Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 27, 2006, 177–194

POSITIVE SOLUTIONS FOR A NONCONVEX ELLIPTIC DIRICHLET PROBLEM WITH SUPERLINEAR RESPONSE

Andrzej Nowakowski — Aleksandra Orpel

Abstract. The existence of bounded solutions of the Dirichlet problem for a ceratin class of elliptic partial differential equations is discussed here. We use variational methods based on the subdifferential theory and the comparison principle for difergence form operators. We present duality and variational principles for this problem. As a consequences of the duality we obtain also the variational principle for minimizing sequences of J which gives a measure of a duality gap between primal and dual functional for approximate solutions.

1. Introduction

The aim of this paper is to show that some class of Dirichlet problems governed by a second order partial differential equation possesses solutions from a known pre-specified interval of the positive axis. We shall consider PDE of elliptic type being a generalization of the membrane equation in the following form

$$\begin{cases} -\mathrm{div}(k(y)\nabla x(y)) = G_x(y, x(y)) \quad \text{for a.e. } y \in \Omega, \\ x|_{\partial\Omega} = 0 \end{cases} \tag{1.1}$$

2000 *Mathematics Subject Classification.* 35J20, 35J25.
Key words and phrases. Nonconvex elliptic Dirichlet problems, positive solutions, duality method, variational method.

with $k \in C^1(\overline{\Omega}, \mathbb{R}_+)$, G_x denoting the derivative with respect to x. We shall assume that G is convex and differentiable with respect to the second variable in some interval and it satisfies the Carathéodory condition. Throughout the paper we shall assume the following conditions:

(Ω) $\Omega \subset \mathbb{R}^n$ is a bounded domain in $\mathbb{R}^n$ having a piecewise $C^{1,1}$ boundary.

(K) $k \in C^1(\overline{\Omega}, \mathbb{R})$, $\overline{k}_0 \geq k(y) \geq k_0 > 0$ for all $y \in \Omega$.

(G1) There exist $\overline{z} \in C_0^1(\overline{\Omega})$, $z_0 \in W_0^{1,2}(\Omega, \mathbb{R}) \cap W^{2,\infty}(\Omega, \mathbb{R})$ such that $0 < z_0(y) \leq \overline{z}(y)$ for all $y \in \Omega$ and

$$G_x(y, \overline{z}) \leq -\operatorname{div}(k(y)\nabla z_0(y)). \tag{1.2}$$

(G2) $G: \Omega \times \mathbb{R} \to \mathbb{R}$ is measurable in y, continuously differentiable and convex with respect to the second variable in some closed neighbourhood $\widetilde{I}$ of the interval $I = [0, \sup_{y\in\Omega} \overline{z}(y)]$ for all $y \in \Omega$, $\int_\Omega G(y, 0)\, dy < \infty$.

(G3) G_x is positive in I for all $y \in \Omega$.

It is worth to note that the assumption (G1) is not very strong, as any function G_x which is superlinear and increasing in some interval satisfies the conditions from (G1). In particular, any polynomial of degree greater than two (eventually shifted) has that property. How to find the functions $\overline{z}$ and z_0 is described in Example 4.2, they may be treated as model functions.

Recently an increasing interest has been observed in investigating the existence of positive solutions of similar problems. It is associated with the fact that a lot of mathematical models of physical and technical phenomena involves nonlinear elliptic problems. The elliptic partial differential equations in the divergence form were discussed e.g. in [12], where $G \in C(\overline{\Omega} \times \mathbb{R})$, in [5], or in [10], where the right-hand side is independent of x and Ω is a bounded n-dimensional polyhedral domain. In [13] the existence of a solution $x \in W_0^{1,p}(\Omega, \mathbb{R}) \cap L^\infty(\Omega, \mathbb{R})$, $p > 1$ for the below PDE

$$-\operatorname{div} A(y, x, Dx) = \mathbf{H}(y, x, Dx) \quad \text{in } \Omega, \tag{1.3}$$

where Ω is an open set in $\mathbb{R}^n$, $n \geq 1$, follows from the existence of the solution of an associated symmetrized semilinear problem. N. Grenon has obtained the results under the assumptions that

(a) $A: \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and $\mathbf{H}: \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ are Carathéodory functions, such that for a.e. $y \in \Omega$, all $x \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$

$$|A(y, x, \xi)| \leq \beta(|x|)|\xi|^{p-1} + b(y), \tag{1.4}$$

$$|\mathbf{H}(y, x, \xi)| \leq \gamma(|x|)\{|\xi|^p + d(y)\}, \tag{1.5}$$

where β, γ are positive and locally bounded, b is a positive element of $L^{p'}(\Omega, \mathbb{R})$, $p' = p/(p-1)$, and $d \in L^1(\Omega, \mathbb{R})$;

(b) $\langle A(y,x,\xi) - A(y,x,\xi'), \xi - \xi' \rangle > 0$ for all $\xi \neq \xi'$ and there exists $\alpha > 0$ such that $\alpha|\xi|^p \leq \langle A(y,x,\xi), \xi \rangle$;

(c) there are nondecreasing $k_i, \theta_i \in C(\mathbb{R}_+, \mathbb{R}_+)$, nonnegative $f_i \in L^q(\Omega, \mathbb{R}_+)$, $\max\{n/p, 1\} < q \leq \infty$, $i = 1,2$ with $\theta_i(0) > 0$ such that

$$\begin{aligned} \mathbf{H}(y,x,\xi) &\leq \alpha\{k_1(x)|\xi|^p + \theta_1(x) f_1(y)\} && \text{for all } x \geq 0, \\ \mathbf{H}(y,x,\xi) &\leq \alpha\{-k_2(-x)|\xi|^p - \theta_2(-x) f_2(y)\} && \text{for all } x \leq 0. \end{aligned}$$

Although for $A(y,x,\xi) = k(y)\xi$ and $\mathbf{H}(y,x,\xi) = G_x(y,x)$, (1.3) gives (1.1), we cannot use the results presented in [13] because we will not assume any additional estimate on G_x like (1.5) and (c).

There are a lot of results concerning the case when k is a constant, among others [14], [15], [28]. In [32] and [27] the existence of a classical solution of (1.1) is discussed under the following assumptions: $G_x(\,\cdot\,,\,\cdot\,) \in C(\Omega \times \mathbb{R}, \mathbb{R})$, G_x satisfies the additional estimate on $\Omega \times \mathbb{R}$ and the following relations between G and G_x holds: there exist $\mu > 0$ and $r \geq 0$ such that for $|x| \geq r$

$$0 < \mu G(y,x) \leq x G_x(y,x). \tag{1.6}$$

The condition similar to (1.6) is used also in [9]. We can find a lot of papers concerning similar problem for G being a polynomial with respect to x (see [30], [26]). Here we point out that weaker assumptions made on G are still sufficient to conclude the existence of a countable set of solutions for (1.1). A natural, and widely used approach to solvability of our problem is to treat (1.1) as the generalized Euler–Lagrange equation for the functional J given by

$$J(x) = \int_\Omega \left\{ \frac{1}{2} k(y) |\nabla x(y)|^2 - G(y, x(y)) \right\} dy \tag{1.7}$$

for $x \in W_0^{1,2}(\Omega, \mathbb{R})$.

