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ON A SECOND ORDER BOUNDARY VALUE PROBLEM WITH SINGULAR NONLINEARITY

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Abstract. In this paper we investigate in a variational setting, the elliptic boundary value problem $-\Delta u = \operatorname{sign} u/|u|^{\alpha+1}$ in Ω, $u = 0$ on $\partial\Omega$, where Ω is an open connected bounded subset of $\mathbb{R}^N$, and $\alpha > 0$. For the positive solution, which is checked as a minimum point of the formally associated functional

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{\alpha}\int_\Omega \frac{1}{|u|^\alpha},$$

we prove dependence on the domain Ω. Moreover, an approximative functional E_ε is introduced, and an upper bound for the sequence of mountain pass points u_ε of E_ε, as $\varepsilon \to 0$, is given. For the onedimensional case, all sign-changing solutions of $-u'' = \operatorname{sign} u/|u|^{\alpha+1}$ are characterized by their nodal set as the mountain pass point and n-saddle points ($n > 1$) of the functional E.

1. Introduction

This paper is concerned with the singular boundary-value equation

$$\begin{cases} -\Delta u(x) = F'(u(x)) & x \in \Omega, \\ u(x) = 0 & x \in \partial\Omega, \end{cases} \tag{1.1}$$

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where Ω is a sufficiently regular bounded subset of $\mathbb{R}^N$, $N \geq 1$, and $F(u) = 1/(\alpha|u|^\alpha)$ with $\alpha > 0$.

In the onedimensional case, this equation comes out from some problems in fluid dynamics and pseudoplastic flow. The boundary value problem

$$\begin{cases} \tau''(v_{\shortparallel}) + \dfrac{v_{\shortparallel}}{\mu\tau(v_{\shortparallel})^\mu} = 0, & 0 < v_{\shortparallel} < 1,\ \mu > 0, \\ \tau'(0) = \tau(1) = 0, \end{cases} \tag{1.2}$$

arises in the investigation of the hydrodynamical equations for the steady flow of an incompressible viscous fluid over a semi-infinite flat plate (see [14]). Here τ is the so-called shear stress, and $v_{\shortparallel}$ is the component of the velocity parallel to the plate. In order to satisfy the above problem both these quantities must be properly normalized. The parameter μ enters in the non-Newtonian relation between the shear stress τ and the gradient of the parallel velocity $v_{\shortparallel}$ along the direction $x_\perp$ perpendicular to the plate,

$$\tau = \text{const} \cdot \left(\frac{\partial v_{\shortparallel}}{\partial x_\perp}\right)^{1/\mu}.$$

For $\mu = 1$ the above relation describes an ordinary Newtonian fluid. When μ is larger or smaller than one the fluid is called 'dilatant' or 'pseudoplastic', respectively. The pseudoplastic case is investigated in [1].

Positive solutions of the N-dimensional problem have been studied by Crandall et al. in [6], in a general setting of second-order elliptic operators and of a nonlinearity $F(x,s)$ which is the primitive of a singular function, $f(x,s)$, in the sense that f is well defined only for $s > 0$, and $\lim_{s\to 0^+} f(x,s) = \infty$, uniformly for $x \in \overline{\Omega}$. Existence and uniqueness of the positive solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ of (1.1) is proved for $\partial\Omega$ of C^3 class and $f \in C^1(\overline{\Omega} \times]0,\infty[)$, by means of the upper-lower solution method.

In a later work by Lazer and McKenna [13], which treats the case $f(x,u) = p(x)u^{-(\alpha+1)}$, is presented a simple proof of the existence and uniqueness of the positive solution $u \in C^{2+\gamma}(\Omega) \cap C(\overline{\Omega})$, $0 < \gamma < 1$, when Ω is of $C^{2+\gamma}$ class. Moreover, it is proved that $u \in H_0^{1,2}(\Omega)$ if and only if $\alpha < 2$.

In the case $f(x,u) = p(x)u^{-(\alpha+1)}$, there exist some other results on the behavior of the gradient ∇u of the solution of the problem (1.1) (see [16], [11]). In [16], a uniform bound for $|\nabla u|$ in Ω, is obtained assuming suitable hypothesis on the function p and on Ω. In this work the solution is obtained as the limit of a sequence of solutions of approximating problems. These solutions are checked as the minimum points of the relative associated functionals.

Moreover, the case $f(x,u) = \lambda q(x,u) + p(x)u^{-(\alpha+1)}$ with q non singular, has been investigated in [4] and recently in [21], showing existence of positive weak solutions in suitable assumptions on the functions q and p.

Sign-changing solutions have been studied lately in [15]. The authors assume that the domain Ω is of C^2 class, and such that $\Omega = \Omega_1 \cup \Omega_2$ with Ω_1 a C^2-subdomain. $\Gamma = \partial\Omega_1$ is called a *free nodal set*. Using the very precise information obtained on the behavior of the positive solution, u, when $u \to 0$, it is shown the existence of two solutions u_1 and u_2 for the problem

$$(1.3)\qquad \begin{aligned} -\Delta u + PV_\Gamma\left(\frac{p(x)}{u^{\alpha+1}}\right) &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega \cup \Gamma, \\ u(x) &\neq 0 \quad \text{in } \Omega \setminus \Gamma, \end{aligned}$$

with $u_1 = -u_2$, $u_1, u_2 \in C^{2,\gamma}(\Omega \setminus \Gamma) \cup C(\overline{\Omega})$, $0 < \gamma < 1$, and PV_Γ is the principal value around Γ, i.e.

$$(PV_\Gamma \varphi, \psi) = \lim_{\varepsilon \to 0} \int_{\Omega \setminus S_\varepsilon} \varphi\psi \, dx$$

for $\varphi \in L^1_{\text{loc}}(\Omega \setminus \Gamma)$, $\psi \in C_0^\infty(\Omega)$ and $S_\varepsilon = \{x \in \Omega : \text{dist}(x, \Gamma) < \varepsilon\}$. This result has been proved in dimension one for $\alpha > 0$ and in more dimensions for $\alpha > 2$.

Essentially, the solution of (1.3) is made by gluing together the positive solution u^{Ω_1} and the negative one, $-u^{\Omega_2}$. As the authors observe, it exists a continuum of solutions when Γ is deformed homeomorphically inside Ω, but in this setting none of this solutions can be distinguished, even in dimension one.

We use a variational approach to study the equation (1.1). We consider the formally associated functional

$$(1.4)\qquad E^\Omega(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \, dx + \frac{1}{\alpha}\int_\Omega \frac{1}{|u|^\alpha} \, dx.$$

It is obvious that E^Ω is not well defined on all $H_0^{1,2}(\Omega)$ because of the singularity on the nonlinear potential. We assume that the open bounded set Ω is such that the set $\mathcal{E}^\Omega = \{u \in H_0^{1,2}(\Omega) : \int_\Omega (1/|u|^\alpha) \, dx < \infty\}$ is not empty. We call Ω admissible if it satisfies this assumption.

In Chapter 2 we prove (see Theorem 2.14) that if Ω is admissible, the functional E^Ω has exactly two minimum points u_+^Ω and $-u_+^\Omega$, with $u_+^\Omega > 0$ on Ω, such that $\pm u_+^\Omega \in H_0^{1,2}(\Omega)$ are solutions of (1.1). We point out weakness of the regularity assumptions on Ω. (see Remark 2.1). Recalling the result of [15], we have that if Ω is of $C^{2+\gamma}$ class, then $\mathcal{E}^\Omega \neq \emptyset$ implies $\alpha < 2$.

In Chapter 3 we give some information on the behavior of the minimum points $u_+^\Omega > 0$ and $-u_+^\Omega$ of the functional E^Ω depending on the set Ω. We have a result of monotony (see Lemma 3.1) and a result of convergence of $u_+^{\Omega_n}$ to u_+^Ω where Ω_n is a non decreasing sequence of admissible subsets, and $\Omega = \bigcup_n \Omega_n$ is an admissible subset (see Lemma 3.3). Moreover, in the case of domains of C^2 class, we prove the continuous dependence of minimum points $\pm u_+^\Omega$ with respect to Ω (see Theorem 3.4).

In Chapter 4 we prove (see Proposition 4.4) the existence of a mountain pass point u_ε for the approximating functional

$$E_\varepsilon^\Omega(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{1}{\alpha}\int_\Omega \frac{1}{(|u|+\varepsilon)^\alpha}\,dx \quad \text{for all } u \in H_0^{1,2}(\Omega), \tag{1.5}$$

which is locally Lipschitz continuous; thus it admits the Clarke's subdifferential. We prefer to consider the functional $E_\varepsilon^\Omega(u)$ as an approximating functional of E^Ω because of the strict convexity of the function $s \mapsto 1/(|s|+\varepsilon)^\alpha$ either for $s > 0$ or $s < 0$. In Theorem 4.9 we prove the boundedness of u_ε in $H_0^{1,2}(\Omega)$ with respect to ε.

In Chapter 5, for the onedimensional case, we show in Theorem 5.4 that u_ε converges to u_0 weakly in $H_0^{1,2}([0,\pi])$, as $\varepsilon \to 0$, where u_0 is a point of mountain pass type for $E^{[0,\pi]}$. The only vanishing point of u_0 is $\pi/2$, and, according to the definition of McKenna and Reichel, u_0 is a "sign-changing solution" of (1.1).

In Theorem 6.6 of Chapter 6, for the onedimensional case we show that a "sign-changing solution" of (1.1), such that the nodal set divides the interval $[0,\pi]$ in equal parts, is characterized by a "variational argument."

