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RADIAL SOLUTIONS OF SEMILINEAR ELLIPTIC EQUATIONS WITH BROKEN SYMMETRY

Anna M. Candela — Giuliana Palmieri — Addolorata Salvatore

Abstract. The aim of this paper is to prove the existence of infinitely many radial solutions of a superlinear elliptic problem with rotational symmetry and non-homogeneous boundary data.

1. Introduction

The development of variational tools in the last thirty years has allowed one to study widely the nonlinear elliptic problem:

$$(\mathcal{P}_{f,h}) \qquad \begin{cases} -\Delta u = g(x,u) + f(x) & \text{in } \Omega, \\ u = h & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 3$, $f \in L^2(\Omega)$, $h \in H^{1/2}(\partial\Omega)$ and $g\colon \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying the following assumptions:

(G_1) there exist $\mu > 2$ and $\varrho \geq 0$ such that

$$\text{if} \quad (x,s) \in \overline{\Omega} \times \mathbb{R},\ |s| \geq \varrho \quad \text{then} \quad 0 < \mu G(x,s) \leq s g(x,s),$$

with $G(x,u) = \int_0^u g(x,s)\,ds$;

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(G_2) there exist $\alpha_0 \in L^\infty(\Omega)$ and $2 < p < 2^*$, $2^* = 2N/(N-2)$, such that

$$|g(x,s)| \le \alpha_0(x)\ (|s|^{p-1} + 1) \quad \text{for all } (x,s) \in \Omega \times \mathbb{R};$$

(G_3) $g(x,-s) = -g(x,s)$ for all $(x,s) \in \Omega \times \mathbb{R}$.

It is well known that problem $(\mathcal{P}_{f,h})$ has a variational structure; thus, its weak solutions are critical points of functional

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \int_\Omega G(x,u)\,dx - \int_\Omega f u\,dx$$

on manifold $M = \{u \in H^1(\Omega) : u = h \text{ on } \partial\Omega\}$.

If $f = h = 0$, functional I is even on the Hilbert space $H_0^1(\Omega)$ so a direct application of the Symmetric Mountain Pass Theorem implies the existence of infinitely many solutions of problem $(\mathcal{P}_{0,0})$ (see [1]).

On the contrary, if either $f \ne 0$ or $h \ne 0$ the problem loses its symmetry and equivariant variational methods cannot be applied. Anyway, some perturbative techniques allow one to obtain partial results. In fact, the existence of infinitely many solutions of $(\mathcal{P}_{f,h})$ has been proved only if exponent p is not much larger than 2 (see [2], [12], [13], [17] if $h = 0$ and [6]–[9], [11] if $h \ne 0$).

Thus, till now, it seems that the loss of eveness does not allow one to prove multiplicity results for any subcritical p without adding some other symmetry conditions. So, in this paper we investigate the existence of infinitely many solutions of $(\mathcal{P}_{f,h})$ under radial assumptions.

More precisely, we consider the problem

$$\begin{cases} -\Delta u = g(x,u) + f(x) & \text{in } B_R, \\ u = \xi & \text{on } \partial B_R, \end{cases} \tag{1.1}$$

where $\Omega = B_R$ is the open ball of radius $R > 0$ and center 0 in $\mathbb{R}^N$, $\xi \in \mathbb{R}$ and $g \in C(\overline{B}_R \times \mathbb{R})$ satisfies also the radial condition

(G_4) $g(x,s) = g(|x|,s)$ for all $(x,s) \in B_R \times \mathbb{R}$.

An elliptic problem with homogeneous boundary data (i.e. $\xi = 0$) and rotational symmetry has already been studied by Struwe in [14], [15]. In these papers condition (G_3) is not required but additional assumptions are made on the smoothness of the nonlinear term and on the growth of its partial derivative g_s. Anyway, under these further hypotheses a direct approach proves that for each $k \in \mathbb{N}$ the given problem admits at least one radial solution with exactly k zeros.

Here, on the contrary, under simpler hypotheses on the nonlinear term, we state the existence of infinitely many radial solutions of (1.1), also if $\xi \ne 0$, by distinguishing them not according to their nodal properties but with respect to their critical levels. So, we can state the following results.

Theorem 1.1. *Let $g: \overline{B}_R \times \mathbb{R} \to \mathbb{R}$ be a continuous function which satisfies hypotheses* (G_1)–(G_4). *Furthermore, assume G differentiable with respect to x such that*

$$|\nabla_x G(x,s)| \leq \alpha_1(x)\ |sg(x,s)| \quad \text{for all } (x,s) \in \overline{B}_R \times \mathbb{R}, \tag{1.2}$$

for a suitable function $\alpha_1 \in L^\infty(B_R)$. Then, taken any radial function $f \in L^2(B_R)$ and $\xi \in \mathbb{R}$, problem (1.1) *has infinitely many weak radial solutions*

(a) *for each $2 < p < 2^*$ if $N \geq 4$,*
(b) *only for $2 < p < 4$ if $N = 3$.*

Moreover, the found solutions are classical if $f: \overline{B}_R \to \mathbb{R}$ is continuous, too.