We see that under the our assumptions, J is not, in general, bounded on $W_0^{1,2}(\Omega, \mathbb{R})$; so that we must look for critical points of (1.7) of "minmax" type or find subsets X and X^d, on which the action functional J or the dual one — J_D is bounded. We shall apply the other approach and choose the special sets over which we will calculate minimum of J and J_D. Our assumptions are not strong enough to use, for example, the Mountain Pass Theorem (see e.g. [18], [32], [27]): G is not sufficiently smooth, we do not assume any additional relations concerning G_x and G, in consequence, J does not satisfied, in general, the (PS)-condition. Of course, we also have the Morse theory and its generalization or the saddle points theorems, but all these methods do not exhaust all critical points of J. Moreover, our approach enables us the numerical characterization of solutions of our problem.

We will develop a duality theory which is similar to the duality presented in [25], where the systems of ODE's is considered. The duality is based on the Fenchel transform, so we recall some properties of the Fenchel conjugate widely discussed, e.g. in [11]. First denote by $\Gamma_0(\mathbb{R}^n)$ the set of all convex and lower semicontinuous functions $f:\mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ which are not identically equal to $+\infty$.

Definition 1.1. Let $f \in \Gamma_0(\mathbb{R}^n)$ and $\langle a,b\rangle := \sum_{i=1}^n a_i b_i$ for all $a,b\in\mathbb{R}^n$. The function $f^*:\mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ given by

$$f^*(u^*) = \sup_{u\in\mathbb{R}^n}\{\langle u^*,u\rangle - f(u)\}$$

is called the Fenchel transform (or conjugate) of f .

Definition 1.2. Let $f \in \Gamma_0(\mathbb{R}^n)$. The set

$$\partial f(u) := \{u^* \in \mathbb{R}^n : \langle u^*, v-u\rangle + f(u) \le f(v) \text{ for all } v\in\mathbb{R}^n\}$$

is called the subdifferential of f at u. If $\partial f(u) \ne \emptyset$ then we say that f is subdifferentiable at u.

Theorem 1.3 ([11, Chapter I, §5]). *Let $f \in \Gamma_0(\mathbb{R}^n)$. Then the following conditions are equivalent*

(a) $f(u) + f^*(u^*) = \langle u^*, u\rangle$,
(b) $u^* \in \partial f(u)$,
(c) $u \in \partial f^*(u^*)$.

Theorem 1.4 ([11, Chapter IX, Proposition 2.1]). *Assume that Ω is a bounded domain in $\mathbb{R}^n$, $f:\Omega\times\mathbb{R}^m \to \mathbb{R}\cup\{+\infty\}$ is a nonnegative Carathéodory function and $1<p<+\infty$. Let F: $L^p(\Omega,\mathbb{R}^m)\to\mathbb{R}\cup\{+\infty\}$ be given by*

$$F(u) = \int_\Omega f(y,u(y))\,dy.$$

Denote by F^* the conjugate function defined as

$$F^*(u^*) = \sup_{u\in L^2(\Omega,\mathbb{R}^m)}\left\{\int_\Omega \langle u(y),u^*(y)\rangle - f(y,u(y))\,dy\right\}$$

for all $u^*\in L^q(\Omega,\mathbb{R}^m)$, with $q=p/(p-1)$. If there exists $u_0\in L^\infty(\Omega,\mathbb{R}^m)$ such that $F(u_0)<+\infty$ then for all $u^*\in L^q(\Omega,\mathbb{R}^m)$,

$$F^*(u^*) = \int_\Omega f^*(y,u^*(y))\,dy,$$

where f^* is the Fenchel transform of f.

Now we recall the relevant theorems from ([12]):

Theorem 1.5. *Let Ω be a bounded domain in $\mathbb{R}^n$ having a piecewise $C^{1,1}$ boundary. Then, if $f \in L^\infty(\Omega, \mathbb{R})$, the Dirichlet problem*

$$\begin{cases} \operatorname{div}(k(y)\nabla u(y)) = f(y) & \text{for a.e. } y \in \Omega, \\ u \in W_0^{1,2}(\Omega, \mathbb{R}), \end{cases}$$

where $k \in C^1(\overline{\Omega}, \mathbb{R})$, $\overline{k}_0 \geq k(y) \geq k_0 > 0$ for all $y \in \Omega$, possesses a unique solution $u \in W_0^{1,2}(\Omega, \mathbb{R}) \cap W^{2,\infty}(\Omega, \mathbb{R})$.

We recall that a function u, weakly differentiable in Ω, satisfies $Qu \geq 0$ $(= 0, \leq 0)$ in Ω if the functions $A(x, u, Du) = \{A^i(x, u, Du)\}_{i=1,\dots,n}$ and $B(x, u, Du)$ are locally integrable in Ω and

$$Q(u, \varphi) = \int_\Omega [A(y, u, Du)\nabla\varphi(y) - B(y, u, Du)\varphi(y)]\, dy \leq 0 \quad (= 0, \ \geq 0)$$

for all non-negative $\varphi \in C_0^1(\overline{\Omega})$, where

$$Qu = \operatorname{div}(A(y, u, Du)) + B(y, u, Du).$$

Theorem 1.6. *Let $u, v \in C_0(\overline{\Omega}) \cap C^1(\Omega)$ satisfy $Qu \geq 0$, $Qv \leq 0$ in Ω and $u \leq v$ on $\partial\Omega$, where A, B are continuous differentiable with respect to the second and third variable in $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n$, Q is elliptic in Ω, B is non-increasing in u. Then, if either*

(a) *A is independent of u; or*
(b) *B is independent of Du*

it follows that $u \leq v$ in Ω.

Now we shall construct the sets of arguments of J and J_D. Let

$$\overline{X} = \{x \in W_0^{1,2}(\Omega, \mathbb{R}),\ x(y) \in I,\ x(y) \leq \overline{z}(y) \text{ on } \Omega \\ \text{and } \operatorname{div}(k\nabla x) \in L^\infty(\Omega, \mathbb{R})\}.$$

Let us note that for each $x \in \overline{X}$ in view of (G2) and (G1) $G_x(y, x(y)) \leq -\operatorname{div}(k(y)\nabla z_0(y)) \in L^\infty$ i.e. $G_x(\,\cdot\,, x(\,\cdot\,)) \in L^\infty$. Define X as the largest subset of $\overline{X}$ having property:

(X) for every $x \in X$ there exists $\widetilde{x} \in X$ such that

$$-\operatorname{div}(k(y)\nabla\widetilde{x}(y)) = G_x(y, x(y)) \quad \text{a.e. on } \Omega. \tag{1.8}$$

Now we shall derive some important facts about $\overline{X}$.