2. Minimum points of the functional E

Let Ω be an open, bounded, connected set in $\mathbb{R}^N$. In the following, given $\alpha > 0$, we consider the functional $E\colon \mathcal{E} \to \mathbb{R}$ defined by

$$E(u) = E^\Omega(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{\alpha}\int_\Omega \frac{1}{|u|^\alpha}, \tag{2.1}$$

where $\mathcal{E}$ is the subset of $H_0^{1,2}(\Omega)$ defined as

$$\mathcal{E} = \mathcal{E}^\Omega = \left\{ u \in H_0^{1,2}(\Omega) : \int_\Omega \frac{1}{|u|^\alpha} < \infty \right\}. \tag{2.2}$$

We can observe that $\mathcal{E}^\omega$ is a cone without internal points such that $0 \notin \mathcal{E}^\Omega$.

Remark 2.1. We can exhibit some cases in which $\mathcal{E}^\Omega \neq \emptyset$.

(a) Let $\Omega =]0,\pi[\times]0,\pi[$. We consider

$$u(x_1,x_2) = (\sin x_1 \cdot \sin x_2)^\beta$$

with $1/2 < \beta < 1/\alpha$, where $\alpha \in]0,2[$. Then, $u \in \mathcal{E}^\Omega$.

(b) Let Ω be of C^2 class. We can consider $\widehat{u} \in H_0^{1,2}(\Omega)$ such that $\widehat{u} > 0$ in Ω and

$$\widehat{u}(x) := \operatorname{dist}(x,\partial\Omega)^\beta, \quad x \in \widehat{\Omega} \tag{2.3}$$

where $\widehat{\Omega} = \{x \in \Omega : \operatorname{dist}(x,\partial\Omega) < \widehat{\rho}\}$, for some $\widehat{\rho} > 0$, with $1/2 < \beta < 1/\alpha$. Then if $\alpha \in]0,2[$, we have $\widehat{u}(x) \in \mathcal{E}^\Omega$.

(c) Next, let $\Omega = \Omega_1 \cap \Omega_2$, where Ω_i are open bounded connected sets of C^1 class such that $\partial\Omega_1 \cap \partial\Omega_2$ is a manifold of codimension 2 made of a finite number of connected components. Then we have that if

$$\overline{u}(x) = \min(u_1(x), u_2(x)),$$

where u_1 on Ω_1 and u_2 on Ω_2 are defined as in (2.3), then $\overline{u} \in \mathcal{E}^\Omega$. We have easily the same result for $\Omega = \bigcap_{i=1}^n \Omega_i$, with Ω_i open bounded connected sets of C^1 class such that $\bigcap_{i=1}^n \partial\Omega_i$ is a manifold of codimension 2 made of a finite number of connected components.

DEFINITION 2.2. The set Ω is called admissible with respect of the functional E^Ω if Ω is an open bounded connected subset of $\mathbb{R}^N$ such that $\mathcal{E}^\Omega \neq \emptyset$.

In the following we assume that Ω is an admissible subset. Moreover, we denote $C_+ = \{u \in H_0^{1,2}(\Omega) : u(x) \geq 0\}$. Then $C_+ \cap \mathcal{E}$ is a convex cone. We set $E_+ = E|_{C_+ \cap \mathcal{E}}$.

LEMMA 2.3. *The following hold*

(a) *E is weakly lower semi-continuous and coercive; so there exists a minimum point of E in $\mathcal{E}$;*
(b) *E_+ has a unique minimum point u_+ in $C_+ \cap \mathcal{E}$;*
(c) $0 \leq \int_\Omega \nabla u_+ \nabla\varphi - \int_\Omega (1/u_+^\alpha)\varphi$, *for all* $\varphi \in H_0^{1,2}(\Omega)$.

PROOF. (a) The coercivity derives from the positivity of $\int 1/|u|^\alpha$. Using the Fatou Lemma, we get the weak lower semicontinuity of the functional $\int 1/|u|^\alpha$ and then the weak lower semicontinuity of E.

(b) By (a) we have the existence of the minimum point of E_+ on $C_+ \cap \mathcal{E}$. Since the real function of the real variable $k(s) = 1/|s|^\alpha$ is strictly convex for $s > 0$ we get

$$0 \leq \int_\Omega \frac{dx}{(tu_1(x) + (1-t)u_2(x))^\alpha} \leq \int_\Omega \frac{t}{(u_1(x))^\alpha}\,dx + \int_\Omega \frac{1-t}{(u_2(x))^\alpha}\,dx < \infty$$

for $t \in [0,1]$ and $u_1, u_2 \in C_+ \cap \mathcal{E}$. Then E_+ is strictly convex on the convex set $C_+ \cap \mathcal{E}$, which implies the uniqueness of the minimum point of E_+ in it.

(c) If $t > 0$, and $\varphi > 0$ with $\varphi \in H_0^{1,2}(\Omega)$, then $u_+ + t\varphi \in C_+ \cap \mathcal{E}$, and we get

$$(2.4)\quad 0 \leq \frac{E(u_+ + t\varphi) - E(u_+)}{t} = \frac{t}{2}\int_\Omega |\nabla\varphi|^2 + \int_\Omega \nabla u_+ \nabla\varphi - \int_\Omega \frac{\varphi}{(u_+ + \vartheta t\varphi)^{\alpha+1}}$$

for $0 < \vartheta = \vartheta(x,t) < 1$. By Fatou Lemma and (2.4) we have

$$\int \frac{\varphi}{u_+^{\alpha+1}} \leq \liminf_{t_n \to 0} \int \frac{\varphi}{(u_+ + \vartheta_n t_n \varphi)^{\alpha+1}} \leq \lim_{t_n \to 0} \frac{t_n}{2}\int |\nabla\varphi|^2 + \int \nabla u_+ \nabla\varphi.$$

Then the thesis follows. $\square$

Remark 2.4. Let us denote by $u_+ \in C_+ \cap \mathcal{E}$ the unique minimum point of E_+ on $C_+ \cap \mathcal{E}$. By the symmetry of E_+ we have that any minimum point w, of E on $\mathcal{E}$ is such that $|w| = u_+$. Indeed

$$E(|w|) = E(w) \leq E(u_+) \leq E(|w|).$$

So $E(w) = E(u_+)$, $|w| = u_+$, and so u_+ is a minimum point of E on all $\mathcal{E}$.

Now let us introduce the perturbed functional $E_\varepsilon : H_0^{1,2}(\Omega) \to \mathbb{R}$ defined by

$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{\alpha}\int_\Omega \frac{1}{(\varepsilon + |u|)^\alpha}. \tag{2.5}$$

We prefer to consider the functional E_ε as an approximating functional of E because of the strict convexity of the function $s \mapsto 1/(\varepsilon + |s|)^\alpha$ either for $s > 0$ or $s < 0$, which gives straightforward the uniqueness of the non negative and non positive minimum point, respectively in the positive and negative cone.

We observe that E_ε is locally Lipschitz. Thus, the functional E_ε admits the Clarke sub-differential (see [3]). We recall its definition and that of the critical point.

Definition 2.5. The sub-differential of a functional f, defined in a Banach space X, is

$$\partial f(u) = \{\xi \in X^* : \langle \xi, \varphi\rangle \leq f^0(u,\varphi) \text{ for all } \varphi \in X\}$$

where

$$f^0(u,\varphi) := \limsup_{w \to u, :\ t \searrow 0} \frac{f(w + t\varphi) - f(w)}{t}.$$

Moreover, $u \in X$ is a critical point for f if $0 \in \partial f(u)$.

Let us now calculate the Clarke sub-differential of our functional E_ε. We consider again $\widetilde{E} = \int_\Omega 1/|u|^\alpha$.

$$\begin{aligned}\widetilde{E}^0(u,\varphi) &= \lim_{w \to u} \sup_{t \searrow 0} \frac{1}{t}(\widetilde{E}(w + t\varphi) - \widetilde{E}(w)) \\ &= \frac{1}{\varepsilon^{\alpha+1}} \int_{\{u=0\}} |\varphi|\, dx - \int_{\{u \neq 0\}} \frac{\operatorname{sign} u}{(\varepsilon + |u|)^{\alpha+1}}\varphi.\end{aligned}$$

So we get

$$\partial E_\varepsilon(u) \ni \xi = u - i^*\bigg(\frac{\operatorname{sign} u}{(\varepsilon + |u|)^{\alpha+1}}\chi_{\{u\neq 0\}}\bigg) - i^*\bigg(\frac{\gamma}{\varepsilon^{\alpha+1}}\chi_{\{u=0\}}\bigg), \tag{2.6}$$

where $\gamma \in \mathbb{R}$ and $|\gamma| \leq 1$. Here, $\chi_{\{u\neq 0\}}(x) = 1$ if $u(x) \neq 0$, and $\chi_{\{u\neq 0\}}(x) = 0$ otherwise. Analogously we define $\chi_{\{u=0\}}(x)$.

By definition, we have that u is a weak critical point for the functional E_ε if it exists $\overline{\gamma} \in [-1,1]$ such that, for all $\varphi \in H_0^{1,2}(\Omega)$

$$0 = \int_\Omega \nabla u \nabla \varphi - \frac{1}{\varepsilon^{\alpha+1}}\int_\Omega \overline{\gamma}\varphi\chi_{\{u=0\}} + \int_\Omega \frac{\operatorname{sign} u}{(\varepsilon + |u|)^{\alpha+1}}\varphi\chi_{\{u\neq 0\}}. \tag{2.7}$$

Remark 2.6. Arguing as in Lemma 2.3 and Remark 2.4 we get that it exists a unique minimum point $u_+^\varepsilon \in C_+^\varepsilon$ for E_ε restricted on the positive cone. By the symmetry of E_ε, u_+^ε is a minimum point of E_ε on the whole space $H_0^{1,2}(\Omega)$, hence u_+^ε is a weak critical point for E_ε; thus it satisfies (2.7).