Obviously, hypothesis (1.2) is trivial if $g(x,s)$ is independent of x while it can be removed if the boundary condition is homogeneous. In fact, if $\xi = 0$ the previous result can be improved as follows:

Theorem 1.2. *Let $g: \overline{B}_R \times \mathbb{R} \to \mathbb{R}$ be a continuous function which satisfies hypotheses* (G_1)–(G_4). *Then, taken any radial function $f \in L^2(B_R)$ problem* (1.1) *with $\xi = 0$ has infinitely many weak radial solutions*

(a) *for each $2 < p < 2^*$ if $N \geq 4$,*
(b) *for $2 < p < \min\{6, 2\mu\}$ if $N = 3$*

(*here, μ is as in* (G_1)). *Moreover, the found solutions are classical if $f: \overline{B}_R \to \mathbb{R}$ is also continuous.*

Remark 1.3. Clearly, the model function $g(x,s) = \alpha_0(x)|s|^{p-2}s$ satisfies the previous assumptions with $\mu = p$ in (G_1) if α_0 is a continuous radial strictly positive function on $\overline{B}_R$. Then, the previous theorems hold and, in particular, they imply the existence of infinitely many radial solutions for each $p \in (2, 2^*)$ not only if $N \geq 4$ but also if $N = 3$, $\xi = 0$.

Remark 1.4. There are some problems which our results apply to while Struwe's theorems in [14], [15] cannot be used. For example, we can consider $g(s) = (a + b\sin^2 s)\ |s|^{p-2}s$ with a, $b > 0$, $2 < p < 2^*$. Obviously, assumptions (G_2)–(G_4) and (1.2) are satisfied. On the other hand, simple calculations imply that hypothesis (G_1) holds taking $2 < \mu < p$ and $b < a(p/\mu - 1)$; hence, Theorems 1.1 and 1.2 apply. On the contrary, the growth estimates on $g'(s)$, required in [14], [15], are not satisfied.

2. Bolle's perturbation arguments

In order to apply the method introduced by Bolle in [5] for dealing with problems with broken symmetry, we recall the main theorem as stated in [6]. Let us point out that in the original version of this theorem the involved functionals have to be of class C^2 but here we assume they are just of class C^1 according to the further paper [10].

The idea is to consider a continuous path of functionals starting from a symmetric functional J_0 and to prove a preservation result for min-max critical levels in order to get critical points also for the end-point functional J_1 (which is the "true" functional of the non-symmetric problem).

Let H be a Hilbert space equipped with the norm $\|\cdot\|$. Assume that $H = H_- \oplus H_+$, where $\dim(H_-) < +\infty$, and let $(e_k)_{k\geq 1}$ be an orthonormal base of H_+. Consider

$$H_0 = H_-, \quad H_{k+1} = H_k \oplus \mathbb{R}e_{k+1} \quad \text{if } k \in \mathbb{N},$$

so $(H_k)_k$ is an increasing sequence of finite dimensional subspaces of H.

Let $J\colon [0,1] \times H \to \mathbb{R}$ be a C^1-functional and, taken any $\theta \in [0,1]$, set $J_\theta = J(\theta,\cdot)\colon H \to \mathbb{R}$ and $J'_\theta(v) = \partial J(\theta,v)/\partial v$.

Let us set

$$\begin{aligned}\Gamma = \{\gamma \in C(H,H) : \gamma \text{ is odd and there exists } L > 0 \\ \text{such that } \gamma(v) = v \text{ if } \|v\| \geq L\},\end{aligned}$$

$$c_k = \inf_{\gamma\in\Gamma} \sup_{v\in H_k} J_0(\gamma(v)).$$

Assume that:

(A_1) J satisfies a weaker form of the classical Palais–Smale condition: any $((\theta_n, v_n))_n \subset [0,1] \times H$ such that

$$(J(\theta_n, v_n))_n \text{ is bounded} \quad \text{and} \quad \lim_{n\to+\infty} J'_{\theta_n}(v_n) = 0 \tag{2.1}$$

converges up to subsequences;

(A_2) for any $b > 0$ there exists $C_b > 0$ such that if $(\theta, v) \in [0,1] \times H$ then

$$|J_\theta(v)| \leq b \;\Rightarrow\; \left|\frac{\partial J}{\partial\theta}(\theta,v)\right| \leq C_b(\|J'_\theta(v)\| + 1)(\|v\| + 1);$$

(A_3) there exist two continuous maps $\eta_1, \eta_2\colon [0,1] \times \mathbb{R} \to \mathbb{R}$, Lipschitz continuous with respect to the second variable, such that $\eta_1(\theta,\,\cdot\,) \leq \eta_2(\theta,\,\cdot\,)$ and if $(\theta, v) \in [0,1] \times H$ then

$$J'_\theta(v) = 0 \;\Rightarrow\; \eta_1(\theta, J_\theta(v)) \leq \frac{\partial J}{\partial\theta}(\theta,v) \leq \eta_2(\theta, J_\theta(v)); \tag{2.2}$$

(A_4) J_0 is even and for each finite dimensional subspace W of H it results

$$\lim_{\substack{v\in W\\ \|v\|\to+\infty}} \sup_{\theta\in[0,1]} J(\theta,v) = -\infty.$$

For $i \in \{1,2\}$, let $\psi_i\colon [0,1] \times \mathbb{R} \to \mathbb{R}$ be the flow associated to η_i, i.e. the solution of problem

$$\begin{cases} \dfrac{\partial \psi_i}{\partial \theta}(\theta, s) = \eta_i(\theta, \psi_i(\theta, s)), \\ \psi_i(0, s) = s. \end{cases}$$

Note that $\psi_i(\theta,\cdot)$ is continuous, non-decreasing on $\mathbb{R}$ and $\psi_1(\theta,\,\cdot\,) \le \psi_2(\theta,\,\cdot\,)$. Set

$$\overline{\eta}_1(s) = \sup_{\theta\in[0,1]} |\eta_1(\theta,s)|, \quad \overline{\eta}_2(s) = \sup_{\theta\in[0,1]} |\eta_2(\theta,s)|.$$

In this framework, the following abstract result can be proved (for more details and the proof, see [5, Theorem 3] and [6, Theorem 2.2]).

