

## SYMMETRIC SYSTEMS OF VAN DER POL EQUATIONS

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**ABSTRACT.** We study the impact of symmetries on the occurrence of periodic solutions in systems of van der Pol equations. We apply the equivariant degree theory to establish existence results for multiple nonconstant periodic solutions and classify their symmetries. The computations of the algebraic invariants in the case of dihedral, tetrahedral, octahedral and icosahedral symmetries for a van der Pol system of equations are included.

### 1. Introduction

Many important problems in physics, chemistry, biology, engineering, etc., can be modelled as dynamical systems with symmetries. Existence of symmetries may have an enormous impact on a dynamical process, which can result in a formation of various patterns exhibiting certain particular symmetric properties. Let us mention symmetric networks of coupled identical oscillators (i.e. the systems with an attracting limit cycle, see for example [1]), including the famous Turing model of a ring of identical oscillators (cf. [27], [10]) with the dihedral symmetries. Such models are related, for example, to appearance of turbulence

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in fluid dynamics (cf. [2]), fluctuations in transmission lines (see [31]), periodic reoccurrence of epidemics, travelling waves in neural networks (cf. [28], [30]), etc. Prediction and classification of the appearing and changing patterns in such systems constitute a complex problem.

In a recent paper by Hirano and Rybicki, an interesting technique was developed based on a direct application of the  $S^1$ -degree theory to study the existence of periodic solutions for second order systems of van der Pol equations (see [12]). The equivariant degree theory is a very effective method for analysis of the occurrence of the Hopf/steady-state bifurcation in dynamical systems (cf. [4]–[6], [8], [16], [18]–[20], [31]), but its applications to the existence of periodic solutions of second order differential equations could not be developed in a standard way. In their paper (cf. [12]), Hirano and Rybicki showed that in spite of technical difficulties related to the fact that the usage of the unknown period as an additional parameter does not always permit to establish *a priori* bounds needed for the standard application of the  $S^1$ -degree theory, it is still possible to use this method.

In our paper we explore the approach introduced by Hirano and Rybicki and extend this method to the class of symmetric van der Pol systems, using the primary  $G$ -equivariant degree theory, where  $G = \Gamma \times S^1$  and  $\Gamma$  denotes the symmetry group of the studied system. Following their idea, we show that this method also works in more general situations allowing the prediction of specific symmetric solutions for the corresponding van der Pol systems. For this purpose, we develop algebraic computational formulae for the primary  $\Gamma \times S^1$ -equivariant degree, where  $\Gamma$  is the dihedral, tetrahedral, octahedral and icosahedral group. Since, for these groups the associated system of van der Pol equations can be explicitly described, it is interesting to observe a tremendous impact of rather small symmetry group on the existence of multiple periodic solutions. The most important advantage of this approach is that, based on the symmetric spectral properties of the linearized system, it is possible to directly detect and classify, according to their symmetry properties, the occurrence of periodic solutions in  $\Gamma$ -symmetric dynamical systems. The main goal of this paper is to set up a standard framework for the van der Pol systems with symmetries.

In Section 2, we introduce several examples of symmetric van der Pol systems based on the geometric symmetries of regular polygons, tetrahedron, octahedron and dodecahedron. In Section 7 we present a series of results describing and classifying the symmetry types of different periodic solutions occurring in these systems, based on the equivariant spectral properties of the linearized systems. The necessary algebraic computations for the considered here groups  $D_n$ ,  $A_4$ ,  $S_4$  and  $A_5$  are presented in Section 6. The equivariant fixed-point setting, introduced in [12], is presented in Section 3 and the computational formulae for the

primary equivariant degree, required for this setting, are discussed in Section 4. Section 5 contains (following exactly the work [12]) the general existence result for symmetric van der Pol systems provided the corresponding coefficient in the equivariant degree is different from zero (Theorem 5.3).

The  $S^1$ -equivariant degree theory was developed in [7] and applied to studying bifurcation phenomenon in [8], [16], [19], [29]. This theory was extended by Ize *et al.* (cf. [13], [14]) to the case of a general  $G$ -equivariant degree theory for an arbitrary compact Lie group  $G$  (see also [24], [21] and [22] where important *nonlinear action techniques* relevant to the equivariant degree were developed). The special features of the equivariant degree, which are related to the so-called *primary  $G$ -equivariant degree*, were developed independently of the work of Ize *et al.* in [9]. For the readers convenience we briefly discuss in Appendix the axiomatic approach to the primary equivariant degree theory (see Section 8). For the equivariant background and jargon used in this paper and for more information about properties, usage and applications of the equivariant degree theory, we refer to the related papers [3]–[6], [17].

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**2. Systems of van der Pol equations with symmetries**

The van der Pol equations are related to the so-called self-excited dynamical systems arising in many models of mechanics, electronics and biology. For more information on van der Pol oscillators and related results, we refer to [11], [23], [25], and [26].

We are interested in systems of coupled identical van der Pol equations of the type

$$(2.1) \quad \begin{cases} \ddot{u}_1 + \varepsilon(u_1^2 - a)\dot{u}_1 + c_{11}u_1 + c_{12}u_2 + \dots + c_{1n}u_n = 0, \\ \ddot{u}_2 + \varepsilon(u_2^2 - a)\dot{u}_2 + c_{21}u_1 + c_{22}u_2 + \dots + c_{2n}u_n = 0, \\ \dots\dots\dots \\ \ddot{u}_n + \varepsilon(u_n^2 - a)\dot{u}_n + c_{n1}u_1 + c_{n2}u_2 + \dots + c_{nn}u_n = 0, \end{cases}$$

where  $a > 0$ ,  $\varepsilon > 0$ , admitting certain ‘‘spatial’’ symmetries. The system (2.1) can be reformulated using the vector ‘‘multiplication’’:

$$uv = \begin{bmatrix} u_1v_1 \\ u_2v_2 \\ \vdots \\ u_nv_n \end{bmatrix}, \quad \text{where } u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \quad \text{and } v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

in the following form

$$(2.2) \quad \ddot{u} + \varepsilon(u^2 - \vec{a})\dot{u} + Cu = 0$$

where

$$\vec{a} = \begin{bmatrix} a \\ a \\ \vdots \\ a \end{bmatrix}, \quad u^2 = \begin{bmatrix} u_1^2 \\ u_2^2 \\ \vdots \\ u_n^2 \end{bmatrix}, \quad C = \begin{bmatrix} c_{11} & c_{12} & c_{13} & \dots & c_{1n} \\ c_{21} & c_{22} & c_{23} & \dots & c_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & c_{n3} & \dots & c_{nn} \end{bmatrix}.$$

In the case of one van der Pol equation (i.e. the case of  $n = 1$ ) the existence of a limit cycle follows from the Poincaré–Bendixon theorem. In other words, such an equation describes a self-exciting oscillator.

There are many possible examples of symmetric van der Pol systems of the type (2.2), where the matrix  $C$  is equivariant with respect to a certain group acting on  $u = (u_1, \dots, u_n)$  by permuting its coordinates. Let us discuss some of them.

EXAMPLE 2.1. We consider a ring of  $n$  identical van der Pol oscillators where the interaction takes place only between the neighbouring oscillators (see Figure 1),

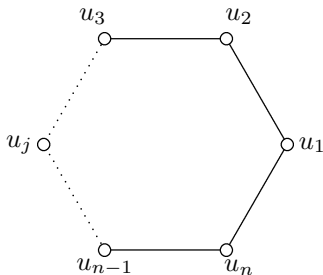


FIGURE 1. System with dihedral symmetries

i.e. in this case the matrix  $C$  is of the type

$$C = \begin{bmatrix} c & d & 0 & \dots & 0 & d \\ d & c & d & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ d & 0 & 0 & \dots & d & c \end{bmatrix}.$$

It is clear that the system (2.2) has the dihedral group  $D_n$  of symmetries.

In the subsequent examples, we present the concrete systems of van der Pol equations modelled on three regular polyhedrons: *tetrahedron*, *octahedron* and *dodecahedron*. In each case, the symmetry group  $\Gamma$  of the system is composed

of the orthogonal symmetries of the corresponding polyhedron. To simplify the presentation, we have considered only those orthogonal symmetries  $T$  for which  $\det T = 1$ . This assumption is not essential, and in the general case, similar results can be easily derived based on the already obtained computations.

**EXAMPLE 2.2.** Let us consider four identical inter-connected van der Pol oscillators having exactly the same linear interaction with all the other oscillators.

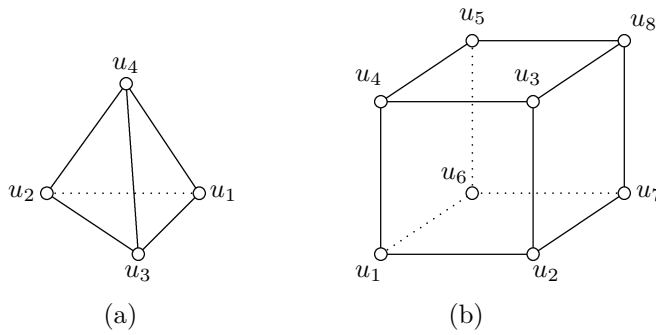


FIGURE 2. (a) System with tetrahedral symmetries, (b) System with octahedral symmetries

In this case, the matrix  $C$  in the system (2.2) can be written as:

$$(2.3) \quad C = \begin{bmatrix} c & d & d & d \\ d & c & d & d \\ d & d & c & d \\ d & d & d & c \end{bmatrix}.$$

The situation is illustrated on Figure 2(a), where the vertices of the tetrahedron symbolize the oscillators and its edges correspond to the connections between the oscillators, indicating the interaction between them. It is clear that this system of differential equations is symmetric with respect to the tetrahedral group  $\mathbb{T} = A_4$ .

**EXAMPLE 2.3.** Suppose that the van der Pol oscillators are arranged in a configuration corresponding to the vertices of a cube. We assume that the interaction takes place between those oscillators that are connected by an edge of the cube (see Figure 2(b)). We assume that all the oscillators are identical.

The eight identical van der Pol oscillators, which are inter-connected like it is illustrated on Figure 2(b), lead to the system of equations with the matrix  $C$

of the following type:

$$(2.4) \quad C = \begin{bmatrix} c & d & 0 & d & 0 & d & 0 & 0 \\ d & c & d & 0 & 0 & 0 & d & 0 \\ 0 & d & c & d & 0 & 0 & 0 & d \\ d & 0 & d & c & d & 0 & 0 & 0 \\ 0 & 0 & 0 & d & c & d & 0 & d \\ d & 0 & 0 & 0 & d & c & d & 0 \\ 0 & d & 0 & 0 & 0 & d & c & d \\ 0 & 0 & d & 0 & d & 0 & d & c \end{bmatrix}.$$

It is clear that the system of van der Pol equations (2.2) is symmetric with respect to the octahedral symmetry group  $\mathbb{O}$  which is isomorphic to the symmetric group  $S_4$ .

EXAMPLE 2.4. Let us consider an arrangement of van der Pol oscillators based on the inter-connections given by the edges of a dodecahedron (see Figure 3).

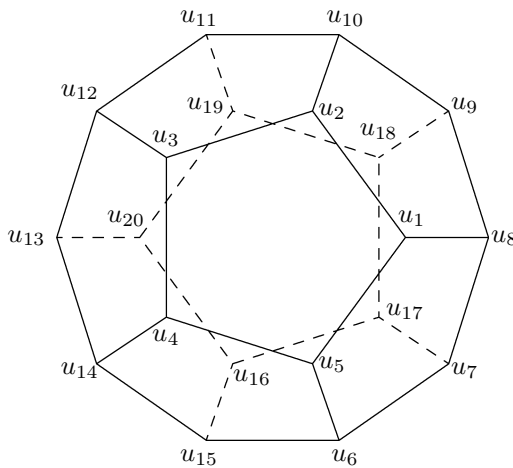


FIGURE 3. System with icosahedral symmetries

It is clear that the group of symmetries of the dodecahedron, which is the icosahedral group  $\mathbb{I}$ , is the symmetry group of the system (2.2). Let us point out that the icosahedral group  $\mathbb{I}$  is isomorphic to the alternating group  $A_5$ . In this case we have the system (2.2) composed of 20 equations, where the matrix  $C$  is

given by:

$$(2.5) \quad C = \begin{pmatrix} c & d & 0 & 0 & d & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ d & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 \\ d & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d \\ 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d \\ 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & d \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & d \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & d & c \end{pmatrix} .$$

Of course, other configurations of the van der Pol oscillators could also be considered, for example based on octahedron, icosahedron or other higher dimensional polyhedra.

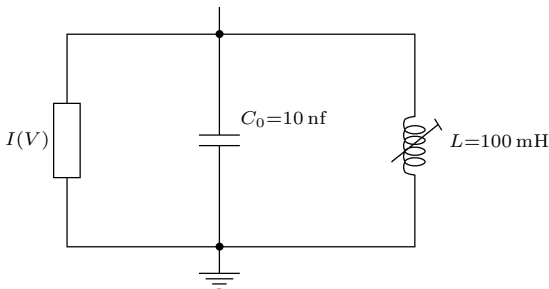


FIGURE 4. The circuit for one van der Pol oscillator

In order to test the obtained in this work results, we believe that it is possible to build an electronic circuit modelling the indicated above systems of van der Pol oscillators. For example, one can use an oscillator which is a parallel inductor-capacitor-resistor (LCR) network. On Figure 4 we show a diagram of such an electronic circuit modelling one van der Pol oscillator. Such a model was studied in [2] to analyze a configuration of three identical van der Pol oscillators, so a similar system with the tetrahedral symmetry group could be also built. However, there may be some technical problems with constructing electronic models for the systems with dihedral, octahedral or icosahedral symmetry groups.

### 3. Reformulation of the problem as an equivariant fixed-point problem with one parameter

In this section we discuss a general strategy based on the application of the equivariant degree allowing to study symmetric periodic solutions to (2.2). The technique presented here was developed in [12].

**3.1. Preliminaries.** Notice that in all the examples discussed above, the space  $V := \mathbb{R}^n$  was an orthogonal representation of a certain finite group  $\Gamma$ , acting on vectors  $u \in \mathbb{R}^n$  by permuting their components, the matrix  $C$  commuted with the action of  $\Gamma$  on  $V$ , and  $\det(C) \neq 0$ . In addition  $C$  was symmetric, i.e.

$$Cu \bullet v = u \bullet Cv, \quad u, v \in \mathbb{R}^n,$$

where  $u \bullet v$  denotes the usual inner product in  $\mathbb{R}^n$ .

By replacing the independent variable  $t$  by  $p\tau/2\pi$ , the equation (2.2) can be rewritten as

$$(3.1) \quad \ddot{u} + \frac{p}{2\pi}\varepsilon(u^2 - \vec{a})\dot{u} + \frac{p^2}{4\pi^2}Cu = 0.$$

Since, we are looking for a periodic solution, the boundary conditions for the system (3.1) are

$$(3.2) \quad u(0) = u(2\pi) \quad \text{and} \quad \dot{u}(0) = \dot{u}(2\pi).$$

Let us put  $\alpha := p/(2\pi)$ , so the equation (3.1) can be rewritten as

$$(3.3) \quad \ddot{u} + \alpha\varepsilon(u^2 - \vec{a})\dot{u} + \alpha^2Cu = 0.$$

Set

$$(3.4) \quad F(u) = \left( \frac{1}{3}u^3 - \vec{a}u \right).$$

Then the equation (3.3) becomes

$$(3.5) \quad \ddot{u} + \alpha\varepsilon \frac{d}{dt}F(u) + \alpha^2Cu = 0.$$

The equation (3.5), together with the periodic boundary conditions (3.2), can be reformulated as a non-linear operator equation in an appropriate Hilbert representation of the group  $G = \Gamma \times S^1$ , where  $\Gamma$  denotes the symmetry group of the system (3.3).

We will need another technical assumption, which is used later to establish *a priori* bounds for the periodic solutions. We will restrict our analysis to the solutions  $u$  of (2.2) satisfying the following additional condition:

$$(3.6) \quad u(t + \pi) = -u(t), \quad \text{for all } t \in \mathbb{R}.$$



Clearly, this technical requirement (originally introduced in [12]) limits the generality of the obtained results, but it seems to be unavoidable under the Hirano–Rybicki treatment of system (2.2). In this way, we transform (2.2) into the following system

$$(3.7) \quad \begin{cases} -\ddot{u} = \alpha\varepsilon(u^2 - \bar{a}^2)\dot{u} + \alpha^2Cu, & u(t) \in V, \\ u(t) = u(t + 2\pi), \dot{u}(t) = \dot{u}(t + 2\pi), u(t + \pi) = -u(t). \end{cases}$$

**3.2. Setting in functional spaces.** Let us introduce the functional spaces, which are appropriate for studying (3.7). First we define the subspace  $\mathbb{H}_o$  of the Sobolev space  $H_{2\pi}^2(\mathbb{R}, V)$  of  $2\pi$ -periodic, twice-differentiable,  $V$ -valued functions, defined as

$$\mathbb{H}_o = \{u \in H_{2\pi}^2(\mathbb{R}, V) : u(t + \pi) = -u(t) \text{ for all } t \in \mathbb{R}\}.$$

We will also identify  $V$  with the space of all constant  $V$ -valued functions. The space  $\mathbb{H}_o$  can be equipped with an inner product, given by

$$\langle u, v \rangle_{\mathbb{H}_o} = \int_0^{2\pi} u(t) \bullet v(t) dt + \int_0^{2\pi} \dot{u}(t) \bullet \dot{v}(t) dt + \int_0^{2\pi} \ddot{u}(t) \bullet \ddot{v}(t) dt.$$

In addition, we define the subspace  $\mathbb{L}_o \subset L^2([0, 2\pi]; V)$  by  $\mathbb{L}_o := L(\mathbb{H}_o)$ , where  $Lu = -\ddot{u}$ . It is clear that  $L: \mathbb{H}_o \rightarrow \mathbb{L}_o$  is an isomorphism. Let us define

$$\mathbb{H} := V \oplus \mathbb{H}_o, \quad \mathbb{L} := V \oplus \mathbb{L}_o.$$

We put

$$K: \mathbb{H} \rightarrow \mathbb{L}, \quad Ku = \frac{1}{2\pi} \int_0^{2\pi} u(t) dt.$$

It is clear that the operator  $K$  is an orthogonal projection on the subspace  $V$  of constant functions and  $L+K: \mathbb{H} \rightarrow \mathbb{L}$  is an isomorphism such that  $(L+K)|_V = \text{id}$  and  $(L+K)|_{\mathbb{H}_o} = L|_{\mathbb{H}_o}$ . Given  $u \in \mathbb{H}$ , denote by  $\bar{u}$  (resp.  $u_o$ ) its orthogonal projection on  $V$  (resp.  $\mathbb{H}_o$ ).

The space  $H_{2\pi}^2(\mathbb{R}; V)$  is a Hilbert representation of the group  $G = \Gamma \times S^1$ , where the element  $(\gamma, e^{i\tau}) \in \Gamma \times S^1$  acts on a function  $u \in H_{2\pi}^2(\mathbb{R}; V)$  by the formula

$$(3.8) \quad (\gamma, e^{i\tau})u(t) = \gamma(u(t + \tau)), \quad \text{for all } t \in \mathbb{R}, \gamma \in \Gamma, e^{i\tau} \in S^1.$$

The  $S^1$ -isotypical components of the space  $H_{2\pi}^2(\mathbb{R}; V)$  are the subspaces  $V_l^c$ ,  $l = 1, 2, \dots$ , and the subspace of constant functions  $V$  (which is the  $S^1$ -fixed-point subspace), where

$$V_l^c = \{\cos lt \cdot a_l + \sin lt \cdot b_l; a_l, b_l \in V\}.$$

A function  $u \in V_l^c$ ,  $u(t) = \cos lt \cdot a_l + \sin lt \cdot b_l$ , can be identified with

$$u(t) = e^{ilt}(x_l + iy_l)$$

where  $x_l = (a_l + b_l)/2$  and  $y_l = (a_l - b_l)/2$ , so the action of  $e^{i\tau} \in S^1$  on  $u(t)$  is simply the complex multiplication by  $e^{il\tau}$ , i.e.  $e^{i\tau} \cdot u(t) = e^{il(t+\tau)}(x_l + iy_l)$ . It is clear that  $V_l^c$  are  $G$ -invariant subspaces of  $H_{2\pi}^2(\mathbb{R}; V)$ ; in addition,  $V_1^c$  (considered as the complex linear space) is  $S^1$ -isomorphic to the complexification of  $V$ . Let  $D(u)(t) = \dot{u}(t)$ , then for  $u(t) = e^{ilt}(x_l + iy_l)$  we have

$$(3.9) \quad D(u) = ilu, \quad \text{and} \quad L(u) = l^2u,$$

so  $L$  and  $D$  preserve  $V_l^c$ ,  $l = 1, 2, \dots$

Notice that  $V_l^c$ ,  $l = 1, 3, 5, \dots$ , are the  $S^1$ -isotypical components of  $\mathbb{H}_o$ .

**3.3. Operator reformulation of the problem (3.6): setting for the equivariant degree treatment.** Let us now reformulate the problem (3.6) as a parameterized  $G$ -equivariant fixed point problem in the space  $\mathbb{H}$ , where  $G = \Gamma \times S^1$ . We consider the following (infinite dimensional) representation of the group  $G$ :

$$\begin{aligned} \mathcal{C}_o := \{u: \mathbb{R} \rightarrow V : u|_{[0, 2\pi]} \in C^1([0, 2\pi], V), \quad u(t + 2\pi) = u(t), \\ u(t + 2\pi) = \dot{u}(t), \quad u(t + \pi) = -u(t) \text{ for all } t \in \mathbb{R}\}, \end{aligned}$$

and  $\mathcal{C} := V \oplus \mathcal{C}_o$ , where  $V$  is identified with the subspace of constant functions. Notice that  $\dot{u} \in \mathbb{L}$  for every function  $u \in \mathcal{C}$ , in particular,  $\dot{u}(t + \pi) = -\dot{u}(t)$  for all  $t \in \mathbb{R}$ , therefore the map  $N: \mathcal{C} \rightarrow L^2([0, 2\pi]; V)$  defined by

$$N(u, \dot{u})(t) = (u^2(t) - \overline{a})\dot{u}(t), \quad t \in \mathbb{R},$$

satisfies

$$N(u, \dot{u})(t + \pi) = -N(u, \dot{u})(t),$$

thus  $N: \mathcal{C} \rightarrow \mathbb{L}$ . It is clear that  $N$  is a continuous map.

We also define the operators:

$$\begin{aligned} j: \mathbb{H} &\hookrightarrow \mathcal{C}, & j(u) &= u, \text{ a.e.} \\ C: \mathcal{C} &\rightarrow \mathbb{L}, & (Cu)(t) &= C(u(t)). \end{aligned}$$

$$(3.10) \quad \begin{array}{ccc} \mathbb{H} & \xrightarrow[L+K]{\approx} & \mathbb{L} \\ & \searrow j & \nearrow N, C \\ & & \mathcal{C} \end{array}$$

The relations between the operators  $L$ ,  $j$ ,  $N$  and  $C$  are illustrated in a diagram (3.10). Notice that the linear operator  $j$  is compact, and  $C$  is a bounded linear operator. In addition, all the above operators are  $G$ -equivariant, where the  $G$ -action on all the above functional spaces is defined by (3.8). The equation (3.7)

can be written in the following operator form:

$$(3.11) \quad L(u) = \alpha \varepsilon N(j(u), D(j(u))) + \alpha^2 C(j(u)), \quad u \in \mathbb{H}.$$

REMARK 3.1. The idea behind a typical usage of the Leray–Schauder degree is based on introducing additional parameters to the original system of differential equations, and allowing its deformation to a linear system. Then, by applying *a priori* bounds to parameterized systems, the existence result can be obtained using the *homotopy property* (cf. Appendix, Theorem 8.3(P3)) of the degree provided that the deformation is *admissible*, i.e. with no zeros on the boundary of the region in question. There is a principal difference between the usage of the Leray–Schauder degree and the equivariant degree (in the setting relevant to one free parameter). Namely, given an equivariant (admissible) map, there is no way to deform it into a linear equivariant one by means of an admissible equivariant deformation — simply equivariant linear maps are never admissible. To overcome this obstacle, Hirano and Rybicki (cf. [12]) used two equivariant deformations: (i) one to connect the original system with a system being “linear up to a cut-off function factor” (the equivariant degree of the corresponding “almost” linear map turns out to be non-zero); (ii) another one to connect the original system with a system giving rise to a non-linear equivariant map of zero equivariant degree. Gluing the two deformations and using the homotopy invariance of the equivariant degree yields the existence of a singular point for the resulting deformation on a *boundary* of the considered region. Combining the last observation with *a priori* bounds and standard cut-off function techniques provides the existence of a periodic solution of the original system.

**3.4. Hirano–Rybicki approach: reduction to a computation of the equivariant degree.** Below we give an exposition of the Hirano–Rybicki idea (see [12]) in the context relevant to our discussion (recall that, in contrast to [12], where the *existence* problem was studied, our main goal is to study *symmetries* of periodic solutions to (3.7)). Following [12], introduce additional parameters  $\delta \in [0, 1]$  and  $\lambda \in \mathbb{R}$  to the equation (3.7):

$$(3.12) \quad \begin{cases} -\ddot{u} = \delta \alpha \varepsilon (u^2 - \vec{a}) \dot{u} + \alpha^2 C u - \lambda \alpha \rho \dot{u}, & u(t) \in V \\ u(t) = u(t + 2\pi), \dot{u}(t) = \dot{u}(t + 2\pi), u(t + \pi) = -u(t) \end{cases}$$

where  $\rho > 0$  is a constant to be specified (see (5.14)).

Assume for a moment that there exists an increasing positive function  $m: \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that every solution  $u_o$  of the system (3.12) for  $\lambda = 0$ , which by the imposed conditions belongs to  $\mathbb{H}_o$ , satisfies the inequality

$$\|u_o\|_{\mathbb{H}_o} \leq m(\alpha)$$

(cf. Lemma 5.1).

Given  $\alpha > 0$ , take  $M > m(\alpha)$ , and choose  $m < m(\alpha)$  to be small enough (cf. proof of Theorem 5.3).

We define  $\theta: \mathbb{R} \rightarrow [0, 1]$  by

$$\eta(t) = \begin{cases} 0 & \text{if } t < m, \\ \frac{t-m}{M-m} & \text{if } m \leq t \leq M, \\ 1 & \text{if } t > M, \end{cases}$$

and put  $\theta(u_o) = \eta(\|u_o\|_{\mathbb{H}_o})$ , where  $u_o \in \mathbb{H}_o$ . We modify the problem (3.12) as follows:

$$(3.13) \quad \begin{cases} -\ddot{u}_o = \delta\alpha\varepsilon(u_o^2 - \bar{a}^2)\dot{u}_o + \alpha^2\theta(u_o)Cu_o - \lambda\alpha\rho\dot{u}_o, & u_o(t) \in V, \\ u_o(t) = u_o(t+2\pi), \dot{u}_o(t) = \dot{u}_o(t+2\pi), & u_o(t+\pi) = -u_o(t). \end{cases}$$

The problem (3.13) can be reformulated as the following parameterized equation in the functional space  $\mathbb{H} = \mathbb{H}_o \oplus V$

$$(3.14) \quad \begin{cases} Lu_o = \delta\alpha\varepsilon N(j(u_o), D(j(u_o))) + \alpha^2\theta(u_o)C(j(u_o)) - \lambda\alpha\rho D(j(u_o)), \\ 0 = \alpha^2\theta(u_o)C\bar{u}. \end{cases}$$

Notice that the equation (3.14) can be written as

$$(3.15) \quad \begin{aligned} (L+K)u &= \delta\alpha\varepsilon N(j(u_o), D(j(u_o))) + \alpha^2\theta(u_o)C(j(u_o)) \\ &\quad + \alpha^2\theta(u_o)C\bar{u} - \lambda\alpha\rho D(j(u_o)) + K(u), \end{aligned}$$

and since  $L+K$  is a  $G$ -equivariant isomorphism, (3.15) is equivalent to

$$(3.16) \quad u = (L+K)^{-1}[\delta\alpha\varepsilon N(j(u_o), D(j(u_o))) + \alpha^2\theta(u_o)C(j(u_o)) + \alpha^2\theta(u_o)C\bar{u} - \lambda\alpha\rho D(j(u_o)) + K(u)].$$

Consequently, the equation (3.16) can be represented as the system of equations

$$(3.17) \quad \begin{cases} u_o = \delta\alpha\varepsilon L^{-1}N(j(u_o), D(j(u_o))) \\ \quad + \alpha^2\theta(u_o)L^{-1}C(j(u_o)) - \lambda\alpha\rho L^{-1}D(j(u_o)), \\ \bar{u} = \alpha^2\theta(u_o)C\bar{u} + \bar{u}. \end{cases}$$

We define  $\tilde{G}(\alpha, \delta, \cdot, \cdot): \mathbb{R} \times \mathbb{H}_o \rightarrow \mathbb{H}_o$ , by

$$(3.18) \quad \begin{aligned} \tilde{G}(\alpha, \delta, \lambda, u_o) &:= \delta\alpha\varepsilon L^{-1}N(j(u_o), D(j(u_o))) \\ &\quad + \alpha^2\theta(u_o)L^{-1}C(j(u_o)) - \lambda\alpha\rho L^{-1}D(j(u_o)), \end{aligned}$$

and  $G(\alpha, \delta, \cdot, \cdot): \mathbb{R} \times \mathbb{H} \rightarrow \mathbb{H}$ , by

$$(3.19) \quad G(\alpha, \delta, \lambda, u) = (\bar{u} + \alpha^2\theta(u_o)C(\bar{u}), \tilde{G}(\alpha, \delta, \lambda, u_o)),$$

where  $u = \bar{u} + u_o$ ,  $\bar{u} \in V$ ,  $u_o \in \mathbb{H}_o$ . Clearly,  $G(\alpha, \delta, \lambda, u)$  is a completely continuous  $G$ -equivariant map.

REMARK 3.2. Notice that, the original van der Pol equation (3.7) corresponds to the case  $\lambda = 0$  and  $\delta = 1$ , except for the nonlinear factor  $\theta(u_o)$  in (3.14). However, if  $\|u_o\|_{\mathbb{H}_o} \geq M$ , then  $\theta(u_o) = 1$  so the solution  $u_o$  of (3.14) is also a solution of (3.7).

