)$ such that $0 \in \partial\varphi(x)$. Then

$$A(x) = u \quad \text{with } u \in L^1(T),\ u(t) \in \partial j(t, x(t)) \text{ a.e. on } T. \tag{3.23}$$

We have $|x'|^{p-2}x' \in W^{-1,p'}(0,b) = W^{1,p}_0(0,b)^*$, $1/p + 1/p' = 1$ (see for example Gasinski–Papageorgiou [5, p. 9]). So, if we denote by $\langle\cdot,\cdot\rangle_0$ the duality pairing between $W^{1,p}_0(0,b)$ and $W^{-1,p'}(0,b)$, for every $\theta \in C^1_c(0,b)$, we have

$$\langle A(x), \theta\rangle = \int_0^b u(t)\theta(t)\,dt,$$

hence (from the definition of the distributional derivative)

$$\langle -(|x'|^{p-2}x')', \theta\rangle_0 = \langle u, \theta\rangle_0,$$

and (since $C^1_c(0,b)$ is dense in $W^{1,p}_0(0,b)$) it follows that

$$-(|x'(t)|^{p-2}x'(t))' = u(t) \quad \text{a.e. on } T. \tag{3.24}$$

Also since $x \in W^{1,p}_{\mathrm{per}}(0,b)$, we have that $x(0) = x(b)$. From (3.24) we infer that

$$|x'(\,\cdot\,)|^{p-2}x'(\,\cdot\,) \in W^{1,1}(0,b) \subseteq C(T),$$

and so $x' \in C(T)$, hence $x \in C^1(T)$.

Let $w \in W^{1,p}_{\mathrm{per}}(0,b)$. Then, by (3.23), we have $\langle A(x), w\rangle = \int_0^b u(t)w(t)\,dt$, that is,

$$\int_0^b |x'(t)|^{p-2}x'(t)w'(t)\,dt = \int_0^b u(t)w(t)\,dt. \tag{3.25}$$

Performing an integration by parts in the integral of the left-hand side of (3.25), we have, by (3.24):

$$\begin{aligned}\int_0^b &|x'(t)|^{p-2}x'(t)w'(t)\,dt \\ &= |x'(b)|^{p-2}x'(b)w(b) - |x'(0)|^{p-2}x'(0)w(0) - \int_0^b (|x'(t)|^{p-2}x'(t))'w(t)\,dt \\ &= |x'(b)|^{p-2}x'(b)w(b) - |x'(0)|^{p-2}x'(0)w(0) - \int_0^b u(t)w(t)\,dt,\end{aligned}$$

hence (see (3.25))

$$|x'(b)|^{p-2}x'(b)w(b) = |x'(0)|^{p-2}x'(0)w(0).$$

Inasmuch as $w \in W^{1,p}_{\mathrm{per}}(0,b)$, we conclude that

$$|x'(b)|^{p-2}x'(b) = |x'(0)|^{p-2}x'(0),$$

therefore $x'(b) = x'(0)$, i.e. $x \in C^1(T)$ solves problem (1.1). □

If we consider the anti-periodic counterpart of problem (1.1), that is

$$\begin{cases} -(|x'(t)|^{p-2}x'(t))' \in \partial j(t, x(t)) & \text{a.e. on } T := [0,b], \\ x(0) = -x(b), \quad x'(0) = -x'(b), & 1 < p < \infty, \end{cases} \tag{3.26}$$

we obtain a result similar to Theorem 3.4. Specifically, we have

Theorem 3.5. *Let conditions* (H_j^1)(i)–(v) *be satisfied* (*with* $\lambda_1 = \lambda_1^a$). *Then problem* (3.26) *has a solution in* $C^1(T)$.