THEOREM 2.1. *There exists $C \in \mathbb{R}$ such that if $k \in \mathbb{N}$ then*

(a) *either J_1 has a critical level $\widetilde{c}_k$ with $\psi_2(1,c_k) < \psi_1(1,c_{k+1}) \le \widetilde{c}_k$,*
(b) *or $c_{k+1} - c_k \le C(\overline{\eta}_1(c_{k+1}) + \overline{\eta}_2(c_k) + 1)$.*

REMARK 2.2. If $\eta_2 \ge 0$ in $[0,1]\times\mathbb{R}$, the function $\psi_2(\,\cdot\,,s)$ is non-decreasing on $[0,1]$. Hence, $c_k \le \widetilde{c}_k$ for all c_k verifying case (a).

3. Variational setting and preliminary lemmas

In order to simplify the variational approach, let us remark that, for fixed $f \in L^2(B_R)$ and $\xi \in \mathbb{R}$, problem (1.1) can be reduced to a new problem with homogeneous boundary conditions. More precisely, $\overline{u}$ is a solution of (1.1) if and only if $\overline{v}$ is a solution of

$$(\mathcal{P}) \qquad \begin{cases} -\Delta v = g(x, v+\xi) + f(x) & \text{in } B_R, \\ v = 0 & \text{on } \partial B_R, \end{cases}$$

where $\overline{u} = \overline{v} + \xi$.

From now on, let (G_1) and (G_2) hold so, by simple calculations, some constants $\beta_i > 0$ exist such that

$$\beta_1|s|^\mu - \beta_2 \le G(x,s) \le \frac{1}{\mu}\, sg(x,s) + \beta_3 \le \beta_4(|s|^p + 1) \tag{3.1}$$

for all $(x,s) \in \overline{B}_R \times \mathbb{R}$. Clearly, it has to be $\mu \le p$.

Thus, studying problem $(\mathcal{P})$ corresponds to looking for critical points of the functional

$$J_1(v) = \frac{1}{2}\int_{B_R} |\nabla v|^2\,dx - \int_{B_R} G(x, v+\xi)\,dx - \int_{B_R} fv\,dx \tag{3.2}$$

on Sobolev space $H_0^1(B_R)$ equipped with the usual norm

$$\|v\|^2 = \int_{B_R} |\nabla v|^2\,dx.$$

Moreover, we denote by $|\cdot|_q$ the usual norm in Lebesgue space $L^q(B_R)$ for any $q \ge 1$.

If, furthermore, g satisfies radial condition (G_4) and f is radial, we just look for radial solutions of $(\mathcal{P})$, that is, critical points of functional (3.2) on the subspace

$$H_r = \{v \in H^1_0(B_R) : v(x) = v(|x|)\}$$

(for some properties of this space, see e.g. [14]).

At last, assume that also condition (G_3) holds.

Now, in order to find multiple critical points of the non-even functional J_1, we want to apply the Bolle's perturbation method. Thus, consider the family of functionals $J\colon [0,1] \times H_r \to \mathbb{R}$ defined as

$$J(\theta, v) = \frac{1}{2}\int_{B_R} |\nabla v|^2\,dx - \int_{B_R} G(x, v + \theta\xi)\,dx - \theta\int_{B_R} fv\,dx. \tag{3.3}$$

Clearly, $J(0,\,\cdot\,)$ is an even functional while $J(1,\,\cdot\,) = J_1$. For simplicity, denote $J_\theta = J(\theta,\,\cdot\,)$.

Classical theorems imply that J is a C^1-functional with

$$\begin{aligned} \frac{\partial J}{\partial \theta}(\theta, v) &= -\xi\int_{B_R} g(x, v + \theta\xi)\,dx - \int_{B_R} fv\,dx, \\ J'_\theta(v)[w] &= \frac{\partial J}{\partial v}(\theta, v)[w] \\ &= \int_{B_R} \nabla v \cdot \nabla w\,dx - \int_{B_R} g(x, v + \theta\xi)w\,dx - \theta\int_{B_R} fw\,dx \end{aligned} \tag{3.4}$$

for all $\theta \in [0,1]$ and $v, w \in H_r$.

From now on, in the proofs of the following lemmas we denote by a_i suitable positive constants.

First of all, we prove a technical lemma which allows one to verify that the functional J in (3.3) satisfies assumptions (A_1)–(A_4) introduced in the previous section.

Lemma 3.1. *Taken any $\delta \in (1/\mu, 1/2)$ two constants $\beta_1(\delta)$, $\beta_2(\delta) > 0$ exist such that for any $(\theta, v) \in [0,1] \times H_r$ it is*

$$\|v\|^2 + |v + \theta\xi|_\mu^\mu \le \beta_1(\delta)(J_\theta(v) - \delta J'_\theta(v)[v]) + \beta_2(\delta), \tag{3.5}$$

$$\begin{aligned} \|v\|^2 + \int_{B_v^+} G(x, v + \theta\xi)\,dx &+ \int_{B_v^+} g(x, v + \theta\xi)(v + \theta\xi)\,dx \\ &\le \beta_1(\delta)\ (J_\theta(v) - \delta J'_\theta(v)[v]) + \beta_2(\delta) \end{aligned} \tag{3.6}$$

where $B_v^+ = \{x \in B_R : |v(x) + \theta\xi| \ge \varrho\}$ (here, ϱ is as in (G_1)*).*