REMARK 3.3. In the case of one free parameter, the simplest equivariant maps (needed for the computations of the equivariant degree) turn out to be the so-called basic maps (see Sections 4 and 6), which on the isotypical components have a form

$$(3.20) \quad f(\lambda, v) = (1 - \|v\| + i\beta\lambda)v, \quad \lambda \in \mathbb{R}, \beta > 0.$$

In Section 4, we will show that the term  $-\lambda\alpha\rho D(j(u_o))$  in the system (3.17) corresponds to the term  $i\beta\lambda v$  in (3.20), while  $(1 - \|v\|)v$  corresponds to  $u_o - \alpha^2\theta(u_o)L^{-1}C(j(u_o))$ , i.e. the basic maps (3.20) “emerge” from the “linearized system” (3.27). However, the “linearized system” can not be connected by an admissible homotopy to the original van der Pol system! The idea of Hirano and Rybicki was based on an observation, that the “breaking” of the homotopy occurs for those solutions  $u_o$  with  $\|u_o\|_{\mathbb{H}_o} = M$ , which are in fact the solutions of the original van der Pol system, thus the existence results still can be obtained (see Section 5).

We define

$$(3.21) \quad \Omega := \{(\lambda, u) \in \mathbb{R} \times \mathbb{H} : \lambda \in (-\lambda_o, \lambda_o), m < \|u_o\|_{\mathbb{H}_o} < M, \|\bar{u}\| < 1\},$$

$$(3.22) \quad \Omega_o := \{(\lambda, u_o) \in \mathbb{R} \times \mathbb{H}_o : \lambda \in (-\lambda_o, \lambda_o), m < \|u_o\|_{\mathbb{H}_o} < M\},$$

$$(3.23) \quad B(0, 1) := \{v \in V : \|v\| < 1\},$$

where  $u = \bar{u} + u_o$ ,  $u_o \in \mathbb{H}_o$ ,  $\bar{u} \in V$  and the constant  $\lambda_o > 0$  is a fixed number, which will be specified later. Notice that the set  $\Omega$  is a product of  $\Omega_o \subset \mathbb{R} \times \mathbb{H}_o$  and  $B(0, 1) \subset V$ . The boundary  $\partial\Omega_o$  is composed of three parts:

$$(3.24) \quad \partial_m := \{(\lambda, u_o) \in \bar{\Omega} : \|u_o\|_{\mathbb{H}_o} = m\},$$

$$(3.25) \quad \partial_M := \{(\lambda, u_o) \in \bar{\Omega} : \|u_o\|_{\mathbb{H}_o} = M\},$$

$$(3.26) \quad \partial_o := \{(\lambda, u_o) \in \bar{\Omega} : |\lambda| = \lambda_o\}.$$

Following [12] it is possible to show that for appropriate values of  $\alpha$ ,  $M$  (see Section 5 for more details) the homotopy  $\tilde{G}(\alpha, \delta, \lambda, u_o)$  (see (3.18)) with respect to  $\delta \in [0, \delta_o]$  (where  $\delta_o$  will be chosen to be large enough) has no fixed points in  $\partial_m \cup \partial_o$ . Notice that for  $\delta = 0$  the equation (3.13) can be written as

$$(3.27) \quad u_o = \alpha^2\theta(u_o)L^{-1}C(j(u_o)) - \lambda\alpha\rho L^{-1}D(j(u_o)), \quad u_o \in \mathbb{H}_o.$$

In addition, the equation (3.27) has no solutions in  $\partial_M$ . Let us put

$$(3.28) \quad \mathcal{F}(\lambda, \bar{u}, u_o) := (\bar{u} + \alpha^2 \theta(u_o) C(\bar{u}), \alpha^2 \theta(u_o) L^{-1} C(j(u_o)) - \lambda \alpha \rho L^{-1} D(j(u_o))) \in V \times \mathbb{H}_o.$$

It is possible to show (cf. Proposition 4.1 and [12, Lemma 3.6]) that the primary equivariant degree

$$(3.29) \quad G\text{-Deg}(\text{id} - \mathcal{F}, \Omega) = \sum_{(H)} n_H(H),$$

is different from zero. In the next section we will reduce the computations of (3.29) to studying the equivariant degrees of the *basic maps* related to irreducible  $\Gamma$ - and  $G$ -representations.

On the other hand, it is possible to apply a  $G$ -equivariant homotopy  $\text{id} - \Psi(s, \lambda, u)$ ,  $s \in [0, 1]$ , to the map  $\text{id} - G(\alpha, \delta_o, \lambda, u)$ , where

$$\begin{aligned} u - \Psi(0, \lambda, u) &= u - G(\alpha, \delta_o, \lambda, u) && \text{for } (\lambda, u) \in \bar{\Omega}, \\ u - \Psi(s, \lambda, u) &\neq 0, && \text{for } (\lambda, u) \in \partial\Omega, \end{aligned}$$

and the map  $\text{id} - \Psi(1, \cdot, \cdot)$  satisfies

$$G\text{-Deg}(\text{id} - \Psi(1, \cdot, \cdot), \Omega) = 0.$$

By using the standard argument, it will follow that for every orbit type  $(H_o)$  in  $\Omega$  for which  $n_{H_o}$  is different from zero, there exist  $\delta > 0$  and  $u \in \partial_M$  satisfying

$$\begin{cases} -\ddot{u}_o = \delta \alpha \varepsilon(u_o^2 - \bar{a}) \dot{u}_o + \alpha^2 \theta(u_o) C u_o - \lambda \alpha \rho \dot{u}_o, & u_o(t) \in V, \\ u_o(t) = u_o(t + 2\pi), \dot{u}_o(t) = \dot{u}_o(t + 2\pi), & u_o(t + \pi) = -u_o(t), \end{cases}$$

and having a symmetry at least  $H_o$ . Since  $u_o \in \partial_M$ , we have  $\theta(u_o) = 1$ , so  $u_o$  is a solution of the equation

$$\begin{cases} -\ddot{u}_o = \delta \alpha \varepsilon(u_o^2 - \bar{a}) \dot{u}_o + \alpha^2 C u_o - \lambda \alpha \rho \dot{u}_o, & u_o(t) \in V, \\ u(t) = u(t + 2\pi), \dot{u}(t) = \dot{u}(t + 2\pi), & u(t + \pi) = -u(t). \end{cases}$$

**3.5. Equivariant degree, dominating orbit types and symmetries of periodic solutions.** Let us recall that the primary equivariant degree (3.29) is an element of the free  $\mathbb{Z}$ -module  $A_1(\Gamma \times S^1)$  generated by the conjugacy classes  $(K^{\varphi, l})$  of the so-called *l-folded  $\varphi$ -twisted* subgroups

$$K^{\varphi, l} := \{(\gamma, z) \in K \times S^1 : \psi(\gamma) = z^l\}$$

of  $\Gamma \times S^1$ , where  $K$  is a subgroup of  $\Gamma$  and  $\varphi: K \rightarrow S^1$  is a homomorphism. In the case of an 1-folded  $\psi$ -twisted subgroup  $K^{\varphi, 1}$ , we will denote it by  $K^\varphi$  and call it simply a *twisted* subgroup of  $\Gamma \times S^1$ . Let us consider an orthogonal representation of  $G = \Gamma \times S^1$  and denote by  $U_o$  the orthogonal complement of the subspace  $U^{S^1} = \{u \in U : zu = u \text{ for all } z \in S^1\}$ . The  $S^1$ -action on  $U_o$  induces a

complex structure on  $U_o$ , which implies that  $U_o$  is a complex  $\Gamma$ -representation. Thus, the isotropy groups  $G_u$  of non-zero vectors  $u$  in  $U_o$  are  $l$ -folded  $\varphi$ -twisted subgroups of  $\Gamma \times S^1$ .

We will denote by  $A(\Gamma)$  the Burnside ring of  $\Gamma$  (cf. Subsection 6.1). It was established (see [5] and [16]) that  $A_1(\Gamma \times S^1)$  has a natural structure of an  $A(\Gamma)$ -module.

DEFINITION 3.4. An orbit type  $(H)$  in  $\mathbb{H}$  is called *dominating*, if  $(H)$  is maximal in the class of all  $\varphi$ -twisted one-folded orbit types in  $\mathbb{H}$  with respect to the usual order relation (see Subsection 6.1).

REMARK 3.5. Let  $(H)$  be a dominating orbit type in  $\mathbb{H}$ . Using the maximality property of  $(H)$  it is easy to see that there exists an irreducible subrepresentation  $\mathcal{V} \subset \mathbb{H}$  and a non-zero vector  $u \in \mathcal{V}$  such that  $G_u = H$ . Consequently, the dominating orbit types in  $\mathbb{H}$  can be easily recognized from the isotypical decomposition of  $\mathbb{H}$  and lattices of isotropies of the corresponding to this decomposition irreducible  $G$ -representations.

In what follows, the dominating orbit types will be used to estimate the minimal number of different periodic solutions (as well as their symmetries) for the system (2.2) (cf. Theorem 5.3).

REMARK 3.6. Assume that the van der Pol system (2.2) admits a non-zero periodic (say,  $p$ -periodic) solution  $u_o$  (for a certain value  $\varepsilon > 0$ ) such that  $G_{u_o} \supset H_o$ , where  $(H_o)$  is a dominating orbit type in  $\mathbb{H}$  with  $H_o = K^\varphi$  for some  $K \subset \Gamma$  and  $\varphi: K \rightarrow S^1$ . Then, by the maximality condition,  $(G_{u_o}) = (K^{\varphi, l})$  with  $l \geq 1$ , and the corresponding orbit  $G(u_o)$ ,  $u_o \in \mathbb{H}_o$ , is composed of exactly  $|G/G_{u_o}|_{S^1}$  different periodic functions (where  $|Y|_{S^1}$  denotes the number of  $S^1$ -orbits in  $Y$ ). It is easy to check that the number of  $S^1$ -orbits in  $G/G_{u_o}$  is  $|\Gamma/K|$  (where  $|X|$  stands for the number of elements in  $X$ ). In addition, it follows from the definition of  $l$ -folding and the  $\Gamma \times S^1$ -action on  $\mathbb{H}$  that  $u_o$  is also a  $p/l$ -periodic function corresponding to another element  $u'_o$  in the space  $\mathbb{H}$  with  $G_{u'_o} = H_o$ . Since  $u'_o$  is also a solution to (2.2), we obtain that (2.2) has at least  $|\Gamma/K|$  different periodic solutions with the orbit type exactly  $(H_o)$ .

Let us point out that the indicated above connections between the equivariant degree (3.29) and the existence of multiple symmetric solutions of the van der Pol equations provide a formal way to establish a *topological classification* of periodic solutions of the system (2.2) according to their symmetries.

In fact,  $G\text{-Deg}(\text{id} - \mathcal{F}, \Omega) = \sum_{(H)} n_H(H)$  can be used as a topological invariant containing the information about the structure of the periodic solutions for the equation (2.2). Since the subspace  $V$  was included as a part of the functional space  $\mathbb{H}$ , it permits to “measure” the additional impact of the positive spectrum of the matrix  $C$  on symmetries of periodic solutions of the system (2.2). We refer

to Section 7, where concrete examples of such classifications are presented for  $\Gamma$  being the dihedral, tetrahedral, octahedral and icosahedral groups. Notice that  $G\text{-Deg}(\text{id} - \mathcal{F}, \Omega)$  provides an instant list of possible symmetries of solutions and (at least for dominating orbit types) the minimal number of different periodic solutions.

#### 4. Computations of the equivariant degree: reduction to basic maps

In this section we reduce the computations of the  $G$ -equivariant degree  $G\text{-Deg}(\text{id} - \mathcal{F}, \Omega)$ , where  $\mathcal{F}$  is defined by (3.28), to the computations of the degrees of basic maps.

**4.1. Finite-dimensional reduction.** We, first, study the solution set for the equation

$$(4.1) \quad u_o = \mathcal{F}_o(\lambda, u_o) \stackrel{\text{def}}{\iff} u_o = \alpha^2 \theta(u_o) L^{-1} C(j(u_o)) - \lambda \alpha \rho L^{-1} D(j(u_o)),$$

$u_o \in \mathbb{H}_o$  (in particular, we will show that the solution set is finite-dimensional). The above equation (4.1) can be rewritten as follows:

$$(4.2) \quad \begin{cases} \ddot{u}_o + \alpha^2 \theta(u_o) C u_o - \lambda \alpha \rho \dot{u}_o = 0, & u_o(t) \in V, \\ u_o(t) = u_o(t + 2\pi), \dot{u}_o(t) = \dot{u}_o(t + 2\pi), & u_o(t + \pi) = -u_o(t). \end{cases}$$

Since the matrix  $C$  is nonsingular, symmetric and  $\Gamma$ -equivariant, it is diagonalizable and every eigenspace is a  $\Gamma$ -invariant subspace. Let  $\sigma(C) = \{\mu_s\}$  denote the spectrum of  $C$  and assume that for every  $v \in V$  we have a decomposition  $v = \sum_s v_s$ , where  $v_s$  is an eigenvector corresponding to the eigenvalue  $\mu_s$ . Then, we can split the system (4.2) into

$$(4.3) \quad \begin{cases} \ddot{u}_s + \alpha^2 \theta(u_o) \mu_s u_s - \lambda \alpha \rho \dot{u}_s = 0, & u_o = \sum_s u_s, \\ u_s(t) = u_s(t + 2\pi), \dot{u}_s(t) = \dot{u}_s(t + 2\pi), & u_s(t + \pi) = -u_s(t). \end{cases}$$

Since (4.3) is the system with constant coefficients, it follows that (4.3) has  $2\pi$ -periodic solutions  $u_s$  satisfying  $u_s(t + \pi) = -u_s(t)$  if and only if

$$(4.4) \quad \alpha^2 \theta(u_o) \mu_s = (2r - 1)^2 \quad \text{and} \quad \lambda = 0$$

for some  $r = 1, 2, \dots$ . By construction, the function  $u_o$  lives in  $\Omega_o$  (see (3.22)), therefore  $\theta(u_o) \in (0, 1)$  (see the definition of  $\theta(\cdot)$  and requirements on  $\Omega_o$ ). From this it follows that (4.4) can be satisfied only if

$$(4.5) \quad \mu_s > \frac{1}{\alpha^2} > 0.$$

By the same argument, the requirement for possible values of  $\alpha$  should be

$$(4.6) \quad \alpha \neq \frac{(2r - 1)}{\sqrt{\mu_s}} \quad \text{for all } \mu_s, r = 1, 2, \dots$$



Bearing in mind the isotypical decomposition of  $\mathbb{H}_o$ , formulae (3.9), (4.4) and (4.5), the solution set to (4.3) satisfies the following equations in  $\mathbb{H}_o$ :

$$(4.7) \quad u_{s,r} - \frac{\theta(u_o)\mu_s\alpha^2}{(2r-1)^2}u_{s,r} = 0,$$

where  $r = 1, 2, \dots$  and  $u_{s,r}(t) = e^{(2r-1)it}(x_r + iy_r)$ , for some  $\mu_s$ -eigenvectors  $x_r$  and  $y_r$  of  $C$ . Thus (4.4) and (4.7) give rise to a non-zero solution for (4.3) if (4.5) and (4.6) are satisfied. In particular, since there are only finitely many such  $r > 0$ , the solution set to (4.7) (and respectively, to (4.3)), is finite-dimensional. Combining the above argument with the suspension property of the equivariant degree (see Appendix, Theorem 8.3(P4), and Remark 8.5) one obtains (see (4.1)) that  $G\text{-Deg}(\text{id} - \mathcal{F}_o, \Omega_o) = G\text{-Deg}(\text{id} - F_o, \Omega_1)$ , where

$$F_o(\lambda, v) = \alpha^2\theta(\|v\|)Av - \lambda Tv, \quad (\lambda, v) \in \mathbb{R} \times U,$$

$U$  is a *finite-dimensional*  $G$ -representation such that  $U^{S^1} = \{0\}$ ,  $\Omega_1 = \Omega_o \cap (\mathbb{R} \times U)$  (see Figure 5 and (3.22)),  $A: U \rightarrow U$  is a  $G$ -equivariant nonsingular linear operator with spectrum (cf. (3.9))

$$(4.8) \quad \sigma(A) = \left\{ \frac{\mu_s}{(2r-1)^2} : r = 1, \dots, k, \mu_s > \frac{1}{\alpha^2} \right\}.$$

The linear operator  $T: U \rightarrow U$  is diagonal with respect to the eigenvectors of  $A$ , with all its diagonal entries being positive multiples of  $i$  (cf. (3.9)). Notice that since  $A$  is  $G$ -equivariant, one may consider  $A$  as a complex linear operator. In particular, the set  $(\text{id} - F_o)^{-1}(0) \cap \Omega_1$  is composed of finitely many  $S^1$ -orbits  $S^1(v_0), \dots, S^1(v_R)$ .

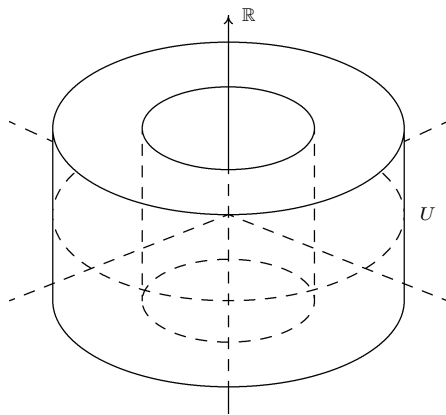


FIGURE 5. The set  $\Omega_1$

**4.2. Isotypical decomposition and basic maps.** In order to compute the  $G$ -degree  $G\text{-Deg}(\text{id} - F_o, \Omega_1)$ , we need to consider the following  $S^1$ -isotypical decomposition  $U = U_1 \oplus \dots \oplus U_k$  of the space  $U$ , where  $U_l$  denotes the isotypical  $S^1$ -component of  $U$  with the  $S^1$ -action given by the complex multiplication  $(\gamma, v) \mapsto \gamma^l \cdot v$ ,  $(\gamma, v) \in S^1 \times U_l$ , and the product “ $\cdot$ ” denotes a complex multiplication. Every subspace  $U_l$  is invariant with respect to the  $\Gamma$ -action. We can consider the  $\Gamma$ -isotypical decomposition of  $U_l$ , which we denote by  $U_l = U_{0,l} \oplus \dots \oplus U_{s,l}$ , where each of the components  $U_{j,l}$ ,  $j = 0, \dots, s$ ,  $l = 1, \dots, k$ , is modelled on the complex irreducible  $\Gamma$ -representation  $\mathcal{V}_j^c$ ,  $j = 1, \dots, s$ , and  $\mathcal{V}_0^c$  being the trivial representation of  $\Gamma$ . It is clear that the space  $\mathcal{V}_j^c$  equipped with the above  $\Gamma \times S^1$ -action is a real irreducible representation of  $G = \Gamma \times S^1$ , which we denote by  $\mathcal{V}_{j,l}$ . Consequently, we obtain the following isotypical decomposition of the space  $U$  with respect to the  $G$ -action:

$$U = \bigoplus_{j,l} U_{j,l}, \quad U_{j,l} \text{ modelled on } \mathcal{V}_{j,l}.$$

For an orthogonal irreducible representation  $\mathcal{V}_{j,l}$  of  $G = \Gamma \times S^1$  such that  $\mathcal{V}_{j,l}^{S^1} = \{0\}$ , we put

$$\mathcal{O} = \{(\lambda, v) \in \mathbb{R} \times \mathcal{V}_{j,l} : 1/2 < \|v\| < 2, -1 < \lambda < 1\}$$

and define the map  $f: \overline{\mathcal{O}} \rightarrow \mathcal{V}_{j,l}$ , by

$$f(\lambda, v) = (1 - \|v\| + i\lambda) \cdot v,$$

where  $(\lambda, v) \in \mathbb{R} \times \mathcal{V}_{j,l}$ . Notice that  $f(\lambda, v) = 0$  if and only if  $1 - \|v\| + i\lambda = 0$ , i.e.  $\lambda = 0$ ,  $\|v\| = 1$ . The map  $f$  will be called a *basic map of the second type*. In what follows, for every  $G$ -irreducible representation  $\mathcal{V}_{j,l}$ , on which  $S^1$  acts non-trivially, we denote by  $(f, \mathcal{O})$  the so-called a  *$\mathcal{V}_{j,l}$ -basic pair*, and we define

$$\text{deg}_{\mathcal{V}_{j,l}} = G\text{-Deg}(f, \mathcal{O}) \in A_1(\Gamma \times S^1).$$

Similarly, let  $\mathcal{V}_j$  be an irreducible representation of  $\Gamma$  and  $\mathcal{B}_j$  be the unit ball in  $\mathcal{V}_j$ . The simplest (in some sense) non-trivial  $\mathcal{B}_j$ -admissible map is  $-\text{id} : \mathcal{V}_j \rightarrow \mathcal{V}_j$ , which we call a *basic map of the first type*. We put

$$\text{deg}_{\mathcal{V}_j} := \Gamma\text{-Deg}(-\text{id}, \mathcal{B}_j) \in A(\Gamma).$$

**4.3. Product formula.** Return to the computations of  $G\text{-Deg}(\text{id} - F_o, \Omega_1)$  and, respectively,  $G\text{-Deg}(\text{id} - \mathcal{F}, \Omega)$  (see (3.28)). Let  $\xi \in \sigma(A)$  be an eigenvalue of the  $G$ -equivariant linear operator  $A: U \rightarrow U$ . Then the eigenspace  $E(\xi) = \{v \in U : Av = \xi v\}$  is a  $G$ -invariant subspace of  $U$ . Clearly, the subspace  $E(\xi)$  can be represented as the direct sum of its  $G$ -isotypical components

$$E(\xi) = \bigoplus_{j,l} E_{j,l}(\xi), \quad E_{j,l}(\xi) \text{ modelled on } \mathcal{V}_{j,l}.$$

We will call the number

$$m_{j,l} = m_{j,l}(\xi) := \dim E_{j,l}(\xi) / \dim \mathcal{V}_{j,l},$$

the  $\mathcal{V}_{j,l}$ -multiplicity of the eigenvalue  $\xi$ . Consider the  $\Gamma$ -equivariant map  $\text{id} - \overline{F}: V \rightarrow V$ ,  $(\text{id} - \overline{F})(\overline{v}) = -\alpha^2 \theta(v_o) C(\overline{v})$ ,  $\overline{v} \in V$  (cf. the second equation in system (3.17)). Let  $\mathcal{B} = B(0,1)$  (cf. (3.23)). Then the  $\Gamma$ -equivariant degree  $\Gamma\text{-deg}(\text{id} - \overline{F}, \mathcal{B}) \in A(\Gamma)$  can be computed as follows: for every eigenvalue  $\mu_o \in \sigma(C)$  such that  $\mu_o > 1/\alpha^2$ , we consider the  $\Gamma$ -isotypical decomposition of the associated with  $\mu_o$  eigenspace  $E(\mu_o) = \bigoplus_j E_j(\mu_o)$ . We put

$$n_j(\mu_o) = \dim E_j(\mu_o) / \dim \mathcal{V}_j.$$

Then we have

$$\Gamma\text{-deg}(\text{id} - \overline{F}, \mathcal{B}) = \prod_{j,s} (\text{deg}_{\mathcal{V}_j})^{n_j(\mu_s)},$$

where the product is taken in the Burnside ring  $A(\Gamma)$  and we assume that  $(\text{deg}_{\mathcal{V}_j})^0 = (\Gamma)$ .<sup>1</sup>

Define  $\text{id} - F: \mathbb{R} \times V \times U \rightarrow V \times U$  by

$$(\text{id} - F)(\lambda, \overline{v}, v) = (-\alpha^2 \theta(v_o) C(\overline{v}), v - \alpha^2 \theta(\|v\|) Av + \lambda T v), \quad (\lambda, \overline{v}, v) \in \mathbb{R} \times V \times U,$$

and put  $\Omega_2 = \Omega \cap (\mathbb{R} \times V \times U)$ . By the argument given in Subsection 4.1,  $G\text{-Deg}(\text{id} - F, \Omega) = G\text{-Deg}(\text{id} - F, \Omega_2)$ . In the statement following below, we present the result for the computation of  $G\text{-Deg}(\text{id} - F, \Omega_2)$ .

**PROPOSITION 4.1.** *Under the notations of this and previous subsections we have*

$$G\text{-Deg}(\text{id} - F, \Omega_2) = \prod_{j,s} (\text{deg}_{\mathcal{V}_j})^{n_j(\mu_s)} \cdot \left[ \sum_{\xi \in \sigma(A)} \sum_{j,l} m_{j,l}(\xi) \text{deg}_{\mathcal{V}_{j,l}} \right]$$

where the product “ $\cdot$ ” denotes the  $A(\Gamma)$  multiplication on the  $\mathbb{Z}$ -module  $A_1(\Gamma \times S^1)$  generated by the twisted conjugacy classes  $(H)$  in  $\Gamma \times S^1$  (cf. [5]).

**PROOF.** Using the homotopy invariance (see Appendix, Theorem 8.3(P3)), we can modify the operator  $A$  (using a small perturbation) in such a way that each eigenvalue  $\xi \in \sigma(A)$  is “simple”, i.e. there exists exactly one  $(j,l)$  such that  $m_{j,l}(\xi) = 1$ . Let us consider an eigenvalue  $\xi \in \sigma(A)$  and suppose that  $E(\xi) = E_{j,l}(\xi)$  for some isotypical component  $U_{j,l}$ . By (4.4) and (4.7), for every eigenvector  $v \in E_{j,l}(\xi)$  we have that  $(\text{id} - F_o)(\lambda, v) = 0$  if  $\lambda = 0$  and  $\alpha^2 \theta(v) \xi = 1$ . Put  $K_{j,l}(\xi) = (\text{id} - F_o)^{-1}(0) \cap \Omega_1 \cap E_{j,l}(\xi)$ . The sets  $K_{j,l}(\xi)$  are compact and it is possible to separate them by choosing small open  $G$ -invariant neighbourhoods  $\Omega_{j,l}(\xi)$  in  $U$ . Notice that for every neighbourhood  $\Omega_{j,l}(\xi)$  the map  $\text{id} - F_o$  is  $G$ -homotopic to a map, which is normal in directions orthogonal

<sup>1</sup>Notice that we always have  $(\text{deg}_{\mathcal{V}_j})^2 = (\Gamma)$ .

to  $E_{j,l}(\xi)$  (see Definition 8.1 and Theorem 8.2 for more details related to the notion of *normality*). Consequently, by the additivity and suspension properties of the  $G$ -equivariant degree (see Appendix, Theorem 8.3, properties (P2), (P4)) we obtain

$$\begin{aligned} G\text{-Deg}(\text{id} - F_o, \Omega_1) &= \sum_{\xi,j,l} G\text{-Deg}(\text{id} - F_o, \Omega_{j,l}(\xi)) \\ &= \sum_{\xi,j,l} G\text{-Deg}((\text{id} - F_o)|_{E_{j,l}(\xi)}, \Omega_{j,l}(\xi) \cap E_{j,l}(\xi)). \end{aligned}$$

On the other hand, it can be easily verified that the map  $(\text{id} - F_o)|_{E_{j,l}(\xi) \cap \Omega_{j,l}(\xi)}$  is  $G$ -homotopic to a basic map on  $\mathcal{V}_{j,l}$ . This reduction to basic maps is fundamental for the computations of the primary degree. Consequently (cf. Appendix, Theorem 8.3(P3)),

$$G\text{-Deg}((\text{id} - F_o)|_{E_{j,l}(\xi)}, \Omega_{j,l}(\xi) \cap E_{j,l}(\xi)) = \text{deg}_{\mathcal{V}_{j,l}}.$$

Therefore, by applying the homotopy and additivity properties again (see Appendix, Theorem 8.3, properties (P3) and (P2)), we get

$$G\text{-Deg}(\text{id} - F_o, \Omega_1) = \sum_{\xi \in \sigma(A)} \sum_{j,l} n_{j,l}(\xi) \text{deg}_{\mathcal{V}_{j,l}}.$$

On the other hand, since  $\text{id} - F$  is a product of two maps  $\text{id} - \overline{F}: V \rightarrow V$  ( $\Gamma$ -equivariant) and  $\text{id} - F_o: \mathbb{R} \times U \rightarrow U$  ( $G$ -equivariant), it follows from the multiplicativity property (see Appendix, Proposition 8.4) that

$$G\text{-Deg}(\text{id} - F, \Omega_2) = \Gamma\text{-deg}(\text{id} - \overline{F}, \mathcal{B}) \cdot G\text{-Deg}(\text{id} - F_o, \Omega_1).$$

Finally, since the map  $\text{id} - \overline{F} = -\alpha^2 \theta(v_o) C: V \rightarrow V$  (notice that since  $\theta(v_o) > 0$ , we can simply consider it to be equal to 1) can be represented by a diagonal-block matrix on the eigenspaces of  $C$ , one has

$$\Gamma\text{-deg}(\text{id} - \overline{F}, \mathcal{B}) = \prod_{j,s} (\text{deg}_{\mathcal{V}_j})^{n_j(\mu_s)},$$

and the result follows.  $\square$

## 5. Existence of symmetric periodic solutions to Van Der Pol systems

Let us recall that we consider the space  $V = \mathbb{R}^n$ , a group  $\Gamma \subset S_n$ , and an  $n \times n$  non-singular matrix  $C$  commuting with the  $\Gamma$ -action on  $V$ . Throughout this section we continue to keep the same notations as in Section 3. We consider a solution to (3.7) as a function living in the  $G$ -space  $\mathbb{H} = \mathbb{H}_o \times V$ , where  $G = \Gamma \times S^1$ ,  $V$  is identified with the  $\Gamma$ -space of constant functions.

As it was indicated in Section 3, in order to provide the equivariant degree treatment to system (3.5) (see also (3.7)), one needs to obtain *a priori* estimates for the solutions.

**5.1. A priori estimates.** The required *a priori* estimates are provided by the two lemmas following below. These statements are parallel to Lemmas 2.2 and 2.3 from [12], where the two-dimensional case was considered (without the symmetry assumption on  $C$ ). Observe that the proofs of Lemmas 5.1 and 5.2 are slight modifications of the arguments presented in [12].

LEMMA 5.1. *There exists an increasing function  $m: \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that for each  $\delta \in (0, 1]$ ,  $\alpha \in \mathbb{R}_+$  and for each solution  $u \in \mathbb{H}_o$  of the system*

$$(5.1) \quad \ddot{u} + \delta\alpha\varepsilon \frac{d}{dt}F(u) + \alpha^2Cu = 0,$$

where  $F$  is given by (3.4), we have  $\|u\|_{\mathbb{H}_o} \leq m(\alpha)$ .