Lemma 2.7. *The set $\mathcal{Z}_\varepsilon = \{x \in \Omega : u_+^\varepsilon(x) = 0\}$ has zero measure. Moreover, it holds*

$$-\Delta u_+^\varepsilon = \frac{1}{(\varepsilon + u_+^\varepsilon)^{\alpha+1}}.$$

Proof. By contradiction let us suppose that $\operatorname{meas}(\mathcal{Z}_\varepsilon) := |\mathcal{Z}_\varepsilon| > 0$. Given ε, we can find two closed subsets F_1 and F_2 such that $F_1 \subset \overset{\circ}{F_2} \subset F_2 \subset \Omega$ and $|F_i \cap \mathcal{Z}_\varepsilon| > 0$ for $i = 1, 2$. We consider the function

$$\chi_\varepsilon(x) = \begin{cases} 1 & x \in \mathcal{Z}_\varepsilon \cap F_1, \\ 0 & \text{otherwise.} \end{cases}$$

We choose $\varphi_n \in H_0^{1,2}(F_2)$ such that for any n, $\varphi_n \geq 0$, $\operatorname{supp} \varphi_n \subset\subset F_2$, and φ_n converges to χ_ε in $L^2(F_2)$. Since $u_+^\varepsilon \in H^{2,2}_{\mathrm{loc}}(\Omega)$ we get

$$\begin{aligned} 0 \leq E_\varepsilon(u_+^\varepsilon + t\varphi_n) - E_\varepsilon(u_+^\varepsilon) &= t \int_{F_2} \bigg(- \Delta u_+^\varepsilon - \frac{1}{(\varepsilon + u_+^\varepsilon)^{\alpha+1}} \bigg) \varphi_n \\ + t^2 \int_{F_2} \bigg(\frac{1}{2} |\nabla \varphi_n|^2 &+ \frac{(\alpha+1)\varphi^2}{(\varepsilon + u_+^\varepsilon + \vartheta t \varphi_n)^{\alpha+2}} \bigg) = tA_n + t^2 B_n \end{aligned}$$

where $t > 0$ and $0 < \vartheta < 1$.

Since $\lim_n A_n = -(1/\varepsilon^\alpha)|\mathcal{Z}_\varepsilon \cap F_1| < 0$, for n large enough we get $A_n < 0$. Then for t small enough we obtain $tA_n + t^2 B_n < 0$. This is a contradiction, so we get $-\Delta u_+^\varepsilon = 1/(\varepsilon + u_+^\varepsilon)^{\alpha+1}$. □

Lemma 2.8. *There exists $a > 0$ such that, for any $\varepsilon > 0$,*

$$a\varphi_1(x) \leq u_+^\varepsilon(x) \quad \textit{for all } x \in \Omega$$

where $\varphi_1(x) > 0$ is an eigenfunction of the first eigenvalue λ_1 of the Laplacian operator $-\Delta$.

Proof. We have $-\Delta(u_+^\varepsilon - a\varphi^1) = H(x) \cdot (u_+^\varepsilon - a\varphi_1) + K(x)$ where

$$H(x) = \begin{cases} \dfrac{(\varepsilon + u_+^\varepsilon)^{-\alpha-1} - (\varepsilon + a\varphi)^{-\alpha-1}}{u_+^\varepsilon - a\varphi_1} & u_+^\varepsilon \neq a\varphi_1, \\ 0 & u_+^\varepsilon = a\varphi_1, \end{cases}$$

and

$$K(x) = (\varepsilon + a\varphi_1)^{-\alpha-1} - a\lambda_1\varphi_1.$$

It is easy to check that the function $H(x) \in L^\infty(\Omega)$ is negative. Moreover, $K(x) \in L^\infty(\Omega)$, and it exists $a > 0$, which does not depend on ε, such that $K(x) > 0$. Then, by the maximum principle we get our claim. □

At this point we obtain the following statement

Lemma 2.9. *It holds* $u_+(x) > 0$ *for any* $x \in \mathring{\Omega}$.

Proof. We have

$$E_\varepsilon(u_+^\varepsilon) \le E_\varepsilon(u_+) \le E(u_+), \quad \text{for all } \varepsilon > 0.$$

Hence u_+^ε is bounded in $H_0^{1,2}(\Omega)$. Thus, it exists a subsequence $u_+^{\varepsilon_k}$ which converges to u weakly in $H_0^{1,2}(\Omega)$ and punctually a.e. Then $u \ge a\varphi_1$. By Fatou Lemma and the weak lower semicontinuity of the norm of $H_0^{1,2}(\Omega)$ we get that

$$E(u) \le \liminf E_\varepsilon(u_+^\varepsilon) \le E(u_+).$$

Then by the uniqueness of the minimum point of E on the convex cone $C_+ \cap \mathcal{E}$ we have that $u = u_+$, so we get the claim. □

Lemma 2.10. *For any* $\varphi \in C_0^\infty(\Omega)$ *it holds*

$$\int_\Omega \nabla u_+ \nabla \varphi - \int_\Omega \frac{\varphi}{u_+^{\alpha+1}} = 0. \tag{2.8}$$

Proof. Given $\varphi \in C_0^\infty(\Omega)$, we can find $\tau > 0$ such that for any t with $t \le |\tau|$, we have $u_+ + t\varphi \in C_+ \cap \mathcal{E}$. It is easy to verify that the real function $t \mapsto E(u_+ + t\varphi)$ for $t \le |\tau|$ is of C^1 class, and $t = 0$ is a minimum point. Then (2.8) follows. □

Remark 2.11. The minimum points of $E = E^\Omega$ on $\mathcal{E} = \mathcal{E}^\Omega$ are exactly $u_+ = u_+^\Omega$ and $-u_+ = -u_+^\Omega$. Indeed, if there exists a sign-changing function w, which is a minimum point of E, by Remark 2.4 follows that $|w| = u_+$. Hence we get $\{x \in \Omega : u_+(x) = 0\} \ne \emptyset$, which contradicts the strict positivity of u_+ proved in Lemma 2.10.

Analogously we get that minimum points of E_ε on $H_0^{1,2}(\Omega)$ are exactly u_+^ε and $-u_+^\varepsilon$.

Remark 2.12. By Lemmas 2.10 and 2.9 we get $\int_\Omega |\nabla u_+|^2 = \int_\Omega 1/u_+^\alpha$. Indeed,

$$\int_\Omega \nabla u_+ \nabla \varphi_n = \int_\Omega \frac{1}{u_+^{\alpha+1}} \varphi_n,$$

where $\varphi_n = (u_+ - 1/n)^+$, and $\operatorname{supp} \varphi_n \subset\subset \Omega$. Since $0 \le \varphi_n \le u_+$ we get the assert by the Lebesgue convergence theorem.

Remark 2.13. Arguing as in the proof of Lemma 2.9 we get

$$E(u_+) = \liminf_{\varepsilon \to 0} E_\varepsilon(u_+^\varepsilon), \qquad u_+^\varepsilon \rightharpoonup u_+ \quad \text{as } \varepsilon \to 0, \qquad \|u_+\| = \liminf_{\varepsilon \to 0} \|u_+^\varepsilon\|.$$

Hence there exists $\varepsilon_k \to 0$ such that $u_{\varepsilon_k} \to u_+$ strongly in $H_0^{1,2}(\Omega)$.

By Remark 2.11 and Lemmas 2.9, 2.10, 2.3 and Remark 2.4 we have the following

THEOREM 2.14. *If Ω is an admissible subset of $\mathbb{R}^N$, then the functional E^Ω defined by*

$$E^\Omega(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{\alpha}\int_\Omega \frac{1}{|u|^\alpha} \quad \textit{for all } u \in \mathcal{E}^\Omega,$$

has exactly two minimum points u_+ and $-u_+$, with $u_+ > 0$ in $\mathring{\Omega}$, and it holds:

$$\int_\Omega \nabla u \nabla \varphi = \int_\Omega \frac{\varphi}{u_+^{\alpha+1}}, \quad \textit{for all } \varphi \in C_0^\infty(\Omega), \quad \operatorname{supp}\varphi \subset\subset \Omega.$$

REMARK 2.15. As we mentioned in the Introduction, in [13] was proved that if $\partial\Omega$ is of $C^{2,\gamma}$ class, $0 < \gamma < 1$, then the unique positive solution $u_+ \in C^2(\Omega) \cap C(\overline{\Omega})$ of (1.1) is in $H_0^{1,2}(\Omega)$ if and only if $\alpha < 2$. Hence by Theorem 2.14 we get that if $\mathcal{E}^\Omega \neq \emptyset$ and $\partial\Omega$ is of $C^{2,\gamma}$ class, then $\alpha < 2$.

3. Dependence of the minimum points of E on the domain

Next we give some information on the behavior of the minimum points u_+ and $-u_+$ of $E = E^\Omega$ with respect to the domain Ω. We recall that for the moment Ω is an open bounded connected subset of $\mathbb{R}^N$ such that $\mathcal{E}^\Omega \neq \emptyset$.