Proof. In this case, the kernel of the negative scalar p-Laplacian with anti-periodic boundary conditions is trivial, so we only need to prove the coercivity of the Euler functional φ on the space

$$W = \{w \in W^{1,p}(0,b) : w(0) = -w(b)\}.$$

This can be accomplished exactly as in the proof of Proposition 3.3, with the mention that condition (H_j^1)(vi) is no longer needed. □

4. Double resonance

In this section our hypotheses on the nonsmooth potential are the following:

(H_j^2) $j: T \times \mathbb{R} \to \mathbb{R}$ is a function such that $j(\cdot, 0) \in L^1(T)$ and

(i) for all $x \in \mathbb{R}$, $t \to j(t,x)$ is measurable;

(ii) for almost all $t \in T$, $x \to j(t,x)$ is locally Lipschitz;

(iii) for almost all $t \in T$, all $x \in \mathbb{R}$ and all $u \in \partial j(t,x)$, we have

$$|u| \leq a(t) + c|x|^{p-1} \quad \text{with } a \in L^1(T)_+,\ c > 0;$$

(iv) there exists $\theta \in L^\infty(T)_+$ such that $\theta(t) \leq \lambda_1$ a.e. on T, with strict inequality on a set of positive measure and

$$0 \leq \liminf_{|x| \to \infty} \frac{pj(t,x)}{|x|^p} \leq \limsup_{|x| \to \infty} \frac{pj(t,x)}{|x|^p} \leq \theta(t)$$

uniformly for almost all $t \in T$;

(v) $\lim_{|x| \to \infty} [ux - pj(t,x)] = \infty$ uniformly for almost all $t \in T$, and all $u \in \partial j(t,x)$;

(vi) $\int_0^b j(t,c)\,dt \to \infty$ as $|c| \to \infty$.

Remark 4.1. (a) As compared to [9] and [5, p. 337], in hypothesis (H_j^2)(v) the limit is now ∞, as opposed to $-\infty$. Furthermore, as opposed to [9], in (H_j^2)(iv), we allow partial interaction with λ_1. Hypothesis (H_j^2)(iv) asymptotically at $\pm\infty$, permits complete resonance with respect to the first eigenvalue and incomplete resonance (nonuniform nonresonance) with respect to the $\lambda_1 > 0$ eigenvalue. So we are dealing with a situation of double resonance in the spectral interval $[0, \lambda_1]$.

(b) The following nonsmooth locally Lipschitz in $x \in \mathbb{R}$ function satisfies hypotheses (H_j^2):

$$j(t,x) = \begin{cases} -\dfrac{1}{r}|x|^r & \text{if } x \leq 0, \\ x^2 \ln(x) & \text{if } x \in (0,1], \\ \dfrac{\theta(t)}{p}x^p - x - (\dfrac{\theta(t)}{p} - 1) & \text{if } x > 1. \end{cases}$$

Here $\theta \in L^\infty(T)_+$ is as in hypothesis (H_j^2)(iv) and $1 \leq r < p$.

The Euler functional $\varphi: W^{1,p}_{\text{per}}(0,b) \to \mathbb{R}$ remains the same, namely

$$\varphi(x) = \frac{1}{p}\|x'\|_p^p - \int_0^b j(t, x(t))\,dt \quad \text{for all } x \in W^{1,p}_{\text{per}}(0,b).$$

We know [5] that φ is locally Lipschitz (in fact it is Lipschitz continuous on bounded sets).

Proposition 4.2. *If hypotheses* (H_j^2) *hold, then* φ *satisfies the nonsmooth* C-*condition.*

Proof. Let $\{x_n\}_{n\geq 1}\subseteq W^{1,p}_{\mathrm{per}}(0,b)$ be a sequence such that

$$\varphi(x_n)\to\beta_0\in\mathbb{R}\quad\text{and}\quad(1+\|x_n\|)m(x_n)\to 0\quad\text{as } n\to\infty.$$

As before (see the proof of Proposition 3.2), we can find $x_n^*\in\partial\varphi(x_n)$ such that $m(x_n)=\|x_n^*\|, n\geq 1$. Then

$$x_n^*=A(x_n)-u_n,$$

where $A\colon W^{1,p}_{\mathrm{per}}(0,b)\to W^{1,p}_{\mathrm{per}}(0,b)^*$ is the nonlinear maximal monotone operator introduced in the proof of Proposition 3.2, and

$$u_n\in L^1(T),\quad u_n(t)\in\partial j(t,x_n(t))\quad\text{a.e. on } T,\ n\geq 1.$$