Proposition 1.7. *$\overline{X} \neq \emptyset$ and $\overline{X}$ has the above property, i.e. $X = \overline{X}$.*

Proof. It is clear that $\overline{z} \in \overline{X}$. Fix any $x \in \overline{X}$. By Theorem 1.5 there exists a unique solution $\overline{x} \in W_0^{1,2}(\Omega, \mathbb{R}) \cap W^{2,\infty}(\Omega, \mathbb{R})$ of the Dirichlet problem for the equation

$$-\operatorname{div}(k(y)\nabla\overline{x}(y)) = G_x(y, x(y)) \quad \text{a.e. on } \Omega.$$

The Sobolev imbedding theorem leads to the conclusion that $\overline{x} \in C_0(\overline{\Omega}) \cap C^1(\Omega)$. Using (G3) ($G_x(y,u)$ is positive for $u \in I$) we obtain, for all nonnegative $\varphi \in C_0^1(\overline{\Omega})$,

$$\begin{aligned}\int_\Omega k(y)\nabla\overline{x}(y)\nabla\varphi(y)\,dy &= -\int_\Omega \operatorname{div}(k(y)\nabla\overline{x}(y))\varphi(y)\,dy \\ &= \int_\Omega G_x(y,x(y))\varphi(y)\,dy \geq 0\end{aligned}$$

and in consequence, by Theorem 1.6, $Q\overline{x} \leq 0$, so that $\overline{x} \geq 0$.

Combining (1.2) with the facts that $G_x(y,\,\cdot\,)$ is increasing and $x \leq \overline{z}$ we infer

$$\operatorname{div}(k\nabla\overline{x}) = -G_x(y,x(y)) \geq -G_x(y,\overline{z}(y)) \geq \operatorname{div}(k\nabla z_0)$$

and further

$$\operatorname{div}(k[\nabla(\overline{x}-z_0)]) \geq 0.$$

Finally we have for all nonnegative $\varphi \in C_0^1(\overline{\Omega})$

$$\int_\Omega k(y)\nabla(\overline{x}-z_0)(y)\nabla\varphi(y)\,dy = -\int_\Omega \operatorname{div}(k(y)[\nabla(\overline{x}-z_0)](y))\varphi(y)\,dy \leq 0$$

which implies $Q(\overline{x}-z_0) \geq 0$.

Applying again Theorem 1.6 we can assert that $\overline{x}-z_0 \leq 0$, so $\overline{x} \leq z_0 \leq \overline{z}$. Summarizing $0 \leq \overline{x} \leq \overline{z}$ and $\overline{x} \in \overline{X}$. Thus $\overline{X}$ has the property X. □

Let

$$\begin{aligned}X^d := \{&p \in W^{1,2}(\Omega,\mathbb{R}) : \text{there exists } x \in X \\ &\text{such that } p(y) = k(y)\nabla x(y) \text{ for a.e. } y \in \Omega \text{ and } \operatorname{div} p \in L^\infty(\Omega,\mathbb{R})\}.\end{aligned}$$

Remark 1.8. For every $x \in X$, there exists $p \in X^d$ satisfying the below relation

$$-\operatorname{div} p(y) = G_x(y,x(y)) \quad \text{for a.e. } y \in \Omega.$$

Proof. Fix $x \in X$. There exists $\widetilde{x} \in X$ such that (1.8) holds. Taking $p(\,\cdot\,) = k(\,\cdot\,)\nabla\widetilde{x}(\,\cdot\,) \in W^{1,2}(\Omega,\mathbb{R})$ we can assert that $p \in X^d$ and, in consequence, the required relation is satisfied. □

2. Duality result

The aim of this section is to develop duality describing the connections between the critical values of J and the dual functional $J_D\colon X^d \to \mathbb{R}$ defined as follows:

$$J_D(p) = \int_\Omega \left\{ -\frac{1}{2k(y)}|p(y)|^2 + G^*(y,-\operatorname{div} p(y))\right\} dy,$$

where $G^*(y,\,\cdot\,)$ ($y\in\Omega$) denotes the Fenchel conjugate of $\widetilde{G}(y,\,\cdot\,)$ and $\widetilde{G}$ is given by

$$\widetilde{G}(y,x)=\begin{cases} G(y,x) & \text{if } x\in I \text{ and } y\in\Omega,\\ \infty & \text{if } x\notin I \text{ and } y\in\Omega.\end{cases}$$

Now we shall use $\widetilde{G}$ to introduce a kind of perturbation of J. We investigate our problem on X only, where $\widetilde{G}(y,x)=G(y,x)$ for $y\in\Omega$, so that we will not change a notation for the functional J containing G or $\widetilde{G}$. We consider the following perturbation $J_x\colon L^2(\Omega,\mathbb{R})\to\mathbb{R}$ of the functional J given by

$$J_x(g)=\int_\Omega\bigg\{-\frac{1}{2}k(y)|\nabla x(y)|^2+\widetilde{G}(y,g(y)+x(y))\bigg\}\,dy,$$

It is clear that $J_x(0)=-J(x)$ for all $x\in X$.

Let us define for every $x\in X$ a type of conjugate $J_x^{\#}\colon X^d\to\mathbb{R}$ of J_x as below

$$\begin{aligned}(2.1)\quad & J_x^{\#}(p)\\ &=\sup_{g\in L^q(\Omega,\mathbb{R})}\int_\Omega\bigg\{\langle g(y),\operatorname{div}p(y)\rangle-\widetilde{G}(y,g(y)+x(y))+\frac{1}{2}k(y)|\nabla x(y)|^2\bigg\}\,dy\\ &=\int_\Omega\bigg\{G^*(y,\operatorname{div}p(y))+\frac{1}{2}k(y)|\nabla x(y)|^2-\langle x(y),\operatorname{div}p(y)\rangle\bigg\}\,dy.\end{aligned}$$

Now we show two auxiliary lemmas.

Lemma 2.1. *For all $p\in X^d$*

$$\sup_{x\in X}(-J_x^{\#}(-p))=-J_D(p). \tag{2.2}$$

Proof. Fix $p\in X^d$. In order to avoid calculation of the conjugate with respect to a nonlinear space (X is not a linear space) we use the special structure of the sets X^d and X. By definition of X^d we infer the existence of $\overline{x}\in X$ satisfying the equality $p(\,\cdot\,)=k(\,\cdot\,)\nabla\overline{x}(\,\cdot\,)$ a.e. on Ω and, in consequence

$$\int_\Omega\bigg\{\langle\nabla\overline{x}(y),p(y)\rangle-\frac{1}{2}k(y)|\nabla\overline{x}(y)|^2\bigg\}\,dy=\int_\Omega\frac{1}{2k(y)}|p(y)|^2\,dy,$$

so that

$$\begin{aligned}(2.3)\quad & \int_\Omega\bigg\{\langle\nabla\overline{x}(y),p(y)\rangle-\frac{1}{2}k(y)|\nabla\overline{x}(y)|^2\bigg\}\,dy\\ &\le\sup_{x\in X}\int_\Omega\bigg\{\langle\nabla x(y),p(y)\rangle-\frac{1}{2}k(y)|\nabla x(y)|^2\bigg\}\,dy\\ &\le\sup_{v\in L^2(\Omega,\mathbb{R}^n)}\int_\Omega\bigg\{\langle v(y),p(y)\rangle-\frac{1}{2}k(y)|v(y)|^2\bigg\}\,dy\\ &=\int_\Omega\frac{1}{2k(y)}|p(y)|^2\,dy=\int_\Omega\bigg\{\langle\nabla\overline{x}(y),p(y)\rangle-\frac{1}{2}k(y)|\nabla\overline{x}(y)|^2\bigg\}\,dy.\end{aligned}$$