PROOF. Let $\delta \in (1/\mu, 1/2)$ and $(\theta, v) \in [0,1] \times H_r$ be fixed. By definition, it is

$$J_\theta(v) - \delta J_\theta'(v)[v] = \left(\frac{1}{2} - \delta\right)\|v\|^2 + \int_{B_R} (\delta g(x, v+\theta\xi)v - G(x, v+\theta\xi))\,dx - (1-\delta)\theta \int_{B_R} fv\,dx. \tag{3.7}$$

Fixed any $s \in (1/\mu, \delta)$ by (3.1) it is quite simple to see that

$$\begin{aligned}\int_{B_R} &(\delta g(x, v+\theta\xi)v - G(x, v+\theta\xi))\,dx \\ &\geq (\mu s - 1)\int_{B_R} G(x, v+\theta\xi)\,dx + \frac{\delta - s}{2}\int_{B_R} g(x, v+\theta\xi)(v+\theta\xi)\,dx \\ &\quad + \frac{\delta - s}{2}\int_{B_R} g(x, v+\theta\xi)\left(v + \theta\xi - \frac{2\theta\xi\delta}{\delta - s}\right)dx - a_1.\end{aligned}$$

If we denote

$$B_+ = \left\{x \in B_R : |v(x) + \theta\xi| \geq \max\left\{\varrho, \frac{2\theta|\xi|\delta}{\delta - s}\right\}\right\}, \quad B_- = B_R \setminus B_+,$$

it is obvious that

$$\left|\int_{B_-} g(x, v+\theta\xi)\left(v + \theta\xi - \frac{2\theta\xi\delta}{\delta - s}\right)dx\right| \leq a_2,$$

while by (G_1) it is easy to check that

$$\int_{B_+} g(x, v+\theta\xi)\left(v + \theta\xi - \frac{2\theta\xi\delta}{\delta - s}\right)dx \geq 0.$$

Hence, we obtain

$$\begin{aligned}(\mu s - 1)\int_{B_R} G(x, v+\theta\xi)\,dx &+ \frac{\delta - s}{2}\int_{B_R} g(x, v+\theta\xi)(v+\theta\xi)\,dx \\ &\leq \int_{B_R} (\delta g(x, v+\theta\xi)v - G(x, v+\theta\xi))\,dx + a_3.\end{aligned} \tag{3.8}$$

On the other hand, by Young inequality, it is

$$(1-\delta)\theta\left|\int_{B_R} fv\,dx\right| \leq \frac{1-2\delta}{4}\|v\|^2 + a_4,$$

thus, by (3.7) and (3.8) it follows

$$\begin{aligned}\frac{1-2\delta}{4}\|v\|^2 &+ (\mu s - 1)\int_{B_R} G(x, v+\theta\xi)\,dx \\ &+ \frac{\delta - s}{2}\int_{B_R} g(x, v+\theta\xi)(v+\theta\xi)\,dx \leq J_\theta(v) - \delta J_\theta'(v)[v] + a_5.\end{aligned}$$

Whence, this last inequality and (3.1) imply (3.5) while (3.6) follows by

$$\left|(\mu s-1)\int_{B_v^-} G(x,v+\theta\xi)\,dx+\frac{\delta-s}{2}\int_{B_v^-} g(x,v+\theta\xi)(v+\theta\xi)\,dx\right|\le a_6,$$

where $B_v^-=B_R\setminus B_v^+$. □

Lemma 3.2. *If $((\theta_n,v_n))_n\subset[0,1]\times H_r$ is a sequence such that* (2.1) *holds, then it converges up to subsequences.*

Proof. It is easy to prove that (3.5) and (2.1) imply $(v_n)_n$ is bounded; hence, it converges weakly in H_r up to subsequences. Thus, the proof follows by (G_2) and standard arguments. □

Lemma 3.3. *For any $b>0$ there exists $C_b>0$ such that if $(\theta,v)\in[0,1]\times H_r$ then*

$$|J_\theta(v)|\le b\ \Rightarrow\ \left|\frac{\partial J}{\partial\theta}(\theta,v)\right|\le C_b(\|J'_\theta(v)\|+1)(\|v\|+1).$$

Proof. By (3.4) it is

$$\left|\frac{\partial J}{\partial\theta}(\theta,v)\right|\le|\xi|\int_{B_R}|g(x,v+\theta\xi)|\,dx+a_1\|v\|.$$

As we can assume $\varrho\ge1$ (without loss of generality), (G_1) implies that

$$\int_{B_R}|g(x,v+\theta\xi)|\,dx\le\int_{B_v^+} g(x,v+\theta\xi)(v+\theta\xi)\,dx+a_2,$$

hence the proof follows by (3.6). □

In order to determine the "control" functions $\eta_i(\theta,s)$, which need in (A_3), we can apply the result stated in [6, Lemma 4.3] in the non-radial case. Anyway, for completeness, here we prove it by using simpler arguments thanks to the radial symmetry.

Lemma 3.4. *There exists a constant $\overline{C}>0$ such that*

$$(\theta,v)\in[0,1]\times H_r,\ J'_\theta(v)=0\ \Rightarrow\ \left|\frac{\partial J}{\partial\theta}(\theta,v)\right|\le\overline{C}(J_\theta^2(v)+1)^{1/4}. \tag{3.9}$$

Proof. Let $(\theta,v)\in[0,1]\times H_r$ be such that $J'_\theta(v)=0$, i.e.

$$(\mathcal{P}_\theta)\qquad\begin{cases}-\Delta v=g(x,v+\theta\xi)+\theta f(x) & \text{in } B_R,\\ v=0 & \text{on } \partial B_R.\end{cases}$$