We have

$$\begin{aligned}(4.1)\qquad |\langle x_n^*,x_n\rangle-p\varphi(x_n)+p\beta_0| &\leq \|x_n^*\|\|x_n\|+|p\beta_0-p\varphi(x_n)|\\ &\leq(1+\|x_n\|)m(x_n)+p|\varphi(x_n)-\beta_0|.\end{aligned}$$

From (4.1) it follows that

$$(4.2)\quad p\varphi(x_n)-\langle x_n^*,x_n\rangle=\int_0^b[u_n(t)x_n(t)-pj(t,x_n(t))]\,dt\to p\beta_0\quad\text{as } n\to\infty.$$

Using (4.2) we will show that the sequence $\{x_n\}_{n\geq 1}\subseteq W^{1,p}_{\mathrm{per}}(0,b)$ is bounded. We proceed by contradiction. Suppose that $\{x_n\}_{n\geq 1}\subseteq W^{1,p}_{\mathrm{per}}(0,b)$ is not bounded. We may assume that $\|x_n\|\to\infty$ as $n\to\infty$. So if $y_n=x_n/\|x_n\|$, $n\geq 1$, we can say (at least for a subsequence) that

$$y_n\to y,\quad\text{weakly in } W^{1,p}_{\mathrm{per}}(0,b)\quad\text{and}\quad y_n\to y\quad\text{in } C(T).$$

Arguing as in the proof of Proposition 3.3, via hypotheses (H_j^2)(iii) and (iv), we can show that

$$(4.3)\qquad h_n(\,\cdot\,):=\frac{j(\,\cdot\,,x_n(\,\cdot\,))}{\|x_n\|^p}\to h(\,\cdot\,),\quad\text{weakly in } L^1(T)$$

and

$$(4.4)\qquad 0\leq h_n(t)\leq\frac{\theta(t)}{p}|y(t)|^p\quad\text{a.e. on } T.$$

From the choice of the sequence $\{x_n\}_{n\geq 1}\subseteq W^{1,p}_{\mathrm{per}}(0,b)$, we have

$$|\varphi(x_n)|\leq M_7\quad\text{for some } M_7>0 \text{ and for all } n\geq 1,$$

hence

$$\frac{1}{p}\|y_n'\|_p^p-\int_0^b\frac{j(t,x_n(t))}{\|x_n\|^p}\,dt\leq\frac{M_7}{\|x_n\|^p}\quad\text{for all } n\geq 1.$$

Passing to the limit as $n \to \infty$ and using (4.3) and (4.4), we obtain

$$\|y'\|_p^p \leq \int_0^b \theta(t)|y(t)|^p\, dt.$$

If $y = 0$, then $y_n \to 0$ in $W^{1,p}_{\mathrm{per}}(0, b)$, which contradicts the fact that $\|y_n\| = 1$ for all $n \geq 1$. Therefore $y \neq 0$. Let $\widehat{T} = \{y \neq 0\}$. Then $|\widehat{T}|_1 > 0$ (where $|\cdot|_1$ stands for the Lebesgue measure on $\mathbb{R}$) and $|x_n(t)| \to \infty$ for all $t \in \widehat{T}$, as $n \to \infty$. We have

$$\int_0^b [u_n(t)x_n(t) - pj(t, x_n(t))]\, dt = \int_{\widehat{T}} [u_n(t)x_n(t) - pj(t, x_n(t))]\, dt + \int_{T\setminus\widehat{T}} [u_n(t)x_n(t) - pj(t, x_n(t))]\, dt. \tag{4.5}$$