(2.3) implies

$$\begin{aligned}&\sup_{x\in X}(-J_x^{\#}(-p))\\&\quad=\sup_{x\in X}\int_\Omega\bigg\{\langle\nabla x(y),p(y)\rangle-\frac{1}{2}k(y)|\nabla x(y)|^2-G^*(y,-\operatorname{div}p(y))\bigg\}\,dy\\&\quad=\int_\Omega\bigg\{\frac{1}{2k(y)}|p(y)|^2-G^*(y,-\operatorname{div}p(y))\bigg\}\,dy=-J_D(p)\end{aligned}$$

what we have claimed. □

Lemma 2.2. *For each $x\in X$*

$$\sup_{p\in X^d}(-J_x^{\#}(-p))=-J(x). \tag{2.4}$$

Proof. To prove this we notice that from Remark 1.8 for each $x\in X$ there exists $\overline{p}\in X^d$ such that for a.e. $y\in\Omega$

$$-\operatorname{div}\overline{p}(y)=G_x(y,x(y)),$$

and further

$$\int_\Omega\{\langle x(y),-\operatorname{div}\overline{p}(y)\rangle-G^*(y,-\operatorname{div}\overline{p}(y))\}\,dy=\int_\Omega G(y,x(y))\,dy.$$

By arguments similar to that in the proof of (2.3), we obtain

$$\begin{aligned}&\sup_{p\in X^d}\int_\Omega\{\langle x(y),-\operatorname{div}p(y)\rangle-G^*(y,-\operatorname{div}p(y))\}\,dy\\&\qquad\qquad=\int_\Omega G^{**}(y,x(y))\,dy=\int_\Omega G(y,x(y))\,dy,\end{aligned}\tag{2.5}$$

where $G^{**}(y,z)=\sup_{x\in\mathbb{R}}\{\langle z,x\rangle-G^*(y,x)\}$ for a.a. $y\in\Omega$ and all $z\in\mathbb{R}$. By (2.5) and (2.1)

$$\begin{aligned}&\sup_{p\in X^d}(-J_x^{\#}(-p))\\&\quad=\sup_{p\in X^d}\int_\Omega\bigg\{\langle x(y),-\operatorname{div}p(y)\rangle-G^*(y,-\operatorname{div}p(y))-\frac{1}{2}k(y)|\nabla x(y)|^2\bigg\}\\&\quad=\int_\Omega\bigg\{-\frac{1}{2}k(y)|\nabla x(y)|^2+G(y,x(y))\bigg\}\,dy=-J(x).\end{aligned}$$ □

Now we have the below duality principle

Theorem 2.3. $\inf_{x\in X}J(x)=\inf_{p\in X^d}J_D(p)$.

Proof. From (2.4) and (2.2)

$$\sup_{x\in X}(-J(x))=\sup_{x\in X}\sup_{p\in X^d}(-J_x^{\#}(-p))=\sup_{p\in X^d}\sup_{x\in X}(-J_x^{\#}(-p))=\sup_{p\in X^d}(-J_D(p)),$$

so that

$$\inf_{x\in X} J(x) = \inf_{p\in X^d} J_D(p). \qquad \square$$

3. Variational principles

This section is devoted to necessary conditions for the existence of the minimizer for (1.7). We present variational principle also for minimizing sequences of functionals J and J_D. This result enables numerical approximation of elements satisfying (1.1).

THEOREM 3.1. *Let $\overline{x} \in X$ satisfy the equality $J(\overline{x}) = \inf_{x\in X} J(x)$. Then there exists $\overline{p} \in X^d$ being a minimizer of J_D:*

$$J_D(\overline{p}) = \inf_{p\in X^d} J_D(p)$$

and such that $-\mathrm{div}\ \overline{p} \in \partial J_{\overline{x}}(0)$ (where $\partial J_{\overline{x}}(0)$ denotes the subdifferential of $J_{\overline{x}}$ at 0). Moreover,

$$J^{\#}_{\overline{x}}(-\overline{p}) + J_{\overline{x}}(0) = 0, \tag{3.1}$$

$$J^{\#}_{\overline{x}}(-\overline{p}) - J_D(\overline{p}) = 0. \tag{3.2}$$

PROOF. Using Remark 1.8 there exists $\overline{p} \in X^d$ such that

$$\int_\Omega \{\langle \overline{x}(y), -\mathrm{div}\,\overline{p}(y)\rangle - G^*(y, -\mathrm{div}\,\overline{p}(y))\}\,dy = \int_\Omega G(y, \overline{x}(y))\,dy.$$

Thus, adding $\int_\Omega\{-(1/2)k(y)|\nabla\overline{x}(y)|^2\}\,dy$ to both sides of the previous assertion we obtain (3.1).

Let $J^*_{\overline{x}}$ denote the Fenchel conjugate of $J_{\overline{x}}$. An easy computation shows that $J^*_{\overline{x}}(-\mathrm{div}\,\overline{p}) = J^{\#}_{\overline{x}}(-\overline{p})$ and, in consequence, by (3.1) and the properties of the subdifferential, we have the inclusion $-\mathrm{div}\,\overline{p} \in \partial J_{\overline{x}}(0)$.

Our task is now to prove that $\overline{p}$ is a minimizer of $J_D\colon X^d \to \mathbb{R}$. Combining the equalities $J_{\overline{x}}(0) = -J(\overline{x})$, (3.1) and Lemma 2.1 we deduce that:

$$-J(\overline{x}) = -J^{\#}_{\overline{x}}(-\overline{p}) \le \sup_{x\in X}(-J^{\#}_{x}(-\overline{p})) = -J_D(\overline{p}).$$

Now Theorem 2.3 leads to the chain of relations

$$J_D(\overline{p}) \le J(\overline{x}) = \inf_{x\in X} J(x) = \inf_{p\in X^d} J_D(p),$$

which is what we have claimed. (3.2) follows from (3.1) and the equalities $J_{\overline{x}}(0) = -J(\overline{x}) = -J_D(\overline{p})$. $\square$

Corollary 3.2. *Assume that $\overline{x} \in X$ is a minimizer of J, then*

$$\begin{cases} -\mathrm{div}[k(y)\nabla\overline{x}(y)] = G_x(y,\overline{x}(y)) & \text{for a.e. } y \in \Omega, \\ \overline{x}|_{\partial\Omega} = 0. \end{cases} \tag{3.3}$$

Proof. By Theorem 3.1 we get the existence of $\overline{p} \in X$ for which (3.1)–(3.2) hold. Hence

$$\int_\Omega \left\{ \frac{1}{2k(y)}|\overline{p}(y)|^2 + \frac{1}{2}k(y)|\nabla\overline{x}(y)|^2 - < \nabla\overline{x}(y), \overline{p}(y) > \right\} dy = 0$$

and

$$\int_\Omega \{G^*(y, -\mathrm{div}\,\overline{p}(y)) + G(y,\overline{x}(y)) - < \overline{x}(y), -\mathrm{div}\,\overline{p}(y) >\}\, dy = 0.$$

Using the properties of the Fenchel conjugate, we obtain for a.e. $y \in \Omega$

$$\begin{aligned} \frac{1}{2k(y)}|\overline{p}(y)|^2 + \frac{1}{2}k(y)|\nabla\overline{x}(y)|^2 - \langle\nabla\overline{x}(y), \overline{p}(y)\rangle &= 0, \\ G^*(y, -\mathrm{div}\,\overline{p}(y)) + G(y,\overline{x}(y)) - \langle\overline{x}(y), -\mathrm{div}\,\overline{p}(y)\rangle &= 0, \end{aligned}$$

so that

$$\overline{p}(y) = k(y)\nabla\overline{x}(y) \quad \text{and} \quad -\mathrm{div}\,\overline{p}(y) = G_x(y,\overline{x}(y))$$

for a.e. $y \in \Omega$. Both equalities imply (3.3). □

Now we prove the numerical version of the above variational principle. We present the result for minimizing sequences that is analogous to the previous theorem.