LEMMA 3.1 (Monotony). *If u_+^1 and u_+^2 are the positive minimum points of the functionals E^{Ω_1} and E^{Ω_2} respectively on the admissible subsets Ω_1 and Ω_2 of Ω, with $\Omega_1 \subset \Omega_2$, and $u_+^1 \equiv 0$ in $\Omega_2 \setminus \Omega_1$, then*

$$u_+^1 \leq u_+^2 \quad \text{a.e. in } \Omega_2.$$

PROOF. Let us consider the positive function $(u_+^1 - u_+^2)^+ \in H_0^1(\Omega_2)$. We can observe that the function $u_+^1 + t(u_+^1 - u_+^2)^+ \in C_+ \cap \mathcal{E} \subset H_0^1(\Omega_2)$, for all $-1 < t$. Moreover, the function $t \mapsto E(u_+^1 + t(u_+^2 - u_+^1)^+)$ is of C^1 class and $t = 0$ is a minimum point. So

$$\int_{\Omega_1} \nabla u_+^1 \nabla (u_+^1 - u_+^2)^+ - \int_{\Omega_1} \frac{(u_+^1 - u_+^2)^+}{(u_+^1)^{\alpha+1}} = 0.$$

Concluding, by (c) of Lemma 2.3 we have

$$\begin{aligned} 0 &\leq \int_{\Omega_2} |\nabla (u_+^1 - u_+^2)^+|^2 = \int_{\Omega_2} \nabla (u_+^1 - u_+^2)^+ \nabla (u_+^1 - u_+^2)^+ \\ &\leq \int_{\Omega_1} \frac{(u_+^1 - u_+^2)^+}{(u_+^1)^{\alpha+1}} - \int_{\Omega_2} \frac{(u_+^1 - u_+^2)^+}{(u_+^2)^{\alpha+1}} \\ &= \int_{\Omega_1} (u_+^1 - u_+^2)^+ \left[\frac{1}{(u_+^1)^{\alpha+1}} - \frac{1}{(u_+^2)^{\alpha+1}} \right] \leq 0. \end{aligned}$$

Then, $(u_+^1 - u_+^2)^+ \equiv 0$. □

Definition 3.2. If $\{\Omega_n\}$ is a sequence of admissible subsets of $\mathbb{R}^N$ such that $\Omega_n \subseteq \Omega_{n+1}$ for any n, and $\Omega = \bigcup_n \Omega_n$ is also an admissible set, we define the function

$$u_n = \begin{cases} u_+^n(x) & x \in \Omega_n, \\ 0 & x \in \Omega \setminus \Omega_n, \end{cases} \tag{3.1}$$

where u_+^n is the minimum point of E^{Ω_n}.

The following result gives a *"weak continuity"* of the map $\{\Omega_n \mapsto u_n\}$; "weak" in the sense that it holds only in the case where Ω_n is a non decreasing sequence of admissible subsets of $\mathbb{R}^N$.

Lemma 3.3. *The sequence $\{u_n\}$ defined in* (3.1), *converges strongly in*$H_0^{1,2}(\Omega)$ *to the positive minimum point u_+ of the functional E^Ω.*

Proof. By Lemma 3.1 we have $u_1 \le u_2 \le \ldots \le u_n \le \ldots \le u_+$. We set

$$u(x) = \sup_n u_n(x);$$

so $u_1 \le u \le u_+$. First we verify that $\|u_n\|$ is bounded. Indeed, by Lemma 2.3(c), since $u_+ \ge u_n > 0$, we have

$$\begin{aligned} \|u_+\|^2 - \|u_n\|^2 &= \langle \nabla u_+ - \nabla u_n, \nabla u_+ - \nabla u_n \rangle_{L^2(\Omega)} \\ &= \|u_+ - u_n\|^2 + 2\int_{\Omega_n} \nabla u_n \nabla (u_+ - u_n) \\ &= \|u_+ - u_n\|^2 + 2\int_{\Omega_n} \frac{u_+ - u_n}{u_n^\alpha} \ge 0. \end{aligned} \tag{3.2}$$

In the same way, if we consider u_{n+1} instead of u_+ we can prove that $\|u_n\|$ is increasing. Then, we can assume that the sequence u_n converges to u weakly in $H_0^{1,2}(\Omega)$, strongly in $L^2(\Omega)$ and punctually a.e. in Ω. Hence

$$\|u\| \le \liminf_{n\to\infty} \|u_n\| \le \|u_+\|. \tag{3.3}$$

Moreover, by Lemma 2.10 for any $\varphi \in C_0^\infty(\Omega)$, for n large enough we get

$$0 = \int_{\Omega_n} \nabla u_n \nabla \varphi - \int_{\Omega_n} \frac{1}{u_n^{\alpha+1}} \varphi.$$

Since the sequence $\{1/u_n^{\alpha+1}\}$ is positive and monotone, by Beppo–Levi Theorem we get

$$0 = \int_\Omega \nabla u \nabla \varphi - \int_\Omega \frac{1}{u^{\alpha+1}} \varphi \quad \text{for all } \varphi \in C_0^\infty(\Omega).$$

Arguing as in Remark 2.12 we get $\int_\Omega |\nabla u|^2 = \int_\Omega 1/u^\alpha$. Then, by (3.3),

$$E^\Omega(u) = \left(\frac{1}{2} + \frac{1}{\alpha}\right) \|u\|^2 \le \left(\frac{1}{2} + \frac{1}{\alpha}\right) \|u_+\|^2 = E^\Omega(u_+).$$

By the uniqueness of the positive minimum point of E^Ω we get $u \equiv u_+$. □

Theorem 3.4 (Continuity of minimum points with respect to the domain). *Let Ω_n be a sequence of C^2 bounded open connected subsets of $\mathbb{R}^N$ such that $\lim_{n\to\infty}\Omega_n = \Omega$, and let $\Omega \subset\subset \Omega^*$, where Ω and Ω^* are of C^2 class. Moreover, let $\alpha < 2$ and let u_+^n and u_+ be respectively the positive minimum points of E^{Ω_n} and E^{Ω}. We define*

$$u_n = \begin{cases} u_+^n & \text{in } \Omega_n, \\ 0 & \text{in } \Omega^*, \end{cases} \qquad u = \begin{cases} u_+ & \text{in } \Omega, \\ 0 & \text{in } \Omega^* \setminus \Omega. \end{cases}$$

Then u_n converges to u in $H_0^{1,2}(\Omega^*)$.

Proof. For a small enough we have $\Omega_{-a} \subset \Omega_n \subset \Omega_a$, for n large, where

$$\Omega_{-a} = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \ge a\}, \quad \Omega_a = \{x \in \Omega^* : \operatorname{dist}(x, \partial\Omega) \le a\} \cup \Omega.$$

By Lemma 3.1 we have $u_+^{-a} < u_+^n < u_+^a$, where $u_+^{\pm a}$ are respectively the positive minimum points of E^{Ω_a} and $E^{\Omega_{-a}}$. By (3.2) we have

$$\|u_+^{-a}\| \le \|u_+^n\| \le \|u_+^a\| .$$

By Lemma 3.1 and Lemma 4.7 (in the following chapter), letting $a \to 0$ we have that u_+^{-a} and u_+^a converge to u in $H_0^{1,2}(\Omega^*)$. Hence u_+^n converges to u. □

4. Boundedness of the mountain pass points u_ε of E_ε

Our aim now is to show the existence of a third critical point of the functional E_ε which changes sign. This will be a mountain pass point for E_ε. Referring to the definition of the (PS) condition for a locally Lipschitz functional we have

Definition 4.1. We say that $E_\varepsilon\colon H_0^{1,2}(\Omega) \to \mathbb{R}$ satisfies the (PS) condition if every sequence $\{u_n\}$ such that

(a) $E_\varepsilon(u_n) \le c < \infty$,
(b) there exists $\gamma_n \in [-1, 1]$ such that

$$u_n - i^*\left[\gamma_n(1-\chi_n) - \frac{\operatorname{sign} u_n}{(\varepsilon + |u_n|)^{\alpha+1}}\chi_n\right] \to 0 \quad \text{in } H^{-1,2}(\Omega), \tag{4.1}$$

where $\chi_n(x) = 1$ if $u_n(x) \ne 0$ and $\chi_n(x) = 0$ if $u_n(x) = 0$,

admits a subsequence which converges strongly in $H_0^{1,2}(\Omega)$.