Because of hypothesis (H_j^2)(v), we can find $M_8 > 0$ such that for almost all $t \in T$, all $x \in \mathbb{R}$ with $|x| \geq M_8$ and all $u \in \partial j(t, x)$, we have

$$ux - pj(t, x) \geq 1. \tag{4.6}$$

On the other hand, as in the proof of Proposition 3.3, we can conclude that

$$|j(t, x)| \leq \widehat{a}(t) + \widehat{c}|x|^p, \quad \text{a.e. on } T, \text{ for all } x \in \mathbb{R},$$

with $\widehat{a} \in L^1(T)_+$ and $\widehat{c} > 0$. So, for almost all $t \in T$, all $x \in \mathbb{R}$ with $|x| \leq M_8$ and all $u \in \partial j(t, x)$, we have

$$ux - pj(t, x) \geq -\alpha(t) \quad \text{for some } \alpha \in L^1(T)_+. \tag{4.7}$$

Thus finally, because of (4.6) and (4.7), we can conclude that

$$ux - pj(t, x) \geq -\alpha(t) \quad \text{a.e. on } T, \text{ for all } x \in \mathbb{R} \text{ and all } u \in \partial j(t, x). \tag{4.8}$$

Returning to (4.5) and using (4.8), we have

$$\int_0^b [u_n(t)x_n(t) - pj(t, x_n(t))]\, dt \geq \int_{\widehat{T}} [u_n(t)x_n(t) - pj(t, x_n(t))]\, dt - \|\alpha\|_1. \tag{4.9}$$

By Fatou's lemma, we have

$$\liminf_{n\to\infty} \int_{\widehat{T}} [u_n(t)x_n(t) - pj(t, x_n(t))]\, dt \geq \int_{\widehat{T}} \liminf_{n\to\infty} [u_n(t)x_n(t) - pj(t, x_n(t))]\, dt.$$

Recall that $|x_n(t)| \to \infty$ for all $t \in \widehat{T}$, as $n \to \infty$. So, hypothesis (H_j^2)(v) implies

$$\int_{\widehat{T}} [u_n(t)x_n(t) - pj(t, x_n(t))]\, dt \to \infty \quad \text{as } n \to \infty,$$

hence (see (4.9))

$$\int_0^b [u_n(t)x_n(t) - pj(t, x_n(t))]\, dt \to \infty \quad \text{as } n \to \infty. \tag{4.10}$$

Comparing (4.2) and (4.10), we reach a contradiction. This means that $\{x_n\}_{n\geq 1} \subseteq W^{1,p}_{\rm per}(0,b)$ is bounded and so we may assume that

$$x_n \to x, \text{ weakly in } W^{1,p}_{\rm per}(0,b) \quad \text{and} \quad x_n \to x \text{ in } C(T).$$

Then, as in the proof of Proposition 3.2, exploiting the maximal monotonicity of A and the Kadec–Klee property of $L^p(T)$, we conclude that $x_n \to x$ in $W^{1,p}_{\rm per}(0,b)$; hence φ satisfies the nonsmooth C-condition. □

As in Section 3, we consider the cone

$$V = \bigg\{ v \in W^{1,p}_{\rm per}(0,b) : \int_0^b |v(t)|^{p-2} v(t)\, dt = 0 \bigg\}.$$

The following inequality, valid on V, will be useful in the sequel.

Lemma 4.3. *If $\theta \in L^\infty(T)_+$ and $\theta(t) \leq \lambda_1$ a.e. on T, with strict inequality on a set of positive measure, then there exists $\xi > 0$ such that*

$$\psi(v) := \|v'\|_p^p - \int_0^b \theta(t)|v(t)|^p dt \geq \xi \|v'\|_p^p, \quad \textit{for all } v \in V.$$

Proof. From Drabek–Manasevich [2] we know that $\psi(v) \geq 0$ for all $v \in V$. Suppose that the lemma is not true. Exploiting the p-homogeneity of ψ, we can find $\{v_n\}_{n\geq 1} \subseteq V$ with $\|v_n'\|_p = 1$ for all $n \geq 1$, such that $\psi(v_n) \downarrow 0$. By virtue of the Poincaré–Wirtinger inequality, we infer that $\{v_n\}_{n\geq 1} \subseteq W^{1,p}_{\rm per}(0,b)$ is bounded, and so we may assume that

$$v_n \to v \quad \text{weakly in } W^{1,p}_{\rm per}(0,b) \quad \text{and} \quad v_n \to v \quad \text{in } C(T).$$