Theorem 3.3. *Let $\{x_n\}_{n\in\mathbb{N}} \subset X$ be a minimizing sequence of $J: X \to \mathbb{R}$. Then for any $n \in \mathbb{N}$ there exists $p_n \in X^d$ satisfying the relations*

$$\begin{gathered} -\mathrm{div}\, p_n \in \partial J_{x_n}(0), \\ \inf_{n\in\mathbb{N}} J_D(p_n) = \inf_{p\in X^d} J_D(p). \end{gathered}$$

Moreover, for all $n \in \mathbb{N}$,

$$J_{x_n}(0) + J^{\#}_{x_n}(-p_n) = 0 \tag{3.4}$$

and for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$,

$$J^{\#}_{x_n}(-p_n) - J_D(p_n) \le \varepsilon, \tag{3.5}$$

$$|J_D(p_n) - J(x_n)| \le \varepsilon. \tag{3.6}$$

Proof. Our proof starts with the observation that $J: X \to \mathbb{R}$ is bounded below. Indeed, from the definition of X we infer that for all $x \in X$ we have $0 \le x \le \overline{z}$ and further, by the convexity of G in I,

$$\int_\Omega G(y,0)\,dy - \int_\Omega G(y,x(y))\,dy \ge \int_\Omega G_x(y,x(y))(0 - x(y))\,dy$$

and so

$$-\int_\Omega G(y,x(y))\,dy \geq -\int_\Omega G_x(y,x(y))x(y)\,dy - \int_\Omega G(y,0)\,dy.$$

Therefore

$$\begin{aligned}(3.7)\quad J(x) &= \int_\Omega \left\{\frac{1}{2}k(y)|\nabla x(y)|^2 - G(y,x(y))\right\} dy \\ &\geq \int_\Omega \frac{1}{2}k(y)|\nabla x(y)|^2\,dy - \int_\Omega G_x(y,\overline{z}(y))\overline{z}(y)\,dy - \int_\Omega G(y,0)\,dy > -\infty.\end{aligned}$$

From the above estimate it is clear that

$$(3.8)\qquad \inf_{n\in\mathbb{N}} J(x_n) =: c > -\infty.$$

Like in the proof of Theorem 3.1 we obtain for any $n \in \mathbb{N}$ the existence of $p_n \in X^d$ satisfying (3.4) and the relation $-\operatorname{div} p_n \in \partial J_{x_n}(0)$. We proceed to show that $\{p_n\}_{n\in\mathbb{N}}$ is a minimizing sequence for $J_D\colon X^d \to \mathbb{R}$. To this end fix $\varepsilon > 0$. Using (3.8) there exists $n_0 \in \mathbb{N}$ such that, for all $n > n_0$, $c + \varepsilon > J(x_n)$, and further, by the equalities $J_{x_n}(0) = -J(x_n)$, (3.4) and (2.4) we may deduce that, for all $n > n_0$,

$$c + \varepsilon > J(x_n) = J^{\#}_{x_n}(-p_n) \geq \inf_{x\in X}(J^{\#}_x(-p_n)) = J_D(p_n).$$

Moreover, Theorem 2.3 leads to the inequality

$$J_D(p_n) \geq \inf_{p\in X^d} J_D(p) = \inf_{x\in X} J(x) = c \qquad \text{for all } n \in \mathbb{N}.$$

Combining both relations we can assert that $\inf_{p\in X^d} J_D(p) = c$ and, in consequence, using again Theorem 2.3, $\{p_n\}_{n\in\mathbb{N}} \subset X$ is a minimizing sequence of J_D on X^d.

(3.5) and (3.6) follow from the last assertion and the fact that $J^{\#}_{x_n}(-p_n) \leq c + \varepsilon$ for all $n > n_0$. □

As a consequence of the previous theorem we can prove the following

COROLLARY 3.4. *Suppose that $\{x_n\}_{n\in\mathbb{N}} \subset X$ is a minimizing sequence for J on X. Then there exists a minimizing sequence $\{p_n\}_{n\in\mathbb{N}} \subset X^d$ with the property*

$$-\operatorname{div} p_n(y) = G_x(y, x_n(y))$$

for a.e. $y \in \Omega$ and every $n \in \mathbb{N}$. Moreover,

$$(3.9)\qquad \lim_{n\to\infty} \int_\Omega \left\{\frac{1}{2k(y)}|p_n(y)|^2 + \frac{1}{2}k(y)|\nabla x_n(y)|^2 - \langle p_n(y), \nabla x_n(y)\rangle\right\} dy = 0.$$

4. The existence of solutions for the Dirichlet problem

This section is devoted to the existence of a solution of (1.1) being a minimizer of J.

Theorem 4.1. *There exists* $x_0 \in X$ *such that*

$$-\operatorname{div}(k(y)\nabla x_0(y)) = G_x(y, x_0(y)) \quad \text{for a.e. } y \in \Omega.$$

Moreover, x_0 *is a minimizer of* J *on* X*:*

$$J(x_0) = \inf_{x \in X} J(x).$$

Proof. By (3.7) we have the lower boundedness of J on X. Taking into account the conditions made on G we see at once that for $\widetilde{a} \in \mathbb{R}$ large enough the set $P_{\widetilde{a}} = \{x \in X : \widetilde{a} \geq J(x)\}$ is not empty. Now we can choose a minimizing sequence $\{x_m\}_{m\in\mathbb{N}} \subset P_{\widetilde{a}}$ for J. According to (3.7) $\{\nabla x_m\}_{m\in\mathbb{N}}$ is bounded in the norm $\|\cdot\|_{L^2(\Omega,\mathbb{R}^n)}$ and further $\{x_m\}_{m\in\mathbb{N}}$ is bounded in $W_0^{1,2}(\Omega,\mathbb{R})$. Thus, going if necessary to a subsequence, we may deduce that $\{x_m\}_{m\in\mathbb{N}}$ tends weakly to some $x_0 \in W_0^{1,2}(\Omega,\mathbb{R})$. So that we can write, by the Rellich–Kondrashov theorem, $x_m \xrightarrow{m\to\infty} x_0$ in $L^2(\Omega,\mathbb{R})$. Thus we can state pointwise convergence of a certain subsequence (still denoted by $\{x_m\}_{m\in\mathbb{N}}$): for a.e. $y \in \Omega$

$$\lim_{m\to\infty} x_m(y) = x_0(y) \tag{4.1}$$

and in consequence

$$0 \leq x_0(y) \leq \overline{z}(y) \tag{4.2}$$

for a.e. $y \in \Omega$, so that $x_0 \in L^\infty(\Omega,\mathbb{R})$. Using Corollary 3.4 there exists a minimizing sequence $\{p_m\}_{m\in N} \subset X^d$ with the property

$$-\operatorname{div} p_m(y) = G_x(y, x_m(y)) \tag{4.3}$$

for a.e. $y \in \Omega$ and every $m \in \mathbb{N}$. Moreover,

$$\lim_{m\to\infty} \int_\Omega \left\{ \frac{1}{2k(y)}|p_m(y)|^2 + \frac{1}{2}k(y)|\nabla x_m(y)|^2 - \langle p_m(y), \nabla x_m(y)\rangle \right\} = 0. \tag{4.4}$$