PROOF. Let us fix  $\alpha \in \mathbb{R}_+$  and  $\delta \in (0, 1]$  and assume that  $u$  is a solution to (5.1). Bearing in mind that  $C$  is symmetric and using integration by parts we have

$$(5.2) \quad \int_0^{2\pi} \ddot{u}(t) \bullet \dot{u}(t) dt = 0 \quad \text{and} \quad \int_0^{2\pi} Cu(t) \bullet \dot{u}(t) dt = 0.$$

Thus, by multiplying (5.1) by  $\dot{u}$  and integrating over  $[0, 2\pi]$  we obtain that

$$\begin{aligned} 0 &= \int_0^{2\pi} \left( \ddot{u} + \delta\alpha\varepsilon \frac{d}{dt}F(u) + \alpha^2Cu \right) \bullet \dot{u} dt \\ &= \delta\alpha\varepsilon \int_0^{2\pi} (u^2 - \vec{a})\dot{u} \bullet \dot{u} dt = \delta\alpha\varepsilon \int_0^{2\pi} (u^2 \bullet \dot{u}^2 - a\dot{u} \bullet \dot{u}) dt. \end{aligned}$$

Therefore,

$$(5.3) \quad \int_0^{2\pi} u^2 \bullet \dot{u}^2 dt = \int_0^{2\pi} a\dot{u} \bullet \dot{u} dt = a\|\dot{u}\|_2^2.$$

Since for  $u \in \mathbb{H}_o$ ,  $u(t) = -u(\pi+t)$ , for each component  $u_k(t)$  of  $u(t)$ , there exists  $s_k \in [0, 2\pi]$  such that  $u_k(s_k) = 0$ . Consequently, using integration by parts one easily obtains for every  $t \in [0, 2\pi]$  satisfying  $t > s_k$ :

$$(5.4) \quad u_k^2(t) \leq 2 \int_{s_k}^t |u_k \dot{u}_k| dt \leq 2 \int_0^{2\pi} |u_k \dot{u}_k| dt.$$

Using (5.3) and (5.4) one obtains (by the Cauchy–Schwartz inequality)

$$\begin{aligned} (5.5) \quad \|u\|_2^2 &= \int_0^{2\pi} u \bullet u dt \leq 4\pi \sum_{k=1}^n \int_0^{2\pi} |u_k \dot{u}_k| dt \\ &\leq 4\pi \sqrt{2n\pi} \left( \int_0^{2\pi} u^2 \bullet \dot{u}^2 \right)^{1/2} dt = 2^{5/2} \pi^{3/2} \sqrt{na} \|\dot{u}\|_2. \end{aligned}$$

On the other hand, if we multiply (5.1) by  $u$  and again integrate over  $[0, 2\pi]$ , we get

$$\begin{aligned} 0 &= \int_0^{2\pi} \left( \ddot{u} + \delta\alpha\varepsilon \frac{d}{dt} F(u) + \alpha^2 C u \right) \bullet u \, dt \\ &= - \int_0^{2\pi} \dot{u} \bullet \dot{u} \, dt - \delta\alpha\varepsilon \int_0^{2\pi} \left( \frac{1}{3} u^3 - \vec{a} u \right) \bullet \dot{u} \, dt + \alpha^2 \int_0^{2\pi} C u \bullet u \, dt \\ &\leq -\|\dot{u}\|_2^2 + \alpha^2 \|C\| \|u\|_2^2, \end{aligned}$$

where  $\|C\|$  denotes the operator norm of  $C$ . So, we obtain

$$(5.6) \quad \|\dot{u}\|_2^2 \leq \alpha^2 \|C\| \|u\|_2^2.$$

Then, by (5.5) and (5.6), we get

$$\|u\|_2^2 \leq 2^{5/2} \pi^{3/2} \sqrt{an} \|\dot{u}\|_2 \leq 2^{5/2} \pi^{3/2} \sqrt{an} \alpha \sqrt{\|C\|} \|u\|_2$$

so

$$(5.7) \quad \|u\|_2 \leq 2^{5/2} \pi^{3/2} \alpha \sqrt{na \|C\|} \quad \text{and} \quad \|\dot{u}\|_2 \leq 2^{5/2} \pi^{3/2} \alpha^2 \|C\| \sqrt{na}.$$

Notice that if  $u \in \mathbb{H}_o$  is a (classical) solution to (5.1), then it is clearly of class  $C^2$ . By multiplying the equation (5.1) by  $\ddot{u}$  and integrating it from 0 to  $2\pi$  we obtain by (5.2) and (5.4)

$$(5.8) \quad \begin{aligned} \|\ddot{u}\|_2^2 &\leq \delta\alpha\varepsilon \int_0^{2\pi} u^2(t) \dot{u}(t) \bullet \ddot{u}(t) \, dt + \alpha^2 \|C\| \|\dot{u}\|_2^2 \\ &\leq 2\delta\alpha\varepsilon \|u\|_2 \|\dot{u}\|_2^2 \|\ddot{u}\|_2 + \alpha^2 \|C\| \|\dot{u}\|_2^2, \end{aligned}$$

so

$$\begin{aligned} \|\ddot{u}\|_2 &\leq \delta\alpha\varepsilon \|u\|_2 \|\dot{u}\|_2^2 + \sqrt{(\delta\alpha\varepsilon \|u\|_2 \|\dot{u}\|_2^2)^2 + \alpha^2 \|C\| \|\dot{u}\|_2^2} \\ &\leq 2\delta\alpha\varepsilon \|u\|_2 \|\dot{u}\|_2^2 + \alpha \sqrt{\|C\|} \|\dot{u}\|_2. \end{aligned}$$

Since the norms  $\|u\|_2$  and  $\|\dot{u}\|_2$  are bounded, it follows from (5.7) that

$$\|\ddot{u}\|_2 \leq 2^{17/2} \pi^{9/2} \delta\alpha^6 \varepsilon \|C\|^{5/2} (na)^{3/2} + 2^{5/2} \pi^{3/2} \alpha^3 \sqrt{na} \|C\|^{3/2}.$$

Therefore, it is to observe that  $\|u\|_{\mathbb{H}_o} \leq m(\alpha)$ , where

$$m(\alpha) := 2^{5/2} \pi^{3/2} \alpha \sqrt{na} \|C\|^{1/2} \sqrt{1 + (\alpha \|C\|^{1/2})^2 + (2^6 \pi \delta \alpha^5 \varepsilon \|C\|^2 na + \alpha^2 \|C\|)^2}.$$

Notice that  $m(\alpha)$  is clearly increasing.  $\square$

LEMMA 5.2. *For every  $\tilde{\alpha} > 0$  there exists  $\delta_1(\tilde{\alpha}) > 0$  such that the equation (5.1) has no non-zero solution in  $\mathbb{H}_o$  for all  $\alpha \in (0, \tilde{\alpha})$  and  $\delta > \delta_1(\tilde{\alpha})$ .*

PROOF. Fix  $\tilde{\alpha} > 0$  and take  $\alpha \in (0, \tilde{\alpha})$ . Let  $m(\cdot)$  be a function provided by Lemma 5.1. Let  $u \in \mathbb{H}_o$  be a solution to (5.1). By multiplying (5.1) by  $\dot{u}$  and integrating it over  $[0, 2\pi]$ , we get

$$(5.9) \quad 0 = \delta\alpha\varepsilon \int_0^{2\pi} (u^2 - \bar{a})\dot{u} \bullet \dot{u} dt.$$

Combining (5.9) with the condition  $u(t + \pi) = -u(t)$  for all  $t$ , and using the standard continuity argument, one can find  $t_0 \in [0, 2\pi]$  and  $k \in \{1, \dots, n\}$  such that  $u_k(t_0) = \sqrt{a}$  and  $\dot{u}_k(t_0) \leq 0$ . Notice, in particular, that  $\|u\|_\infty \geq \sqrt{a}$ . Since  $u(t) = -u(t - \pi)$ ,  $\dot{u}(t) = -\dot{u}(t - \pi)$ , and  $F(u)$  is an odd function, we have

$$\begin{aligned} 0 &= \int_{t_0-\pi}^{t_0} \left( \ddot{u} + \delta\alpha\varepsilon \frac{d}{dt} F(u) + \alpha^2 C u \right) dt \\ &= \dot{u}(t_0) - \dot{u}(t_0 - \pi) + \delta\alpha\varepsilon F(u)|_{t_0-\pi}^{t_0} + \alpha^2 \int_{t_0-\pi}^{t_0} C u dt \\ &= 2\dot{u}(t_0) + 2\delta\alpha\varepsilon F(u(t_0)) + \alpha^2 \int_{t_0-\pi}^{t_0} C u dt. \end{aligned}$$

Consequently,

$$\begin{aligned} 0 &\geq \dot{u}_k(t_0) = -\delta\alpha\varepsilon \left( \frac{a^{3/2}}{3} - a^{3/2} \right) - \frac{\alpha^2}{2} \int_{t_0-\pi}^{t_0} (Cu)_k dt \\ &\geq \alpha \left( \frac{2}{3} \delta\varepsilon a^{2/3} - \frac{\tilde{\alpha}}{2} \int_{t_0-\pi}^{t_0} \|C\| \|u(t)\|_\infty dt \right) \\ &\geq \alpha \left( \frac{2}{3} \delta\varepsilon a^{3/2} - \frac{\tilde{\alpha}}{2} \sqrt{\pi} \|C\| \|u\|_{\mathbb{H}_o} \right) \geq \alpha \left( \frac{2}{3} \delta\varepsilon a^{3/2} - \frac{\tilde{\alpha}}{2} \sqrt{\pi} \|C\| m(\tilde{\alpha}) \right), \end{aligned}$$

where  $\|u(t)\|_\infty$  stands for  $\max\{|u_1(t)|, \dots, |u_n(t)|\}$ . Therefore, it is sufficient to take

$$\delta_1 = \frac{3\tilde{\alpha}\sqrt{\pi}\|C\|m(\tilde{\alpha})}{4\varepsilon a^{3/2}}. \quad \square$$

**5.2. Existence result: formulation.** Assume that

$$(5.10) \quad \Sigma(C) := \{\mu \in \sigma(C) : \mu > 0\} \neq \emptyset.$$

Take a function  $m : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  provided by Lemma 5.1 and choose  $\alpha > 0$  such that

$$(5.11) \quad \alpha^2 \neq \frac{(2r-1)^2}{\mu} \quad \text{for all } r = 1, \dots, k, \mu \in \Sigma(C)$$

(cf. (4.8)) and

$$(5.12) \quad \mu > \frac{1}{\alpha^2} \quad \text{for all } \mu \in \Sigma(C).$$

Let  $J: \mathbb{H}_o \rightarrow C(S^1; V)$  be the natural injection. We choose  $m > 0$  such that  $m < \sqrt{a}/\|J\|$ . Then for every  $u \in \mathbb{H}_o$  such that  $\|u\|_{\mathbb{H}_o} \leq m$  we have

$$\|u\|_{\infty} = \|J(u)\|_{\infty} \leq \|J\| \|u\|_{\mathbb{H}_o} \leq \|J\| m < \sqrt{a}.$$

Notice that for any solution  $u$  of the equation (3.7) we have  $\|u\|_{\infty} > \sqrt{a}$  (see the proof of Lemma 5.2), thus there is no solution  $u$  such that  $\|u\|_{\mathbb{H}_o} = m$ . Next, we choose  $M > m(\alpha)$  and the numbers  $\lambda_o$  and  $\delta_o$  to be large enough in order to have

$$\lambda_o - \delta_o > \delta_1(\alpha), \quad [0, \lambda_o] \subset \{\lambda : \lambda \geq \delta_1(\alpha)\} \cup \left\{ \lambda : 0 \leq \lambda \leq \frac{\delta_o m^2}{m(\alpha)^2} \right\}.$$

Next, we define  $\Omega$ ,  $\Omega_o$ ,  $\partial_m$ ,  $\partial_M$  and  $\partial_o$  according to formulae (3.21), (3.22) and (3.24)–(3.26).

We are now in a position to formulate the existence theorem providing a general framework for the classification of periodic solutions to (2.1) according to their symmetries.

**THEOREM 5.3.** *Assume (5.10) is satisfied and choose  $\alpha > 0$  satisfying (5.11) and (5.12).*

- (a) *Suppose that for a certain orbit type  $(H_o)$  in  $\Omega$  (see (3.21)), the coefficient  $n_{H_o}$  of the equivariant degree (3.29) is non-zero. Then the van der Pol system of equations (2.2) has a periodic solution  $u$  such that  $G_u \supset H_o$ .*
- (b) *If in addition, the orbit type  $(H_o)$  is dominating in  $\mathbb{H}$  (see Definition 3.4), then the system (2.2) has at least  $|G/H_o|_{S^1}$  different periodic solutions with symmetries exactly  $(H_o)$ , where  $|X|_{S^1}$  stands for the number of different  $S^1$ -orbits in  $X$ .*

**PROOF.** (a) The idea of the proof of the first part of Theorem 5.3 is based on the following fact: *Let  $\text{id} - \mathcal{F}^t$  be a homotopy of two equivariant maps  $\text{id} - \mathcal{F}^0$  and  $\text{id} - \mathcal{F}^1$  such that  $G\text{-Deg}(\text{id} - \mathcal{F}^j, \Omega) = \sum n_H^j(H)$ ,  $j = 0, 1$ . If  $n_{H_o}^0 \neq n_{H_o}^1$ , then there exists  $t \in (0, 1)$  such that the map  $\mathcal{F}^t$  has a fixed point in  $\partial\Omega^{H_o}$ .* As the arguments used in this proof are very close to the ones given in [12], we present only a sketch of the proof.

Let

$$\mathcal{S}_o := \{(\lambda, u) : u = \tilde{G}(\alpha, \delta, \lambda, u) \text{ for some } \delta \in [0, \delta_o]\},$$

where  $\tilde{G}(\alpha, \delta, \cdot, \cdot): \mathbb{R} \times \mathbb{H}_o \rightarrow \mathbb{H}_o$  is defined by (3.18). By using exactly the same arguments as in [12] we can show that

$$\mathcal{S}_o \cap (\partial_o \cup \partial_m) = \emptyset.$$



Notice that if  $(\lambda, u) \in \mathcal{S}_o \cap \partial_M$ , then  $\theta(u) = 1$  and  $\delta + \lambda > 0$  and (by Lemma 3.5 in [12]) the function  $w = \sqrt{\delta/(\delta + \lambda)}u$  satisfies the equation

$$(5.13) \quad \ddot{w} + (\delta + \lambda)\alpha\varepsilon(w^2 - \vec{a})\dot{w} + \alpha^2 Cw = 0.$$

In particular, that means the function  $w$  is a  $2\pi$ -periodic solution of equation (3.7) with  $\varepsilon$  replaced by  $(\delta + \lambda)\varepsilon$ .

Following [12], define the parameterized nonlinear operators  $F_s: V \rightarrow V$  by

$$F_s(u) := \left( \frac{1}{3}u^3 - (1-s)\vec{a}u \right), \quad s \in [0, 1],$$

and consider the following family of parameterized differential equations

$$(5.14) \quad \begin{cases} -\ddot{u} = \delta_o\alpha\varepsilon \frac{d}{dt}F_s(u) + \alpha^2\theta(u_o)Cu - \lambda\alpha\varepsilon a\dot{u}, & u(t) \in V, \\ u(t) = u(t + 2\pi), \quad \dot{u}(t) = \dot{u}(t + 2\pi), \quad u(t + \pi) = -u(t). \end{cases}$$

Again, we can reformulate the above system using the setting in the functional space  $\mathbb{H}_o$ . We define  $\tilde{H}(\alpha, s, \cdot, \cdot): \mathbb{R} \times \mathbb{H}_o \rightarrow \mathbb{H}_o$  by

$$\begin{aligned} \tilde{H}(\alpha, s, \lambda, u) := & \delta_o\alpha\varepsilon L^{-1}N_s(j(u), D(j(u))) \\ & + \alpha^2\theta(u_o)L^{-1}C(j(u)) - \lambda\alpha\varepsilon aL^{-1}D(j(u)), \end{aligned}$$

and  $H(\alpha, s, \cdot, \cdot): \mathbb{R} \times \mathbb{H} \rightarrow \mathbb{H}$ , by

$$H(\alpha, s, \lambda, u) = (\bar{u} + \alpha^2\theta(u_o)C(\bar{u}), \tilde{H}(\alpha, s, \lambda, u_o)), \quad u = \bar{u} + u_o, \quad \bar{u} \in V, \quad u_o \in \mathbb{H}_o,$$

where

$$N_s(u, \dot{u}) = \frac{d}{dt}F_s(u) = \frac{d}{dt} \left( \frac{1}{3}u^3 - (1-s)\vec{a}u \right) = (u^2 - (1-s)\vec{a})\dot{u}, \quad u \in \mathbb{H}_o.$$

The map  $\text{id} - H(\alpha, s, \cdot, \cdot)$  is a  $G$ -equivariant homotopy. Using Lemmas 5.1 and 5.2 and the same argument as in [12], one can show that:

- the homotopy  $\text{id} - \tilde{H}(\alpha, s, \cdot, \cdot)$  has no zeros in the set  $\partial_o \cup \partial_m$ ;
- the map  $\text{id} - \tilde{H}(\alpha, 1, \cdot, \cdot)$  has no zeros in  $\overline{\Omega}_o$  (in particular, by the existence property of the  $G$ -equivariant degree (see Appendix, Theorem 8.3(P1)),  $G\text{-Deg}(\text{id} - H(\alpha, 1, \cdot, \cdot), \Omega) = 0$ ).

Next, we can define the following  $G$ -equivariant homotopy  $\text{id} - \Psi(\tau, \cdot, \cdot)$  by

$$\Psi(\tau, \lambda, u) := \begin{cases} G(\alpha, 2\tau\delta_o, \lambda, u) & \text{for } (\lambda, u) \in \overline{\Omega}, \quad \tau \in [0, 1/2], \\ H(\alpha, 2\tau - 1, \lambda, u) & \text{for } (\lambda, u) \in \overline{\Omega}, \quad \tau \in [1/2, 1]. \end{cases}$$

As it was explained in Section 4, the solution set to the equation (4.7) is non-empty only if conditions (4.5) and (4.6) are satisfied (cf. the formulation of Theorem 5.3). Therefore, these conditions are necessary for the equivariant degree (3.29) to be different from zero. Assume (according to the Theorem 5.3 conditions) that  $n_{H_o} \neq 0$ . Suppose that  $u - \Psi(\tau, \lambda, u) \neq 0$  for all  $(\lambda, u) \in \partial\Omega$ .

Then, by the homotopy property of the  $G$ -equivariant degree (see Appendix, Theorem 8.3(P3)), we would also have that the  $(H_o)$ -coefficient of  $G\text{-Deg}(\text{id} - H(\alpha, 1, \cdot, \cdot), \Omega_o)$  is non-zero, what is impossible.

Since for  $u = \bar{u} + u_o \in V \times \mathbb{H}_o$ , the equation  $u = \Psi(\tau, \lambda, u)$  implies that  $-\alpha^2 \theta(u_o) C(\bar{u}) = 0$ , thus  $\bar{u} = 0$ , therefore, there exists  $(\lambda, u) = (\lambda, u_o) \in \partial\Omega_o$  such that  $u = \Psi(\tau, \lambda, u)$  for some  $\tau \in [0, 1]$ . However, the equation  $u - \Psi(\tau, \lambda, u) = 0$  has no solution  $(\lambda, u)$  in  $\partial_o \cup \partial_m$ . Consequently, it has a solution  $u$  in  $\partial_M$ . By applying a standard transformation, we obtain a solution for the equation (3.7), for the value of  $\varepsilon$  replaced by another (appropriate) value, with the period equal to  $2\pi$ .

(b) Assume now that  $(H_o)$  is a dominating orbit type in  $\mathbb{H}$  (cf. Definition 3.4). Then, by applying the arguments that were explained in Remark 3.6, we obtain the existence of at least  $|G/H_o|_{S^1}$  different periodic solutions to the equation (3.5), which means that there are at least  $|G/H_o|_{S^1}$  different periodic solutions (with symmetries  $(H_o)$ ) to the equation (2.2).  $\square$

## 6. Computations for special cases of group $\Gamma$

In this section we consider several particular cases of the group  $\Gamma$ , for which we present the computations of the  $A(\Gamma)$ -module structure of  $A_1(\Gamma \times S^1)$  (including the multiplication tables), the isotropy lattices for the irreducible representations of  $\Gamma \times S^1$ , and in addition, the computations of the  $\Gamma \times S^1$ -degree of the related basic maps (see [5], [6] and [15], [16] for the corresponding background). These computations are used in Section 7 to classify the symmetry types of the periodic solutions of the van der Pol systems with  $\Gamma$ -symmetries. In fact, the information contained in Section 6 is more general than it is needed for the applications in Section 7 and may be used for further investigation of the van der Pol systems as well as for other nonlinear problems with symmetries.

**6.1. Preliminaries.** Let us explain shortly, how to obtain an explicit multiplication table for the Burnside ring  $A(\Gamma)$  of a finite<sup>2</sup> group  $\Gamma$ . For a subgroup  $H$  of  $\Gamma$  we denote by  $(H)$  the conjugacy class of  $H$ , and for two subgroups  $H$  and  $K$  of  $\Gamma$ , we write  $(H) \leq (K)$  if  $H \subset g^{-1}Kg$  for some  $g \in \Gamma$ . The relation  $\leq$  defines a partial order on the set  $\Phi(\Gamma)$  of all the conjugacy classes  $(H)$  in  $\Gamma$ . The Burnside ring  $A(\Gamma)$  is the  $\mathbb{Z}$ -module generated by  $\Phi(\Gamma)$  with the multiplication defined on the generators by the following formula:

$$(6.1) \quad (H) \cdot (K) = \sum_{(L) \in \Phi(\Gamma)} n_L(H, K) \cdot (L),$$

<sup>2</sup>In this paper, to simplify our presentation, we consider only the case of a finite group  $\Gamma$ , but this technique can also be applied in a similar way to any compact Lie group by adopting the concept of so-called “bi-orientable” groups (cf. [5], [6], [9], [24]).

where  $n_L(H, K)$  is an integer representing the number of  $(L)$ -orbits in the set  $\Gamma/H \times \Gamma/K$ . The numbers  $n_L := n_L(H, K)$  can be computed using a simple recurrence formula. For two subgroups  $H$  and  $L$ , such that  $(H) \geq (L)$ , we define the set

$$N(L, H) = \{g \in \Gamma : gLg^{-1} \subset H\}$$

and the number (see [10], [21])

$$n(L, H) = \left| \frac{N(L, H)}{N(H)} \right|,$$

where  $|X|$  denotes the number of elements in the set  $X$ . In the case  $(H)$  and  $(L)$  are not comparable, we will simply put  $n(L, H) = 0$ . The partial order on the set  $\Phi(\Gamma)$  can be extended to a total order, which we will also denote by  $\leq$ . Then we have

$$(6.2) \quad n_L = \frac{1}{|W(L)|} \left[ n(L, H) |W(H)| n(L, K) |W(K)| - \sum_{(\tilde{L}) > (L)} n(L, \tilde{L}) n_{\tilde{L}} |W(\tilde{L})| \right],$$

where  $W(L) = N(L)/L$  denotes the Weyl group of  $L$  in  $\Gamma$ .

Recall that the  $\mathbb{Z}$ -module  $A_1(\Gamma \times S^1)$  is generated by all the conjugacy classes of twisted subgroups  $(H^{\varphi, l})$  in  $\Gamma \times S^1$ . The  $A(\Gamma)$ -multiplication on the generators  $(K) \in A(\Gamma)$  and  $(H^{\varphi, l}) \in A_1(\Gamma \times S^1)$ , is defined by the formula

$$(K) \cdot (H^{\varphi, l}) = \sum_{(L)} n_L \cdot (L^{\varphi, l}),$$

where the numbers  $n_L$  are computed using the recurrence formula

$$(6.3) \quad n_L = \frac{1}{|W(L^{\varphi, l})/S^1|} \left[ n(L, K) |W(K)| n(L^{\varphi, l}, H^{\varphi, l}) \left| \frac{W(H^{\varphi, l})}{S^1} \right| \right. \\ \left. - \sum_{(\tilde{L}) > (L)} n(L^{\varphi, l}, \tilde{L}^{\varphi, l}) n_{\tilde{L}} \left| \frac{W(\tilde{L}^{\varphi, l})}{S^1} \right| \right].$$

Let  $\mathcal{V}_j$  be an irreducible representation of the group  $\Gamma$ ,  $\mathcal{B}_j$  the unit ball in  $\mathcal{V}_j$ , and  $-\text{id} : \mathcal{V}_j \rightarrow \mathcal{V}_j$  the *basic map* of the first type corresponding to the representation  $\mathcal{V}_j$ . Then the equivariant degree  $\text{deg}_{\mathcal{V}_j} = \Gamma\text{-deg}(-\text{id}, \mathcal{B}_j)$  can be computed using a recurrence formula. More precisely, suppose that the partial order of the orbit types  $(L)$  in  $\mathcal{V}_j$  is extended to a total order. It is clear that  $(\Gamma)$  is the maximal element. Then  $\text{deg}_{\mathcal{V}_j} = \sum_{(L)} n_L \cdot (L)$ , where

$$(6.4) \quad n_L = \frac{1}{|W(L)|} \left[ (-1)^{k_L} - \sum_{(\tilde{L}) > (L)} n(L, \tilde{L}) n_{\tilde{L}} |W(\tilde{L})| \right],$$

$k_L = \dim \mathcal{V}_j^L$  and  $n_{\Gamma} = 1$ .

Recall that if  $\mathcal{V}_j^c$  is a complex irreducible representation of  $\Gamma$ , then by using the complex structure on  $\mathcal{V}_j^c$ , we define the action of  $S^1$ , by  $(z, v) \mapsto z^l \cdot v$ , where

$z \in S^1$ ,  $v \in \mathcal{V}_j^c$ ,  $l$  is a positive integer and  $\cdot$  denotes the complex multiplication. In this way, we obtain an irreducible representation  $\mathcal{V}_{j,l}$  of the group  $G := \Gamma \times S^1$ . We consider the space  $\mathbb{R} \times \mathcal{V}_{j,l}$ , the set  $\mathcal{O}_{j,l} := \{(\lambda, v) \in \mathbb{R} \times \mathcal{V}_{j,l} : 1/2 < \|v\| < 2, -1 < \lambda < 1\}$  and the basic map

$$f_{j,l}(\lambda, v) = (1 - \|v\| + i\lambda) \cdot v, \quad (\lambda, v) \in \mathbb{R} \times \mathcal{V}_{j,l}.$$

The primary equivariant degree  $\deg_{\mathcal{V}_{j,l}} = G\text{-Deg}(f_{j,l}, \mathcal{O}_{j,l})$  can also be computed by applying a recurrence formula. First, we extend the partial order of the orbit types  $(L)$  (where  $L$  is a twisted subgroup) to a total order. Then  $\deg_{\mathcal{V}_{j,l}} = \sum_{(L)} n_L \cdot (L)$ , where

$$(6.5) \quad n_L = \frac{1}{|W(L)/S^1|} \left[ \frac{k_L}{2} - \sum_{(\tilde{L}) > (L)} n_{(L, \tilde{L})} n_{\tilde{L}} \left| \frac{W(\tilde{L})}{S^1} \right| \right],$$

and  $k_L = \dim \mathcal{V}_{j,l}^L$ . Notice that it is sufficient to compute the basic degree  $\deg_{\mathcal{V}_{j,1}} = \sum_{(L)} n_L \cdot (L)$  for the irreducible representation  $\mathcal{V}_{j,1}$ . In the general case of the representation  $\mathcal{V}_{j,l}$  we have that

$$(6.6) \quad \deg_{\mathcal{V}_{j,l}} = \sum_{(L)} n_L \Psi_l^{-1}(L),$$

where  $\Psi_l: \Gamma \times S^1 \rightarrow \Gamma \times S^1$  is the homomorphism  $\Psi_l(g, z) = (g, z^l)$  for  $g \in \Gamma$ ,  $z \in S^1$ .

In the remaining parts of this section, we present the computations of the multiplication tables and the basic degrees for the dihedral, tetrahedral, octahedral and icosahedral groups.

**6.2. Computations for dihedral group  $D_N$ .** Let us illustrate the computations of the  $G$ -equivariant degree of basic pairs in the case of the group  $\Gamma = D_N$  of order  $2N$  (see [16] for more information) composed of the rotations  $1, \xi, \dots, \xi^{N-1}$  of the complex plane ( $\xi$  is the multiplication by  $e^{2\pi i/N}$ ) and the reflections  $\kappa, \kappa\xi, \dots, \kappa\xi^{N-1}$ , with  $\kappa = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ .

We start with a list of the subgroups of  $D_N$ :

$$\begin{aligned} \mathbb{Z}_k &:= \{1, \gamma, \dots, \gamma^{k-1}\}, \\ D_{k,j} &:= \{1, \gamma, \gamma^2, \dots, \gamma^{k-1}, \kappa\xi^j, \kappa\xi^j\gamma, \dots, \kappa\xi^j\gamma^{k-1}\}, \end{aligned}$$

where  $k \in \mathbb{Z}_+$  is such that  $k|N$ ,  $\gamma := e^{2\pi i/k}$  and  $j = 0, \dots, N/k - 1$ .

Notice that  $(D_{k,j}) = (D_{k,j'})$  for  $j \equiv j' \pmod{2}$ , and  $(D_{k,0}) = (D_{k,1})$  if and only if  $N/k$  is odd. We denote  $D_k := D_{k,0}$  for all  $k$  with  $k|N$ , and  $\tilde{D}_{k'} := D_{k',1}$  for  $k'$  such that  $N/k'$  is even (i.e.  $2k'|N$ ). Thus, the Burnside ring  $A(D_N)$  is generated by  $(\mathbb{Z}_k)$ ,  $(D_k)$  and  $(\tilde{D}_{k'})$  for all  $k$  and  $k'$  with  $k|N$  and  $2k'|N$ .