Lemma 4.2. *E_ε satisfies the* (PS) *condition.*

Proof. Let $\{u_n\}$ be a (PS) sequence. Then since $E_\varepsilon(u_n)$ is bounded, we have that $\{u_n\}$ is bounded in $H_0^{1,2}(\Omega)$. Then we can assume that it converges to a function u, weakly in $H_0^{1,2}(\Omega)$ and strongly in $L^2(\Omega)$. Moreover, we can assume that $\gamma_n \to \gamma$. Next we set

$$v_n = \gamma_n(1-\chi_n) - \frac{\operatorname{sign} u_n}{(\varepsilon + |u_n|)^{\alpha+1}}\chi_n .$$

We have that $\{v_n\}$ is bounded in $L^2(\Omega)$. Thus we can assume that $\{v_n\}$ converges to a function v weakly in $L^2(\Omega)$. Recalling that $\{u_n\}$ is a (PS) sequence, for all $\varphi \in H_0^{1,2}(\Omega)$, we get

$$0 = \lim_n (\langle u_n, \varphi\rangle_{H_0^{1,2}(\Omega)} - \langle v_n, \varphi\rangle_{L^2(\Omega)}) = \langle u, \varphi\rangle_{H_0^{1,2}(\Omega)} - \langle v, \varphi\rangle_{L^2(\Omega)}.$$

Then, respectively, for $\varphi = u_n$ and $\varphi = u$, we have

$$0 = \lim_n (\|u_n\|^2_{H_0^{1,2}(\Omega)} - \langle v_n, u_n\rangle_{L^2(\Omega)}) = \lim_n \|u_n\|^2_{H_0^{1,2}(\Omega)} - \langle v, u\rangle_{L^2(\Omega)}, \tag{4.2}$$

$$0 = \lim_n (\langle u_n, u\rangle_{H_0^{1,2}(\Omega)} - \langle v_n, u\rangle_{L^2(\Omega)}) = \|u\|^2_{H_0^{1,2}(\Omega)} - \langle v, u\rangle_{L^2(\Omega)}. \tag{4.3}$$

By (4.2) and (4.3), $\lim_n \|u_n\|^2_{H_0^{1,2}(\Omega)} = \|u\|^2_{H_0^{1,2}(\Omega)}$. Hence the claim. □

Lemma 4.3. *There exists $\rho > 0$ such that $E_\varepsilon(u) > E_\varepsilon(u_+^\varepsilon)$, for all u with $\|u - u_+^\varepsilon\| = \rho$, where $u_+^\varepsilon \in C_+$ is the minimum point of E_ε.*

Proof. The proof is based on an argument of De Figuerido–Solimini which we adopt for functionals which admits Clarke's sub-differential. We suppose by contradiction that, for all $\rho > 0$,

$$\inf_{u \in H_0^{1,2}(\Omega)} \{E_\varepsilon(u) : u \in H_0^{1,2}(\Omega) \,\|u - u_+^\varepsilon\} = \rho\| = E_\varepsilon(u_+^\varepsilon).$$

We consider E_ε restricted to $\mathcal{R} = \{u : 0 < \rho - \delta < \|u - u_+^\varepsilon\| < \rho + \delta\}$. Let u_n be such that $\|u_n - u_+^\varepsilon\| = \rho$ and $E_\varepsilon(u_n) \le E_\varepsilon(u_+^\varepsilon) + 1/n$. Now we apply the Ekeland Variational Principle and obtain a sequence v_n such that

$$\begin{cases} E_\varepsilon(v_n) \le E_\varepsilon(u_n) & \|u_n - v_n\| \le 1/n, \\ E_\varepsilon(v_n) \le E_\varepsilon(u) + \|v_n - u\|/n & \text{for all } u \in \mathcal{R}. \end{cases}$$

Let us choose $u = v_n + t\varphi$, where $\operatorname{supp}\varphi \subset \{x \in \Omega : v_n(x) \neq 0\}$. Then

$$\begin{aligned} A(v_n, \varphi) &:= \limsup_{v \to \varphi,\, t \searrow 0} \frac{E_\varepsilon(v_n + tv) - E_\varepsilon(v_n)}{t} \\ &= \langle v_n, \varphi\rangle_{H_0^{1,2}(\Omega)} - \frac{1}{n} \int_{\{v_n \neq 0\}} \frac{\operatorname{sign} v_n}{(\varepsilon + v_n)^{\alpha+1}}\, \varphi, \end{aligned}$$

since $\int_{\{v_n(x)=0\}} |\varphi| = 0$. Moreover, since $E_\varepsilon(v_n) \le E_\varepsilon(v_n + t\varphi) + (t/n)\|\varphi\|$, we have

$$\begin{cases} -A(v_n, \varphi) \le \|\varphi\|/n, \\ A(v_n, \varphi) = -A(v_n, -\varphi) \le \|-\varphi\|/n. \end{cases}$$

So

$$\frac{|A(v_n, \varphi)|}{\|\varphi\|} \le \frac{1}{n} \quad \text{for all } \varphi, \quad \operatorname{supp}\varphi \subset \{v_n(x) \neq 0\}.$$

We notice that the map $\xi : \varphi \mapsto A(v_n, \varphi)$, $\varphi \in H_0^{1,2}(\Omega)$, belongs to $\partial E_\varepsilon(v_n) \subset H^{-1,2}(\Omega)$. So if $\xi_n \in \partial E_\varepsilon(v_n)$, and $\|\xi\| = \min_n \|\xi_n\|$, then

$$\|\xi_n\| \le \frac{|A(v_n, \varphi)|}{\|\varphi\|} \le \frac{1}{n}.$$

Using the (PS)-condition we get that $v_n \to v$ in $H_0^{1,2}(\Omega)$, hence $E_\varepsilon(v) = E_\varepsilon(u_+^\varepsilon)$. Moreover, $0 \in \partial E_\varepsilon(v)$ and $\|v - u_+^\varepsilon\| = \rho$. But this is a contradiction since by Remark 2.11 we know that u_+^ε and $u_-^\varepsilon = -u_+^\varepsilon$ are the only minimum points of E_ε. □

PROPOSITION 4.4. *We have that*

$$c_\varepsilon = \min_{\gamma\in\Gamma_\varepsilon} \max_{u\in\gamma} E_\varepsilon(u)$$

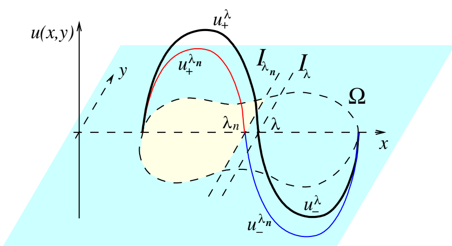
is a weak critical point for the functional E_ε *where*

$$\Gamma_\varepsilon = \{\gamma \in C([0,1], H_0^{1,2}(\Omega)) : \gamma(0) = u_+^\varepsilon,\ \gamma(1) = u_-^\varepsilon\}.$$

PROOF. By Lemmas 4.2 and 4.3, using for example the Deformation Theorem for nonsmooth functionals proved in [5], we get the existence of a weak critical point u_ε for E_ε. □

The following steps consist on showing that the set $\{u_\varepsilon\}_{\varepsilon>0}$ of the mountain pass point for the perturbed functional E_ε is bounded in $H_0^{1,2}(\Omega)$. For this purpose we build a continuous path from u_+^ε to u_-^ε. We can connect u_+^ε with u_+, and u_- with u_-^ε by segments, so it suffices to construct only a continuous path which connects u_+ with u_-. In the following Ω is a bounded open connected subset of $\mathbb{R}^n$ with boundary C^2, and $\alpha = 2$. We can assume that $0 \le x_1 \le 1$ for any $x = (x_1, \dots, x_N) \in \Omega$. We slice Ω with an hyperplane $I_\lambda = \{x : x_1 = \lambda\}$. To simplify, we assume that $\Omega \cap I_\lambda$ is connected.

DEFINITION 4.5. For $0 \le \lambda \le 1$ we set $\Omega_\lambda = \{x \in \Omega : 0 \le x_1 \le \lambda\}$ with $\Omega_0 = \emptyset$ and $\Omega_1 = \Omega$. We define u_+^λ such as to be equal to the positive minimum point of E^{Ω_λ} on Ω_λ, and $u_+^\lambda \equiv 0$ in $\Omega \setminus \Omega_\lambda$.



Moreover, we define $\widetilde{u}_+^\lambda$ to be equal to the positive minimum point of $E^{\Omega\setminus\Omega_\lambda}$ on $\Omega \setminus \Omega_\lambda$, and $\widetilde{u}_+^\lambda \equiv 0$ on Ω_λ. Finally

$$u_\lambda = \begin{cases} u_+^\lambda & \text{for } x \in \Omega_\lambda, \\ -\widetilde{u}_+^\lambda & \text{for } x \in \Omega \setminus \Omega_\lambda, \end{cases} \tag{4.4}$$

and we call $\widetilde{\gamma}$ the path $\lambda \mapsto u_\lambda$.

Here $u_0 = -u_+ = u_-$ and $u_1 = u_+$, where u_+ is the positive minimum point of E^Ω. We observe also that since Ω is of C^2 class, by Remark 2.1 we have that Ω_λ and $\Omega \setminus \Omega_\lambda$ are admissible subsets, so the function u_λ is well defined.

Lemma 4.6. *When $\lambda \to 0$, then u_+^λ converges to* 0.

Proof. By Remark 2.12 we have

$$\left(\frac{1}{2}+\frac{1}{\alpha}\right)\|u_+^\lambda\|^2 = E^{\Omega_\lambda}(u_+^\lambda) = \min_{v \in H_0^1(\Omega_\lambda)} E^{\Omega_\lambda}(v).$$

If we consider the function

$$d_\lambda(x) = \min[(\operatorname{dist}(x, \partial\Omega))^\beta, (\lambda - x_1)^\beta] \in H_0^{1,2}(\Omega_\lambda)$$

where $1/2 < \beta < 1/\alpha$, it is easy to see that $E^{\Omega_\lambda}(d_\lambda) \to 0$ when $\lambda \to 0$. □

Lemma 4.7. *Let $\lambda_n \searrow \lambda_0 \in \,]0,1[$ as $n \to \infty$. If we denote by u_n the function such that $u_n|_{\Omega_{\lambda_n}} \equiv u_+^{\lambda_n}$ and by u^0 the function such that $u^0|_{\Omega_{\lambda_0}} \equiv u_+^{\lambda_0}$, then u_n converges to u^0 strongly in $H_0^{1,2}(\Omega)$.*