Hence, by (3.4) it is

$$\begin{aligned}\frac{\partial J}{\partial\theta}(\theta,v) &= -\,\xi\int_{B_R} g(x,v+\theta\xi)\,dx-\int_{B_R} fv\,dx\\ &=\xi\int_{B_R}(\Delta v+\theta f)\,ds-\int_{B_R} fv\,dx\\ &=\xi\int_{\partial B_R}\frac{\partial v}{\partial\nu}\,d\sigma+\theta\xi\int_{B_R} f\,dx-\int_{B_R} fv\,dx.\end{aligned}$$

Since $v=v(|x|)$ then

$$\int_{\partial B_R}\frac{\partial v}{\partial\nu}\,d\sigma=\dot v(R)\,\omega_{N-1},$$

where $\omega_{N-1}=\int_{\partial B_R} d\sigma$. Thus,

$$\frac{\partial J}{\partial\theta}(\theta,v)=\dot v(R)\xi\omega_{N-1}+\theta\xi\int_{B_R} f\,dx-\int_{B_R} fv\,dx. \tag{3.10}$$

As $\mu>2$, by (3.5) it is

$$\left|\int_{B_R} fv\,dx\right|\le|f|_2\;|v|_2\le a_1|f|_2(|v+\theta\xi|_\mu+a_2)\le a_3(J^2_\theta(v)+1)^{1/(2\mu)}. \tag{3.11}$$

On the other hand, in order to give an estimate on $\dot v(R)$, let us remark that as v is a radial solution of $(\mathcal P_\theta)$ then as one-dimensional function $v=v(\rho)$ (with $\rho=|x|$) it solves

$$(\mathcal P_{\theta,r})\qquad\begin{cases}-\ddot v-\dfrac{N-1}{\rho}\dot v=g(\rho,v+\theta\xi)+\theta f(\rho) & \text{if } \rho\in(0,R],\\ v(R)=0.\end{cases}$$

By multiplying by $\rho^N\dot v$ and integrating on $[0,R]$, simple calculations give

$$\begin{aligned}-\,\dot v^2(R)\frac{R^N}{2}-\frac{N-2}{2}\int_0^R\dot v^2\rho^{N-1}\,d\rho&=-N\int_0^R G(\rho,v+\theta\xi)\rho^{N-1}\,d\rho\\ &+\;G(R,v(R)+\theta\xi)R^N-\int_0^R\frac{\partial G}{\partial\rho}(\rho,v+\theta\xi)\rho^N\,d\rho+\theta\int_0^R f\dot v\rho^N\,d\rho;\end{aligned}$$

then

$$\begin{aligned}\dot v^2(R)\frac{R^N}{2}\le{}& N\int_0^R G(\rho,v+\theta\xi)\rho^{N-1}\,d\rho+|G(R,\theta\xi)|\;R^N\\ &+\int_0^R\left|\frac{\partial G}{\partial\rho}(\rho,v+\theta\xi)\right|\rho^N d\rho+\int_0^R|f\dot v|\rho^N\,d\rho\\ \le{}&\frac{N}{\omega_{N-1}}\int_{B_R} G(x,v+\theta\xi)\,dx+|G(R,\theta\xi)|R^N\\ &+\frac{R}{\omega_{N-1}}\int_{B_R}|\nabla_x G(x,v+\theta\xi)|\,dx+\frac{R}{\omega_{N-1}}\int_{B_R}|f||\nabla v|\,dx.\end{aligned}$$

Let us remark that by definition it is

$$\int_{B_R} G(x, v + \theta\xi)\, dx = \frac{1}{2}\|v\|^2 - J_\theta(v) - \theta \int_{B_R} fv\, dx$$

while by (G_1) and (1.2) it is

$$\int_{B_R} |\nabla_x G(x, v + \theta\xi)|\, dx \le a_4 \int_{B_v^+} g(x, v + \theta\xi)(v + \theta\xi)\, dx + a_5.$$

Hence, (3.11), Lemma 3.1 with $J'_\theta(v) = 0$ and simple calculations imply

$$|\dot v(R)| \le a_6 (J_\theta^2(v) + 1)^{1/4}. \tag{3.12}$$

Thus (3.10)–(3.12) yield the conclusion. □

Remark 3.5. If $\xi = 0$, the result in Lemma 3.4 can be improved by avoiding assumption (1.2). In fact, in this case (3.12) is not more necessary while only (3.11) is needed, so (3.9) can be replaced by

$$\left|\frac{\partial J}{\partial \theta}(\theta, v)\right| \le a_3 (J_\theta^2(v) + 1)^{1/(2\mu)}.$$

Lemma 3.6. *For each finite dimensional subspace W of H_r it results*

$$\lim_{\substack{v \in W \\ \|v\| \to +\infty}} \sup_{\theta \in [0,1]} J(\theta, v) = -\infty.$$

Proof. Since (3.1) implies that $J(\theta, v) \le a_1\|v\|^2 - a_2|v|_\mu^\mu + a_3$, up to some suitable positive constants a_i's, then the conclusion follows by $\mu > 2$ and the equivalence of all norms in a finite dimensional space. □

Remark 3.7. Let us point out that all the lemmas in this section can be obtained even if no radial assumption holds (see e.g. [6], [8]).

4. Growth estimates and proof of the result

In order to apply Theorem 2.1 we need a sequence of finite dimensional subspaces of H_r for introducing a suitable sequence of min-max levels.