On the other hand, the twisted subgroups in  $D_N \times S^1$  (besides those we mentioned above), can be classified (up to conjugacy classes) as follows:

$$\begin{aligned} \mathbb{Z}_k^{tr} &:= \{(1, 1), (\gamma, \gamma^r), \dots, (\gamma^{k-1}, \gamma^{(k-1)r})\}, \quad r = 1, \dots, k-1, \\ D_{k,j}^z &:= \{(1, 1), (\gamma, 1), \dots, (\gamma^{k-1}, 1), (\kappa\xi^j, -1), (\kappa\xi^j\gamma, -1), \dots, (\kappa\xi^j\gamma^{k-1}, -1)\}, \end{aligned}$$

where  $k|N$  and  $j = 0, \dots, N/k - 1$ . Since  $(\mathbb{Z}_k^{tr}) = (\mathbb{Z}_k^{t_{k-r}})$ , we will consider  $(\mathbb{Z}_k^{tr})$  only for  $r < k/2$ . Observe that  $(D_{k,j}^z) = (D_{k,j'}^z)$  if and only if  $(D_{k,j}) = (D_{k,j'})$ , therefore we will employ similar notations as before,  $D_k^z := D_{k,0}^z$  for all  $k|N$  and  $\tilde{D}_{k'}^z := D_{k',1}^z$  for  $2k'|N$ .

In addition, we have

$$\begin{aligned} \mathbb{Z}_{2m}^d &:= \{(1, 1), (\gamma, -1), (\gamma^2, 1), \dots, (\gamma^{2m-2}, 1), (\gamma^{2m-1}, -1)\}, \\ D_{2m,s}^d &:= \{(1, 1), (\gamma, -1), (\gamma^2, 1), \dots, (\gamma^{2m-1}, -1), \\ &\quad (\kappa\xi^s, 1), (\kappa\xi^s\gamma, -1), \dots, (\kappa\xi^s\gamma^{2m-1}, -1)\}; \end{aligned}$$

where  $2m|N$  and  $s = 0, \dots, N/(2m) - 1, N/(2m)$ .

Here, we notice that  $(D_{2m,s}^d) = (D_{2m,s'}^d)$  if and only if  $s \equiv s' \pmod{2}$ . We denote  $D_{2m}^d := D_{2m,0}^d$  for all  $2m|N$ ,  $\tilde{D}_{2m}^d := D_{2m,1}^d$  for  $4m|N$  and  $D_{2m}^{\hat{d}} := D_{2m,N/(2m)}^d$  for  $2m|N$  but  $4m \nmid N$ . That is,

$$\begin{aligned} D_{2m}^d &:= \{(1, 1), (\gamma, -1), (\gamma^2, 1), \dots, (\gamma^{2m-1}, -1), \\ &\quad (\kappa, 1), (\kappa\gamma, -1), \dots, (\kappa\gamma^{2m-1}, -1)\}; \\ \tilde{D}_{2m}^d &:= \{(1, 1), (\gamma, -1), (\gamma^2, 1), \dots, (\gamma^{2m-1}, -1), \\ &\quad (\kappa\xi, 1), (\kappa\xi\gamma, -1), \dots, (\kappa\xi\gamma^{2m-1}, -1)\}; \\ D_{2m}^{\hat{d}} &:= \{(1, 1), (\gamma, -1), (\gamma^2, 1), \dots, (\gamma^{2m-1}, -1), \\ &\quad (\kappa, -1), (\kappa\gamma, 1), \dots, (\kappa\gamma^{2m-1}, 1)\}. \end{aligned}$$

Therefore,  $A_1(D_N \times S^1)$  is generated by (besides the generators of  $A(D_N)$ )  $(\mathbb{Z}_k^{tr})$  (for  $k|N$ ,  $r = 1, 2, \dots$ , and  $r < k/2$ ),  $(D_k^z)$  (for  $k|N$ ),  $(\tilde{D}_{k'}^z)$  (for  $2k'|N$ ),  $(\mathbb{Z}_{2m}^d)$  (for  $2m|N$ ),  $(D_{2m}^d)$  (for  $2m|N$ ),  $(\tilde{D}_{2m'}^d)$  (for  $4m'|N$ ) and  $(D_{2m}^{\hat{d}})$  (for  $2m'|N$  but  $4m' \nmid N$ ).

The Weyl groups of all the conjugacy classes representatives are listed in Table 1 and the numbers  $n(L, H)$  in Table 2.

By applying the numbers  $n(L, H)$  and the formula (6.3) it is easy to establish the complete  $A(D_N)$ -module multiplication table for  $A_1(D_N \times S^1)$ , in the case  $N$  is odd. As an example, we present in Table 3 the multiplication formulae for the  $A(D_N)$ -module  $A_1(D_N \times S^1)$  in the cases  $N = 3$  and  $5$ .<sup>3</sup> In Table 4 we present a multiplication table<sup>4</sup> for the  $A(D_4)$ -module  $A_1(D_4 \times S^1)$ .

<sup>3</sup>We use here the notation  $\mathbb{Z}_3^t := \mathbb{Z}_3^{t_1}$ .

<sup>4</sup>We use there the notation  $\mathbb{Z}_2^-$  instead of  $\mathbb{Z}_2^{t_1}$  and  $\mathbb{Z}_4^t$  instead of  $\mathbb{Z}_4^{t_1}$ .

| $H$                 | $N(H)$                    | $W(H)$                        | Conditions         | # of Conjug. |
|---------------------|---------------------------|-------------------------------|--------------------|--------------|
| $\mathbb{Z}_k$      | $D_N \times S^1$          | $D_{N/k} \times S^1$          | $k N$              | 1            |
| $\mathbb{Z}_k^{tr}$ | $\mathbb{Z}_N \times S^1$ | $\mathbb{Z}_{N/k} \times S^1$ | $k N, 0 < r < k/2$ | 2            |
| $\mathbb{Z}_{2m}^d$ | $D_N \times S^1$          | $D_{N/(2m)} \times S^1$       | $2m N$             | 1            |
| $D_k$               | $D_{2k} \times S^1$       | $\mathbb{Z}_2 \times S^1$     | $2k N$             | $N/(2k)$     |
| $\tilde{D}_k$       | $D_{2k} \times S^1$       | $\mathbb{Z}_2 \times S^1$     | $2k N$             | $N/(2k)$     |
| $D_k$               | $D_k \times S^1$          | $\mathbb{Z}_1 \times S^1$     | $k N, 2k \nmid N$  | $N/k$        |
| $D_k^z$             | $D_{2k} \times S^1$       | $\mathbb{Z}_2 \times S^1$     | $2k N$             | $N/(2k)$     |
| $\tilde{D}_k^z$     | $D_{2k} \times S^1$       | $\mathbb{Z}_2 \times S^1$     | $2k N$             | $N/(2k)$     |
| $D_k^z$             | $D_k \times S^1$          | $\mathbb{Z}_1 \times S^1$     | $k N, 2k \nmid N$  | $N/k$        |
| $D_{2m}^d$          | $D_{2m} \times S^1$       | $\mathbb{Z}_1 \times S^1$     | $4m N$             | $N/(2m)$     |
| $\tilde{D}_{2m}^d$  | $D_{2m} \times S^1$       | $\mathbb{Z}_1 \times S^1$     | $4m N$             | $N/(2m)$     |
| $D_{2m}^d$          | $D_{2m} \times S^1$       | $\mathbb{Z}_1 \times S^1$     | $2m N, 4m \nmid N$ | $N/2m$       |
| $D_{2m}^{\hat{d}}$  | $D_{2m} \times S^1$       | $\mathbb{Z}_1 \times S^1$     | $2m N, 4m \nmid N$ | $N/(2m)$     |

TABLE 1. Representatives of conjugacy classes of twisted subgroups in  $D_N \times S^1$ 

| $L$                 | $H$                 | $n(L, H)$ | Conditions  | $L$                 | $H$                | $n(L, H)$ | Conditions                            |
|---------------------|---------------------|-----------|---|---------------------|--------------------|-----------|---------------------------------------|
| $\mathbb{Z}_k$      | $\mathbb{Z}_l$      | 1         | $k l N$   | $\mathbb{Z}_{2m}^d$ | $D_{2n}^d$         | $N/(2n)$  | $2m 2n N, 2m \nmid n, 4n \nmid N$     |
| $\mathbb{Z}_k$      | $\mathbb{Z}_l^{tr}$ | 2         | $k l N, k r, 0 < r < l/2$   | $\mathbb{Z}_{2m}^d$ | $D_{2n}^{\hat{d}}$ | $N/(2n)$  | $2m 2n N, 2m \nmid n, 4n \nmid N$     |
| $\mathbb{Z}_k$      | $\mathbb{Z}_{2n}^d$ | 1         | $k n, 2n N$   | $D_k$               | $D_l$              | 1         | $k l N$                               |
| $\mathbb{Z}_k^{ts}$ | $\mathbb{Z}_l^{tr}$ | 1         | $k l N, k \nmid r, 0 < r < l/2,$<br>$0 < s < k, s \equiv r \pmod{k}, 2s \neq k$ | $D_k$               | $D_{2n}^d$         | 1         | $k n, 2n N$                           |
| $\mathbb{Z}_{2m}^d$ | $\mathbb{Z}_l^{tr}$ | 2         | $2m l N, m r, 2m \nmid r, 0 < r < l/2$  | $\tilde{D}_k$       | $D_l$              | 1         | $2k l N, 2l \nmid N$                  |
| $\mathbb{Z}_{2m}^d$ | $\mathbb{Z}_{2n}^d$ | 1         | $2m 2n N, 2m \nmid n$   | $\tilde{D}_k$       | $\tilde{D}_l$      | 1         | $2k 2l N$                             |
| $\mathbb{Z}_k$      | $D_l$               | $N/(2l)$  | $k l N, 2l N$   | $\tilde{D}_k$       | $\tilde{D}_{2n}^d$ | 1         | $k n, 4n N$                           |
| $\mathbb{Z}_k$      | $\tilde{D}_l$       | $N/(2l)$  | $k l N, 2l N$   | $\tilde{D}_k$       | $D_{2n}^{\hat{d}}$ | 1         | $k n, 2n N, 4n \nmid N$               |
| $\mathbb{Z}_k$      | $D_l$               | $N/l$     | $k l N, 2l \nmid N$   | $D_k^z$             | $D_l^z$            | 1         | $k l N$                               |
| $\mathbb{Z}_k$      | $D_l^z$             | $N/(2l)$  | $k l N, 2l N$   | $D_k^z$             | $D_{2n}^z$         | 1         | $k n, 4n N$                           |
| $\mathbb{Z}_k$      | $\tilde{D}_l^z$     | $N/(2l)$  | $k l N, 2l N$   | $D_k^z$             | $D_{2n}^{\hat{z}}$ | 1         | $k n N, 2n N, 4n \nmid N$             |
| $\mathbb{Z}_k$      | $D_l^z$             | $N/l$     | $k l N, 2l \nmid N$   | $\tilde{D}_k^z$     | $D_l^z$            | 1         | $2k l N, 2l \nmid N$                  |
| $\mathbb{Z}_k$      | $D_{2n}^d$          | $N/(2n)$  | $k n N, 4n N$   | $\tilde{D}_k^z$     | $\tilde{D}_l^z$    | 1         | $k l N, 2l N$                         |
| $\mathbb{Z}_k$      | $\tilde{D}_{2n}^d$  | $N/(2n)$  | $k n N, 4n N$   | $\tilde{D}_k^z$     | $D_{2n}^z$         | 1         | $k n, 2n N, 4n \nmid N$               |
| $\mathbb{Z}_k$      | $D_{2n}^d$          | $N/(2n)$  | $k n N, 2n N, 4n \nmid N$   | $\tilde{D}_k^z$     | $\tilde{D}_{2n}^z$ | 1         | $k n, 4n N$                           |
| $\mathbb{Z}_k$      | $D_{2n}^{\hat{d}}$  | $N/(2n)$  | $k n N, 2n N, 4n \nmid N$   | $D_{2m}^d$          | $D_{2n}^d$         | 1         | $m n N, 2m \nmid n, 2n N$             |
| $\mathbb{Z}_{2m}^d$ | $D_{2n}^d$          | $N/(2n)$  | $2m 2n N, 2m \nmid n, 4n N$   | $\tilde{D}_{2m}^d$  | $\tilde{D}_{2n}^d$ | 1         | $m n N, 2m \nmid n, 4n N$             |
| $\mathbb{Z}_{2m}^d$ | $\tilde{D}_{2n}^d$  | $N/(2n)$  | $2m 2n N, 2m \nmid n, 4n N$   | $D_{2m}^{\hat{d}}$  | $D_{2n}^{\hat{d}}$ | 1         | $m n N, 2m \nmid n, 2n N, 4n \nmid N$ |

TABLE 2. Numbers  $n(L, H)$  for twisted subgroups in  $D_N \times S^1$

|           |                   |            |          |           |               |                    |                |           |               |
|-----------|-------------------|------------|----------|-----------|---------------|--------------------|----------------|-----------|---------------|
| $(D_3)$   | $(D_1)$           | $(Z_3)$    | $(Z_1)$  | $N = 3$   | $(D_5)$       | $(D_1)$            | $(Z_5)$        | $(Z_1)$   | $N = 5$       |
| $(D_3)$   | $(D_1)$           | $(Z_3)$    | $(Z_1)$  | $(D_3)$   | $(D_5)$       | $(D_1)$            | $(Z_5)$        | $(Z_1)$   | $(D_5)$       |
| $(D_1)$   | $(D_1) + (Z_1)$   | $(Z_1)$    | $3(Z_1)$ | $(D_1)$   | $(D_1)$       | $(D_1) + 2(Z_1)$   | $(Z_1)$        | $5(Z_1)$  | $(D_1)$       |
| $(Z_3)$   | $(Z_1)$           | $2(Z_3)$   | $2(Z_1)$ | $(Z_3)$   | $(Z_5)$       | $(Z_1)$            | $2(Z_5)$       | $2(Z_1)$  | $(Z_5)$       |
| $(Z_1)$   | $3(Z_1)$          | $2(Z_1)$   | $6(Z_1)$ | $(Z_1)$   | $(Z_1)$       | $5(Z_1)$           | $2(Z_1)$       | $10(Z_1)$ | $(Z_1)$       |
| $(Z_3^t)$ | $(Z_1)$           | $2(Z_3^t)$ | $2(Z_1)$ | $(Z_3^t)$ | $(Z_5^{t_1})$ | $(Z_1)$            | $2(Z_5^{t_1})$ | $2(Z_1)$  | $(Z_5^{t_1})$ |
| $(D_3^z)$ | $(D_1^z) + (Z_1)$ | $(Z_3)$    | $3(Z_1)$ | $(D_3^z)$ | $(Z_5^{t_2})$ | $(Z_1)$            | $2(Z_5^{t_2})$ | $2(Z_1)$  | $(Z_5^{t_2})$ |
| $(D_1^z)$ | $(D_1^z) + (Z_1)$ | $(Z_1)$    | $3(Z_1)$ | $(D_1^z)$ | $(D_5^z)$     | $(D_1^z)$          | $(Z_5)$        | $(Z_1)$   | $(D_5^z)$     |
|           |                   |            |          |           | $(D_1^z)$     | $(D_1^z) + 2(Z_1)$ | $(Z_1)$        | $5(Z_1)$  | $(D_1^z)$     |

TABLE 3.  $A(D_N)$ -multiplication table for  $A_1(D_N \times S^1)$  for  $N = 3, 5$

|                   |            |                    |                    |                                   |            |            |          |                   |
|-------------------|------------|--------------------|--------------------|-----------------------------------|------------|------------|----------|-------------------|
| $(D_4)$           | $(D_2)$    | $(\tilde{D}_2)$    | $(D_1)$            | $(\tilde{D}_1)$                   | $(Z_4)$    | $(Z_2)$    | $(Z_1)$  | $N = 4$           |
| $(D_4)$           | $(D_2)$    | $(\tilde{D}_2)$    | $(D_1)$            | $(\tilde{D}_1)$                   | $(Z_4)$    | $(Z_2)$    | $(Z_1)$  | $(Z_4)$           |
| $(D_2)$           | $2(D_2)$   | $(Z_2)$            | $2(D_1)$           | $(Z_1)$                           | $(Z_2)$    | $2(Z_2)$   | $2(Z_1)$ | $(D_2)$           |
| $(\tilde{D}_2)$   | $(Z_2)$    | $2(\tilde{D}_2)$   | $(Z_1)$            | $2(\tilde{D}_1)$                  | $(Z_2)$    | $2(Z_2)$   | $2(Z_1)$ | $(\tilde{D}_2)$   |
| $(D_1)$           | $2(D_1)$   | $(Z_1)$            | $2(D_1) + (Z_1)$   | $2(Z_1)$                          | $(Z_1)$    | $2(Z_1)$   | $4(Z_1)$ | $(D_1)$           |
| $(\tilde{D}_1)$   | $(Z_1)$    | $2(\tilde{D}_1)$   | $2(Z_1)$           | $2(\tilde{D}_1) + (Z_1)$          | $(Z_1)$    | $2(Z_1)$   | $4(Z_1)$ | $(\tilde{D}_1)$   |
| $(Z_4)$           | $(Z_2)$    | $(Z_2)$            | $(Z_1)$            | $(Z_1)$                           | $2(Z_4)$   | $2(Z_2)$   | $2(Z_1)$ | $(Z_4)$           |
| $(Z_2)$           | $2(Z_2)$   | $2(Z_2)$           | $2(Z_1)$           | $2(Z_1)$                          | $2(Z_2)$   | $4(Z_2)$   | $4(Z_1)$ | $(Z_2)$           |
| $(Z_1)$           | $2(Z_1)$   | $2(Z_1)$           | $4(Z_1)$           | $4(Z_1)$                          | $2(Z_1)$   | $4(Z_1)$   | $8(Z_1)$ | $(Z_1)$           |
| $(D_4^z)$         | $(D_2^z)$  | $(\tilde{D}_2^z)$  | $(D_1^z)$          | $(\tilde{D}_1^z)$                 | $(Z_4)$    | $(Z_2)$    | $(Z_1)$  | $(D_4^z)$         |
| $(D_4^d)$         | $(D_2^d)$  | $(\tilde{D}_2^d)$  | $(D_1^d)$          | $(\tilde{D}_1^d)$                 | $(Z_4^d)$  | $(Z_2)$    | $(Z_1)$  | $(D_4^d)$         |
| $(D_4^d)$         | $(D_2)$    | $(D_2^z)$          | $(D_1)$            | $(\tilde{D}_1^z)$                 | $(Z_4^d)$  | $(Z_2)$    | $(Z_1)$  | $(D_4^d)$         |
| $(D_2^d)$         | $2(D_2^d)$ | $(Z_2^-)$          | $(D_1^z) + (D_1)$  | $(Z_1)$                           | $2(Z_2^-)$ | $2(Z_2^-)$ | $2(Z_1)$ | $(D_2^d)$         |
| $(\tilde{D}_2^d)$ | $(Z_2^-)$  | $2(\tilde{D}_2^d)$ | $(Z_1)$            | $(\tilde{D}_1^z) + (\tilde{D}_1)$ | $2(Z_2^-)$ | $2(Z_2^-)$ | $2(Z_1)$ | $(\tilde{D}_2^d)$ |
| $(D_2^z)$         | $2(D_2^z)$ | $(Z_2)$            | $2(D_1^z)$         | $(Z_1)$                           | $(Z_2)$    | $2(Z_2)$   | $2(Z_1)$ | $(D_2^z)$         |
| $(\tilde{D}_2^z)$ | $(Z_2)$    | $2(\tilde{D}_2^z)$ | $(Z_1)$            | $2(\tilde{D}_1^z)$                | $(Z_2)$    | $2(Z_2)$   | $2(Z_1)$ | $(\tilde{D}_2^z)$ |
| $(D_1^z)$         | $2(D_1^z)$ | $(Z_1)$            | $2(D_1^z) + (Z_1)$ | $2(Z_1)$                          | $(Z_1)$    | $2(Z_1)$   | $4(Z_1)$ | $(D_1^z)$         |
| $(\tilde{D}_1^z)$ | $(Z_1)$    | $2(\tilde{D}_1^z)$ | $2(Z_1)$           | $2(\tilde{D}_1^z) + (Z_1)$        | $(Z_1)$    | $2(Z_1)$   | $4(Z_1)$ | $(\tilde{D}_1^z)$ |
| $(Z_4^t)$         | $(Z_2^-)$  | $(Z_2^-)$          | $(Z_1)$            | $(Z_1)$                           | $2(Z_4^t)$ | $2(Z_2^-)$ | $2(Z_1)$ | $(Z_4^t)$         |
| $(Z_4^d)$         | $(Z_2^-)$  | $(Z_2^-)$          | $(Z_1)$            | $(Z_1)$                           | $2(Z_4^d)$ | $2(Z_2)$   | $2(Z_1)$ | $(Z_4^d)$         |
| $(Z_2^-)$         | $2(Z_2^-)$ | $2(Z_2^-)$         | $2(Z_1)$           | $2(Z_1)$                          | $2(Z_2^-)$ | $4(Z_2^-)$ | $4(Z_1)$ | $(Z_2^-)$         |

TABLE 4. Multiplication table for the  $A(D_4)$ -module  $A_1(D_4 \times S^1)$

Let us point out that general multiplication tables for  $A(D_N)$ -module  $A_1(D_N \times S^1)$  can be established, however because of a large number of different cases being involved, it may be difficult to use it practically. For example, we have such a (partial) information presented in Table 5, which was taken from [16].

|                                 | $(D_k^z)$<br>$2k \nmid N$                   | $(D_k^z)$<br>$2k   N$                        | $(\tilde{D}_k^z)$<br>$2k \nmid N$                   | $(\tilde{D}_k^z)$<br>$2k   N$                        | $(\mathbb{Z}_k^d)$<br>$2   k$    |
|---------------------------------|---|--|---|--|----------------------------------|
| $(D_r)$<br>$2r \nmid N$         | $(D_l^z) + \frac{Nl-kr}{2kr}(\mathbb{Z}_l)$ | $(D_l^z) + \frac{Nl-kr}{2kr}(\mathbb{Z}_l)$  | $\frac{Nl}{2rk}(\mathbb{Z}_l)$                      | $\frac{Nl}{2rk}(\mathbb{Z}_l)$                       | $\frac{Nl}{rk}(\mathbb{Z}_l^d)$  |
| $(D_r)$<br>$2r   N$             | $(D_l^z) + \frac{Nl-kr}{2kr}(\mathbb{Z}_l)$ | $(D_l^z) + \frac{Nl-2kr}{2kr}(\mathbb{Z}_l)$ | $\frac{Nl}{2rk}(\mathbb{Z}_l)$                      | $\frac{Nl}{2rk}(\mathbb{Z}_l)$                       | $\frac{Nl}{rk}(\mathbb{Z}_l^d)$  |
| $(\tilde{D}_r)$<br>$2r \nmid N$ | $\frac{Nl}{2rk}(\mathbb{Z}_l)$              | $\frac{Nl}{2rk}(\mathbb{Z}_l)$               | $(\tilde{D}_l^z) + \frac{Nl-kr}{2kr}(\mathbb{Z}_l)$ | $(\tilde{D}_l^z) + \frac{Nl-kr}{2kr}(\mathbb{Z}_l)$  | $\frac{Nl}{rk}(\mathbb{Z}_l^d)$  |
| $(\tilde{D}_r)$<br>$2r   N$     | $\frac{Nl}{2rk}(\mathbb{Z}_l)$              | $\frac{Nl}{2rk}(\mathbb{Z}_l)$               | $(\tilde{D}_l^z) + \frac{Nl-kr}{2kr}(\mathbb{Z}_l)$ | $(\tilde{D}_l^z) + \frac{Nl-2kr}{2kr}(\mathbb{Z}_l)$ | $\frac{Nl}{rk}(\mathbb{Z}_l^d)$  |
| $(\mathbb{Z}_r)$                | $\frac{Nl}{rk}(\mathbb{Z}_l)$               | $\frac{Nl}{rk}(\mathbb{Z}_l)$                | $\frac{Nl}{rk}(\mathbb{Z}_l)$                       | $\frac{Nl}{rk}(\mathbb{Z}_l)$                        | $\frac{Nl}{2rk}(\mathbb{Z}_l^d)$ |

TABLE 5. Partial multiplication table for  $A(D_N)$ -module  $A_1(D_N \times S^1)$   
( $l = \gcd(k, l)$ ,  $r | N$  and  $k | N$ )

Now, let us describe all the real irreducible representations of  $D_N$  and compute the degrees of the corresponding basic maps of the first type.

- (a0) Clearly, there is a one-dimensional trivial representation  $\mathcal{V}_0$ . In this case we have  $\deg_{\mathcal{V}_0} = -(D_N)$ .
- (a1) For every integer number  $1 \leq j < N/2$ , there is an orthogonal representation  $\mathcal{V}_j$  of  $D_N$  on  $\mathbb{C}$  given by

$$\begin{aligned} \gamma z &:= \gamma^j \cdot z, \quad \text{for } \gamma \in \mathbb{Z}_N \text{ and } z \in \mathbb{C}; \\ \kappa z &:= \bar{z}, \end{aligned}$$

where  $\gamma^j \cdot z$  denotes the usual complex multiplication. Put  $h = \gcd(j, N)$  and  $m = N/h$ . In this case we have the following degrees of the basic maps:

$$\begin{aligned} \deg_{\mathcal{V}_j} &= (D_N) - 2(D_h) + (\mathbb{Z}_h) && \text{if } m \text{ is odd,} \\ \deg_{\mathcal{V}_j} &= (D_N) - (D_h) - (\tilde{D}_h) + (\mathbb{Z}_h) && \text{if } m \text{ is even.} \end{aligned}$$

Put  $j_N := [(N+1)/2]$ .

- (a2) There is a representation  $\mathcal{V}_{j_N}$  given by the homomorphism  $c: D_N \rightarrow \mathbb{Z}_2 \subset O(1)$ , such that  $\ker c = \mathbb{Z}_N$ . The corresponding degree of the basic map is  $\deg_{\mathcal{V}_{j_N}} = (D_N) - (\mathbb{Z}_N)$ .
- (a3) For  $N$  even, there is an irreducible representation  $\mathcal{V}_{j_N+1}$  given by  $d: D_N \rightarrow \mathbb{Z}_2 \subset O(1)$  such that  $\ker d = D_{N/2}$ , for which the degree of the corresponding basic map is  $\deg_{\mathcal{V}_{j_N+1}} = (D_N) - (D_{N/2})$ .
- (a4) For  $N$  divisible by 4, there is an irreducible representation  $\mathcal{V}_{j_N+2}$  associated with  $\hat{d}: D_N \rightarrow \mathbb{Z}_2 \subset O(1)$  such that  $\ker \hat{d} = \tilde{D}_{N/2}$ . The degree



of the basic map corresponding to this representation is  $\deg_{\mathcal{V}_{j_N+2}} = (D_N) - (\tilde{D}_{N/2})$ .

Next, we will discuss orthogonal irreducible  $D_N \times S^1$ -representations and degrees of the corresponding basic maps of the second type. Denote by  $\mathcal{V}_j^c$ ,  $j = 0, \dots, j_N + 2$ , the complexification of  $\mathcal{V}_j$  (when defined). Since all  $\mathcal{V}_j$  are of real type, we have the following possibilities for  $\mathcal{V}_j^c$ :

- (b0) The representation  $\mathcal{V}_0^c$  defined on  $\mathbb{C}$ ;
- (b1) The representations  $\mathcal{V}_j^c$  for  $1 \leq j < j_N$ , defined on  $\mathbb{C} \oplus \mathbb{C}$  by

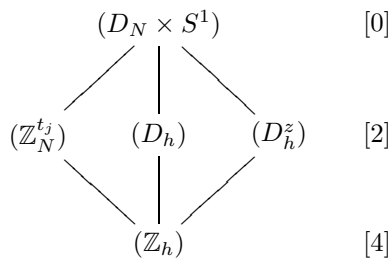
$$\begin{aligned} \gamma(z_1, z_2) &:= (\gamma^j \cdot z_1, \gamma^{-j} \cdot z_2), \quad \text{for } \gamma \in \mathbb{Z}_N, \text{ and } z_1, z_2 \in \mathbb{C}, \\ \kappa(z_1, z_2) &:= (z_2, z_1); \end{aligned}$$

- (b2) The representation  $\mathcal{V}_{j_N}^c$  defined by  $c: D_N \rightarrow \mathbb{Z}_2 \subset U(1)$ , such that  $\ker c = \mathbb{Z}_N$ ;
- (b3) In the case when  $N$  is even, the representation  $\mathcal{V}_{j_N+1}^c$  given by  $d: D_N \rightarrow \mathbb{Z}_2 \subset U(1)$ , such that  $\ker d = D_{N/2}$ ;
- (b4) In the case when  $N$  is divisible by 4, the representation  $\mathcal{V}_{j_N+2}^c$  given by  $\hat{d}: D_N \rightarrow \mathbb{Z}_2 \subset U(1)$ , such that  $\ker \hat{d} = \tilde{D}_{N/2}$ .

For  $l = 1, 2, \dots$ , we define the action of  $S^1$  on  $\mathcal{V}_j^c$  by  $zv = z^l \cdot v$ , for  $z \in S^1$  and  $v \in \mathcal{V}_j^c$ , where the product “ $\cdot$ ” is the usual complex multiplication. Obtained in this way a real representation of  $D_N \times S^1$ , which we denote by  $\mathcal{V}_{j,l}$ , is irreducible. For each of the representations  $\mathcal{V}_{j,1}$  of  $D_N \times S^1$ , we can compute the degrees  $\deg_{\mathcal{V}_{j,1}}$  of the associated basic maps of the second type on  $\mathcal{V}_{j,1}$ . The equivariant degrees  $\deg_{\mathcal{V}_{j,l}}$ , for any arbitrary  $l$ , can be determined from the degrees  $\deg_{\mathcal{V}_{j,1}}$  in a standard way (cf. (6.6)).

We have the following cases (the numbers located on the right side of the isotropy lattice denote the (real) dimension of the corresponding fixed-point space).