By the assumptions concerning G we get, from (4.3)

$$\lim_{m\to\infty}(-\operatorname{div} p_m(y)) = \lim_{m\to\infty} G_x(y, x_m(y)) = G_x(y, x_0(y))$$

and further $\{\operatorname{div} p_m\}_{m\in\mathbb{N}}$ is bounded in $L^\infty(\Omega,\mathbb{R})$. In consequence, we derive the existence of $A > 0$ such that, for $m \in \mathbb{N}$,

$$\left| \int_\Omega \langle \operatorname{div} p_m(y), x_m(y)\rangle \, dy \right| < A.$$

Taking into account (4.4), (4.5) and the boundedness of $\{\nabla x_m\}_{m\in\mathbb{N}}$ in $L^2(\Omega,\mathbb{R}^n)$ it follows that $\{p_m\}_{m\in\mathbb{N}}$ is bounded in $L^2(\Omega,\mathbb{R}^n)$. Thus, by the boundedness of $\{\operatorname{div} p_m\}_{m\in\mathbb{N}}$ in $L^\infty(\Omega,\mathbb{R}) \subset L^2(\Omega,\mathbb{R})$, there exists a subsequence still denoted by $\{p_m\}_{m\in\mathbb{N}}$ weakly convergent to some p_0 in $L^2(\Omega,\mathbb{R}^n)$ and such that $\operatorname{div} p_m \overset{m\to\infty}{\rightharpoonup} z$, where $z \in L^2(\Omega,\mathbb{R})$. Now, we show that $\operatorname{div} p_0 = z$ in $L^2(\Omega,\mathbb{R})$. From the above we have the following chain of relations

$$\begin{aligned}\int_\Omega \langle p_0(y), \nabla h(y)\rangle\,dy &= \lim_{m\to\infty}\int_\Omega \langle p_m(y), \nabla h(y)\rangle\,dy \\ &= -\lim_{m\to\infty}\int_\Omega \langle \operatorname{div} p_m(y), h(y)\rangle\,dy = -\int_\Omega \langle z(y), h(y)\rangle\,dy\end{aligned}$$

for any $h \in C_0^\infty(\Omega,\mathbb{R})$, hence

$$\int_\Omega (\langle p_0(y), \nabla h(y)\rangle + \langle z(y), h(y)\rangle)\,dy = 0,$$

for all $h \in C_0^\infty(\Omega,\mathbb{R})$ and finally, by the Euler–Lagrange lemma, $\operatorname{div} p_0(y) = z(y)$ for a.e. $y \in \Omega$. Thus, by $\operatorname{div} p_m \overset{m\to\infty}{\rightharpoonup} \operatorname{div} p_0$ in $L^2(\Omega,\mathbb{R})$ and $x_m \overset{m\to\infty}{\to} x_0$ in $L^2(\Omega,\mathbb{R})$, we can write

$$\lim_{m\to\infty}\int_\Omega \langle \operatorname{div} p_m(y), x_m(y)\rangle\,dy = \int_\Omega \langle \operatorname{div} p_0(y), x_0(y)\rangle\,dy. \tag{4.6}$$

On account of (4.1), the assumption (G2) (the continuity), we obtain

$$\liminf_{m\to\infty}\int_\Omega G(y, x_m(y))\,dy = \int_\Omega G(y, x_0(y))\,dy.$$

Moreover, we know that $\{\operatorname{div} p_m\}_{m\in\mathbb{N}}$ tends weakly to $\operatorname{div} p_0$ in $L^2(\Omega,\mathbb{R})$ and $L^2(\Omega,\mathbb{R}) \ni z \to \int_\Omega G^*(y, z(y))\,dy$ is weakly lower semicontinuous. This implies

$$\liminf_{m\to\infty}\int_\Omega G^*(y, -\operatorname{div} p_m(y))\,dy \ge \int_\Omega G^*(y, -\operatorname{div} p_0(y))\,dy. \tag{4.7}$$

Combining (4.2), (4.7) and (4.3) we see that

$$\int_\Omega \{G^*(y, -\operatorname{div} p_0(y)) + G(y, x_0(y)) + \langle \operatorname{div} p_0(y), x_0(y)\rangle\}\,dy \le 0. \tag{4.8}$$

Thus, by the properties of the Fenchel transform, we have the equality in (4.8), and, in consequence, for a.e. $y \in \Omega$

$$G^*(y, -\operatorname{div} p_0(y)) + G(y, x_0(y)) + \langle \operatorname{div} p_0(y), x_0(y)\rangle = 0.$$

Finally, we obtain

$$-\operatorname{div} p_0(y) = G_x(y, x_0(y)) \quad \text{for a.e. } y \in \Omega. \tag{4.9}$$

Now using (3.9) and (4.6) we can assert that

$$\begin{aligned}0 &= \lim_{m\to\infty}\int_\Omega \left\{\frac{1}{2k(y)}|p_m(y)|^2 + \frac{1}{2}k(y)|\nabla x_m(y)|^2 - \langle p_m(y), \nabla x_m(y)\rangle\right\}dy \\ &\ge \int_\Omega \left[\frac{1}{2k(y)}|p_0(y)|^2 + \frac{1}{2}k(y)|\nabla x_0(y)|^2 - \langle p_0(y), \nabla x_0(y)\rangle\right]dy.\end{aligned}$$

On account of the last relation, analysis similar to that in the proof of (4.9) shows that

$$p_0(y) = k(y)\nabla x_0(y) \quad \text{for a.e. } y \in \Omega. \tag{4.10}$$

Combining (4.2), (4.9), (4.10) and the relation $\operatorname{div} p_0 \in L^\infty(\Omega, \mathbb{R})$ we derive that $x_0 \in \overline{X}$. (4.9) and (4.10) imply

$$-\operatorname{div}(k(y)\nabla x_0(y)) = G_x(y, x_0(y))$$

for a.e. $y \in \Omega$. Summarizing, the last equality yields $x_0 \in X$.