To this aim, let us denote by $\lambda_1 < \lambda_2 \le \lambda_3 \ldots$ the eigenvalues of $-\Delta$ acting on the radial functions with homogeneous boundary conditions, i.e. the eigenvalues of problem

$$\begin{cases} -\phi'' - \dfrac{N-1}{\rho}\,\phi' = \lambda\phi & \text{in } (0, R], \\ \phi(R) = 0. \end{cases}$$

As it is well known, classical arguments and the boundary condition imply that

$$\lambda_k = \left(\frac{j_\nu^k}{R}\right)^2 \quad \text{for each } k \ge 1,$$

where $(j_\nu^k)_k$ is the sequence of the positive zeros of Bessel function $\mathcal{J}_\nu$ and $\nu = (N-2)/2$. Moreover, it is

$$\lambda_k \sim \left(\frac{\pi}{R}\right)^2 k^2 \quad \text{for } k \text{ large enough.}$$

Now, if $(u_k)_k$ is an orthonormal basis of eigenfunctions associated to $(\lambda_k)_k$, let us define a sequence of finite dimensional subspaces of H_r as

$$H_1 = \mathbb{R}u_1, \quad H_{k+1} = H_k \oplus \mathbb{R}u_{k+1} \quad \text{if } k \geq 1,$$

and a corresponding sequence of min-max levels as

$$c_k = \inf_{\gamma\in\Gamma} \sup_{v\in H_k} J_0(\gamma(v)) \quad \text{for each } k \geq 1,$$

where Γ is as in Section 2 with $H = H_r$.

Let us remark that by the lemmas in the previous section the path of functionals $(J_\theta)_{\theta\in[0,1]}$ satisfies assumptions (A_1)–(A_4) of Theorem 2.1 with

$$-\eta_1(\theta, s) = \eta_2(\theta, s) = \overline{C}\ (s^2+1)^{1/4} \tag{4.1}$$

or better, by Remark 3.5,

$$-\eta_1(\theta, s) = \eta_2(\theta, s) = \overline{C}\ (s^2+1)^{1/(2\mu)} \quad \text{if } \xi = 0. \tag{4.2}$$

Thus, we have just to estimate the growth of c_k's. It is known that there exists a constant $M > 0$ such that

$$c_k \geq M\ k^{2p/(N(p-2))} \quad \text{for } k \text{ large enough.} \tag{4.3}$$

This inequality follows by a result due to Tanaka (cf. [17]) and a suitable estimate of the number of the negative eigenvalues of the operator $-\Delta + V(x)$ in $H_0^1(B_R)$, briefly $N_-(-\Delta + V(x))$, taken a potential $V = V(x)$ not necessarily radial (see, e.g. [6], [8], [9]).

Here, we want to improve (4.3) by exploiting the radial symmetry of our problem; hence, we give a better estimate of $N_-(-\Delta + V(x))$ in the space of radial functions H_r. Such a result is already known in $\mathbb{R}^N$ (see [3, Theorem 2.5.1 and Section 3.3]) but for completeness we give here the proof in the open sphere B_R.

LEMMA 4.1. *Let $N \geq 3$ and $V \in L^1(B_R)$, $V(x) = V(|x|)$. Then, there exists a constant $c_R > 0$, depending only on the radius R, such that*

$$N_-(-\Delta + V(x)) \leq c_R \int_0^R \rho |V_-(\rho)|\, d\rho$$

with $\rho = |x|$ *and* $V_-(\rho) = \min\{0, V(\rho)\}$.

Proof. By the radial symmetry of the problem, λ is an eigenvalue of $-\Delta + V(x)$ in H_r with eigenfunction φ if and only if it is an eigenvalue of

$$-\frac{d^2}{d\rho^2} + \frac{(N-1)(N-3)}{4\rho^2} + V(\rho)$$

in $H_0^1([0,R])$ with eigenfunction $\widetilde{\varphi} = \rho^{(N-1)/2}\varphi$. Then, it is

$$N_-(-\Delta + V(x)) = N_-\bigg(-\frac{d^2}{d\rho^2} + \frac{(N-1)(N-3)}{4\rho^2} + V(\rho)\bigg),$$

where $N \geq 3$ implies

$$N_-\bigg(-\frac{d^2}{d\rho^2} + \frac{(N-1)(N-3)}{4\rho^2} + V(\rho)\bigg) \leq N_-\bigg(-\frac{d^2}{d\rho^2} + V_-(\rho)\bigg).$$

Thus, without loss of generality, we can restrict to estimate

$$N_-\bigg(-\frac{d^2}{d\rho^2} + V(\rho)\bigg)$$

in the further assumption $V \leq 0$, $V \not\equiv 0$.

Firstly, on one-dimensional Sobolev space $H_0^1([0,R])$ let us introduce the family of operators

$$T_\tau(y) = -\ddot{y} + \tau V(\rho)y, \quad \tau \in [0,1].$$

Clearly, for each $\tau \in [0,1]$ the spectrum of T_τ is a sequence of simple eigenvalues

$$\lambda_1(\tau) < \lambda_2(\tau) < \ldots < \lambda_n(\tau) < \ldots.$$

Taken $k = N_-(-d^2/d\rho^2 + V(\rho))$, we can assume $k > 0$ (otherwise the proof is trivial). So, $\lambda_k(1) < 0 \leq \lambda_{k+1}(1)$ and there exists $\mu_0 > 0$ small enough such that

$$\lambda_k(1) < -\mu_0^2. \tag{4.4}$$

By [3, Theorem S1.3.1] it follows that for any $n \in \mathbb{N}$ function $\lambda_n(\tau)$ is analytic and

$$\lambda_n'(\tau) = \int_0^R V(\rho)|\psi_n(\tau,\rho)|^2\,d\rho,$$

with $\psi_n(\tau,\rho)$ normalized eigenfunction of T_τ corresponding to eigenvalue $\lambda_n(\tau)$. By the way, since V is negative, it is $\lambda_n'(\tau) < 0$. Hence, (4.4) and $\lambda_k(0) > 0$ imply that there exists a unique $\tau_k \in (0,1)$ such that $\lambda_k(\tau_k) = -\mu_0^2$ while $\lambda_k(\tau) < -\mu_0^2$ for all $\tau \in (\tau_k, 1]$. Thus, by the previous arguments, there exist exactly k numbers $\tau_1, \ldots, \tau_k \in (0,1)$ such that for each $n \in \{1, \ldots, k\}$ the equation

$$-\ddot{y} + \mu_0^2 y = -\tau_n V(\rho)y \tag{4.5}$$

admits non trivial solutions in $H_0^1([0,R])$.