- (i1) The case  $m$  is odd:



$$\deg_{\mathcal{V}_{j,1}} = (\mathbb{Z}_N^{t_j}) + (D_h) + (D_h^z) - (\mathbb{Z}_h), \quad 0 < j < N/2.$$

(i2) The case  $m \equiv 2 \pmod{4}$ :

$$\begin{array}{ccc}
 & (D_N \times S^1) & [0] \\
 & \swarrow \quad \downarrow \quad \searrow & \\
 (\mathbb{Z}_N^{t_j}) & (D_{2h}^d) & (D_{2h}^{\hat{d}}) & [2] \\
 & \swarrow \quad \downarrow \quad \searrow & \\
 & (\mathbb{Z}_{2h}^d) & [4]
 \end{array}$$

$$\deg_{\mathcal{V}_{j,1}} = (\mathbb{Z}_N^{t_j}) + (D_{2h}^d) + (D_{2h}^{\hat{d}}) - (\mathbb{Z}_{2h}^d), \quad 0 < j < N/2.$$

(i3) The case  $m \equiv 0 \pmod{4}$ :

$$\begin{array}{ccc}
 & (D_N \times S^1) & [0] \\
 & \swarrow \quad \downarrow \quad \searrow & \\
 (\mathbb{Z}_N^{t_j}) & (D_{2h}^d) & (\tilde{D}_{2h}^d) & [2] \\
 & \swarrow \quad \downarrow \quad \searrow & \\
 & (\mathbb{Z}_{2h}^d) & [4]
 \end{array}$$

$$\deg_{\mathcal{V}_{j,1}} = (\mathbb{Z}_N^{t_j}) + (D_{2h}^d) + (\tilde{D}_{2h}^d) - (\mathbb{Z}_{2h}^d), \quad 0 < j < N/2.$$

(i4) In the case of irreducible two-dimensional representations  $\mathcal{V}_{j_N,1}$ ,  $\mathcal{V}_{j_N+1,1}$  and  $\mathcal{V}_{j_N+2,1}$ , we have the following lattices of the isotropy groups and the corresponding basic map degrees:

$$\begin{array}{ccccc}
 (D_N \times S^1) & (D_N \times S^1) & (D_N \times S^1) & & [0] \\
 \downarrow & \downarrow & \downarrow & & \\
 (D_N^z) & (D_N^d) & (D_N^{\hat{d}}) & & [2] \\
 \mathcal{V}_{j_N,1} & \mathcal{V}_{j_N+1,1} & \mathcal{V}_{j_N+2,1} & &
 \end{array}$$

$$\deg_{\mathcal{V}_{j_N,1}} = (D_N^z), \quad \deg_{\mathcal{V}_{j_N+1,1}} = (D_N^d), \quad \deg_{\mathcal{V}_{j_N+2,1}} = (D_N^{\hat{d}}).$$

**6.3. Computations for the tetrahedral group  $\mathbb{T}$ .** Let us consider the tetrahedral group  $\mathbb{T}$ , which denotes the group of symmetries of a regular tetrahedron in the Euclidean space  $\mathbb{R}^3$ . It is easy to notice that  $\mathbb{T}$  is isomorphic to

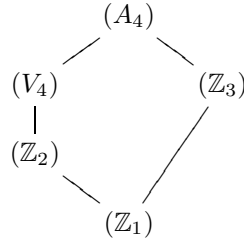


DIAGRAM 1

the alternating group  $A_4$  of order 12 with the lattice of the conjugacy classes shown on the Diagram 1.

There are four conjugate subgroups isomorphic to  $\mathbb{Z}_3$ , which correspond to the rotations of each of the four sides of the tetrahedron, which are:  $\{(1), (123), (132)\}$ ;  $\{(1), (124), (142)\}$ ;  $\{(1), (134), (143)\}$ ; and  $\{(1), (234), (243)\}$ . There is also the Klein subgroup  $V_4$ , which is isomorphic to  $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ . It is composed of the identity transformation and the elements  $(12)(34)$ ,  $(13)(24)$  and  $(14)(23)$ , corresponding to the transposition of two pairs of vertices of the tetrahedron (there is only one subgroup in the conjugacy class of  $(V_4)$ ). Each of these three non-trivial transformations generates a subgroup of order 2, i.e. isomorphic to  $\mathbb{Z}_2$ . All these three subgroups belong to the same conjugacy class  $(\mathbb{Z}_2)$ . Finally, there is the trivial subgroup  $(\mathbb{Z}_1)$ . In addition, we have that  $N(V_4) = A_4$ ,  $N(\mathbb{Z}_3) = \mathbb{Z}_3$ ,  $N(\mathbb{Z}_2) = V_4$ , so  $W(V_4) = \mathbb{Z}_3$ ,  $W(\mathbb{Z}_3) = \mathbb{Z}_1$  and  $W(\mathbb{Z}_2) = \mathbb{Z}_2$ .

| $H = K^\varphi$                | $K$            | $\varphi(K)$   | $\text{Ker } \varphi$ | $N(H)$                    | $W(H)$                    | Comments           |
|--------------------------------|----------------|----------------|-----------------------|---------------------------|---------------------------|--------------------|
| $A_4$                          | $A_4$          | $\mathbb{Z}_1$ | $A_4$                 | $A_4 \times S^1$          | $S^1$                     |                    |
| $V_4$                          | $V_4$          | $\mathbb{Z}_1$ | $V_4$                 | $A_4 \times S^1$          | $\mathbb{Z}_3 \times S^1$ |                    |
| $\mathbb{Z}_3$                 | $\mathbb{Z}_3$ | $\mathbb{Z}_1$ | $\mathbb{Z}_3$        | $\mathbb{Z}_3 \times S^1$ | $S^1$                     |                    |
| $\mathbb{Z}_2$                 | $\mathbb{Z}_2$ | $\mathbb{Z}_1$ | $\mathbb{Z}_2$        | $V_4 \times S^1$          | $\mathbb{Z}_2 \times S^1$ |                    |
| $\mathbb{Z}_1$                 | $\mathbb{Z}_1$ | $\mathbb{Z}_1$ | $\mathbb{Z}_1$        | $A_4 \times S^1$          | $A_4 \times S^1$          |                    |
| $A_4^{t_k}, k = 1, 2$          | $A_4$          | $\mathbb{Z}_3$ | $V_4$                 | $A_4 \times S^1$          | $S^1$                     |                    |
| $V_4^-$                        | $V_4$          | $\mathbb{Z}_2$ | $\mathbb{Z}_2$        | $V_4 \times S^1$          | $S^1$                     |                    |
| $\mathbb{Z}_2^-$               | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_1$        | $V_4 \times S^1$          | $\mathbb{Z}_2 \times S^1$ |                    |
| $\mathbb{Z}_3^{t_k}, k = 1, 2$ | $\mathbb{Z}_3$ | $\mathbb{Z}_3$ | $\mathbb{Z}_1$        | $\mathbb{Z}_3 \times S^1$ | $S^1$                     | $\varphi(g) = g^k$ |

TABLE 6. Twisted subgroups of  $A_4 \times S^1$

All the one-folded twisted subgroups of the group  $A_4 \times S^1$ , i.e. the subgroups of the type  $K^\varphi$ , are described in Table 6. The lattice of the conjugacy classes

for the twisted subgroups in  $A_4 \times S^1$  and the  $A(A_4)$ -multiplication table for the generators of  $A_1(A_4 \times S^1)$  is shown in Table 7.

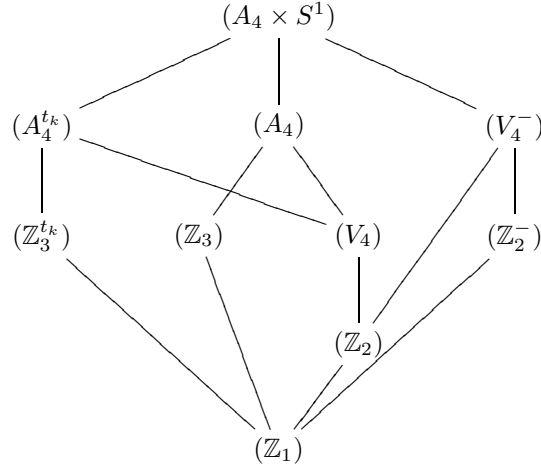


DIAGRAM 2

| $(A_4)$               | $(V_4)$             | $(\mathbb{Z}_3)$                       | $(\mathbb{Z}_2)$                      | $(\mathbb{Z}_1)$   |                       |
|-----------------------|---------------------|--|---------------------------------------|--------------------|-----------------------|
| $(A_4)$               | $(V_4)$             | $(\mathbb{Z}_3)$                       | $(\mathbb{Z}_2)$                      | $(\mathbb{Z}_1)$   | $(A_4)$               |
| $(V_4)$               | $3(V_4)$            | $(\mathbb{Z}_1)$                       | $3(\mathbb{Z}_2)$                     | $3(\mathbb{Z}_1)$  | $(V_4)$               |
| $(\mathbb{Z}_3)$      | $(\mathbb{Z}_1)$    | $(\mathbb{Z}_3) + (\mathbb{Z}_1)$      | $2(\mathbb{Z}_1)$                     | $4(\mathbb{Z}_1)$  | $(\mathbb{Z}_3)$      |
| $(\mathbb{Z}_2)$      | $3(\mathbb{Z}_2)$   | $2(\mathbb{Z}_1)$                      | $2(\mathbb{Z}_2) + 2(\mathbb{Z}_1)$   | $6(\mathbb{Z}_1)$  | $(\mathbb{Z}_2)$      |
| $(\mathbb{Z}_1)$      | $3(\mathbb{Z}_1)$   | $4(\mathbb{Z}_1)$                      | $6(\mathbb{Z}_1)$                     | $12(\mathbb{Z}_1)$ | $(\mathbb{Z}_1)$      |
| $(A_4^{tk})$          | $(V_4)$             | $(\mathbb{Z}_3^{tk})$                  | $(\mathbb{Z}_2)$                      | $(\mathbb{Z}_1)$   | $(A_4^{tk})$          |
| $(V_4^-)$             | $3(V_4^-)$          | $(\mathbb{Z}_1)$                       | $2(\mathbb{Z}_2^-) + 2(\mathbb{Z}_2)$ | $3(\mathbb{Z}_1)$  | $(V_4^-)$             |
| $(\mathbb{Z}_3^{tk})$ | $(\mathbb{Z}_1)$    | $(\mathbb{Z}_3^{tk}) + (\mathbb{Z}_1)$ | $2(\mathbb{Z}_1)$                     | $4(\mathbb{Z}_1)$  | $(\mathbb{Z}_3^{tk})$ |
| $(\mathbb{Z}_2^-)$    | $3(\mathbb{Z}_2^-)$ | $2(\mathbb{Z}_1)$                      | $2(\mathbb{Z}_2^-) + 2(\mathbb{Z}_1)$ | $6(\mathbb{Z}_1)$  | $(\mathbb{Z}_2^-)$    |

TABLE 7. Conjugacy classes of twisted subgroups in  $A_4 \times S^1$  and  $A(A_4)$ -module multiplication table for  $A_1(A_4 \times S^1)$

| $L$            | $H$              | $n(L, H)$ | $L$            | $H$            | $n(L, H)$ | $L$                 | $H$        | $n(L, H)$ | $L$              | $H$                 | $n(L, H)$ |
|----------------|------------------|-----------|----------------|----------------|-----------|---------------------|------------|-----------|------------------|---------------------|-----------|
| $\mathbb{Z}_1$ | $A_4$            | 1         | $\mathbb{Z}_1$ | $V_4$          | 1         | $\mathbb{Z}_1$      | $A_4^{tk}$ | 1         | $\mathbb{Z}_1$   | $V_4^-$             | 3         |
| $\mathbb{Z}_2$ | $A_4$            | 1         | $\mathbb{Z}_2$ | $V_4$          | 1         | $V_4$               | $A_4^{tk}$ | 1         | $\mathbb{Z}_2$   | $V_4^-$             | 1         |
| $\mathbb{Z}_3$ | $A_4$            | 1         | $\mathbb{Z}_1$ | $\mathbb{Z}_3$ | 4         | $\mathbb{Z}_2$      | $A_4^{tk}$ | 1         | $\mathbb{Z}_2^-$ | $V_4^-$             | 2         |
| $V_4$          | $A_4$            | 1         | $\mathbb{Z}_1$ | $\mathbb{Z}_2$ | 3         | $\mathbb{Z}_3^{tk}$ | $A_4^{tk}$ | 1         | $\mathbb{Z}_1$   | $\mathbb{Z}_3^{tk}$ | 4         |
| $\mathbb{Z}_1$ | $\mathbb{Z}_2^-$ | 3         |                |                |           |                     |            |           |                  |                     |           |

TABLE 8. Numbers  $n(L, H)$  for twisted subgroups in  $A_4 \times S^1$  ( $k = 1, 2$ )

To compute basic degrees of the first type, we describe real irreducible  $A_4$ -representations. We have the one-dimensional trivial representation  $\mathcal{V}_0$ . Using the homomorphisms  $\varphi: A_4 \rightarrow A_4/V_4 \simeq \mathbb{Z}_3$ , we obtain the two-dimensional representations  $\mathcal{V}_1, \mathcal{V}_2$  (associated with the  $\mathbb{Z}_3$ -actions on  $\mathbb{R}^2 \simeq \mathbb{C}$  given by  $\gamma z = \gamma^k \cdot z$ ,  $k = 1, 2$ ), which, in fact, are equivalent. There is also one three-dimensional natural representation  $\mathcal{V}_3$  of  $A_4$  (see the Diagrams 3 and 4).

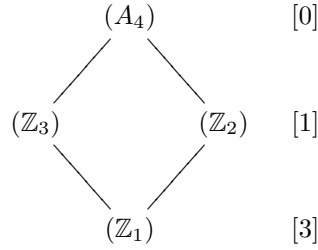


DIAGRAM 3. Representation  $\mathcal{V}_3$

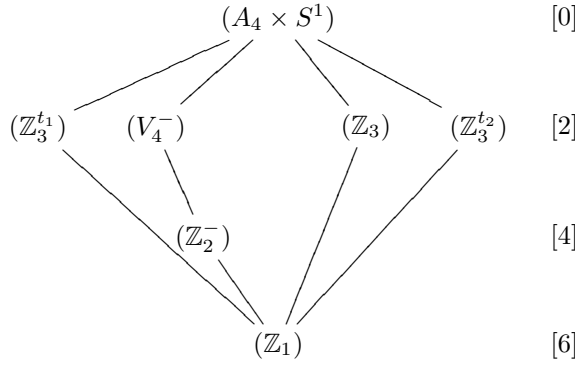


DIAGRAM 4. Representation  $\mathcal{V}_{3,1}$

The computation of the first type basic degrees, related to the representations  $\mathcal{V}_0, \mathcal{V}_1, \mathcal{V}_2$  and  $\mathcal{V}_3$  is a straightforward application of the formula (6.4).

$$\deg_{\mathcal{V}_0} = -(A_4), \quad \deg_{\mathcal{V}_1} = \deg_{\mathcal{V}_2} = (A_4), \quad \deg_{\mathcal{V}_3} = (A_4) - 2(\mathbb{Z}_3) - (\mathbb{Z}_2) + (\mathbb{Z}_1).$$

For the complexifications  $\mathcal{V}_j^c$  of the representations  $\mathcal{V}_j, j = 0, \dots, 3$ , we define the  $l$ -th action of  $S^1, l = 1, 2, \dots$ , by  $zv := z^l \cdot v$ , where  $z \in S^1 \subset \mathbb{C}, v \in \mathcal{V}_j$ , and “ $\cdot$ ” is the complex multiplication, which leads to the irreducible real representation  $\mathcal{V}_{j,l}$  of  $A_4 \times S^1$ . Let us discuss the isotropy lattices for the representations  $\mathcal{V}_{j,1}, j = 0, 1, 2, 3$  (computations for the general case of an arbitrary  $l \geq 1$  are standard — see (6.6)). Here, it should be pointed out that although  $\mathcal{V}_1 \cong \mathcal{V}_2$ ,

the representations  $\mathcal{V}_{1,1}$  and  $\mathcal{V}_{2,1}$  are not equivalent. Of course, the only twisted orbit type for  $\mathcal{V}_{0,1}$  is  $(A_4)$ . There is also only one twisted isotropy class  $(A_4^{t_2})$  in  $\mathcal{V}_{1,1}$  and for  $\mathcal{V}_{2,1}$ , the only twisted isotropy class is  $(A_4^{t_1})$ , which are, in both cases, determined by the homomorphism  $\varphi_j: A_4 \xrightarrow{\varphi} A_4/V_4 \simeq \mathbb{Z}_3 \xrightarrow{\gamma \rightarrow \gamma^j} \mathbb{Z}_3$ ,  $j = 1, 2$ .

To obtain the lattice for  $\mathcal{V}_{3,1}$ , consider the action of  $A_4$  on  $\mathbb{C}^4$  permuting the coordinates of the vectors  $\vec{z} = \langle z_1, z_2, z_3, z_4 \rangle$  and let  $S^1$  act by the complex multiplication. The subspace  $\{\langle z, z, z, z \rangle : z \in \mathbb{C}\}$  is the fixed-point subspace for the action of  $A_4$ , and its complement is equivalent to the representation  $\mathcal{V}_3^c$ . Let us choose the following basis in this subspace:  $\vec{v}_1 = \langle 1, -1, 1, -1 \rangle$ ,  $\vec{v}_2 = \langle 1, 1, -1, -1 \rangle$ , and  $\vec{v}_3 = \langle -1, 1, 1, -1 \rangle$ . Notice that the vectors  $\vec{v}_1$ ,  $\vec{v}_2$ , and  $\vec{v}_3$  have the isotropy groups (with respect to  $G = A_4 \times S^1$ ) belonging to the class  $(V_4^-)$ . Indeed:

$$\begin{aligned} G_{\vec{v}_1} &= \{((1), 1), ((13)(24), 1), ((12)(34), -1), ((14)(23), -1)\}, \\ G_{\vec{v}_2} &= \{((1), 1), ((12)(34), 1), ((13)(24), -1), ((14)(23), -1)\}, \\ G_{\vec{v}_3} &= \{((1), 1), ((14)(23), 1), ((12)(34), -1), ((13)(24), -1)\}. \end{aligned}$$

Next, notice that the vectors  $\vec{x} = \vec{v}_1 + \vec{v}_2 = \langle 0, 2, 0, -2 \rangle$  and  $\vec{y} = \vec{v}_1 - \vec{v}_2 = \langle 2, 0, -2, 0 \rangle$  have the isotropy group  $H = \{((1), 1), ((13)(24), -1)\}$ , which belongs to the class  $(\mathbb{Z}_2^-)$ . The vectors  $\vec{v}_1 + \vec{v}_2 + \vec{v}_3$ ,  $\vec{v}_1 + \vec{v}_2 - \vec{v}_3$ ,  $\vec{v}_1 - \vec{v}_2 - \vec{v}_3$ , and  $-\vec{v}_1 + \vec{v}_2 - \vec{v}_3$  have the isotropy group belonging to the class of the subgroup  $(\mathbb{Z}_3)$ . Let  $\omega = e^{2\pi i/3}$ . Then the vectors  $\vec{w}_1^1 = \langle 1, \omega, \omega^2, 0 \rangle$ ,  $\vec{w}_2^1 = \langle 1, \omega, 0, \omega^2 \rangle$ ,  $\vec{w}_3^1 = \langle 1, 0, \omega, \omega^2 \rangle$ , and  $\vec{w}_4^1 = \langle 0, 1, \omega, \omega^2 \rangle$  have the isotropy groups belonging to the class  $(\mathbb{Z}_3^{t_1})$ , and  $\vec{w}_1^2 = \langle 1, \omega^2, \omega, 0 \rangle$ ,  $\vec{w}_2^2 = \langle 1, \omega^2, 0, \omega \rangle$ ,  $\vec{w}_3^2 = \langle 1, 0, \omega^2, \omega \rangle$ , and  $\vec{w}_4^2 = \langle 0, 1, \omega^2, \omega \rangle$  have the isotropy groups belonging to the class  $(\mathbb{Z}_3^{t_2})$ . The isotropy lattices for  $\mathcal{V}_3$  (as the representation of  $A_4$ ) and  $\mathcal{V}_{3,1}$  (as the representation of  $A_4 \times S^1$ ) are shown on the diagram above.

Finally, we can list all the  $A_4 \times S^1$ -degrees of the basic mappings of the second type associated with these representations:

$$\begin{aligned} \deg_{\mathcal{V}_{0,1}} &= (A_4), & \deg_{\mathcal{V}_{1,1}} &= (A_4^{t_2}), \\ \deg_{\mathcal{V}_{2,1}} &= (A_4^{t_1}), & \deg_{\mathcal{V}_{3,1}} &= (\mathbb{Z}_3^{t_1}) + (\mathbb{Z}_3^{t_2}) + (V_4^-) + (\mathbb{Z}_3) - (\mathbb{Z}_1). \end{aligned}$$

**6.4. Computations for the octahedral group  $\mathbb{O}$ .** Let us consider the octahedral group  $\mathbb{O}$  of symmetries of the cube. Since the group  $\mathbb{O}$  is isomorphic to  $S_4$ , it is easy to classify the conjugacy classes of the subgroups in  $S_4$ . First of all, we have the subgroup  $A_4$  and all its subgroups are also subgroups of  $S_4$ , including  $V_4$ ,  $\mathbb{Z}_3$ ,  $\mathbb{Z}_2$  and  $\mathbb{Z}_1$ . In addition, there are other subgroups:  $D_4$ ,  $\mathbb{Z}_4$ ,  $D_3$ ,  $D_2$  and  $D_1$ . The conjugacy classes of these subgroups are illustrated by Diagram 5.

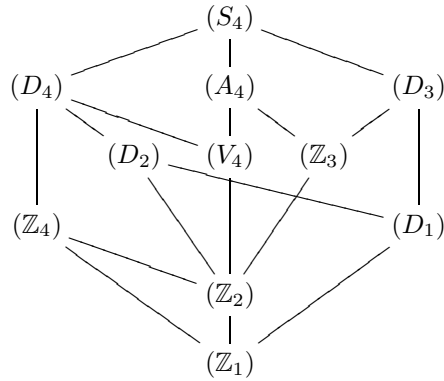


DIAGRAM 5. Lattice of conjugacy classes in  $S_4$

Let us notice that the subgroup  $D_4$ , which is composed of the elements  $\{(1), (1324), (12)(34), (1423), (34), (14)(23), (12), (13)(24)\}$  has the normalizer  $N(D_4) = D_4$ . The conjugacy class  $(D_4)$  contains three elements, corresponding to the symmetry subgroups of the three pairs of parallel faces of the cube. The normalizer of the subgroup  $A_4$  is  $N(A_4) = S_4$ . The group  $D_3$  consists of the elements  $\{(1), (123), (132), (12), (23), (13)\}$ . The conjugacy class  $(D_3)$  contains four subgroups corresponding to the symmetries of the cube around each of four pairs of opposite vertices of the cube. In addition  $N(D_3) = D_3$ . The subgroup  $\mathbb{Z}_4$  consisting of the rotations belonging to  $D_4$ , has the normalizer  $N(\mathbb{Z}_4) = D_4$ . There are three subgroups in the conjugacy class  $(\mathbb{Z}_4)$ , which correspond to the rotations of the three pairs of the parallel faces of the cube. The normalizer of  $V_4$  is  $N(V_4) = S_4$ , the subgroup  $D_2$  has the normalizer  $N(D_2) = D_4$ , and the subgroup  $D_1$  has the normalizer  $N(D_1) = D_2$ . Finally, the subgroup  $\mathbb{Z}_2$  has the normalizer  $N(\mathbb{Z}_2) = D_4$ . Notice that  $D_3$  is isomorphic to  $S_3$  and that the two dihedral subgroups of  $D_4$  of order 4 are  $V_4$  and  $D_2$ .

The twisted subgroups  $K^\varphi$  of  $S_4 \times S^1$  are listed in Table 9 (notice that the original subgroups  $H$  of  $S_4$  can be easily “read” from the first column of Table 9, and the numbers  $n(L, H)$  for the twisted subgroups are given in Table 10. The multiplication table for the  $A(S_4)$ -module  $A_1(S_4 \times S^1)$  (and implicitly for the Burnside ring  $A(S_4)$ ) is given in Table 11.

There are exactly five real (and also complex) irreducible representations of  $S_4$ : the trivial representation  $\mathcal{V}_0$ , the one-dimensional representation  $\mathcal{V}_1$  corresponding to the homomorphism  $\varphi: S_4 \rightarrow \mathbb{Z}_2 \subset O(1)$ , where  $\ker \varphi = A_4$ , the two-dimensional representation  $\mathcal{V}_2$  corresponding to the homomorphism  $\psi: S_4 \rightarrow S_4/V_4 = S_3 \simeq D_3 \subset O(2)$ , and two different three-dimensional representations of  $S_4$ , one of them being the natural representation  $\mathcal{V}_3$  of  $S_4$ , while the other  $\mathcal{V}_4$

| $H = K^\varphi$   | $\text{Ker } \varphi$ | $\varphi(K)$   | $N(H)$                    | $W(H)$  | Comments         |
|-------------------|-----------------------|----------------|---------------------------|---|------------------|
| $S_4^-$           | $A_4$                 | $\mathbb{Z}_2$ | $S_4 \times S_4$          | $S^1$   |                  |
| $D_4^z$           | $\mathbb{Z}_4$        | $\mathbb{Z}_2$ | $D_4 \times S^1$          | $S^1$   |                  |
| $D_4^d$           | $D_2$                 | $\mathbb{Z}_2$ | $D_4 \times S^1$          | $S^1$   |                  |
| $D_4^{\tilde{d}}$ | $V_4$                 | $\mathbb{Z}_2$ | $D_4 \times S^1$          | $S^1$   |                  |
| $A_4^t$           | $V_4$                 | $\mathbb{Z}_3$ | $A_4 \times S^1$          | $S^1$   |                  |
| $D_3^z$           | $\mathbb{Z}_3$        | $\mathbb{Z}_2$ | $D_3 \times S^1$          | $S^1$   |                  |
| $D_2^z$           | $\mathbb{Z}_2$        | $\mathbb{Z}_2$ | $D_4 \times S^1$          | $\mathbb{Z}_2 \times S^1$                     |                  |
| $D_2^d$           | $D_1$                 | $\mathbb{Z}_2$ | $D_2 \times S^1$          | $S^1$   |                  |
| $V_4^-$           | $\mathbb{Z}_2$        | $\mathbb{Z}_2$ | $D_4 \times S^1$          | $\mathbb{Z}_2 \times S^1$                     |                  |
| $D_1^z$           | $\mathbb{Z}_1$        | $\mathbb{Z}_2$ | $D_2 \times S^1$          | $\mathbb{Z}_2 \times S^1$                     |                  |
| $\mathbb{Z}_4^-$  | $\mathbb{Z}_2$        | $\mathbb{Z}_2$ | $D_4 \times S^1$          | $\mathbb{Z}_2 \times S^1$                     |                  |
| $\mathbb{Z}_4^c$  | $\mathbb{Z}_1$        | $\mathbb{Z}_4$ | $\mathbb{Z}_4 \times S^1$ | $S^1$   | $\varphi(g) = g$ |
| $\mathbb{Z}_3^t$  | $\mathbb{Z}_1$        | $\mathbb{Z}_3$ | $\mathbb{Z}_3 \times S^1$ | $S^1$   | $\varphi(g) = g$ |
| $\mathbb{Z}_2^-$  | $\mathbb{Z}_1$        | $\mathbb{Z}_2$ | $D_4 \times S^1$          | $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \times S^1$ |                  |

TABLE 9. Twisted subgroups in  $S_4 \times S^1$ 

| $L$               | $H$              | $n(L, H)$ | $L$              | $H$              | $n(L, H)$ | $L$              | $H$               | $n(L, H)$ | $L$              | $H$              | $n(L, H)$ |
|-------------------|------------------|-----------|------------------|------------------|-----------|------------------|-------------------|-----------|------------------|------------------|-----------|
| $A_4$             | $S_4^-$          | 1         | $\mathbb{Z}_2^-$ | $\mathbb{Z}_4^z$ | 2         | $D_2^z$          | $D_4^{\tilde{d}}$ | 1         | $D_2^z$          | $D_4^z$          | 1         |
| $V_4$             | $S_4^-$          | 1         | $\mathbb{Z}_1$   | $\mathbb{Z}_4^z$ | 6         | $\mathbb{Z}_4^-$ | $D_4^{\tilde{d}}$ | 1         | $D_2$            | $D_4^z$          | 1         |
| $\mathbb{Z}_3$    | $S_4^-$          | 1         | $\mathbb{Z}_4^-$ | $D_4^d$          | 1         | $V_4$            | $D_4^{\tilde{d}}$ | 3         | $\mathbb{Z}_4$   | $D_4^z$          | 1         |
| $\mathbb{Z}_2$    | $S_4^-$          | 1         | $V_4^-$          | $D_4^d$          | 1         | $\mathbb{Z}_2$   | $D_4^{\tilde{d}}$ | 3         | $V_4^-$          | $D_4^z$          | 1         |
| $\mathbb{Z}_1$    | $S_4^-$          | 1         | $\mathbb{Z}_2^-$ | $D_4^d$          | 2         | $D_1^z$          | $D_4^{\tilde{d}}$ | 1         | $\mathbb{Z}_2^-$ | $D_4^z$          | 2         |
| $D_4^{\tilde{d}}$ | $S_4^-$          | 1         | $D_2$            | $D_4^d$          | 1         | $\mathbb{Z}_1$   | $D_4^{\tilde{d}}$ | 3         | $D_1^z$          | $D_4^z$          | 1         |
| $\mathbb{Z}_4^-$  | $S_4^-$          | 1         | $D_1$            | $D_4^d$          | 1         | $V_4$            | $A_4^t$           | 2         | $\mathbb{Z}_1$   | $D_4^z$          | 3         |
| $V_4^-$           | $S_4^-$          | 1         | $\mathbb{Z}_2$   | $D_4^d$          | 1         | $\mathbb{Z}_3^t$ | $A_4^t$           | 1         | $\mathbb{Z}_2$   | $\mathbb{Z}_4^-$ | 2         |
| $\mathbb{Z}_2^-$  | $S_4^-$          | 1         | $\mathbb{Z}_1$   | $D_4^d$          | 3         | $\mathbb{Z}_2$   | $A_4^t$           | 2         | $\mathbb{Z}_1$   | $\mathbb{Z}_4^-$ | 3         |
| $\mathbb{Z}_2^-$  | $V_4^-$          | 2         | $\mathbb{Z}_3$   | $D_3^z$          | 1         | $\mathbb{Z}_1$   | $A_4^t$           | 1         | $\mathbb{Z}_2^-$ | $D_2^d$          | 2         |
| $\mathbb{Z}_2$    | $V_4^-$          | 1         | $D_1^z$          | $D_3^z$          | 2         | $D_1^z$          | $D_2^z$           | 1         | $D_1^z$          | $D_2^d$          | 1         |
| $\mathbb{Z}_1$    | $\mathbb{Z}_3^t$ | 8         | $\mathbb{Z}_1$   | $D_3^z$          | 4         | $\mathbb{Z}_2$   | $D_2^z$           | 1         | $D_1$            | $D_2^d$          | 1         |
| $V_4$             | $A_4$            | 1         | $V_4$            | $D_4$            | 3         | $\mathbb{Z}_3$   | $D_3$             | 1         | $\mathbb{Z}_2$   | $\mathbb{Z}_4$   | 1         |
| $\mathbb{Z}_3$    | $A_4$            | 1         | $\mathbb{Z}_4$   | $D_4$            | 1         | $D_1$            | $D_3$             | 2         | $\mathbb{Z}_1$   | $\mathbb{Z}_4$   | 3         |
| $\mathbb{Z}_2$    | $A_4$            | 1         | $D_2$            | $D_4$            | 1         | $\mathbb{Z}_1$   | $D_3$             | 4         | $\mathbb{Z}_1$   | $\mathbb{Z}_3$   | 4         |
| $\mathbb{Z}_1$    | $A_4$            | 1         | $D_1$            | $D_4$            | 1         | $\mathbb{Z}_2$   | $V_4$             | 1         | $D_1$            | $D_2$            | 1         |
| $\mathbb{Z}_1$    | $\mathbb{Z}_3^t$ | 1         | $\mathbb{Z}_2$   | $D_4$            | 3         | $\mathbb{Z}_1$   | $V_4$             | 3         | $\mathbb{Z}_2$   | $D_2$            | 1         |
| $\mathbb{Z}_1$    | $D_1^z$          | 6         | $\mathbb{Z}_1$   | $D_4$            | 3         | $\mathbb{Z}_1$   | $D_1$             | 6         | $\mathbb{Z}_1$   | $D_2$            | 1         |
| $\mathbb{Z}_1$    | $\mathbb{Z}_2^-$ | 1         | $\mathbb{Z}_1$   | $\mathbb{Z}_2$   | 3         |                  |                   |           |                  |                  |           |