To prove the assertion $\inf_{x\in X} J(x) = J(x_0)$, it is sufficient to note that

$$\begin{aligned}\inf_{x\in X} J(x) &= \liminf_{m\to\infty} \int_\Omega \left\{\frac{1}{2}k(y)|\nabla x_m(y)|^2 - G(y, x_m(y))\right\} dy \\ &\geq \int_\Omega \frac{1}{2}k(y)|\nabla x_0(y)|^2\, dy - \int_\Omega G(y, x_0(y))\, dy = J(x_0). \qquad \square\end{aligned}$$

Example 4.2. Define $\Omega = \{y = (y_1, y_2) \in \mathbb{R}^2 : y_1, y_2 \in [0, 3]\}$ and

$$\begin{cases} 4.5y_1 - 6y_1^2 & y_1 \in [0, 0.75],\ y_2 \in [0, 3], \\ 4.5(y_1 - 0.75) - 6(y_1 - 0.75)^2 & y_1 \in [0.75, 1.5],\ y_2 \in [0, 3], \\ 4.5(y_1 - 1.5) - 6(y_1 - 1.5)^2 & y_1 \in [1.5, 2.25],\ y_2 \in [0, 3], \\ 4.5(y_1 - 2.25) - 6(y_1 - 2.25)^2 & y_1 \in [2.25, 3],\ y_2 \in [0, 3]. \end{cases}$$

Let us put

$$G_x(y, x) = x + \frac{1}{2}x^6\left(\sin\frac{1}{2}x\right)^2 + \exp\left(-\frac{1}{x^2}\right)$$

and $k(y) = 1$ for all $y \in \Omega$. Then

$$G_{xx}(y, x) = 1 + 3x^5\sin^2\frac{1}{2}x + \frac{1}{4}x^6\sin x + \frac{2\exp(-1/x^2)}{x^3}.$$

We easily check that $G_{xx}(y, x) > 0$ for x from some neighbourhood $\widetilde{I}$ of $I = [0, 4.5]$. That means $G(y, x)$ (the primitive of G_x) is convex in $\widetilde{I}$ and $G_x(y, x) > 0$ in I. However, G is not convex in $[-1, 6]$ and G_x does not satisfy condition (c) in Section 1. The last means that we cannot apply the results of [13]. Let us define

$$\begin{aligned} z^0(y) = h(y)\exp\left(-\frac{1}{(10y_1)^2}\right)&\exp\left(-\frac{1}{(10(3-y_1))^2}\right) \\ &\cdot\exp\left(-\frac{1}{(10y_2)^2}\right)\exp\left(-\frac{1}{(10(3-y_2))^2}\right),\end{aligned}$$

for $y_1, y_2 \in [0,3]$. Now define

$$\overline{z}(y) = \exp\bigg(-\frac{1}{(10y_1)^2}\bigg)\exp\bigg(-\frac{1}{(10(3-y_1))^2}\bigg) \cdot \exp(-\frac{1}{(10y_2)^2})\exp\bigg(-\frac{1}{(10(3-y_2))^2}\bigg),$$

for $y_1, y_2 \in [0,3]$ Since $0 \le h(y) < 1$ and $h_{y_1y_1} = -12$, $y_1, y_2 \in [0,3]$ we easily check that assumption (G1) is satisfied for the above functions $z_0(y)$ and $\overline{z}(y)$. Therefore, we can apply the above theorem to our example to get the existence of positive solution.

5. The existence of a countable set of solutions

Let us introduce the following conditions:

(G1a) there exist $\{a_i\}_{i\in\mathbb{N}} \in C_0^1(\overline{\Omega})$, $\{\overline{a}_i\}_{i\in N} \in W_0^{1,2}(\Omega,\mathbb{R}) \cap W^{2,\infty}(\Omega,\mathbb{R})$ such that for all $y \in \Omega$ and all $i \in \mathbb{N}$,

$$0 < a_i(y) \le \overline{a}_i(y), \qquad G_x(y,a_i) \ge -\operatorname{div}(k(y)\nabla\overline{a}_i(y));$$

(G1b) there exist $\{b_i\}_{i\in\mathbb{N}} \in C_0^1(\overline{\Omega})$, $\{\overline{b}_i\}_{i\in N} \in W_0^{1,2}(\Omega,\mathbb{R}) \cap W^{2,\infty}(\Omega,\mathbb{R})$ such that for all $y \in \Omega$ and all $i \in \mathbb{N}$,

$$0 < \overline{b}_i(y) \le b_i(y), \qquad G_x(y,b_i) \le -\operatorname{div}(k(y)\nabla\overline{b}_i(y));$$

(G1c) for each $i \in \mathbb{N}$, $a_i < b_i < a_{i+1}$;

(G2a) for each $i \in I$, $G:\Omega \times \mathbb{R} \to \mathbb{R}$ is measurable in y, is continuously differentiable and convex with respect to the second variable in some closed neighbourhood $\widetilde{I}_i$ of the interval

$$I_i = [0, \sup_{y\in\Omega} b_i(y)] \quad \text{for all } y \in \Omega, \qquad \bigg|\int_\Omega G(y,a_i(y))\,dy\bigg| < \infty;$$

(G3a) G_x is positive in I_i for each $i \in \mathbb{N}$.

Now for each $i \in \mathbb{N}$ we shall construct the sets of arguments of J and J_D. Let

$$\overline{X}_i = \{x \in W_0^{1,2}(\Omega,\mathbb{R}) \cap W^{2,\infty}(\Omega,\mathbb{R}) : a_i(y) \le x(y) \le b_i(y) \text{ on } \Omega \text{ and } \operatorname{div}(k\nabla x) \in L^\infty(\Omega,\mathbb{R})\}.$$

Define X_i as the largest subset of $\overline{X}_i$ having property:

(X_i) for every $x \in X_i$, there exists $\widetilde{x} \in X_i$ such that

$$-\operatorname{div}(k(y)\nabla\widetilde{x}(y)) = G(y,x(y))$$

Now we shall derive some important facts about X_i. First of all we show that $X_i \ne \emptyset$.

Lemma 5.1. *Assume that conditions* (Ω), (K), (G1a)–(G3a) *are satisfied.* $\overline{X}_i$ *has the above property i.e.* $X_i = \overline{X}_i \neq \emptyset$.

Proof. Fix $x \in \overline{X}_i$. As in the proof of Proposition 1.7 we infer the existence of $\overline{x} \in W_0^{1,2}(\Omega,\mathbb{R}) \cap W^{2,\infty}(\Omega,\mathbb{R})$ being a unique solution of the Dirichlet problem for the equation

$$-\mathrm{div}\,(k(y)\nabla\overline{x}(y)) = G_x(y,x(y)) \quad \text{a.e. on } \Omega.$$

Applying (G1a) we get what follows

$$-\mathrm{div}(k\nabla\overline{x}) = G_x(y,x(y)) \geq G_x(y,a_i(y)) \geq -\mathrm{div}(k\nabla\overline{a}_i)$$

and further for all nonnegative $\varphi \in C_0^1(\overline{\Omega})$

$$\begin{aligned}\int_\Omega k(y)[\nabla\{\overline{x}(y) - \overline{a}_i(y)\}]\nabla\varphi(y)\,dy \\ = -\int_\Omega \mathrm{div}(k(y)[\nabla\{\overline{x}(y) - \overline{a}_i(y)\}])\varphi(y)\,dy \geq 0\end{aligned}$$

and, in consequence,

$$Q(\overline{x} - \overline{a}_i) \leq 0$$

so that, by Theorem 1.6, $\overline{x} \geq \overline{a}_i \geq a_i$.