Consider the differential operator $\mathcal{L} = -d^2/d\rho^2 + \mu_0^2$ defined on $H_0^1([0,R])$. It is well-known that it is invertible with

$$\mathcal{L}^{-1} f = \int_0^R K(\rho,\sigma) f(\sigma)\, d\sigma,$$

where by classical arguments it is

$$K(\rho,\sigma) = \begin{cases} \dfrac{\sinh(\mu_0(R-\sigma))\ \sinh(\mu_0\rho)}{\mu_0 \sinh(\mu_0 R)} & \text{for } 0 \le \rho < \sigma, \\ \dfrac{\sinh(\mu_0\sigma)\ \sinh(\mu_0(R-\rho))}{\mu_0 \sinh(\mu_0 R)} & \text{for } \sigma < \rho \le R. \end{cases}$$

Clearly, equation (4.5) becomes

$$\mathcal{L}^{-1}(|V(\rho)|y) = \frac{1}{\tau_n}\, y,$$

whence, the new integral operator $\widetilde{\mathcal{L}} f = \mathcal{L}^{-1}(|V(\rho)|f)$, with kernel

$$K_1(\rho,\sigma) = K(\rho,\sigma)|V(\rho)|,$$

has k eigenvalues $1/\tau_n \in (1,+\infty)$. Since $\widetilde{\mathcal{L}}$ is a positive trace class operator, it has a discrete spectrum of positive simple eigenvalues $\nu_1,\dots,\nu_n,\dots$ such that

$$\sum_{n=1}^{+\infty} \nu_n = \int_0^R K_1(\rho,\rho)\, d\rho.$$

Thus, the previous arguments and simple calculations imply that

$$\begin{aligned} k &\le \sum_{n=1}^{k} \frac{1}{\tau_n} = \sum_{\nu_n \ge 1} \nu_n \le \sum_{n=1}^{+\infty} \nu_n = \int_0^R K_1(\rho,\rho)\, d\rho \\ &= \int_0^R \frac{\sinh(\mu_0(R-\rho))\ \sinh(\mu_0\rho)}{\mu_0 \sinh(\mu_0 R)}\ |V(\rho)|\, d\rho \\ &\le \frac{1}{\mu_0} \int_0^R \sinh(\mu_0\rho)\ |V(\rho)|\, d\rho \le c_R \int_0^R \rho\ |V(\rho)|\, d\rho \end{aligned}$$

where $c_R = \sinh(\mu_0 R)/(\mu_0 R) > 0$ since $\sinh t/t$ is increasing in $(0,\mu_0 R]$. □

PROOF OF THEOREM 1.1. As already remarked, Theorem 2.1 applies. Now, in order to state the existence of infinitely many solutions of (1.1), by Remark 2.2 and (4.3) the last step is proving that case (b) cannot occur for all k large enough where, here, by (4.1) condition (b) becomes

$$c_{k+1} - c_k \le \overline{C}_1 (c_{k+1}^{1/2} + c_k^{1/2} + 1) \tag{4.6}$$

for a suitable constant $\overline{C}_1 > 0$.

In fact, if (4.6) holds for all k large enough by [2, Lemma 5.3] a constant $\overline{C}_2 > 0$ and an integer k_0 exist such that

$$c_k \le \overline{C}_2\ k^2 \quad \text{for all } k \ge k_0. \tag{4.7}$$

On the other hand, by (3.1) it is

$$J_0(v) \geq \frac{1}{2}\ \|v\|^2 - \overline{C}_3|v|_p^p - \overline{C}_4$$

for suitable positive constants $\overline{C}_3$, $\overline{C}_4$, so it is

$$c_k \geq b_k - \overline{C}_4, \tag{4.8}$$

where

$$b_k = \inf_{\gamma\in\Gamma}\sup_{v\in H_k} K(\gamma(v)), \quad K(v) = \frac{1}{2}\ \|v\|^2 - \overline{C}_3|v|_p^p.$$

Thus, by [17, Theorem B] it follows that for all $k \in \mathbb{N}$ there exists $v_k \in H_r$, critical point of K, such that

$$K(v_k) \leq b_k, \tag{4.9}$$

and its large Morse index is greater or equal than k, i.e.

$$K''(v_k) = -\Delta - \overline{C}_3 p(p-1)|v_k|^{p-2} \quad \text{has at least } k \text{ non-positive eigenvalues.}$$

Hence, by (4.9) and Lemma 4.1 with $V(x) = -\overline{C}_3 p(p-1)|v_k|^{p-2}$ we obtain

$$k \leq N_-(-\Delta - \overline{C}_3 p(p-1)|v_k|^{p-2}) \leq \overline{C}_5 \int_0^R \rho|v_k(\rho)|^{p-2}d\rho,$$

which implies by Hölder inequality that

$$k \leq \overline{C}_5\bigg(\int_0^R \rho^l d\rho\bigg)^{2/p}\bigg(\int_0^R \rho^{N-1}|v_k(\rho)|^p d\rho\bigg)^{(p-2)/p}, \tag{4.10}$$

with $l = N-1-p(N-2)/2$.