TABLE 10. Numbers  $n(L, H)$  for twisted subgroups in  $S_4 \times S^1$



| $(D_4)$             | $(A_4)$    | $(D_3)$             | $(D_2)$             | $(V_4)$    | $(Z_4)$             | $(Z_3)$             | $(Z_2)$             | $(D_1)$                    |           |
|---------------------|------------|---------------------|---------------------|------------|---------------------|---------------------|---------------------|----------------------------|-----------|
| $(D_4)$             | $(V_4)$    | $(D_1)$             | $(D_2) + (Z_2)$     | $3(V_4)$   | $(Z_4) + (Z_2)$     | $(Z_1)$             | $3(Z_2)$            | $(D_1) + (Z_1)$            | $3(Z_1)$  |
| $(V_4)$             | $2(A_4)$   | $(Z_3)$             | $(Z_2)$             | $2(V_4)$   | $(Z_2)$             | $2(Z_3)$            | $2(Z_2)$            | $(Z_1)$                    | $2(Z_1)$  |
| $(D_1)$             | $(Z_3)$    | $(D_3) + (D_1)$     | $2(D_1)$            | $(Z_1)$    | $(Z_1)$             | $(Z_3) + (Z_1)$     | $2(Z_1)$            | $2(D_1) + (Z_1)$           | $4(Z_1)$  |
| $(D_2) + (Z_2)$     | $(Z_2)$    | $2(D_1)$            | $2(D_2) + (Z_1)$    | $3(Z_2)$   | $(Z_2) + (Z_1)$     | $2(Z_1)$            | $2(Z_2) + 2(Z_1)$   | $2(D_1) + 2(Z_1)$          | $6(Z_1)$  |
| $3(V_4)$            | $(V_4)$    | $(Z_1)$             | $3(Z_2)$            | $6(V_4)$   | $3(Z_2)$            | $2(Z_1)$            | $6(Z_2)$            | $3(Z_1)$                   | $6(Z_1)$  |
| $(Z_4) + (Z_2)$     | $2(Z_2)$   | $(Z_1)$             | $(Z_2) + (Z_1)$     | $3(Z_2)$   | $2(Z_4) + (Z_1)$    | $2(Z_1)$            | $2(Z_2) + 2(Z_1)$   | $3(Z_1)$                   | $6(Z_1)$  |
| $(Z_1)$             | $2(Z_3)$   | $(Z_3) + (Z_1)$     | $2(Z_1)$            | $2(Z_1)$   | $2(Z_1)$            | $2(Z_3) + 2(Z_1)$   | $4(Z_1)$            | $4(Z_1)$                   | $8(Z_1)$  |
| $3(Z_2)$            | $2(Z_2)$   | $2(Z_1)$            | $2(Z_2) + 2(Z_1)$   | $6(Z_2)$   | $2(Z_2) + 2(Z_1)$   | $4(Z_1)$            | $4(Z_2) + 4(Z_1)$   | $6(Z_1)$                   | $12(Z_1)$ |
| $(D_1) + (Z_1)$     | $(Z_1)$    | $2(D_1) + (Z_1)$    | $2(D_1) + 2(Z_1)$   | $3(Z_1)$   | $3(Z_1)$            | $4(Z_1)$            | $6(Z_1)$            | $2(D_1) + 5(Z_1)$          | $12(Z_1)$ |
| $3(Z_1)$            | $2(Z_1)$   | $4(Z_1)$            | $6(Z_1)$            | $6(Z_1)$   | $6(Z_1)$            | $8(Z_1)$            | $12(Z_1)$           | $12(Z_1)$                  | $24(Z_1)$ |
| $(D_4^+)$           | $(V_4^-)$  | $(D_4^+)$           | $(D_2^+) + (Z_2^-)$ | $3(V_4^-)$ | $(Z_4) + (Z_2^-)$   | $(Z_1)$             | $(Z_2) + 2(Z_2^-)$  | $(D_1^+) + (Z_1)$          | $3(Z_1)$  |
| $(D_4^+)$           | $(V_4^-)$  | $(D_1)$             | $(D_2) + (Z_2^-)$   | $3(V_4^-)$ | $(Z_4^-) + (Z_2^-)$ | $(Z_1)$             | $(Z_2) + 2(Z_2^-)$  | $(D_1) + (Z_1)$            | $3(Z_1)$  |
| $(D_4^+)$           | $(V_4)$    | $(D_1^+)$           | $(D_2^+) + (Z_2)$   | $3(V_4)$   | $(Z_4^-) + (Z_2)$   | $(Z_1)$             | $3(Z_2)$            | $(D_1^+) + (Z_1)$          | $3(Z_1)$  |
| $(V_4)$             | $2(A_4^+)$ | $(Z_3^+)$           | $(Z_2)$             | $2(V_4)$   | $(Z_2)$             | $2(Z_3)$            | $2(Z_2)$            | $(Z_1)$                    | $2(Z_1)$  |
| $(D_1^+)$           | $(Z_3)$    | $(D_3^+) + (D_1^+)$ | $2(D_1^+)$          | $(Z_1)$    | $(Z_1)$             | $(Z_3) + (Z_1)$     | $2(Z_1)$            | $2(D_1^+) + (Z_1)$         | $4(Z_1)$  |
| $(D_2^+) + (Z_2)$   | $(Z_2)$    | $2(D_1^+)$          | $2(D_2^+) + (Z_1)$  | $3(Z_2)$   | $(Z_2) + (Z_1)$     | $2(Z_1)$            | $2(Z_2) + 2(Z_1)$   | $2(D_1^+) + 2(Z_1)$        | $6(Z_1)$  |
| $(D_2^+)$           | $(Z_2^-)$  | $(D_1) + (D_1^+)$   | $(Z_2) + 2(Z_2^-)$  | $6(V_4^-)$ | $(Z_2) + 2(Z_2^-)$  | $2(Z_1)$            | $2(Z_2) + 4(Z_2^-)$ | $(D_1) + (D_1^+) + 2(Z_1)$ | $6(Z_1)$  |
| $3(V_4^-)$          | $2(V_4^-)$ | $(Z_1)$             | $(Z_2^-) + (Z_1)$   | $3(Z_2)$   | $2(Z_4^+) + (Z_1)$  | $2(Z_1)$            | $2(Z_2^-) + 2(Z_1)$ | $3(Z_1)$                   | $6(Z_1)$  |
| $(Z_4^+) + (Z_2^-)$ | $(Z_2^-)$  | $(Z_1)$             | $2(Z_1)$            | $2(Z_1)$   | $2(Z_4^+) + 2(Z_1)$ | $2(Z_3^+) + 2(Z_1)$ | $4(Z_1)$            | $3(Z_1)$                   | $6(Z_1)$  |
| $(Z_1)$             | $2(Z_3^+)$ | $(Z_3^+) + (Z_1)$   | $2(Z_1)$            | $2(Z_1)$   | $2(Z_1)$            | $4(Z_1)$            | $4(Z_1)$            | $4(Z_1)$                   | $8(Z_1)$  |
| $3(Z_2^-)$          | $2(Z_2^-)$ | $2(Z_1)$            | $2(Z_2^-) + 2(Z_1)$ | $6(Z_2^-)$ | $2(Z_2^-) + 2(Z_1)$ | $4(Z_1)$            | $4(Z_2^-) + 4(Z_1)$ | $6(Z_1)$                   | $12(Z_1)$ |
| $(D_4^+)$           | $(A_4)$    | $(D_3^+)$           | $(D_2^+) + 2(Z_1)$  | $(V_4)$    | $(Z_4^-)$           | $(Z_3)$             | $(Z_2)$             | $(D_1^+)$                  | $(Z_1)$   |
| $(D_1^+) + (Z_1)$   | $(Z_1)$    | $2(D_1^+) + (Z_1)$  | $2(D_1^+) + 2(Z_1)$ | $3(Z_1)$   | $3(Z_1)$            | $4(Z_1)$            | $6(Z_1)$            | $2(D_1^+) + 5(Z_1)$        | $12(Z_1)$ |
| $(Z_1^-) + (Z_2)$   | $(Z_2)$    | $(Z_1)$             | $(Z_2) + (Z_1)$     | $3(Z_2)$   | $2(Z_1^-) + (Z_1)$  | $2(Z_1)$            | $2(Z_2) + 2(Z_1)$   | $3(Z_1)$                   | $6(Z_1)$  |

TABLE 11.  $A(S_4)$ -multiplication table for  $A_1(S_4 \times S^1)$

being the tensor product  $\mathcal{V}_1 \otimes \mathcal{V}_3$  of the natural three-dimensional representation with the non-trivial one-dimensional representation.

It is easy to notice that the representation  $\mathcal{V}_0$  contains the orbit type  $(S_4)$ , the representation  $\mathcal{V}_1$  (where  $S_4$  acts as  $\mathbb{Z}_2$ ) has only the orbit types  $(S_4)$  (with the dimension 0 of fixed-point space) and  $(A_4)$  (with  $\dim \mathcal{V}_1^{A_4} = 1$ ), and the representation  $\mathcal{V}_2$  (where  $S_4$  acts as  $D_3$  on  $\mathbb{R}^2$ ) contains the orbit types  $(S_4)$  (with zero-dimensional fixed-point space),  $(D_4)$  (with  $\dim \mathcal{V}_2^{D_4} = 1$ ) and  $(V_4)$  (with  $\dim \mathcal{V}_2^{V_4} = 2$ ).

In order to describe the orbit types in the natural representation, it is convenient to describe this representation exactly in the same way, as it was done for the group  $A_4$ . Namely, we consider the representation  $\mathbb{R}^4$  of the group  $S_4$ , where the group  $S_4$  acts by permuting the coordinates of the vectors  $\vec{v} = \langle x_1, x_2, x_3, x_4 \rangle$ . In the space  $\mathbb{R}^4$  we choose the orthogonal basis  $\vec{v}_0 = \langle 1, 1, 1, 1 \rangle$ ,  $\vec{v}_1 = \langle 1, -1, 1, -1 \rangle$ ,  $\vec{v}_2 = \langle 1, 1, -1, -1 \rangle$ , and  $\vec{v}_3 = \langle -1, 1, 1, -1 \rangle$ . It is easy to observe that the subspace  $\text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$  is equivalent to the representation  $\mathcal{V}_3$ . Let us notice that the isotropy of the vector  $\vec{x} = \vec{v}_1 + \vec{v}_2 + \vec{v}_3$  is the subgroup  $D_3$ . The vector  $\vec{v}_2$  has the isotropy group  $D_2 = \{(1), (12)(34), (12), (34)\}$ , and the vector  $\vec{v}_1 + \vec{v}_2$  has the isotropy group  $D_1 = \{(1), (23)\}$ . Finally, the vector  $\vec{v}_1 + 2\vec{v}_2 + 3\vec{v}_3$  has the isotropy group  $\mathbb{Z}_1$ . By considering the isotropy groups of the elements  $\alpha\vec{v}_1 + \beta\vec{v}_2 + \gamma\vec{v}_3$  we conclude that there are only the following orbit types in the representation  $\mathcal{V}_3$ :  $(S_4)$ ,  $(D_3)$ ,  $(D_2)$ ,  $(D_1)$  and  $(\mathbb{Z}_1)$ . The representation  $\mathcal{V}_4$  can be described in a similar way as the representation  $\mathcal{V}_3$ . Namely, we consider again the space  $\mathbb{R}^4$  where the group  $A_4$  acts as before, i.e. it permutes the coordinates of the vectors  $\vec{x}$  in  $\mathbb{R}^4$ . In the case  $\sigma \in S_4$  is an odd permutation, we define  $\sigma\langle x_1, x_2, x_3, x_4 \rangle = -\langle x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)} \rangle$ . Then again, the subspace spanned by the vectors  $\vec{v}_1$ ,  $\vec{v}_2$  and  $\vec{v}_3$  is equivalent to the representation  $\mathcal{V}_4$ . Notice that the isotropy group of the vector  $\vec{v}_2$  is  $\mathbb{Z}_4 = \{(1), (1324), (12)(34), (1423)\}$ . The vector  $\vec{v}_1 + \vec{v}_2 + \vec{v}_3$  has the isotropy group  $\mathbb{Z}_3$ , and the vector  $\vec{v}_1 + \vec{v}_2 = \langle 2, 0, 0, -2 \rangle$  has the isotropy group  $D_1 = \{(1), (14)\}$ . For each of the representations  $\mathcal{V}_0$ ,  $\mathcal{V}_1$ ,  $\mathcal{V}_2$ ,  $\mathcal{V}_3$ , and  $\mathcal{V}_4$ , we compute the element  $\text{deg}_{\mathcal{V}_j} \in A(S_4)$ ,  $j = 0, \dots, 4$ :

$$\begin{aligned} \text{deg}_{\mathcal{V}_0} &= -(S_4), & \text{deg}_{\mathcal{V}_1} &= (S_4) - 2(D_4), & \text{deg}_{\mathcal{V}_2} &= (S_4) - 2(D_4) + (V_4), \\ \text{deg}_{\mathcal{V}_3} &= (S_4) - 2(D_3) - (D_2) + 3(D_1) - (\mathbb{Z}_1), \\ \text{deg}_{\mathcal{V}_4} &= (S_4) - (\mathbb{Z}_4) - (D_1) - (\mathbb{Z}_3) + (\mathbb{Z}_1). \end{aligned}$$

Now, we can describe the irreducible representations of the group  $S_4 \times S^1$ . We consider the complexifications  $\mathcal{V}_j^c$  of the representations  $\mathcal{V}_j$ ,  $j = 0, \dots, 4$ , and define the  $S^1$ -action on  $\mathcal{V}_j^c$  by  $\gamma \vec{v} = \gamma^l \cdot \vec{v}$ , where  $l = 0, 1, \dots$ ;  $\gamma \in S^1$ ,  $\vec{v} \in \mathcal{V}_j^c$ . We will denote the obtained irreducible  $S_4 \times S^1$ -representation by  $\mathcal{V}_{j,l}$ ,  $j = 0, \dots, 4$  and  $l = 0, 1, \dots$ .

The representation  $\mathcal{V}_{0,1}$  contains two orbit types:  $(S_4 \times S^1)$  and  $(S_4)$ , so we have  $\deg_{\mathcal{V}_{0,1}} = (S_4)$ . For the representations  $\mathcal{V}_{1,1}$  there are also two classes of the isotropy groups:  $(S_4 \times S^1)$  and  $(S_4^-)$ , so we have  $\deg_{\mathcal{V}_{1,1}} = (S_4^-)$ . In the case of the representations  $\mathcal{V}_{2,1}$ , we have the lattice of the isotropy groups (see Diagram 6).

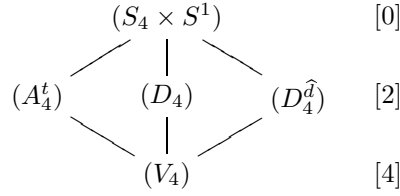


DIAGRAM 6. Isotropy lattice for  $\mathcal{V}_{2,1}$

For the representation  $\mathcal{V}_{2,1}$  we obtain that the corresponding  $S_4 \times S^1$ -degree of the basic map is

$$\deg_{\mathcal{V}_{2,1}} = (A_4^t) + (D_4) + (D_4^d) - (V_4).$$

Now, let us consider the representation  $\mathcal{V}_{3,1}$  of  $S_4 \times S^1$ , which is obtained by taking the complexification of  $\mathcal{V}_3$  and defining the action of  $S^1$  by complex multiplication. The isotropy lattice for the natural representation  $\mathcal{V}_{3,1}$  of  $S_4 \times S^1$  is shown in the Diagram 7.

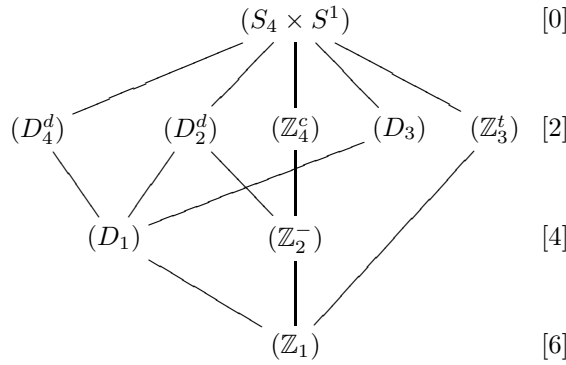


DIAGRAM 7. Isotropy lattice for  $\mathcal{V}_{3,1}$

Notice that the isotropy group  $G_x$  of  $x = \vec{v}_1$  is represented by  $D_4^d$ , for  $x = \vec{v}_1 + \vec{v}_2$  it is  $D_2^d$ , for  $x = \vec{v}_1 + \vec{v}_2 + \vec{v}_3$  it is  $D_3$ , for  $x = (1+i)\vec{v}_1 - (1-i)\vec{v}_3$  it is  $\mathbb{Z}_4^c$ , for  $x = \vec{v}_1 + 2\vec{v}_2$  it is  $\mathbb{Z}_2^-$ , for  $x = \vec{v}_1 + \gamma\vec{v}_2 + \gamma^2\vec{v}_3$ , where  $\gamma \in S^1$  is the third root of 1, it is  $\mathbb{Z}_3^t$ , and finally for  $x = \vec{v}_1 + 2\vec{v}_2 + 3\vec{v}_3$ , it is  $\mathbb{Z}_1$ .

By applying the standard computational formulae, we obtain the following value of the  $G$ -equivariant degree for the basic map on the representation  $\mathcal{V}_{3,1}$ :

$$\deg_{\mathcal{V}_{3,1}} = (D_4^d) + (D_2^d) + (D_3) + (\mathbb{Z}_4^c) + (\mathbb{Z}_3^t) - (\mathbb{Z}_2^-) - (D_1).$$

Taking the complexification of  $\mathcal{V}_4$  and defining the action of  $z \in S^1$  by the complex multiplication, we obtain the irreducible representation  $\mathcal{V}_{4,1}$  of the group  $S_4 \times S^1$  (see Diagram 8).

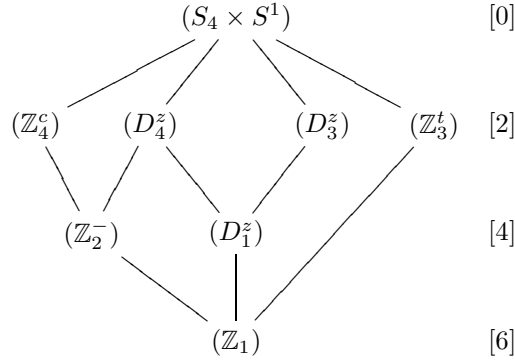


DIAGRAM 8. Isotropy lattice for  $\mathcal{V}_{4,1}$

Notice that the isotropy group  $G_x$  of  $x = \vec{v}_1$  is represented by  $D_4^z$ , for  $x = \vec{v}_1 + \vec{v}_2$  it is  $D_2^d$ , for  $x = \vec{v}_1 + \vec{v}_2 + \vec{v}_3$  it is  $D_3^z$ , for  $x = (1-i)\vec{v}_1 - (1+i)\vec{v}_3$  it is  $\mathbb{Z}_4^c$ , for  $x = \vec{v}_1 + 2\vec{v}_2$  it is  $\mathbb{Z}_2^-$ , for  $x = \vec{v}_1 + \gamma\vec{v}_2 + \gamma^2\vec{v}_3$ , where  $\gamma \in S^1$  is the third root of 1, it is  $\mathbb{Z}_3^t$ , for  $x = 2\vec{v}_1 + \vec{v}_2 + 2\vec{v}_3$  it is  $D_1^z$ , and finally for  $x = \vec{v}_1 + 2\vec{v}_2 + 3\vec{v}_3$ , it is  $\mathbb{Z}_1$ . By applying the standard computational formulas, we obtain the following value of the  $G$ -equivariant degree for the basic map on the representation  $\mathcal{V}_{4,1}$

$$\deg_{\mathcal{V}_{4,1}} = (D_4^z) + (D_2^d) + (\mathbb{Z}_4^c) + (D_3^z) + (\mathbb{Z}_3^t) - (\mathbb{Z}_2^-) - (D_1^z).$$

**6.5. Computations for the icosahedral group  $\mathbb{I}$ .** Let us consider the icosahedral group  $\mathbb{I}$ , which is isomorphic to the alternating group of five elements  $A_5$ . The group  $A_5$  has 60 elements. The conjugacy classes of the subgroups of  $A_5$  can be classified as follows: there are 15 elements in the conjugacy class of the subgroup  $\mathbb{Z}_2 := \{(1), (12)(34)\}$ , 10 elements in the conjugacy class of the subgroup  $\mathbb{Z}_3 := \{(1), (123), (132)\}$ , 5 elements in the conjugacy class of the subgroup  $V_4 := \{(1), (12)(34), (13)(24), (23)(14)\}$ , 6 elements in the conjugacy class of  $\mathbb{Z}_5 := \{(1), (12345), (13524), (14253), (15324)\}$ , 10 elements in the conjugacy class of  $D_3 := \{(1), (123), (132), (12)(45), (13)(45), (23)(45)\}$ , 5 elements in the conjugacy class of  $A_4 := \{(1), (12)(34), (123), (132), (13)(24), (14)(23), (124),$

$(142), (134), (143), (234), (243)\}$ , and 6 elements in the conjugacy class of the subgroup  $D_5 := \{(1), (12345), (13524), (15432), (14253), (12)(35), (13)(54), (14)(23), (15)(24), (25)(34)\}$ .

The lattice of the conjugacy classes of subgroups in  $A_5$  is shown in the Diagram 9.

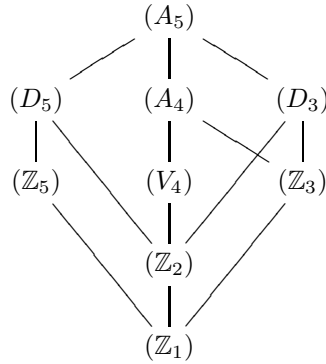


DIAGRAM 9. Lattice of the conjugacy classes for  $A_5$

Let us list, up to conjugacy class, the twisted subgroups  $K^\varphi$  of  $A_5 \times S^1$ . There are 15 elements in the conjugacy class of the subgroup  $\mathbb{Z}_2^- := \{((1), 1), ((12)(34), -1)\}$ , 15 elements in the conjugacy class of  $V_4^- := \{((1), 1), ((12)(34), -1), ((13)(24), -1), ((23)(14), 1)\}$ , 12 elements in the conjugacy class of  $\mathbb{Z}_5^{t_k} := \{((1), 1), ((12345), \xi^{2k}), ((13524), \xi^{3k}), ((14253), \xi^{4k}), ((15324), \xi^k)\}$ , where  $k = 1, 2$  and  $\xi = e^{2\pi i/5}$ , 20 elements in the conjugacy classes of  $\mathbb{Z}_3^t := \{((1), 1), ((123), \gamma), ((132), \gamma^2)\}$ , where  $\gamma = e^{2\pi i/3}$ , 10 elements in the conjugacy class of  $D_3^\xi := \{((1), 1), ((123), 1), ((132), 1), ((12)(45)), ((13)(45), -1), ((23)(45), -1)\}$ , 5 elements in the conjugacy classes of  $A_4^{t_1} := \{((1), 1), ((12)(34), 1), ((123), \gamma), ((132), \gamma^2), ((13)(24), 1), ((14)(23), 1), ((124), \gamma^2), ((142), \gamma), ((134), \gamma), ((143), \gamma^2), ((234), \gamma^2), ((243), \gamma)\}$  and  $A_4^{t_2} := \{((1), 1), ((12)(34), 1), ((123), \gamma^2), ((132), \gamma), ((13)(24), 1), ((14)(23), 1), ((124), \gamma), ((142), \gamma^2), ((134), \gamma^2), ((143), \gamma), ((234), \gamma), ((243), \gamma^2)\}$ , 6 elements in the conjugacy class of  $D_5^\xi := \{((1), 1), ((12345), 1), ((13524), 1), ((154323), 1), (14253), 1), ((12)(35), -1), ((13)(54), -1), ((14)(23), -1), ((15)(24), -1), ((25)(34), -1)\}$ .

All the twisted one-folded subgroups of  $A_5 \times S^1$  (up to their conjugacy class), their normalizers and Weyl groups, are listed in Table 12. The lattice of the conjugacy classes of the twisted subgroups in  $A_5 \times S^1$  is shown in the Diagram 10. The numbers  $n(L, H)$  for twisted subgroups in  $A_5 \times S^1$  are listed in Table 13. These numbers are used for the computations of the multiplication table (see Table 14) and the equivariant degrees of basic maps.

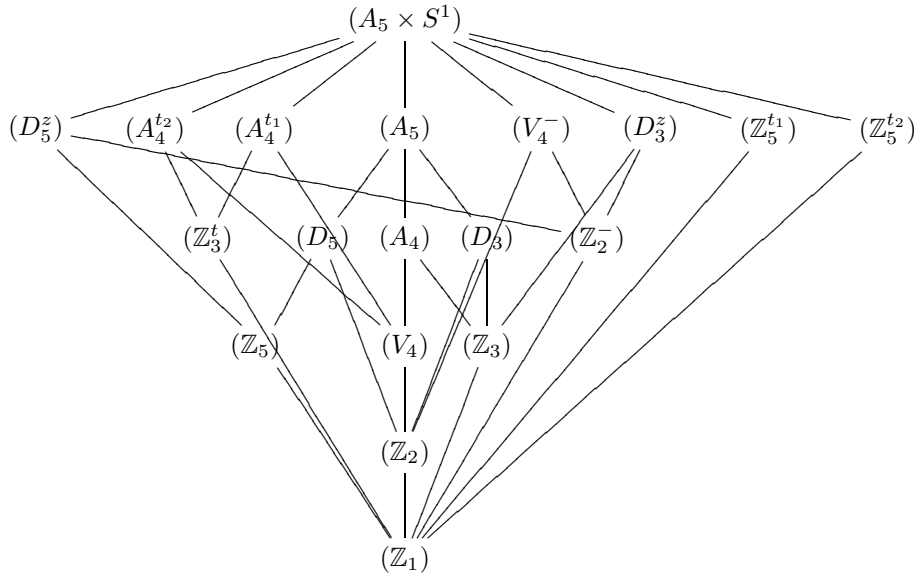
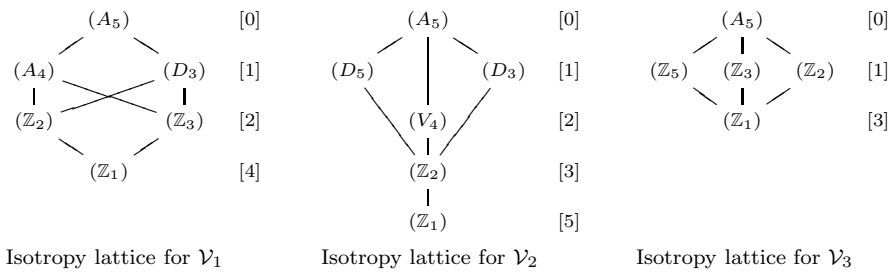


DIAGRAM 10. Conjugacy classes of twisted subgroups in  $A_5 \times S^1$

By applying the standard techniques, one can verify that there are exactly 4 irreducible representations of  $A_5$ :  $\mathcal{V}_0$  — the trivial representation,  $\mathcal{V}_1$  — the natural 4-dimensional representation of  $A_5$ ,  $\mathcal{V}_2$  — the 5-dimensional representation of  $A_5$ , and one 3-dimensional representations  $\mathcal{V}_3$ . In order to compute degrees of the basic maps of the first type, we need to analyze the isotropy groups for these representations. Clearly, for the representation  $\mathcal{V}_0$  we have the basic degree  $\text{deg}_{\mathcal{V}_0} = -(A_5)$ .