Proof. By Lemma 3.1 we have $u_1 \geq u_2 \geq \ldots \geq u_n \geq \cdots \geq u^0$. We set $u(x) = \inf_n u_n(x)$. Analogously to Lemma 3.3 we can show that the sequence $\{u_n\}$ is decreasing, hence bounded. Then we can assume that u_n converges to u weakly in $H_0^{1,2}(\Omega)$ and punctually a.e. in Ω. Arguing again as in the proof of Lemma 3.3, by the monotony of $\{u_n\}$ we get $\int_{\Omega_{\lambda_0}} |\nabla u|^2 = \int_{\Omega_{\lambda_0}} 1/u^\alpha$. Then since $1/u(x) \leq 1/u^0(x)$ for $x \in \Omega_{\lambda_0}$ we obtain

$$E^{\Omega_{\lambda_0}}(u) = \left(\frac{1}{2}+\frac{1}{\alpha}\right)\int_{\Omega_{\lambda_0}} \frac{1}{u^\alpha} \leq \left(\frac{1}{2}+\frac{1}{\alpha}\right)\int_{\Omega_{\lambda_0}} \frac{1}{(u^0)^\alpha} = E^{\Omega_0}(u^0).$$

By the uniqueness of the positive minimum point of $E^{\Omega_{\lambda_0}}$ we get $u = u^0$. So $\|u^0\| \leq \lim_n \|u_n\|$. Moreover, being u_n the minimum point of $E^{\Omega_{\lambda_n}}$ we get

$$E^{\Omega_{\lambda_n}}(u_n) \leq E^{\Omega_{\lambda_0}}(u^0) + E^{\Omega_{\lambda_n} - \Omega_{\lambda_0}}(\widetilde{u}_+^n) = \left(\frac{1}{2}+\frac{1}{\alpha}\right)(\|u^0\|^2 + \|\widetilde{u}_+^n\|^2)$$

where $\widetilde{u}_+^n$ is the positive minimum point of $E^{\Omega_{\lambda_n} - \Omega_{\lambda_0}}$. In the same way as in the previous Lemma we have that $\widetilde{u}_+^n$ converges to 0. Then $\lim_n \|u_n\|^2 \leq \|u^0\|^2$. So $u_n \to u^0$ strongly in $H_0^{1,2}(\Omega)$. □

At this point, by (4.4) and Lemmas 3.3, 4.7 and 4.6, we get the continuity of the path

$$\widetilde{\gamma}(\lambda) = u_\lambda, \tag{4.5}$$

which links u_+ with u_-, is continuous.

Remark 4.8. Let $\gamma_1(t) = tu^\varepsilon_+ + (1-t)u_+$ where $0 \le t \le 1$. Since the segment $\gamma_1 = [u^\varepsilon_+, u_+]$ is connected in the convex cone C_+ of the positive functions, and E_ε is strictly convex in C_+, we get

$$E_\varepsilon(tu^\varepsilon_+ + (1-t)u_+) \le E_\varepsilon(u_+) \le E(u_+) \quad \text{for all } t \in [0,1].$$

If we consider $\gamma_2(t) = tu^\varepsilon_- + (1-t)u_-$ ($\gamma_2 = [u^\varepsilon_-, u_-]$) with $0 \le t \le 1$, analogously we get

$$E_\varepsilon(tu^\varepsilon_- + (1-t)u_-) \le E_\varepsilon(u_-) \le E(u_-) \quad \text{for all } t \in [0,1].$$

Theorem 4.9. *The set $\{u_\varepsilon\}$ of the mountain pass points for the perturbed functional E_ε is bounded in $H^{1,2}_0(\Omega)$. When $\varepsilon_k \to 0$ there exists a subsequence $\{u_{\varepsilon_k}\}$ which converges to u_0 weakly in $H^{1,2}_0(\Omega)$. Moreover, $E(u_0) \le \max_{\widetilde{\gamma}} E$, where the path $\widetilde{\gamma}$ is defined by* (4.4).

Proof. *Step 1.* $E_\varepsilon(u_\varepsilon) \le \max_{\widetilde{\gamma}} E$.

We consider the path $\gamma_\varepsilon = [u^\varepsilon_+, u_+] \cup \widetilde{\gamma} \cup [u_-, u^\varepsilon_-]$. By Remark 4.8 and by the definition of the path $\widetilde{\gamma}$ (see (4.4)), we get

$$E_\varepsilon(u_\varepsilon) \le \max_{\gamma_\varepsilon} E_\varepsilon \le \max_{\widetilde{\gamma}} E_\varepsilon \le \max_{\widetilde{\gamma}} E.$$

Hence $\|u_\varepsilon\|$ is bounded.

Step 2. If $\varepsilon_2 < \varepsilon_1$ then $E_{\varepsilon_1}(u_{\varepsilon_1}) \le E_{\varepsilon_2}(u_{\varepsilon_2})$.

Indeed by the convexity of E_ε on $[u^{\varepsilon_1}_+, u^{\varepsilon_2}_+]$ and $[u^{\varepsilon_1}_-, u^{\varepsilon_2}_-]$ we get

$$E_{\varepsilon_1}(u_{\varepsilon_1}) \le \max_{[u^{\varepsilon_1}_+, u^{\varepsilon_2}_+] \cup \gamma_{\varepsilon_2} \cup [u^{\varepsilon_1}_-, u^{\varepsilon_2}_-]} E_{\varepsilon_1} \le \max_{\widetilde{\gamma}_{\varepsilon_2}} E_{\varepsilon_1} \le \max_{\widetilde{\gamma}_{\varepsilon_2}} E_{\varepsilon_2}$$

for any path γ_{ε_2} from $u^{\varepsilon_2}_+$ to $u^{\varepsilon_2}_-$. Hence the claim.

Step 3. There exists a subsequence $\{u_{\varepsilon_k}\}$ such that $u_{\varepsilon_k} \rightharpoonup u_0$ weakly in $H^{1,2}_0(\Omega)$ and $E(u_0) \le \max_{\widetilde{\gamma}} E$.

By Step 1 we get the boundedness of $\|u_\varepsilon\|$. Hence we get the first claim. So we can assume that ε_k is decreasing to 0 and $u_{\varepsilon_k} \rightharpoonup u_0$ weakly in $H^{1,2}_0(\Omega)$. By Fatou's Lemma, by Step 2, and by Step 1 we get

$$E(u_0) \le \liminf_k E_{\varepsilon_k}(u_{\varepsilon_k}) = \lim_k E_{\varepsilon_k}(u_{\varepsilon_k}) \le \max_{\widetilde{\gamma}} E. \qquad \square$$

Lemma 4.10. *If w_ε is a weak critical point of E^Ω_ε, we get*

$$-\Delta w_\varepsilon = \frac{\operatorname{sign} w_\varepsilon}{(\varepsilon + |w_\varepsilon|)^{\alpha+1}} \chi_{\{w_\varepsilon \neq 0\}}$$

with $w_\varepsilon \in H^{2,2}(\Omega) \cap C^1(\overline{\Omega})$.

Proof. By (2.7) we have

$$-\Delta w_\varepsilon = \frac{\gamma}{(\varepsilon^{\alpha+1})} \chi_{\{w_\varepsilon = 0\}} + \frac{\operatorname{sign} w_\varepsilon}{(\varepsilon + |w_\varepsilon|)^{\alpha+1}} \chi_{\{w_\varepsilon \neq 0\}}$$

for some γ such that $|\gamma| < 1$. So $w_\varepsilon \in H^{2,2}(\Omega) \cap C^1(\overline{\Omega})$. Since $-\Delta w_\varepsilon(x) = 0$ for all x such that $w_\varepsilon(x) = 0$, If we suppose that $\operatorname{meas}(\{x : w_\varepsilon(x) = 0\}) > 0$, we get that $0 = \gamma/(\varepsilon^{\alpha+1})$. □

Remark 4.11. We consider the open set $\Omega_+^\varepsilon = \{x : u_\varepsilon(x) > 0\}$. Then the restriction $\widetilde{u}_\varepsilon$ of the weak critical point u_ε on Ω_+^ε coincides with the positive minimum point of the functional $E_\varepsilon^{\Omega_+^\varepsilon}$. Indeed $\widetilde{u}_\varepsilon \in H_0^{1,2}(\Omega_+^\varepsilon)$ is a positive solution of the equation $-\Delta u = 1/(\varepsilon + u)^{\alpha+1}$ on Ω_+^ε, and by the maximum principle the positive solution of the previous equation is unique, hence the claim. Then by regularity we get that $\widetilde{u}_\varepsilon \in C^2(\Omega_+^\varepsilon)$.

Remark 4.12. Let u_ε be a critical point for E_ε with $E_\varepsilon(u_\varepsilon) > E_\varepsilon(u_+^\varepsilon)$. Then u_ε changes sign. By contradiction, we have $-\Delta u_\varepsilon = (1/(\varepsilon + u_\varepsilon)^{\alpha+1})\chi_{\{u_\varepsilon \neq 0\}}$, if $u_\varepsilon \geq 0$. If $\omega \subset\subset \Omega$ with $\partial\omega$ smooth, we get $u_\varepsilon \in C^2(\omega)$, and by the strong maximum principle $u_\varepsilon > 0$ on ω. Then, $u_\varepsilon > 0$ on Ω, and $-\Delta u_\varepsilon = 1/(\varepsilon + u_\varepsilon)^{\alpha+1}$. If $u_\varepsilon + \varphi \geq 0$ we get

$$E_\varepsilon(u_\varepsilon + \varphi) - E_\varepsilon(u_\varepsilon) = \frac{1}{2}\int |\nabla\varphi|^2 + (\alpha+1)\int \frac{\varphi^2}{(\varepsilon + u_\varepsilon + \vartheta\varphi)^{\alpha+2}} \geq 0$$

with $0 < \vartheta < 1$. Hence $u_\varepsilon \neq u_\varepsilon^+$ is a minimum point of E_ε on the cone of positive functions. By uniqueness on Remark 2.6 this is a contradiction.