As $l > -1$ and $K'(v_k) = 0$, (4.10) implies

$$k \leq \overline{C}_6\bigg(\int_\Omega |v_k(x)|^p dx\bigg)^{(p-2)/p} \leq \overline{C}_7\ b_k^{(p-2)/p}. \tag{4.11}$$

Whence, by (4.8) and (4.11) it is

$$c_k \geq \overline{C}_8 k^{p/(p-2)} \quad \text{if } k \text{ is large enough.} \tag{4.12}$$

Hence, (4.12) is in contradiction with (4.7) if

$$\frac{p}{p-2} > 2 \Leftrightarrow 2 < p < 4,$$

inequality which gives a further upper bound on a subcritical p only if $N = 3$.

At last, if $f \in C(\overline{B}_R)$, we claim that each weak radial solution u of (1.1) is classical. In fact, standard regularity results imply that $u \in C^{1,\alpha}(\overline{B}_R)$ with $\alpha \in (0,1)$ (see, e.g. [16, Appendix B]). Moreover, since u is radial, it satisfies the scalar equation

$$-\ddot{u} - \frac{N-1}{\rho}\dot{u} = g(\rho, u) + f \quad \text{in } (0, R],$$

thus, as $g(\cdot, u) + f \in C([0, R])$, $\ddot{u}$ is continuous not only in $(0, R]$ but also in 0 (see [4, Lemma 4.1]). $\square$

PROOF OF THEOREM 1.2. If $\xi = 0$ by (4.2) inequality (4.6) can be improved as

$$c_{k+1} - c_k \le \overline{C}_1(c_{k+1}^{1/\mu} + c_k^{1/\mu} + 1).$$

Hence, (4.7) becomes

$$c_k \le \overline{C}_2\ k^{\mu/(\mu-1)} \quad \text{for all } k \ge k_0. \tag{4.13}$$

On the other hand, (4.12) still holds, so it contradicts (4.13) if

$$\frac{\mu}{\mu - 1} < \frac{p}{p-2}, \quad \text{i.e. } p < 2\mu,$$

which is always true for all $p < 2^*$ when $N \ge 4$ since $2\mu > 4$. $\square$

REFERENCES

[1] A. AMBROSETTI AND P.H. RABINOWITZ, *Dual variational methods in critical point theory and applications*, J. Funct. Anal. **14** (1973), 349–381.

[2] A. BAHRI AND H. BERESTYCKI, *A perturbation method in critical point theory and applications*, Trans. Amer. Math. Soc. **267** (1981), 1–32.

[3] F. A. BEREZIN AND M. A. SHUBIN, *The Schrödinger equation*, Kluwer Academic Publishers, Dordrecht, 1991.

[4] H. BERESTYCKI AND P. L. LIONS, *Nonlinear scalar field equations, Part* I, Arch. Rational Mech. Anal. **82** (1983), 313–345.

[5] P. BOLLE, *On the Bolza Problem*, J. Differential Equations **152** (1999), 274–288.

[6] P. BOLLE, N. GHOUSSOUB AND H. TEHRANI, *The multiplicity of solutions in non-homogeneous boundary value problems*, Manuscripta Math. **101** (2000), 325–350.

[7] A. M. CANDELA AND A. SALVATORE, *Multiplicity results of an elliptic equation with non-homogeneous boundary conditions*, Topol. Methods Nonlinear Anal. **11** (1998), 1–18.

[8] A. M. CANDELA, A. SALVATORE AND M. SQUASSINA, *Semilinear elliptic systems with lack of symmetry*, Dynam. Contin. Discrete Impuls. Systems **10** (2003), 181–192.

[9] C. CHAMBERS AND N. GHOUSSOUB, *Deformation from symmetry and multiplicity of solutions in non-homogeneous problems*, Discrete Contin. Dynam. Systems **8** (2001), 267–281.

[10] M. CLAPP, Y. DING AND S. HERNÁNDEZ-LINARES, *Strongly indefinite functionals with perturbed symmetries and multiple solutions of non symmetric elliptic systems*, Preprint.

[11] M. CLAPP, S. HERNÁNDEZ-LINARES AND E. HERNÁNDEZ-MARTÍNEZ, *Linking-preserving perturbations of symmetric functionals*, J. Differential Equations **185** (2002), 181–199.

[12] P. H. RABINOWITZ, *Multiple critical points of perturbed symmetric functionals,*, Trans. Amer. Math. Soc. **272** (1982), 753–769.

[13] M. STRUWE, *Infinitely many critical points for functionals which are not even and applications to superlinear boundary value problems*, Manuscripta Math. **32** (1980), 335–364.

[14] ______, *Infinitely many solutions of superlinear boundary value problems with rotational symmetry*, Arch. Math. **36** (1981), 360–369.

[15] ———, *Superlinear elliptic boundary value problems with rotational symmetry*, Arch. Math. **39** (1982), 233–240.

[16] ———, *Variational methods. Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems*, 3rd Edition, Springer, Berlin, 1996.

[17] K. Tanaka, *Morse indices at critical points related to the Symmetric Mountain Pass Theorem and applications*, Comm. Partial Differential Equations **14** (1989), 99–128.

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Anna M. Candela, Giuliana Palmieri and Addolorata Salvatore
Dipartimento di Matematica
Università degli Studi di Bari
Via E. Orabona 4
70125 Bari, ITALY

E-mail address: candela@dm.uniba.it, palmieri@dm.uniba.it, salvator@dm.uniba.it