Based on the isotropy lattices for the representation  $\mathcal{V}_1$ ,  $\mathcal{V}_2$ , and  $\mathcal{V}_3$  (see the Diagrams 11), we compute the basic degrees:

$$\begin{aligned} \text{deg}_{\mathcal{V}_1} &= (A_5) - 2(A_4) - 2(D_3) + 3(\mathbb{Z}_2) + 3(\mathbb{Z}_3) - 2(\mathbb{Z}_1), \\ \text{deg}_{\mathcal{V}_2} &= (A_5) - 2(D_5) - 2(D_3) + 3(\mathbb{Z}_2) - (\mathbb{Z}_1), \\ \text{deg}_{\mathcal{V}_3} &= (A_5) - (\mathbb{Z}_5) - (\mathbb{Z}_3) - (\mathbb{Z}_2) + (\mathbb{Z}_1). \end{aligned}$$



DIAGRAMS 11

| $H = K^\varphi$  | $\text{Im } \varphi$ | $\text{Ker } \varphi$ | $N(H)$                    | $W(H)$                    | Comments   |
|------------------|----------------------|-----------------------|---------------------------|---------------------------|------------|
| $A_5$            | $\mathbb{Z}_1$       | $A_5$                 | $A_5 \times S^1$          | $S^1$                     |            |
| $D_5$            | $\mathbb{Z}_1$       | $D_5$                 | $D_5 \times S^1$          | $S^1$                     |            |
| $A_4$            | $\mathbb{Z}_1$       | $A_4$                 | $A_4 \times S_1$          | $S^1$                     |            |
| $D_3$            | $\mathbb{Z}_1$       | $D_3$                 | $D_3 \times S_1$          | $S^1$                     |            |
| $\mathbb{Z}_5$   | $\mathbb{Z}_1$       | $\mathbb{Z}_5$        | $D_5 \times S_1$          | $\mathbb{Z}_2 \times S^1$ |            |
| $V_4$            | $\mathbb{Z}_1$       | $V_4$                 | $A_4 \times S_1$          | $\mathbb{Z}_3 \times S^1$ |            |
| $\mathbb{Z}_3$   | $\mathbb{Z}_1$       | $\mathbb{Z}_3$        | $D_3 \times S_1$          | $\mathbb{Z}_2 \times S^1$ |            |
| $\mathbb{Z}_2$   | $\mathbb{Z}_1$       | $\mathbb{Z}_2$        | $V_4 \times S_1$          | $\mathbb{Z}_2 \times S^1$ |            |
| $\mathbb{Z}_1$   | $\mathbb{Z}_1$       | $\mathbb{Z}_1$        | $A_5 \times S_1$          | $A_5 \times S^1$          |            |
| $V_4^-$          | $\mathbb{Z}_2$       | $\mathbb{Z}_2$        | $V_4 \times S^1$          | $S^1$                     |            |
| $D_5^z$          | $\mathbb{Z}_2$       | $\mathbb{Z}_5$        | $D_5 \times S^1$          | $S^1$                     |            |
| $A_4^{t_k}$      | $\mathbb{Z}_3$       | $V_4$                 | $A_4 \times S^1$          | $S^1$                     | $k = 1, 2$ |
| $D_3^z$          | $\mathbb{Z}_2$       | $\mathbb{Z}_3$        | $D_3 \times S^1$          | $S^1$                     |            |
| $\mathbb{Z}_3^t$ | $\mathbb{Z}_3$       | $\mathbb{Z}_1$        | $\mathbb{Z}_3 \times S^1$ | $S^1$                     |            |
| $\mathbb{Z}_5^k$ | $\mathbb{Z}_5$       | $\mathbb{Z}_1$        | $\mathbb{Z}_5 \times S^1$ | $S^1$                     | $k = 1, 2$ |
| $\mathbb{Z}_2^-$ | $\mathbb{Z}_2$       | $\mathbb{Z}_1$        | $V_4 \times S^1$          | $\mathbb{Z}_2 \times S^1$ |            |

TABLE 12. Twisted subgroups  $H$  in  $A_5 \times S^1$

| $L$              | $H$         | $n(L, H)$ | $L$              | $H$            | $n(L, H)$ | $L$              | $H$     | $n(L, H)$ | $L$            | $H$                  | $n(L, H)$ |
|------------------|-------------|-----------|------------------|----------------|-----------|------------------|---------|-----------|----------------|----------------------|-----------|
| $\mathbb{Z}_5$   | $D_5^z$     | 1         | $\mathbb{Z}_2$   | $A_4^{t_2}$    | 1         | $\mathbb{Z}_3$   | $A_5$   | 1         | $\mathbb{Z}_1$ | $D_3^z$              | 10        |
| $\mathbb{Z}_2^-$ | $D_5^z$     | 2         | $\mathbb{Z}_1$   | $A_4^{t_2}$    | 5         | $\mathbb{Z}_2$   | $A_5$   | 1         | $\mathbb{Z}_1$ | $\mathbb{Z}_5^{t_k}$ | 12        |
| $\mathbb{Z}_1$   | $D_5^z$     | 6         | $\mathbb{Z}_3^t$ | $A_4^{t_1}$    | 1         | $\mathbb{Z}_1$   | $A_5$   | 1         | $\mathbb{Z}_1$ | $\mathbb{Z}_2^-$     | 15        |
| $\mathbb{Z}_5$   | $D_5$       | 1         | $V_4$            | $A_4^{t_1}$    | 1         | $\mathbb{Z}_2^-$ | $V_4^-$ | 2         | $V_4$          | $A_4$                | 1         |
| $\mathbb{Z}_2$   | $D_5$       | 2         | $\mathbb{Z}_2$   | $A_4^{t_1}$    | 1         | $\mathbb{Z}_2$   | $V_4^-$ | 1         | $\mathbb{Z}_3$ | $A_4$                | 2         |
| $\mathbb{Z}_1$   | $D_5$       | 6         | $\mathbb{Z}_1$   | $A_4^{t_1}$    | 5         | $\mathbb{Z}_1$   | $V_4^-$ | 15        | $\mathbb{Z}_2$ | $A_4$                | 1         |
| $\mathbb{Z}_3^t$ | $A_4^{t_2}$ | 1         | $A_4$            | $A_5$          | 1         | $\mathbb{Z}_3$   | $D_3$   | 1         | $\mathbb{Z}_1$ | $A_4$                | 1         |
| $V_4$            | $A_4^{t_2}$ | 1         | $V_4$            | $A_5$          | 1         | $\mathbb{Z}_2$   | $D_3$   | 2         | $\mathbb{Z}_1$ | $\mathbb{Z}_3^t$     | 20        |
| $\mathbb{Z}_1$   | $D_3$       | 10        | $\mathbb{Z}_1$   | $\mathbb{Z}_5$ | 6         | $\mathbb{Z}_3$   | $D_3^z$ | 1         | $\mathbb{Z}_1$ | $\mathbb{Z}_3$       | 10        |
| $\mathbb{Z}_2^-$ | $D_3^z$     | 2         | $\mathbb{Z}_2$   | $V_4$          | 1         | $\mathbb{Z}_1$   | $V_4$   | 5         | $\mathbb{Z}_1$ | $\mathbb{Z}_2$       | 15        |

TABLE 13. Numbers  $n(L, H)$  for twisted subgroups in  $A_5 \times S^1$

|           |                     |                    |                           |                     |                     |                   |           |           |
|-----------|---------------------|--------------------|---------------------------|---------------------|---------------------|-------------------|-----------|-----------|
| $(A_5)$   | $(D_5)$             | $(A_4)$            | $(D_3)$                   | $(Z_5)$             | $(V_4)$             | $(Z_2)$           | $(Z_1)$   | $(A_5)$   |
| $(A_5)$   | $(D_5)$             | $(A_4)$            | $(D_3)$                   | $(Z_5)$             | $(V_4)$             | $(Z_2)$           | $(Z_1)$   | $(A_5)$   |
| $(D_5)$   | $(D_5) + (Z_2)$     | $(Z_2)$            | $2(Z_2)$                  | $(Z_5) + (Z_1)$     | $3(Z_2)$            | $2(Z_1)$          | $6(Z_1)$  | $(D_5)$   |
| $(A_4)$   | $(Z_2)$             | $(A_4) + (Z_3)$    | $(Z_3) + (Z_2)$           | $(Z_1)$             | $(V_4) + (Z_1)$     | $2(Z_3) + (Z_1)$  | $5(Z_1)$  | $(A_4)$   |
| $(D_3)$   | $2(Z_2)$            | $(Z_3) + (Z_2)$    | $(D_3) + (Z_2) + (Z_1)$   | $2(Z_1)$            | $3(Z_2) + (Z_1)$    | $2(Z_2) + 4(Z_1)$ | $10(Z_1)$ | $(D_3)$   |
| $(Z_5)$   | $(Z_5) + (Z_1)$     | $(Z_1)$            | $2(Z_1)$                  | $2(Z_5) + 2(Z_1)$   | $3(Z_1)$            | $4(Z_1)$          | $12(Z_1)$ | $(Z_5)$   |
| $(V_4)$   | $3(Z_2)$            | $(V_4) + (Z_1)$    | $3(Z_2) + (Z_1)$          | $3(Z_1)$            | $3(V_4) + 3(Z_1)$   | $5(Z_1)$          | $15(Z_1)$ | $(V_4)$   |
| $(Z_3)$   | $2(Z_1)$            | $2(Z_3) + (Z_1)$   | $(Z_3) + 3(Z_1)$          | $4(Z_1)$            | $5(Z_1)$            | $2(Z_3) + 6(Z_1)$ | $10(Z_1)$ | $(Z_3)$   |
| $(Z_2)$   | $2(Z_2) + 2(Z_1)$   | $(Z_2) + 2(Z_1)$   | $2(Z_2) + 4(Z_1)$         | $6(Z_1)$            | $3(Z_2) + 6(Z_1)$   | $10(Z_1)$         | $30(Z_1)$ | $(Z_2)$   |
| $(Z_1)$   | $6(Z_1)$            | $5(Z_1)$           | $10(Z_1)$                 | $12(Z_1)$           | $15(Z_1)$           | $20(Z_1)$         | $60(Z_1)$ | $(Z_1)$   |
| $(A_4^+)$ | $(Z_2)$             | $(A_4^+) + (Z_3)$  | $(Z_2) + (Z_3)$           | $(Z_1)$             | $(V_4) + (Z_1)$     | $2(Z_3) + (Z_1)$  | $5(Z_1)$  | $(A_4^+)$ |
| $(A_4^+)$ | $(Z_2)$             | $(A_4^+) + (Z_3)$  | $(Z_2) + (Z_3)$           | $(Z_1)$             | $(V_4) + (Z_1)$     | $2(Z_3) + (Z_1)$  | $5(Z_1)$  | $(A_4^+)$ |
| $(D_5^-)$ | $(D_5^-) + (Z_2)$   | $(Z_2)$            | $2(Z_2)$                  | $(Z_5) + (Z_1)$     | $3(Z_2)$            | $2(Z_1)$          | $6(Z_1)$  | $(D_5^-)$ |
| $(V_4^-)$ | $2(Z_2) + (Z_2)$    | $(V_4^-) + (Z_1)$  | $2(Z_2) + (Z_2) + (Z_1)$  | $3(Z_1)$            | $3(V_4^-) + 3(Z_1)$ | $5(Z_1)$          | $15(Z_1)$ | $(V_4^-)$ |
| $(D_3^-)$ | $2(Z_2)$            | $(Z_3) + (Z_2)$    | $(D_3^-) + (Z_2) + (Z_1)$ | $2(Z_1)$            | $3(Z_2) + (Z_1)$    | $(Z_3) + 3(Z_1)$  | $10(Z_1)$ | $(D_3^-)$ |
| $(Z_5^+)$ | $(Z_5^+) + (Z_1)$   | $(Z_1)$            | $2(Z_1)$                  | $2(Z_5^+) + 2(Z_1)$ | $3(Z_1)$            | $4(Z_1)$          | $12(Z_1)$ | $(Z_5^+)$ |
| $(Z_5^+)$ | $(Z_5^+) + (Z_1)$   | $(Z_1)$            | $2(Z_1)$                  | $2(Z_5^+) + 2(Z_1)$ | $3(Z_1)$            | $4(Z_1)$          | $12(Z_1)$ | $(Z_5^+)$ |
| $(Z_3^+)$ | $2(Z_1)$            | $(Z_1)$            | $2(Z_1)$                  | $4(Z_1)$            | $5(Z_1)$            | $10(Z_1)$         | $20(Z_1)$ | $(Z_3^+)$ |
| $(Z_2^-)$ | $2(Z_2^-) + 2(Z_1)$ | $(Z_2^-) + 2(Z_1)$ | $2(Z_2^-) + 4(Z_1)$       | $6(Z_1)$            | $3(Z_2^-) + 6(Z_1)$ | $10(Z_1)$         | $30(Z_1)$ | $(Z_2^-)$ |

TABLE 14.  $A(A_5)$ -module multiplication table for  $A_1(A_5 \times S^1)$ . The upper half of the table describes the multiplication in the Burnside ring  $A(A_5)$



Finally, let us present the computations of the basic  $A_5 \times S^1$ -degrees for the representations  $\mathcal{V}_{1,1}$ ,  $\mathcal{V}_{2,1}$ ,  $\mathcal{V}_{3,1}$  and  $\mathcal{V}_{4,1}$ . In the case of the representations  $\mathcal{V}_{1,1}$  and  $\mathcal{V}_{2,1}$  we have the lattices of twisted subgroups given in Diagrams 12 and 13.

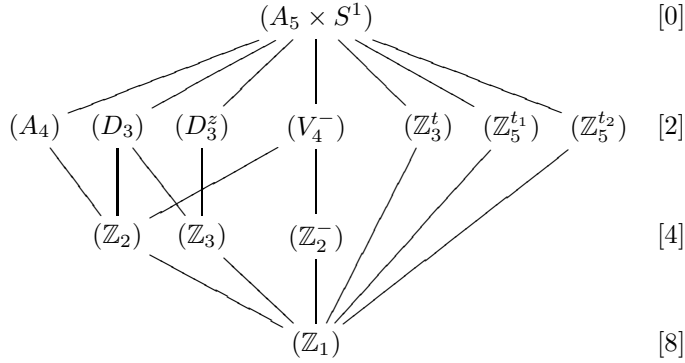


DIAGRAM 12. Twisted isotropy lattice for  $\mathcal{V}_{1,1}$

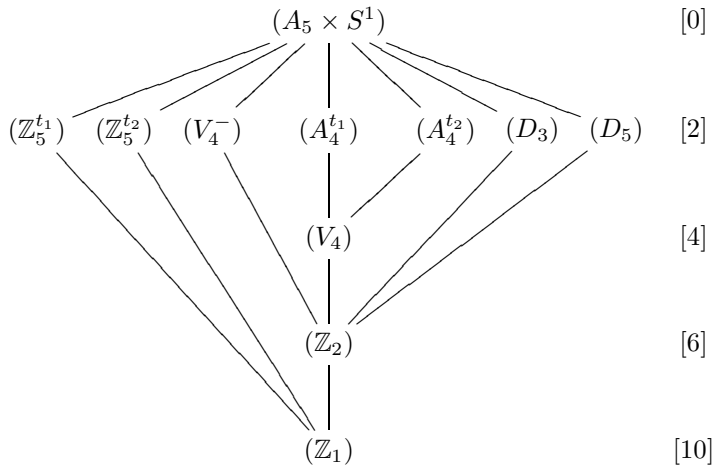
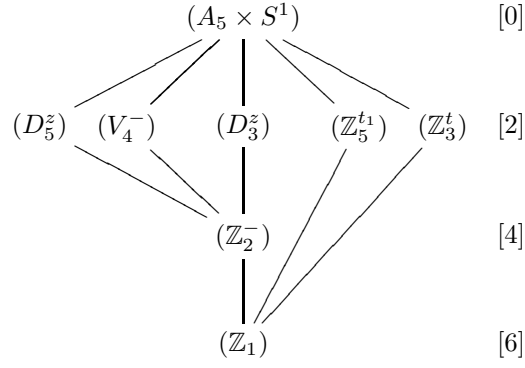
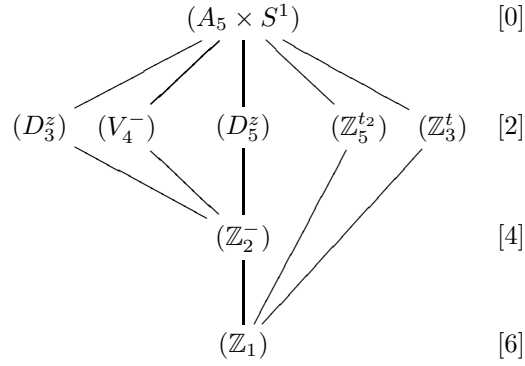


DIAGRAM 13. Twisted isotropy lattice for  $\mathcal{V}_{2,1}$

The basic degrees for these representations are given by:

$$\begin{aligned} \deg_{\mathcal{V}_{1,1}} &= (A_4) + (D_3) + (D_3^z) + (V_4^-) \\ &\quad + (Z_3^t) + (Z_5^{t1}) + (Z_5^{t2}) - (Z_2) - (Z_3) - (Z_2^-), \\ \deg_{\mathcal{V}_{2,1}} &= (D_5) + (D_3) + (A_4^{t1}) + (A_4^{t2}) + (V_4^-) + (Z_5^{t1}) + (Z_5^{t2}) - 2(Z_2). \end{aligned}$$

For the representations  $\mathcal{V}_{3,1}$  and  $\mathcal{V}_{4,1}$ , we have the isotropy lattice of twisted subgroups given in Diagrams 14 and 15:

DIAGRAM 14. Twisted isotropy lattice for  $\mathcal{V}_{3,1}$ DIAGRAM 15. Twisted isotropy lattice for  $\mathcal{V}_{4,1}$ 

The basic degrees for these two representations are equal to:

$$\begin{aligned}
\deg_{\mathcal{V}_{3,1}} &= (D_5^z) + (V_4^-) + (D_3^z) + (\mathbb{Z}_5^{t_1}) + (\mathbb{Z}_3^t) - 2(\mathbb{Z}_2^-), \\
\deg_{\mathcal{V}_{4,1}} &= (D_5^z) + (V_4^-) + (D_3^z) + (\mathbb{Z}_5^{t_2}) + (\mathbb{Z}_3^t) - 2(\mathbb{Z}_2^-).
\end{aligned}$$

## 7. Conclusions and applications

In this section we will show how the existence Theorem 5.3 in compliance with the computations related to the basic maps presented in Section 6, and the equivariant degree product formula provided by Proposition 4.1, allow to study symmetric periodic solutions to concrete van der Pol equations. In particular, we will discuss the Examples 2.1–2.4.

Let us recall that for each of the discussed systems of van der Pol equations, we have the following associated  $G$ -equivariant degree ( $G = \Gamma \times S^1$ ):

$$G\text{-Deg}(\text{id} - \mathcal{F}(0, \cdot, \cdot), \Omega) = \sum_{(L^{\varphi, l})} n_{L^{\varphi, l}}(L^{\varphi, l}),$$

where  $(L^{\varphi,l})$  are the generators of  $A_1(\Gamma \times S^1)$ ,  $L \subset \Gamma$ , and  $\mathcal{F}$  is given in (3.28). Although the entire value of the degree  $G\text{-Deg}(\text{id} - \mathcal{F}(0, \cdot, \cdot), \Omega)$  should be considered as the complete *equivariant invariant* classifying the solutions of the corresponding equations, in order to simplify our exposition, we will restrict our computations to the coefficients  $n_{L^{\varphi,1}}$ , which will be called *first coefficients* and the corresponding part of the equivariant degree will be denoted by  $G\text{-Deg}(\text{id} - \mathcal{F}(0, \cdot, \cdot), \Omega)_1$ . As it follows from Theorem 5.3(a), if  $n_{L^{\varphi,1}} \neq 0$ , then system (3.7) has *at least one* periodic solution  $u$  with symmetry  $G_u \supset L^{\varphi,1}$ . However (see Theorem 5.3(b)), only dominating orbit types occurring in eigenspaces relevant to suitable eigenvalue of  $C$  give a possibility to estimate a *precise* number of periodic solutions with the corresponding symmetry (see Remarks 3.5 and 3.6).

In addition, we will assume here, that the value of the parameter  $\alpha$  was always chosen in the most favorable way, i.e. the conditions (5.11) and (5.12) are satisfied.

**7.1. Conclusions for the dihedral group  $D_N$ .** Let us consider again the system describing the ring of identical van der Pol oscillators, which was discussed in Example 2.1. This system has the group of symmetries  $\Gamma = D_N$ . Let us describe explicitly the  $D_N$ -action on  $V = \mathbb{R}^N$ , its isotypical decomposition and the spectrum of the linear operator  $C$ . We denote by  $\xi := e^{2\pi i/N}$  the generator of  $\mathbb{Z}_N$ . Notice that  $\xi$  acts on a vector  $\vec{x} = (x_0, x_1, \dots, x_{N-1})$  by sending the  $k$ -th coordinate of  $\vec{x}$  to the  $k + 1 \pmod{N}$  coordinate. As we are dealing with first coefficients only, take  $U := V^c = \mathbb{C}^N$  — the complexification of the above  $D_N$ -representation  $V$ . Notice that we have the following  $\mathbb{Z}_N$ -isotypical decomposition of  $U$

$$U = \tilde{U}_0 \oplus \tilde{U}_1 \oplus \dots \oplus \tilde{U}_{N-1},$$

where  $\tilde{U}_j = \text{span}(\langle 1, \xi^j, \xi^{2j}, \dots, \xi^{(N-1)j} \rangle)$ . Since  $\kappa$  sends  $\tilde{U}_j$  onto  $\tilde{U}_{-j}$  (where  $-j$  is taken  $\pmod{N}$ ), the  $D_N$ -isotypical components of  $U$  are

$$U_0 = \tilde{U}_0, \quad U_j := \tilde{U}_j \oplus \tilde{U}_{-j}, \quad 0 < j < N/2,$$

and, in addition, if  $N$  is even there is also the component  $U_{N/2} := \tilde{U}_{N/2}$ .

It is easy to check that the isotypical component  $U_j$ ,  $0 \leq j < N/2$ , is equivalent to the irreducible representation  $\mathcal{V}_j^c$  of  $D_N$ , and  $U_{N/2}$  (for  $N$  even) is equivalent to  $\mathcal{V}_{j_{N+1}}^c$ . The subspace  $U_j$  is also an eigenspace of the matrix  $C$  corresponding to the eigenvalue  $\lambda_j := c + 2d \cos(2\pi j/N)$  (where we consider  $C$  acting on  $\mathbb{C}^N$ ). Then, by Proposition 4.1, we have:

$$(7.1) \quad G\text{-Deg}(\text{id} - \mathcal{F}(0, \cdot, \cdot), \Omega)_1 = \prod_{\lambda_j \in \Sigma(C)} \text{deg}_{\mathcal{V}_j} \cdot \left[ \sum_{\lambda_j \in \Sigma(C)} \text{deg}_{\mathcal{V}_{j,1}} \right],$$

where  $\Sigma(C)$  is defined in (5.10). Moreover, for an eigenvalue  $\lambda_j > 0$  the values of  $\text{deg}_{\mathcal{V}_j}$  and  $\text{deg}_{\mathcal{V}_{j,1}}$  are listed in Table 15, where  $h = \text{gcd}(j, N)$ .

| $\lambda_j > 0$   | $\deg v_j$                                       | $\deg v_{j,1}$   |
|---|--|--|
| $j = 0$   | $-(D_N)$   | $(D_N)$  |
| $0 < j < N/2$<br>$m$ is odd                               | $(D_N) - 2(D_h) + (\mathbb{Z}_h)$                | $(\mathbb{Z}_N^t) + (D_h) + (D_h^z) - (\mathbb{Z}_h)$                      |
| $0 < j < N/2$<br>$N$ is even<br>and $m \equiv 0 \pmod{4}$ | $(D_N) - (D_h) - (\tilde{D}_h) + (\mathbb{Z}_h)$ | $(\mathbb{Z}_N^t) + (D_{2h}^d) + (\tilde{D}_{2h}^d) - (\mathbb{Z}_{2h}^d)$ |
| $0 < j < N/2$<br>$N$ is even<br>and $m \equiv 2 \pmod{4}$ | $(D_N) - (D_h) - (\tilde{D}_h) + (\mathbb{Z}_h)$ | $(\mathbb{Z}_N^t) + (D_{2h}^d) + (D_{2h}^{\hat{d}}) - (\mathbb{Z}_{2h}^d)$ |
| $j = j_N + 1$<br>$N$ is even                              | $(D_N) - (D_{N/2})$                              | $(D_N^d)$  |

TABLE 15. Values of  $\deg v_j$  and  $\deg v_{j,1}$  corresponding to  $\lambda_j > 0$ , where  $j_N = [(N+1)/2]$ ,  $h = \gcd(N, j)$  and  $m = N/h$

Let us illustrate these results for the particular cases  $N = 3, 4$  and  $5$ .

In the case  $N = 3$ , the spectrum  $\sigma(C)$  of the matrix  $C$  is  $\{\lambda_0 = c + 2d, \lambda_1 = c - d\}$  and the dominating orbit types (occurring in  $V^c$ ) are  $(\mathbb{Z}_3^t)$ ,  $(D_3)$  and  $(D_3^z)$  (see Remark 3.5). If a coefficient  $n_L \neq 0$  is standing by a dominating orbit type, then there is an orbit of periodic solutions of the system (2.2) composed of exactly  $|G/L|_{S^1}$  periodic solutions (see Remark 3.6). In particular, for the orbit type  $(\mathbb{Z}_3^t)$  there are 2 distinct periodic solutions, for  $(D_3^z)$  there are 3 periodic solutions, and 1 periodic solution for  $(D_3)$ . If  $n_{D_3} = 0$ , then still one more periodic solution can be detected as long as  $n_L \neq 0$  for some  $(L) < (D_3)$ . The lower estimates for the number of periodic solutions for the equation (2.2) in the case  $N = 3$  are summarized in the Table 16.

| $\Sigma(C)$         | $G\text{-Deg}(\text{id} - \mathcal{F}(0, \cdot, \cdot), \Omega)_1$ | Minimal # of solutions |
|---------------------|--|------------------------|
| $\emptyset$         | 0  | 0                      |
| $\{c - d\}$         | $(\mathbb{Z}_3^t) - (D_3^z) - (D_1) + 3(\mathbb{Z}_1)$             | 6                      |
| $\{c + 2d\}$        | $-(D_3)$   | 1                      |
| $\{c + 2d, c - d\}$ | $-(\mathbb{Z}_3^t) + (D_3^z) - (D_3) + 3(D_1) - 2(\mathbb{Z}_1)$   | 6                      |

TABLE 16. Possible cases for  $N = 3$

In the case  $N = 4$ , the spectrum  $\sigma(C)$  of the matrix  $C$  is  $\{\lambda_0 = c + 2d, \lambda_1 = c, \lambda_2 = c - 2d\}$  and the dominating orbit types (occurring in  $V^c$ ) are  $(\mathbb{Z}_4^t)$ ,  $(D_4^d)$ ,  $(D_2^d)$ ,  $(\tilde{D}_2^d)$  and  $(D_4)$ . For the orbit type  $(\mathbb{Z}_4^t)$  there are 2 distinct periodic

solutions, for  $(D_4^d)$  there is 1 periodic solutions, for  $(D_2^d)$  and  $(\tilde{D}_2^d)$  there are 2 periodic solutions, and there is 1 periodic solution for  $(D_4)$ . We also have<sup>5</sup>

$$\begin{aligned} \deg_{\mathcal{V}_0} &= -(D_4), & \deg_{\mathcal{V}_1} &= (D_4) - (D_1) - (\tilde{D}_1) + (\mathbb{Z}_1), \\ \deg_{\mathcal{V}_3} &= (D_4) - (D_2), & \deg_{\mathcal{V}_{0,1}} &= (D_4), \\ \deg_{\mathcal{V}_{1,1}} &= (\mathbb{Z}_4^t) + (D_2^d) + (\tilde{D}_2^d) - (\mathbb{Z}_2^-), & \deg_{\mathcal{V}_{3,1}} &= (D_4^d). \end{aligned}$$

The lower estimates for the number of periodic solutions for the equation (2.2) in the case  $N = 4$  are summarized in the Table 17.

| $\Sigma(C)$             | $G\text{-Deg}(\text{id} - \mathcal{F}(0, \cdot, \cdot), \Omega)_1$  | Minimal # of Solutions |
|-------------------------|---|------------------------|
| $\{c + 2d\}$            | $-(D_4)$  | 1                      |
| $\{c - 2d\}$            | $(D_4^d)$   | 1                      |
| $\{c + 2d, c\}$         | $-(\mathbb{Z}_4^t) - (\tilde{D}_2^d) - (D_2^d) - (D_4) + (\mathbb{Z}_2^-) + (D_1^z) + (\tilde{D}_1^z) + 2(D_1) + 2(\tilde{D}_1) - 3(\mathbb{Z}_1)$            | 7                      |
| $\{c - 2d, c\}$         | $(D_4^d) + (D_2^d) + (\tilde{D}_2^d) + (\mathbb{Z}_4^t) - (\mathbb{Z}_2^-) - (D_1^z) - 2(\tilde{D}_1^z) - 2(D_1) - (\tilde{D}_1) + 3(\mathbb{Z}_1)$           | 8                      |
| $\{c + 2d, c - 2d, c\}$ | $-(D_4) - (D_4^d) - (D_2^d) - (\tilde{D}_2^d) - (\mathbb{Z}_4^t) + (\mathbb{Z}_2^-) + (D_1^z) + 2(\tilde{D}_1^z) + 3(D_1) + 2(\tilde{D}_1) - 4(\mathbb{Z}_1)$ | 8                      |

 TABLE 17. Possible cases for  $N = 4$ 

| $\Sigma(C)$  | $G\text{-Deg}(\text{id} - \mathcal{F}(0, \cdot, \cdot), \Omega)_1$                           | Minimal # of solutions |
|--|--|------------------------|
| $\emptyset$  | 0  | 0                      |
| $\{c + 2d\}$   | $-(D_5)$   | 1                      |
| $\{c - 2d\frac{\sqrt{5}+1}{4}\}$                                     | $(\mathbb{Z}_5^{t_2}) - (D_1^z) - (D_1) + (\mathbb{Z}_1)$                                    | 8                      |
| $\{c + 2d\frac{\sqrt{5}-1}{4}, c - 2d\frac{\sqrt{5}+1}{4}\}$         | $(\mathbb{Z}_5^{t_1}) + (\mathbb{Z}_5^{t_2}) + 2(D_1^z) + 2(D_1) - 2(\mathbb{Z}_1)$          | 10                     |
| $\{c + 2d\frac{\sqrt{5}-1}{4}, c + 2d\}$                             | $-(\mathbb{Z}_5^{t_1}) + (D_1^z) - (D_5) + 3(D_1) - 6(\mathbb{Z}_1)$                         | 8                      |
| $\{c + 2d, c + 2d\frac{\sqrt{5}-1}{4}, c - 2d\frac{\sqrt{5}+1}{4}\}$ | $-(\mathbb{Z}_5^{t_1}) - (\mathbb{Z}_5^{t_2}) - (D_5) - 2(D_1^z) - 2(D_1) + 2(\mathbb{Z}_1)$ | 10                     |

 TABLE 18. Possible cases for  $N = 5$ 

In the case  $N = 5$ , the spectrum  $\sigma(C)$  of the matrix  $C$  is  $\{\lambda_0 = c + 2d, \lambda_1 = c + 2d(\sqrt{5} - 1)/4, \lambda_3 = c - 2d(\sqrt{5} + 1)/4\}$  and the dominating orbit types are  $(\mathbb{Z}_5^{t_1})$ ,  $(\mathbb{Z}_5^{t_2})$ ,  $(D_5)$  and  $(D_1^z)$ . We have the following equivariant degrees of the

<sup>5</sup>Notice that for  $N = 4$  we have  $\tilde{D}_2^d = D_2^{\hat{d}}$  (cf. Table 15).

basic maps related to the eigenspaces of  $C$

$$\begin{aligned} \deg_{\mathcal{V}_0} &= -(D_5), & \deg_{\mathcal{V}_1} &= (D_5) - 2(D_1) + (\mathbb{Z}_1), \\ \deg_{\mathcal{V}_2} &= (D_5) - 2(D_1) + (\mathbb{Z}_1) & \deg_{\mathcal{V}_{0,1}} &= (D_5), \\ \deg_{\mathcal{V}_{1,1}} &= (\mathbb{Z}_5^{t_1}) + (D_1^z) + (D_1) - (\mathbb{Z}_1), & \deg_{\mathcal{V}_{2,1}} &= (\mathbb{Z}_5^{t_2}) + (D_1^z) + (D_1) - (\mathbb{Z}_1). \end{aligned}$$

For the orbit types  $(\mathbb{Z}_5^{t_1})$  and  $(\mathbb{Z}_5^{t_2})$  there are 2 distinct periodic solutions, for  $(D_1^z)$  there are 5 periodic solutions, and 1 periodic solution for  $(D_5)$ . The lower estimates for the number of periodic solutions for the equation (2.2) in the case  $N = 5$  are summarized in the Table 18.