5. Mountain pass points for E_ε in the onedimensional case

In this chapter we assume $\Omega = [0,\pi]$. Let u_ε be a weak critical point of the functional E_ε^Ω. We define the *nodal set* of the function u_ε as

$$\mathcal{Z}_\varepsilon := \{x \in \,]0,\pi[\, : u_\varepsilon(x) = 0\}. \tag{5.1}$$

Firstly we will characterize the nodal set $\mathcal{Z}_\varepsilon$ of the weak critical points of E_ε. Next we will show that for the mountain pass points we have $\#\mathcal{Z}_\varepsilon = 1$.

Lemma 5.1. *It holds*

- (a) $\#\mathcal{Z}_\varepsilon < \infty$ *and the elements of* $\mathcal{Z}_\varepsilon$ *divide the interval* $[0,\pi]$ *in* $\nu_\varepsilon + 1$ *equal parts, where* $\nu_\varepsilon = \#\mathcal{Z}_\varepsilon$.
- (b) *If* u_ε *is a mountain pass point of* E_ε*, then there exists a sequence* ε_k *convergent to zero such that* u_{ε_k} *converges to* u_0 *uniformly and the integer* ν_{ε_k} *is constant for* ε_k *small enough.*

Proof. (a) Given ε, we consider u_ε. If $u_\varepsilon > 0$ for $x \in \,]a,b[$ with $u_\varepsilon(a) = u_\varepsilon(b) = 0$, then $-u_\varepsilon''(x) = 1/(\varepsilon + u_\varepsilon(x))^{\alpha+1}$ for $x \in \,]a,b[$. Hence $(u_\varepsilon'(x))^2 - 2/\alpha(u_\varepsilon(x) + \varepsilon)^{-\alpha}$ is a constant on $]a,b[$. So

$$0 < u_\varepsilon'(a) = -u_\varepsilon'(b) = \sqrt{\frac{2}{\alpha}\left[\frac{1}{\varepsilon^\alpha} - \frac{1}{(\varepsilon + M_\varepsilon)^\alpha}\right]}$$

where M_ε is the maximum for u_ε on $[a,b]$. Then u_ε changes sign and there exist c such that $u_\varepsilon(x) < 0$ for $x \in \,]b,c[$ and $u_\varepsilon(c) = 0$. It is easy to see that $c = 2b-a$ and $u_\varepsilon(x) = -u_\varepsilon(x+a-b)$ for $b < x < 2b-a$. So we have (a).

(b) By Theorem 4.9 there exists a sequence $\varepsilon_k \to 0$ such that $u_{\varepsilon_k} \to u_0$ uniformly. Hence π/ν_{ε_k} is a vanishing point of u_{ε_k}. Moreover, π/ν_{ε_k} is bounded by the uniform convergence of u_{ε_k}; this implies that ν_{ε_k} is constant for ε_k small enough. □

In the following u_ε is a mountain pass point of E_ε.

LEMMA 5.2. *It is false that* $\#\,\mathcal{Z}_\varepsilon$ *is an odd integer larger or equal than* 3.

PROOF. By contradiction we assume that $\#\mathcal{Z}_\varepsilon \geq 3$. We define the following function for $|t| \leq 1$

$$w_{\varepsilon,t} = \begin{cases} (1+t)^{2/(\alpha+2)} u_\varepsilon\bigg(\dfrac{x}{1+t}\bigg) & 0 \leq x \leq (1+t)B, \\ -(1-t)^{2/(\alpha+2)} u_\varepsilon\bigg(\dfrac{2B-x}{1-t}\bigg) & (1+t)B \leq x \leq 2B, \\ u_\varepsilon(x) & 2B \leq x \leq \pi. \end{cases} \tag{5.2}$$

$$w_{\varepsilon,-1} = \begin{cases} -2^{2/(\alpha+2)} u_\varepsilon\bigg(B - \dfrac{x}{2}\bigg) & 0 \leq x \leq 2B, \\ u_\varepsilon(x) & 2B \leq x \leq \pi. \end{cases} \tag{5.3}$$

$$w_{\varepsilon,+1} = \begin{cases} 2^{2/(\alpha+2)} u_\varepsilon\bigg(\dfrac{x}{2}\bigg) & 0 \leq x \leq 2B, \\ u_\varepsilon(x) & 2B \leq x \leq \pi. \end{cases} \tag{5.4}$$

We will show first that, for $|t| \leq 1$,

$$E_\varepsilon^\Omega(w_{\varepsilon,t}) < E_\varepsilon^\Omega(u_\varepsilon). \tag{5.5}$$

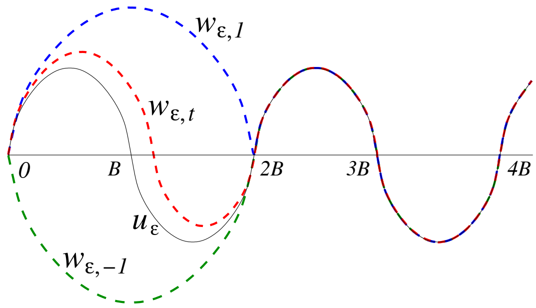
By (5.2) we have

$$E_\varepsilon^\Omega(w_{\varepsilon,t}) = E_\varepsilon^{[0,(1+t)B]}(w_{\varepsilon,t}) + E_\varepsilon^{[(1+t)B,2B]}(w_{\varepsilon,t}) + E_\varepsilon^{[2B,\pi]}(u_\varepsilon),$$

so, it suffices to show that $E_\varepsilon^{[0,2B]}(w_{\varepsilon,t}) < E_\varepsilon^{[0,2B]}(u_\varepsilon)$. Now by a changing variable argument we get

$$\begin{aligned} E_\varepsilon^{[0,(1+t)B]}(w_{\varepsilon,t}) = {} & \frac{1}{2}(1+t)^{(2-\alpha)/(2+\alpha)} \int_0^B (u_\varepsilon'(\xi))^2 \, d\xi \\ & + \frac{1}{\alpha} \int_0^B \frac{(1+t)\, d\xi}{(\varepsilon + (1+t)^{2/(\alpha+2)} u_\varepsilon(\xi))^\alpha} \end{aligned} \tag{5.6}$$

$$\begin{aligned} E_\varepsilon^{[(1+t)B,2B]}(w_{\varepsilon,t}) = {} & \frac{1}{2}(1-t)^{(2-\alpha)/(2+\alpha)} \int_0^B (u_\varepsilon'(\xi))^2 \, d\xi \\ & + \frac{1}{\alpha} \int_0^B \frac{(1-t)\, d\xi}{(\varepsilon + (1-t)^{2/(\alpha+2)} u_\varepsilon(\xi))^\alpha} \end{aligned} \tag{5.7}$$



Let us define $\varphi(t) = E_\varepsilon^{[0,(1+t)B]}(w_{\varepsilon,t}) + E_\varepsilon^{[(1+t)B,2B]}(w_{\varepsilon,t})$ for $|t| < 1$. By (5.6), (5.7) and by the symmetry of u_ε with respect to the point B, we get $\varphi(0) = E_\varepsilon^{[0,2B]}(u_\varepsilon)$. By calculating $\varphi'(t)$ and $\varphi''(t)$ we have that $\varphi'(0) = 0$ and $\varphi''(0) < 0$. Hence we have

$$E_\varepsilon^{[0,2B]}(w_{\varepsilon,t}) < E_\varepsilon^{[0,2B]}(u_\varepsilon), \quad \text{for } 0 < |t| < 1, \tag{5.8}$$

which implies (5.5). By (5.3) and (5.4) we have also $E_\varepsilon^{\Omega}(w_{\varepsilon,\pm 1}) < E_\varepsilon^{\Omega}(u_\varepsilon)$.

Next we define the following function for $|\tau| < 1$

$$v_{\varepsilon,\tau} = \begin{cases} u_\varepsilon(x) & 0 \le x \le 2B, \\ (1+\tau)^{2/(\alpha+2)} u_\varepsilon\left(\dfrac{x-2B}{1+\tau}\right) & 2B \le x \le (3+\tau)B, \\ -(1-\tau)^{2/(\alpha+2)} u_\varepsilon\left(\dfrac{4B-x}{1-\tau}\right) & (3+\tau)B \le x \le 4B, \\ u_\varepsilon(x) & 4B \le x \le \pi. \end{cases} \tag{5.9}$$

$$v_{\varepsilon,-1} = \begin{cases} u_\varepsilon(x) & 0 \le x \le 2B, \\ -2^{2/(\alpha+2)} u_\varepsilon\left(2B - \dfrac{x}{2}\right) & 2B \le x \le 4B, \\ u_\varepsilon(x) & 4B \le x \le \pi. \end{cases} \tag{5.10}$$

$$v_{\varepsilon,+1} = \begin{cases} u_\varepsilon(x) & 0 \le x \le 2B, \\ 2^{2/(\alpha+2)} u_\varepsilon\left(B - \dfrac{x}{2}\right) & 2B \le x \le 4B, \\ u_\varepsilon(x) & 4B \le x \le \pi. \end{cases} \tag{5.11}$$

Arguing as in the previous case, we consider $\widetilde{\varphi}(\tau) = E_\varepsilon^{I_3}(v_\varepsilon,\tau) + E_\varepsilon^{I_4}(v_\varepsilon,\tau)$ for $|\tau| < 1$, where $I_3 = [2B, 3B + B\tau]$ e $I_4 = [3B + B\tau, 4B]$, and we see that $\tau = 0$ is the unique strict maximum point for $\widetilde{\varphi}$. Moreover, by (5.10) and (5.11) we get

$$E_\varepsilon^{[2B,4B]}(v_\varepsilon, \pm 1) < E_\varepsilon^{[2B,4B]}(u_\varepsilon). \tag{5.12}$$