The functional ψ is weakly lower semicontinuous on $W^{1,p}_{\rm per}(0,b)$. So, $\psi(v) \leq \lim_{n\to\infty} \psi(v_n) = 0$, which implies implies that

$$\|v'\|_p^p \leq \int_0^b \theta(t)|v(t)|^p\, dt \leq \lambda_1 \|v\|_p^p. \tag{4.11}$$

Since $v \in V$, from (4.11) it follows that $\|v'\|_p^p = \lambda_1 \|v\|_p^p$ (see Drabek–Manasevich [2]), hence $v = 0$ or v is an eigenfunction corresponding to $\lambda_1 > 0$.

If $v = 0$, then $v_n' \to 0$ in $L^p(T)$, a contradiction to the fact that $\|v_n'\|_p = 1$ for all $n \geq 1$. So v is an eigenfunction for $\lambda_1 > 0$, hence $v(t) \neq 0$ a.e. on T. Then, from the first inequality in (4.11) and the hypothesis on θ, we have

$$\|v'\|_p^p < \lambda_1 \|v\|_p^p,$$

a contradiction to the variational characterization of $\lambda_1 > 0$. □

Using this lemma we can prove the coercivity of $\varphi|_V$.

Proposition 4.4. *If hypotheses* (H_j^2) *hold, then* $\varphi|_V$ *is coercive.*

Proof. Recall that

$$|j(t,x)| \le \widehat{a}(t) + \widehat{c}|x|^p \quad \text{with } \widehat{a} \in L^1(T)_+,\ \widehat{c} > 0.$$

Combining this growth condition with hypothesis (H_j^2)(iv), given $\varepsilon > 0$ we can find $a_\varepsilon \in L^1(T)_+$ such that

$$j(t,x) \le \frac{1}{p}(\theta(t)+\varepsilon)|x|^p + a_\varepsilon(t), \quad \text{a.e. on } T, \text{ for all } x \in \mathbb{R}. \tag{4.12}$$

Then if $v \in V$, we have (see (4.12)):

$$\begin{aligned} \varphi(v) &= \frac{1}{p}\|v'\|_p^p - \int_0^b j(t,v(t))\,dt \\ &\ge \frac{1}{p}\|v'\|_p^p - \frac{1}{p}\int_0^b \theta(t)|v(t)|^p\,dt - \frac{\varepsilon}{p}\|v\|_p^p - \|a_\varepsilon\|_1 \\ &\ge \frac{1}{p}\bigg(\xi - \frac{\varepsilon}{\lambda_1}\bigg)\|v'\|_p^p - \|a_\varepsilon\|_1 \end{aligned} \tag{4.13}$$

(see Lemma 4.3 and recall the variational characterization of $\lambda_1 > 0$). If we choose $\varepsilon \le \lambda_1\xi$, from (4.13) and the Poincaré–Wirtinger inequality, we conclude that $\varphi|_V$ is coercive. $\square$

Now we can state our second existence theorem for problem (1.1).

Theorem 4.5. *If hypotheses* (H_j^2) *hold, then problem* (1.1) *has a solution* $x \in C^1(T)$.

Proof. Proposition 4.4 implies that

$$-\infty < m_V = \inf_V \varphi.$$

Also, due to hypothesis (H_j^2)(vi), we can find $c > 0$ large enough such that

$$\varphi(\pm c) < m_V.$$

As before (see the proof of Theorem 3.4), we consider the sets

$$D_0 = \{-c,\ +c\}, \quad D = \{x \in W^{1,p}_{\mathrm{per}}(0,b) : -c \le x(t) \le c \text{ for all } t \in T\},$$

and V. The sets D_0 and V link in $W^{1,p}_{\mathrm{per}}(0,b)$ through D. Because of Proposition 4.2, we can apply Theorem 2.4 and obtain $x \in W^{1,p}_{\mathrm{per}}(0,b)$ such that $0 \in \partial\varphi(x)$. As in the proof of Theorem 3.4, we check that $x \in C^1(T)$ and that it solves problem (1.1). $\square$

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