The same arguments apply to the $\overline{x}$ and $\overline{b}_i$ (assumption (G1b)) give the below chain of relations

$$-\mathrm{div}(k\nabla\overline{x}) = G_x(y,x(y)) \leq G_x(y,b(y)) \leq -\mathrm{div}(k\nabla\overline{b}_i).$$

Thus $-\mathrm{div}(k[\nabla(\overline{x} - \overline{b}_i)]) \leq 0$ and for all nonnegative $\varphi \in C_0^1(\overline{\Omega})$ we have

$$\int_\Omega k(y)\nabla(\overline{x} - \overline{b}_i)(y)\nabla\varphi(y)\,dy = -\int_\Omega \mathrm{div}(k[\nabla(\overline{x} - \overline{b}_i)])\varphi(y)\,dy \leq 0$$

which implies

$$Q(\overline{x} - \overline{b}_i) \geq 0.$$

Applying again Theorem 1.6 we can assert that $\overline{x} - \overline{b}_i \leq 0$, so $\overline{x} \leq \overline{b}_i \leq b_i$. Summarizing $a_i \leq \overline{x} \leq b_i$ and $\overline{x} \in \overline{X}_i$. Thus $\overline{X}_i$ has the property X_i. Finally $\overline{X}_i \subset X_i$. $\square$

Applying Theorem 4.1 for the sequence of nonempty sets X_i give the following main result

Theorem 5.2. *Under hypotheses* (Ω), (K) *and* (G1a)–(G3a) *for each* $i \in \mathbb{N}$, *there exists* $x_i > 0$ *being a solution for* (1.1) *and* $x_i \neq x_j$ *for* $i \neq j$. *Moreover, for all* $i \in \mathbb{N}$, *the element* x_i *is a minimizer of* J *on the set* X_i.

References

[1] R. A. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.

[2] J. Aubin and I. Ekeland, *Applied Nonlinear Analysis*, John Wiley, New York, 1984.

[3] S. Ahmad, A. C. Lazer and J. L. Paul, *Elementary critical point theory and perturbations of elliptic boundary value problems at resonance*, Indiana Univ. Math. J. **25** (1976), 933–944.

[4] A. Ambrosetti and P. H. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Funct. Anal. **14** (1973), 349–381.

[5] A. Benkirane and A. Elmahi, *A strongly nonlinear elliptic equation having natural growth terms and L^1 data*, Nolinear Anal. Ser. A: Theory Methods **39** (2000), no. 4, 403–411.

[6] H. Brezis, *Opérateurs Maximaux Monotones*, North–Holand, Amsterdam, 1973.

[7] L. Cesari, *Optimization — Theory and Applications*, Springer–Verlag, New York, 1983.

[8] K. C. Chang, *Variational methods for nondifferentiable functionals and their applications to partial differential equations*, J. Math. Anal. Appl. **80** (1981), 102–129.

[9] M. Degiovanni and S. Zani, *Multiple solutions of semilinear elliptic equations with one-sided growth conditions, Nonlinear operator theory*, Math. Comput. Modelling **32** (2000), no. 11–13, 1377–1393.

[10] C. Ebmeyer and J. Frehse, *Mixed boundary value problems for nonlinear elliptic equations with p-structure in nonsmooth domains*, Differential Integral Equations **14** (2001), no. 7, 801–820.

[11] I. Ekeland and R. Temam, *Convex Analysis and Variational Problems*, North–Holand, Amsterdam, 1976.

[12] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer–Verlag, 1983.

[13] N. Grenon, *Existence and comparison results for quasilinear elliptic equations with critical growth in the gradient*, J. Differential Equations **171** (2001), no. 1, 1–23.

[14] L. Huang and Y. Li, *Multiple solutions of an elliptic equation*, Fuijan Shifan Daxue Xuebao Ziran Kexue Ban **16** (2000), no. 3, 15–19.

[15] I. Kuzin and S. Pohozaev, *Entire solutions of semilinear elliptic equations*, Progr. Nonlinear Differential Equations Appl. **33** (1997).

[16] J. L. Lions and E. Magenes, *Problémes aux limites non homogénes et applications*, Dunod, Paris, 1968.

[17] P. Magrone, *On a class of semilinear elliptic equations with potential changing sign*, Dynam. System Appl. **9** (2000), no. 4, 459–467.

[18] J. Mawhin, *Metody Wariacyjne dla Nieliniowych Problemów Dirichleta*, Wydawnictwo Naukowo Techniczne, Warszawa, 1994.

[19] J. Mawhin and M. Willem M., *Critical Points Theory and Hamiltonian Systems*, Springer–Verlag, New York, 1989.

[20] Ch. B. Morrey, *Multiple Integrals in the Calculus of Variations*, Springer–Verlag, Berlin, 1966.

[21] J. Milnor, *Morse Theory*, Princeton Univ. Press, Princeton, N.J., 1963.

[22] A. Nowakowski, *Metody wariacyjne dla nieliniowych problemów Dirichleta*, Chapter 6, Wydawnictwo Naukowo-Techniczne, 1994.

[23] A. Nowakowski A. and A. Rogowski, *Periodic solutions in the calculus of variations and differential equations*, Nonlinear Anal. **21** (1993), 537–546.

[24] ———, *Dependence on parameters for the Dirichlet problem with superlinear nonlinearities*, Topol. Methods Nonlinear Anal. **16** (2000), 145–160.

[25] ______, *Multiple positive solutions for a nonlinear Dirichlet problem with nonconvex vector-valued response*, Proc. Roy. Soc. Edinburgh Sec. A **135** (2005), no. 1.

[26] Z. Peihao Z. and Z. Chengkui, *On the infinitely many positive solutions of a supercritical elliptic problem*, Nonlinear Anal., Ser. A: Theory Methods **44** (2001), no. 1, 123–139.

[27] P. H. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations.*, CBMS (1986), AMS, University of Miami.

[28] B. Ricceri, *Existence and location of solutions to the Dirichlet problem for a class of nonlinear elliptic equations*, Appl. Math. Lett. **14** (2001), no. 2, 143–148.

[29] M. Struwe, *Infinitely many critical points for functionals which are not even and applications to superlinear boundary problems*, Manuscripta Math. **32** (1980), 335–364.

[30] Y. Sun and S. Wu, *On a nonlinear elliptic equation with sublinear term at the origin*, Acta Math. Sci. Ser. A Chin. Ed. **20** (2000), no. 4, 461–467.

[31] Z. Wang, *Nonlinear boundary value problems with concave nonlinearities near the origin*, NoDEA Nonlinear Differentian Equations Appl. **8** (2001), no. 1, 15–33.

[32] M. Willem, *Minimax Theorems, Progress in Nonlinear Differential Equations and Their Applications*, vol. 24, Birkhäuser, Basel, Boston, Berlin, 1996.

[33] X. Xu, *The boundary value problem for nonlinear elliptic equations in annular domains*, Acta Math. Sci. Ser. A Chin. Ed., suppl. **20** (2000), 675–683.

Manuscript received January 27, 2005

Andrzej Nowakowski and Aleksandra Orpel
Faculty of Mathematics
University of Łódź
Banacha 22
90-238 Łódź, Poland

E-mail address: annowako@math.uni.lodz.pl, orpela@math.uni.lodz.pl