**7.2. Conclusions for the tetrahedral group  $\mathbb{T}$ .** Let us consider the system of van der Pol oscillators with the tetrahedral symmetry group, which was studied in Example 2.2. Here, the tetrahedral group  $A_4$  acts on the space  $V = \mathbb{R}^4$  by permuting the coordinates of vectors. The subspace  $V_0$  of the fixed-points of this action is spanned by the vector  $\langle 1, 1, 1, 1 \rangle$ , and its orthogonal complement  $V_3$  is the natural three-dimensional representation of  $A_4$ , which was in Section 6 denoted by  $\mathcal{V}_3$ . These two subspaces are the eigenspaces of the matrix  $C$ : the subspace  $V_0$  corresponds to the eigenvalue  $c + 3d$  and  $V_3$  to the eigenvalue  $c - d$ . The dominating orbit types in  $V^c$  are  $(A_4)$ ,  $(\mathbb{Z}_3^{t_1})$ ,  $(\mathbb{Z}_3^{t_2})$ , and  $(V_4^-)$ . For non-zero first coefficient corresponding to the orbit type  $(A_4)$  there is at least one periodic solution, for  $(\mathbb{Z}_3^{t_1})$  or  $(\mathbb{Z}_3^{t_2})$  — at least 4 periodic solutions, and for  $(V_4^-)$  there exist at least 3 periodic solutions.

In order to compute the equivariant degree  $A_4 \times S^1$ -Deg( $\text{id} - \mathcal{F}(0, \cdot, \cdot), \Omega$ ) $_1$ , we apply the computational formula similar to (7.1). Depending on the set  $\Sigma(C)$ , we need the basic degrees:  $\deg_{\mathcal{V}_0} \in A(A_4)$  (if  $c + 3d > 0$ ), degree  $\deg_{\mathcal{V}_3} \in A(A_4)$  (if  $c - d > 0$ ),  $\deg_{\mathcal{V}_{0,1}} \in A_1(A_4 \times S^1)$  (if  $c + 3d > 0$ ),  $\deg_{\mathcal{V}_{3,1}} \in A_1(A_4 \times S^1)$  (if  $c - d > 0$ ). The related to this formula basic degrees are presented in Table 19.

| Rep.                | Basic Degrees $\deg_{\mathcal{V}_j}$ or $\deg_{\mathcal{V}_{j,1}}$                        | Eigenvalue of $C$ |
|---------------------|---|-------------------|
| $\mathcal{V}_0$     | $-(A_4)$  | $c + 3d > 0$      |
| $\mathcal{V}_3$     | $(A_4) - 2(\mathbb{Z}_3) - (\mathbb{Z}_2) + (\mathbb{Z}_1)$                               | $c - d > 0$       |
| $\mathcal{V}_{0,1}$ | $(A_4)$   | $c + 3d > 0$      |
| $\mathcal{V}_{3,1}$ | $(\mathbb{Z}_3^{t_1}) + (\mathbb{Z}_3^{t_2}) + (V_4^-) + (\mathbb{Z}_3) + (\mathbb{Z}_1)$ | $c - d > 0$       |

TABLE 19

By using the established multiplication tables for the  $A(A_4)$ -module  $A_1(A_4 \times S^1)$ , and applying the computational formula similar to (7.1), we obtain first coefficients of the equivariant degrees  $A_4 \times S^1$ -Deg( $\text{id} - \mathcal{F}(0, \cdot, \cdot), \Omega$ ) (see Table 20).

| $\Sigma(C)$     | $A_4 \times S^1\text{-Deg}(\text{id} - \Psi(0, \cdot, \cdot), \Omega)_1$  | # Solutions |
|-----------------|---|-------------|
| $c + 3d$        | $-(A_4)$  | 1           |
| $c - d$         | $-(\mathbb{Z}_3^{t_1}) - (\mathbb{Z}_3^{t_2}) + (V_4^-) - (\mathbb{Z}_3) - (\mathbb{Z}_2) + 2(\mathbb{Z}_1)$                              | 12          |
| $c + 3d, c - d$ | $(\mathbb{Z}_3^{t_1}) + (\mathbb{Z}_3^{t_2}) - (A_4) - (V_4^-) + 3(\mathbb{Z}_3) + 2(\mathbb{Z}_2^-) + 2(\mathbb{Z}_2) - 3(\mathbb{Z}_1)$ | 12          |

TABLE 20

**7.3. Conclusions for the octahedral group  $\mathbb{O}$ .** Let us discuss the system of van der Pol equations described in Example 2.3. Here we have the group  $S_4$  is acting on the eight-dimensional space  $V := \mathbb{R}^8$  by permuting the coordinates of the vectors in the same way as the symmetries of the cube in  $\mathbb{R}^3$  permutes the eight vertices of the cube. It can be easily verified, that the representation  $V$  can be decomposed into a direct sum of four subspaces:

$$V = V_0 \oplus V_1 \oplus V_3^1 \oplus V_3^2,$$

where

$$\begin{aligned} V_0 &= \text{span}\{\langle 1, 1, 1, 1, 1, 1, 1, 1 \rangle\}, \\ V_1 &= \text{span}\{\langle 1, -1, 1, -1, 1, -1, 1, -1 \rangle\}, \\ V_3^1 &= \text{span}\{\langle 1, 1, -1, -1, 1, -1, -1, 1 \rangle, \\ &\quad \langle 1, -1, 1, -1, -1, 1, -1, 1 \rangle, \langle -1, 1, 1, -1, 1, 1, -1, -1 \rangle\}, \\ V_3^2 &= \text{span}\{\langle 1, -1, -1, 1, 1, 1, -1, -1 \rangle, \\ &\quad \langle 1, 1, 1, 1, -1, -1, -1, -1 \rangle, \langle -1, -1, 1, 1, 1, -1, -1, 1 \rangle\}. \end{aligned}$$

Notice that these subspaces are irreducible representations of  $S_4$ , where  $V_3^1$  is equivalent to the natural three-dimensional representation  $\mathcal{V}_3$  of  $S_4$ , and  $V_3^2$  is equivalent to the another three-dimensional irreducible representation  $\mathcal{V}_4$  of  $S_4$ . The subspace  $V_0$  is the fixed-point space of the action of  $S_4$ . The subspaces  $V_0, V_1, V_3^1$  and  $V_3^2$  are eigenspaces for the matrix  $C$ . Indeed, it is easy to check that:

| Subspace | Eigenvalue of $C$ | Type of representation         | Dimension |
|----------|-------------------|--------------------------------|-----------|
| $V_0$    | $c + 3d$          | trivial                        | 1         |
| $V_1$    | $c - 3d$          | representation $\mathcal{V}_1$ | 1         |
| $V_3^1$  | $c + d$           | natural $\mathcal{V}_3$        | 3         |
| $V_3^2$  | $c - d$           | representation $\mathcal{V}_4$ | 3         |

TABLE 21

| Rep.                | Basic degree $\deg \mathcal{V}_j$ or $\deg \mathcal{V}_{j,1}$                                  | Eigenvalue of $C$ |
|---------------------|--|-------------------|
| $\mathcal{V}_0$     | $-(S_4)$   | $c + 3d > 0$      |
| $\mathcal{V}_1$     | $(S_4) - 2(D_4)$   | $c - 3d > 0$      |
| $\mathcal{V}_3$     | $(S_4) - 2(D_3) - (D_2) + 3(D_1) - (\mathbb{Z}_1)$   | $c + d > 0$       |
| $\mathcal{V}_4$     | $(S_4) - (\mathbb{Z}_4) - (D_1) - (\mathbb{Z}_3) + (\mathbb{Z}_1)$                             | $c - d > 0$       |
| $\mathcal{V}_{0,1}$ | $(S_4)$  | $c + 3d > 0$      |
| $\mathcal{V}_{1,1}$ | $(S_4^-)$  | $c - 3d > 0$      |
| $\mathcal{V}_{3,1}$ | $(D_4^d) + (D_2^d) + (D_3) + (\mathbb{Z}_4^c) + (\mathbb{Z}_3^t) - (\mathbb{Z}_2^-) - (D_1)$   | $c + d > 0$       |
| $\mathcal{V}_{4,1}$ | $(D_4^z) + (D_2^d) + (D_3^z) + (\mathbb{Z}_4^c) + (\mathbb{Z}_3^t) - (\mathbb{Z}_2^-) - (D_1)$ | $c - d > 0$       |

TABLE 22

In order to compute the equivariant degree  $S_4 \times S^1\text{-Deg}(\text{id} - \Psi(0, \cdot, \cdot), \Omega)$ , we will apply the computational formula similar to (7.1). All the related to this formula degrees of the basic maps are presented in the Table 22.

Let us list the dominating orbit types:  $(S_4)$  (orbit contains one periodic solution),  $(S_4^-)$  (orbit contains one periodic solution),  $(D_4^d)$  (orbit contains 3 periodic solutions),  $(D_4^{\hat{d}})$  (orbit contains 3 periodic solutions),  $(D_2^d)$  (orbit contains 6 periodic solutions),  $(\mathbb{Z}_4^c)$  (orbit contains 6 periodic solutions),  $(\mathbb{Z}_4^-)$  (orbit contains 6 periodic solutions),  $(\mathbb{Z}_3^t)$  (orbit contains 8 periodic solutions), and  $(D_4^z)$  (orbit contains 3 periodic solutions).

By using the above equivariant degrees of the basic maps, as well as the multiplications table for the  $A(S_4)$ -module  $A_1(S_4 \times S^1)$  we obtain the following values of  $S_4 \times S^1\text{-Deg}(\text{id} - \Psi(0, \cdot, \cdot), \Omega)_1$ , for all the possible distributions of the eigenvalues of the matrix  $C$ .

**7.4. Conclusions for the icosahedral group  $\mathbb{I}$ .** Finally we consider the system of van der Pol equations with icosahedral symmetry group described in Example 2.4. Here we have the group  $A_5$  acting on the twenty-dimensional space  $V := \mathbb{R}^{20}$  by permuting the coordinates of the vectors in the same way as the symmetries in  $\mathbb{R}^3$  permutes the vertices of the dodecahedron. It can be verified, that the matrix  $C$ , defined by (2.5) in Example 2.5 has the following eigenvalues:

$$\sigma(C) := \{\lambda_0 = c + 3d, \lambda_1 = c - 2d, \lambda_2 = c + d, \lambda_3 = c + \sqrt{5}d\}$$

and there is the following decomposition of  $V$  into the eigenspaces of  $C$ :

$$V = V_0 \oplus V_1 \oplus V_2 \oplus V_3,$$



| $\Sigma(C)$                    | $S_4 \times S^1\text{-Deg}(\text{id} - \mathcal{F}(0, \cdot, \cdot), \Omega)_1$  | # Sol. |
|--------------------------------|--|--------|
| $c + 3d$                       | $-(S_4)$   | 1      |
| $c - 3d$                       | $(S_4^-) - 2(D_4^d)$   | 4      |
| $c + d, c + 3d$                | $-(S_4) - 4(D_4^d) - 4(D_3) - 4(D_2^d) - 4(Z_4^c) - 4(Z_3^t) + 4(D_1) + 4(Z_2^-)$  | 18     |
| $c - d, c - 3d$                | $(S_4^-) - 2(D_4^d) - (D_4^z) + (D_3^z) - (D_2^d) + (Z_4^c) + (Z_4^-) + (Z_4) - 2(V_4^-) - (Z_3^t) - 2(Z_3) + 2(D_1^z) + (D_1) + 5(Z_2^-) + 4(Z_2) - 4(Z_1)$   | 34     |
| $c - d, c + d, c + 3d$         | $-(S_4) - (D_4^d) - (D_4^z) + (D_3^z) + 3(D_3) + 2(D_2^d) + (D_2^z) + 2(D_2) + 2(Z_4^c) + (Z_4^-) + 2(Z_4) - 2(Z_3^t) - 3(Z_3) - (D_1^z) - 3(D_1) - 2(Z_2^-) - 3(Z_2) + 3(Z_1)$  | 33     |
| $c + d, c - d, c - 3d$         | $(S_4^-) - 2(D_4^d) - (D_4^z) - (D_4^-) - 3(D_3^z) - (D_3) + 2(D_2^d) + 2(D_2^z) + (D_2) + 2(Z_4^c) + 2(Z_4^-) + (Z_4) - 4(V_4^-) + 2(Z_3^t) + 3(Z_3) + 3(D_1^z) + (D_1) - 2(Z_2^-) - 3(Z_2) - 3(Z_1)$                       | 36     |
| $c - 3d, c - d, c + d, c + 3d$ | $-(S_4^-) - (S_4) + 2(D_4^d) + (D_4^z) + (D_4^-) + 2(D_4) + 3(D_3^z) + 3(D_3) - 2(D_2^d) - 2(D_2^z) - 2(D_2) - 2(Z_4^c) - 2(Z_4^-) - 2(Z_4) + 4(V_4^-) - 2(Z_3^t) - 4(Z_3) - 3(D_1^z) - 3(D_1) + 2(Z_2^-) + 4(Z_2) + 4(Z_1)$ | 37     |

TABLE 23

where  $V_0$  is a one dimensional subspace of  $V$ , with a trivial action of  $A_5$  (i.e.  $V_0 = V^{A_5}$ ), and  $V_1 \simeq \mathcal{V}_1 \oplus \mathcal{V}_1$ ,  $V_2 \simeq \mathcal{V}_2$ ,  $V_3 \simeq \mathcal{V}_3 \oplus \mathcal{V}_3$ , where  $\mathcal{V}_1$ ,  $\mathcal{V}_2$  and  $\mathcal{V}_3$  are irreducible representations of  $A_5$ , which were discussed in Section 6.

In order to compute the equivariant degree  $A_5 \times S^1\text{-Deg}(\text{id} - \Psi(0, \cdot, \cdot), \Omega)_1$ , we will apply the computational formula:

$$(7.2) \quad G\text{-Deg}(\text{id} - \Psi(0, \cdot, \cdot), \Omega)_1 = \prod_{\lambda_j \in \Sigma(C)} \text{deg}_{\mathcal{V}_j}^{m_{j,1}} \cdot \left[ \sum_{\lambda_j \in \Sigma(C)} m_{j,1} \text{deg}_{\mathcal{V}_{j,1}} \right],$$

where  $m_{j,1}$  denotes the  $\mathcal{V}_{j,1}$ -multiplicity of the eigenvalue  $\lambda_j$ , which is 2 in the case of  $\lambda_1$  and  $\lambda_3$ .

We need the basic degrees  $\text{deg}_{\mathcal{V}_j} \in A(A_5)$  and  $\text{deg}_{\mathcal{V}_{j,1}} \in A_1(A_5 \times S^1)$  (in the case the eigenvalue corresponding to the irreducible representation  $\mathcal{V}_j$  is positive). All the related to this formula degrees of the basic maps are presented in the following table:

| Rep.                | Basic degree $\text{deg}_{\mathcal{V}_j}$ or $\text{deg}_{\mathcal{V}_{j,1}}$               | Eigenvalue of $C$   |
|---------------------|---|---------------------|
| $\mathcal{V}_0$     | $-(A_5)$  | $c + 3d > 0$        |
| $\mathcal{V}_1$     | $(A_5) - 2(A_4) - 2(D_3) + 3(Z_2) + 3(Z_3) - 2(Z_1)$  | $c - 2d > 0$        |
| $\mathcal{V}_2$     | $(A_5) - 2(D_5) - 2(D_3) + 3(Z_2) - (Z_1)$  | $c + d > 0$         |
| $\mathcal{V}_3$     | $(A_5) - (Z_5) - (Z_3) - (Z_2) + (Z_1)$   | $c + \sqrt{5}d > 0$ |
| $\mathcal{V}_{0,1}$ | $(A_5)$   | $c + 3d > 0$        |
| $\mathcal{V}_{1,1}$ | $(A_4) + (D_3) + (D_3^z) + (V_4^-) + (Z_3^t) + (Z_5^t) + (Z_5^t) - (Z_2) - (Z_3) - (Z_2^-)$ | $c - 2d > 0$        |
| $\mathcal{V}_{2,1}$ | $(D_5) + (D_3) + (A_4^t) + (A_4^t) + (Z_5^t) + (Z_5^t) + (V_4^-) - 2(Z_2)$                  | $c + d > 0$         |
| $\mathcal{V}_{3,1}$ | $(D_5^z) + (V_4^-) + (D_3^z) + (Z_5^t) + (Z_3^t) - 2(Z_2^-)$                                | $c + \sqrt{5}d > 0$ |

TABLE 24

Let us list the dominating orbit types:  $(A_4^{t_1})$  and  $(A_4^{t_2})$  (orbit contains 5 periodic solutions),  $(A_5)$  (orbit contains 1 periodic solution),  $(V_4^-)$  (orbit contains 15 periodic solutions),  $(D_3^z)$  (orbit contains 6 periodic solutions),  $(D_3^z)$  (orbit contains 10 periodic solutions),  $(\mathbb{Z}_5^{t_1})$  and  $(\mathbb{Z}_5^{t_2})$  (orbit contains 12 periodic solutions).

By using the above equivariant degrees of the basic maps, as well as the multiplications table for the  $A(A_5)$ -module  $A_1(A_5 \times S^1)$  we obtain the following values of  $A_5 \times S^1$ -Deg(id -  $\mathcal{F}(0, \cdot, \cdot), \Omega)_1$ , for the possible distributions of the eigenvalues of the matrix  $C$ .

| $\Sigma(C)$                            | $A_5 \times S^1$ -Deg(id - $\mathcal{F}(0, \cdot, \cdot), \Omega)_1$  | # Sol. |
|--|---|--------|
| $c + 3d$                               | $-(A_5)$  | 1      |
| $c - 2d$                               | $2(A_4) + 2(D_3^z) + 2(D_3) + 2(\mathbb{Z}_5^{t_1}) + 2(\mathbb{Z}_5^{t_2})$<br>$+2(V_4^-) + 2(\mathbb{Z}_3^t) - 2(\mathbb{Z}_3) - 2(\mathbb{Z}_2^-) - 2(\mathbb{Z}_2)$   | 45     |
| $c + 3d, c + \sqrt{5}d$                | $-(A_5) - 2(D_3^z) - 2(D_3^z) - 2(\mathbb{Z}_5^{t_1}) - 2(V_4^-)$<br>$-2(\mathbb{Z}_3^t) + 4(\mathbb{Z}_2^-)$   | 29     |
| $c - 2d, c + d$                        | $(A_4^{t_1}) + (A_4^{t_2}) + 2(A_4) - (D_5) - 2(D_3^z) - 3(D_3)$<br>$-3(\mathbb{Z}_5^{t_1}) - 3(\mathbb{Z}_5^{t_2}) + 3(V_4^-) - 6(\mathbb{Z}_3^t) - 2(\mathbb{Z}_3)$<br>$-4(\mathbb{Z}_2^-) - 3(\mathbb{Z}_2) + 7(\mathbb{Z}_1)$                             | 45     |
| $c + \sqrt{5}d, c + 3d, c + d$         | $-(A_5) - (A_4^{t_1}) - (A_4^{t_2}) + 2(D_3^z) + 3(D_5)$<br>$+2(D_3^z) + 3(D_3) + 3(\mathbb{Z}_5^{t_1}) + (\mathbb{Z}_5^{t_2}) - 3(V_4^-)$<br>$+6(\mathbb{Z}_3^t) + 2(\mathbb{Z}_2^-) - 4(\mathbb{Z}_1)$  | 51     |
| $c + d, c - 2d, c + \sqrt{5}d$         | $(A_4^{t_1}) + (A_4^{t_2}) + 2(A_4) - (D_5^z) - (D_5) - 3(D_3^z)$<br>$-3(D_3) + 4(\mathbb{Z}_5^{t_1}) + 3(\mathbb{Z}_5^{t_2}) + 2(\mathbb{Z}_5) + 4(V_4^-)$<br>$+3(\mathbb{Z}_3^t) + 4(\mathbb{Z}_3) + 4(\mathbb{Z}_2^-) + 4(\mathbb{Z}_2) - 8(\mathbb{Z}_1)$ | 51     |
| $c + 3d, c - 2d, c + \sqrt{5}d, c + d$ | $-(A_5) - (A_4^{t_1}) - (A_4^{t_2}) - 2(A_4) + 2(D_3^z) + 3(D_5)$<br>$+4(D_3^z) + 5(D_3) + 5(\mathbb{Z}_5^{t_1}) + 3(\mathbb{Z}_5^{t_2}) - 5(V_4^-)$<br>$+8(\mathbb{Z}_3^t) + 2(\mathbb{Z}_3) + 4(\mathbb{Z}_2^-) + 2(\mathbb{Z}_2) - 8(\mathbb{Z}_1)$        | 51     |

TABLE 25

REMARKS. Computations of the equivariant degrees, which were applied to estimate of the number of periodic solutions of the above systems of van der Pol equations, were done based using Maple 8. The Maple worksheets, containing the complete multiplication tables and the equivariant degrees of the basic maps for the groups  $D_3 \times S^1$ ,  $D_4 \times S^1$ ,  $D_5 \times S^1$ ,  $A_4 \times S^1$ ,  $S_4 \times S^1$  and  $A_5 \times S^1$ , are available at the web site at: <http://www.math.ualberta.ca/~wkrawcew/degree> or <http://krawcewicz.net/degree>.

### 8. Appendix. Primary equivariant degree with one free parameter

For simplicity, we assume that  $\Gamma$  is a finite group.<sup>6</sup> Let  $V$  be an orthogonal representation of  $G = \Gamma \times S^1$ ,  $\Omega$  an open bounded  $G$ -invariant subset of  $\mathbb{R} \times V$ , and  $f: \mathbb{R} \times V \rightarrow V$  an  $\Omega$ -admissible map, i.e.  $f$  is a continuous  $G$ -equivariant and  $f(x) \neq 0$  for all  $x \in \partial\Omega$ . We will call such a pair  $(f, \Omega)$  an *admissible pair*.

DEFINITION 8.1. Let  $(f, \Omega)$  be an admissible pair. Then  $f$  is said to be *normal* in  $\Omega$ , if for every  $\alpha = (H)$  such that  $H = G_{x_o}$  for a certain  $x_o \in f^{-1}(0) \cap \Omega$ , the following condition is satisfied:

- for all  $x \in f^{-1}(0) \cap \Omega_H$  there exists  $\delta_x > 0$  such that for all  $w \in \nu_x(\Omega_{(H)})$

$$\text{if } \|w\| < \delta_x \text{ then } f(x + w) = f(x) + w = w,$$

where  $\nu(\Omega_{(H)})$  denotes the normal bundle to the submanifold  $\Omega_{(H)}$  in  $\mathbb{R} \times V$ .

We say that  $f$  is *regular normal* if:

- $f$  is of class  $C^1$ ,
- $f$  is normal in  $\Omega$ ,
- for every orbit type  $(H)$  in  $\Omega$  zero is a regular value of

$$f_H := f|_{\Omega_H}: \Omega_H \rightarrow V^H.$$

THEOREM 8.2 (Regular Normal Approximation Theorem). *Let  $(f, \Omega)$  be an admissible pair. Then, for every  $\eta > 0$  there exists a regular normal (in  $\Omega$ )  $G$ -equivariant map  $\tilde{f}: V \rightarrow W$  such that*

$$\sup_{x \in \Omega} \|\tilde{f}(x) - f(x)\| < \eta.$$

We consider the set  $\Phi_1(G) := \{(H) : H \subset G, \dim W(H) = 1\}$  (obviously,  $\Phi_1(G)$  consists of conjugacy classes of twisted subgroups) and denote by  $A_1(G)$  the free  $\mathbb{Z}$ -module generated by the symbols  $(H) \in \Phi_1(G)$ , i.e.  $A_1(G) = \mathbb{Z}[\Phi_1(G)]$ . An element  $\alpha \in A_1(G)$  will be written as a finite sum

$$\alpha = \sum_{(H) \in \Phi_1(G)} n_H(H) = n_{H_1}(H_1) + \dots + n_{H_r}(H_r).$$

The statement following below provides an axiomatic approach to the *primary*  $G$ -equivariant degree.

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<sup>6</sup>The equivariant degree as well as its part — the primary equivariant degree, is defined for any compact Lie group. For more information and precise definitions we refer to [3], [9], [13], [14], [16], [24].

THEOREM 8.3. *There exists a unique function, denoted by  $G\text{-Deg}$ , assigning to each admissible pair  $(f, \Omega)$  an element  $G\text{-Deg}(f, \Omega) \in A_1(G)$  satisfying the following properties:*

- (P1) (Existence) *If  $G\text{-Deg}(f, \Omega) = \sum_{(H)} n_H(H)$  is such that  $n_H \neq 0$  for some  $(H) \in \Phi_1(G)$ , then there exists  $x \in \Omega$  with  $f(x) = 0$  and  $G_x \supset H$ .*
- (P2) (Additivity) *Assume that  $\Omega_1$  and  $\Omega_2$  are two  $G$ -invariant open disjoint subsets of  $\Omega$  such that  $f^{-1}(0) \cap \Omega \subset \Omega_1 \cup \Omega_2$ . Then*

$$G\text{-Deg}(f, \Omega) = G\text{-Deg}(f, \Omega_1) + G\text{-Deg}(f, \Omega_2).$$

- (P3) (Homotopy) *Suppose that  $f: [0, 1] \times \mathbb{R} \times V \rightarrow V$  is an  $\Omega$ -admissible  $G$ -equivariant homotopy (i.e.  $f_t := f(t, \cdot, \cdot)$  is  $\Omega$ -admissible for  $t \in [0, 1]$ ). Then*

$$G\text{-Deg}(f_t, \Omega) = \text{constant}.$$

- (P4) (Suspension) *Suppose that  $W$  is another orthogonal  $G$ -representation and let  $U$  be an open, bounded  $G$ -invariant neighborhood of 0 in  $W$ . Then*

$$G\text{-Deg}(f \times \text{id}, \Omega \times U) = G\text{-Deg}(f, \Omega).$$

- (P5) (Normalization) *Suppose  $f$  is regular normal and  $f(x_o) = 0$  for some  $x_o \in \Omega$  with  $(G_{x_o}) = (H) \in \Phi_1(G)$ . Let  $\mathcal{U}_{G(x_o)}$  be an invariant tube around the orbit  $G(x_o)$ ,  $S_{x_o}$  a positively oriented slice to  $W(H)(x_o)$  in  $\mathbb{R} \oplus V^H$  and  $f^{-1}(0) \cap \mathcal{U}_{G(x_o)} = G(x_o)$ . Then*

$$G\text{-Deg}(f, \mathcal{U}_{G(x_o)}) = (\text{sign } \det(Df(x_o)|_{S_{x_o}})) \cdot (H).$$

- (P6) (Elimination) *Suppose  $f$  is normal in  $\Omega$  and  $\Omega_H \cap f^{-1}(0) = \emptyset$  for every  $(H) \in \Phi_1(G)$ . Then  $G\text{-Deg}(f, \Omega) = 0$ .*

The following *multiplicativity property* of the primary degree is very useful.

PROPOSITION 8.4. *The  $\mathbb{Z}$ -module  $A_1(G)$  admits a natural structure of an  $A(\Gamma)$ -module, where  $A(\Gamma)$  denotes the Burnside ring. Assume, in addition, that  $(f, \Omega)$  is an admissible pair in  $V$ ,  $W$  is an orthogonal representation of  $\Gamma$ ,  $U$  is an open  $\Gamma$ -invariant subset of  $W$  and  $g: W \rightarrow W$  is a  $\Gamma$ -equivariant map such that  $g(v) \neq 0$  for all  $v \in \partial\Omega$ . Then we have:*

- (P7) (Multiplicativity) *The product map  $f \times g: \mathbb{R} \times V \times W \rightarrow V \times W$  is  $\Omega \times U$ -admissible, and*

$$G\text{-Deg}(f \times g, \Omega \times U) = \Gamma\text{-Deg}(g, U) \cdot G\text{-Deg}(f, \Omega),$$

where  $\Gamma\text{-Deg}(g, U) \in A(\Gamma)$  denotes the  $\Gamma$ -equivariant degree (without free parameters, cf. [19]) of  $g$  in  $U$  and the multiplication “ $\cdot$ ” is the  $A(\Gamma)$ -module multiplication.

The computational formulae (6.2)–(6.5), used in Section 6, were derived using the regular normal approximations of equivariant maps and the recurrence formula for the primary degree with one free parameter (see [16]).

REMARK 8.5. In a standard way using the suspension property one can extend the above equivariant degree theory to the case of maps  $\text{id} - F$ , where  $F: \mathbb{R} \times \mathbb{E} \rightarrow \mathbb{E}$  is a completely continuous map and  $\mathbb{E}$  stands for a Banach  $G$ -representation (see, for instance, [13]–[15] and [19]).

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