To simplify some notation in the following we consider the case $\#\mathcal{Z}_\varepsilon = 3$, and u_ε positive in $[0, B]$. Since $u_\varepsilon \in C^1([0,\pi])$ it is easy to verify that the application $\Gamma\colon Q \to H_0^{1,2}([0,\pi])$ where $Q = \{(t,\tau) : |t| \le 1, |\tau| \le 1\}$, defined by

$$\Gamma(t,\tau) = w_{\varepsilon,t}|_{[0,2B]} + v_{\varepsilon,\tau}|_{[2B,4B]}$$

is continuous. Here $w_{\varepsilon,t}|_{[0,2B]}$ is the restriction of $w_{\varepsilon,t}$ to the interval $[0, 2B]$ and zero on the interval $[2B,\pi]$. Analogously we define $v_{\varepsilon,\tau}|_{[2B,4B]}$. Then we have that

$$E_\varepsilon^\Omega(\Gamma(t,\tau)) < E_\varepsilon^\Omega(u_\varepsilon), \quad \text{for all } (t,\tau) \in Q \setminus \{0,0\}.$$

Indeed, $E_\varepsilon^\Omega(\Gamma(t,\tau)) = E_\varepsilon^{[0,2B]}(w_{\varepsilon,t}) + E_\varepsilon^{[2B,4B]}(v_{\varepsilon,\tau})$, then by (5.8) and (5.12) we get the claim.

Next we consider the continuous path $t \mapsto \Gamma(t,t)$, $t \in [0,1]$, which links the positive function $\Gamma(1,1)$ to u_ε in $H_0^{1,2}([0,\pi])$. Moreover, the map $\{\lambda \mapsto \lambda u_+^\varepsilon + (1-\lambda)\Gamma(1,1)\}$, with $0 \le \lambda \le 1$, is in the cone of the positive functions C_+. By the convexity of E_ε^Ω on C_+ and by the fact that u_+^ε is the positive minimum point of E_ε^Ω we get $E_\varepsilon^\Omega(\lambda u_+^\varepsilon + (1-\lambda)\Gamma(1,1)) < E_\varepsilon^\Omega(\Gamma(1,1)) < E_\varepsilon^\Omega(u_\varepsilon)$.

Analogously we build a continuous path from u_ε to u_-^ε such that u_ε is the maximum point of E_ε^Ω on this path. So finally, since u_ε is a strict maximum point for $E_\varepsilon|_{\Gamma(Q\setminus\{(0,0)\})}$, it is clear that we can build a path from u_-^ε to u_+^ε such that the maximum of E_ε^Ω on this path is strictly smaller than $E_\varepsilon^\Omega(u_\varepsilon)$. And this is a contradiction since u_ε is a mountain pass point. □

We can argue analogously in the cases in which $\#\mathcal{Z}_\varepsilon$ is an even integer larger than 2. Indeed we have the following

Lemma 5.3. *It is false that $\#\mathcal{Z}_\varepsilon$ is an even integer larger or equal than 2.*

Proof. Let us suppose $\# Z_\varepsilon = 2$ and $u_\varepsilon > 0$ in $]0, B[$. Here $B = \pi/3$. We define

$$\Gamma_\varepsilon(t,\tau) = \begin{cases} (1+t)^{2/(\alpha+2)} u_\varepsilon\left(\dfrac{x}{1+t}\right) & 0 \le x \le (1+t)B, \\ -(1-t+\tau)^{2/(\alpha+2)} u_\varepsilon\left(\dfrac{(2+\tau)B - x}{1-t+\tau}\right) & \\ & (1+t)B \le x \le (2+\tau)B, \\ (1-\tau)^{2/(\alpha+2)} u_\varepsilon\left(\dfrac{3B-x}{1-\tau}\right) & (2+\tau)B \le x \le 3B. \end{cases} \tag{5.13}$$

$$\Gamma_\varepsilon(-1,1) = -3^{2/(\alpha+2)} u_\varepsilon\left(B - \frac{x}{3}\right) \quad 0 \le x \le 3B,$$

$$\Gamma_\varepsilon\left(\frac{1}{2}, -\frac{1}{2}\right) = \begin{cases} \left(\dfrac{3}{2}\right)^{2/(\alpha+2)} u_\varepsilon\left(\dfrac{2}{3}x\right) & 0 \le x \le \dfrac{3}{2}B, \\ \left(\dfrac{3}{2}\right)^{2/(\alpha+2)} u_\varepsilon\left(B - \dfrac{2}{3}x\right) & \dfrac{3}{2}B \le x \le 3B, \end{cases}$$

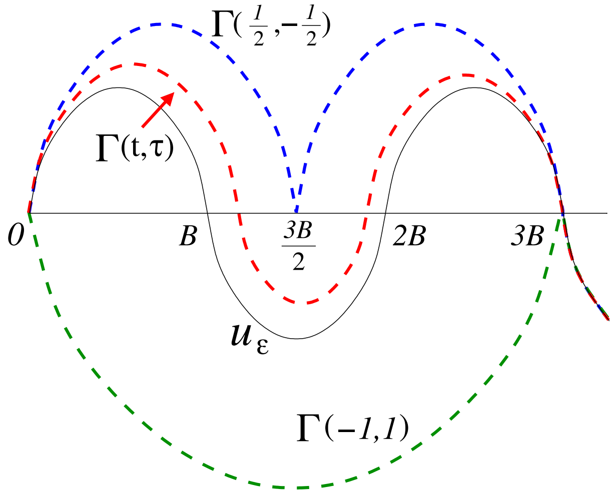
where $(t,\tau)\in\widetilde{Q}$. Here

$$\widetilde{Q}=\{(t,\tau):|t|<1,\ |\tau|<1,\ \tau>t-1\}\cup\{(-1,1),(1/2,-1/2)\}.$$

Since $\alpha<2$ and $u_\varepsilon\in C^1([0,\pi])$, is is easy to see that $\Gamma_\varepsilon:\widetilde{Q}\to H_0^{1,2}([0,\pi])$ is continuous. By calculations of the same type as in those of Lemma 5.2 we can verify that $(0,0)$ is the unique maximum point of E_ε on $\Gamma_\varepsilon(\widetilde{Q})$ since

$$\begin{aligned}E_\varepsilon(\Gamma_\varepsilon(t,\tau))=&[(1+t)^{(2-\alpha)/(2+\alpha)}+(1-\tau)^{(2-\alpha)/(2+\alpha)}+(1-t+\tau)^{(2-\alpha)/(2+\alpha)}]\\&\cdot\int_0^B(u_\varepsilon'(\xi))^2\,d\xi+\frac{1+t}{\alpha}\int_0^B\frac{d\xi}{(\varepsilon+(1+t)^{2/(2+\alpha)}u_\varepsilon(\xi))^\alpha}\\&+\frac{1-\tau}{\alpha}\int_0^B\frac{d\xi}{(\varepsilon+(1-\tau)^{2/(2+\alpha)}u_\varepsilon(\xi))^\alpha}\\&+\frac{1-t+\tau}{\alpha}\int_0^B\frac{d\xi}{(\varepsilon+(1-t+\tau)^{2/(2+\alpha)}u_\varepsilon(\xi))^\alpha}.\end{aligned}$$

Moreover, we can consider the segment which links the positive function $\Gamma_\varepsilon(1/2,-1/2)$ to the u_+^ε in the cone C_+ of the positive functions of $H_0^{1,2}([0,\pi])$, and the segment which links the negative function $\Gamma_\varepsilon(-1,1)$ to the u_-^ε in the cone of the negative functions. So we can build a path from u_+^ε to u_-^ε such that the maximum of E_ε^Ω on this path is strictly smaller than $E_\varepsilon^\Omega(u_\varepsilon)$.



This is a contradiction since u_ε is a mountain pass point. By Lemma 5.2 and the previous argument we can prove that $\# Z_\varepsilon$ is not an even integer. □

At this point we can characterize variationally the function u_0 which was found as the weak limit in $H_0^{1,2}([0,\pi])$, as $\varepsilon\to0$, of the sequence of mountain pass points $\{u_\varepsilon\}$ (see Theorem 4.9).

THEOREM 5.4. *The function* u_0 (*defined in the Theorem* 4.9) *is such that* $u_0|_{[0,\pi/2]} = u_+^{[0,\pi/2]}$, $u_0|_{[\pi/2,\pi]} = -u_+^{[\pi/2,\pi]}$, *where* $u_+^{[0,\pi/2]}$ *and* $u_+^{[\pi/2,\pi]}$ *are respectively the positive minimum points of* $E^{[0,\pi/2]}$ *and* $E^{[\pi/2,\pi]}$. *Moreover,*

$$E^{[0,\pi]}(u_0) = \inf_{\gamma\in\mathcal{A}} \max_{\gamma} E^{[0,\pi]}$$

where $\mathcal{A} = \{\gamma\colon [0,1] \to \mathcal{E}^{[0,\pi]}$ *is continuous* $\gamma(0) = u_+, \gamma(1) = -u_+\}$.

PROOF. *Step 1.* u_0 changes sign and the only vanishing point in $]0,\pi[$ is $\pi/2$. The restriction of u_0 either to $]0,\pi/2[$ or $]\pi/2,\pi[$ is of C^2 class and it satisfies the equation $-u_0'' = 1/|u_0|^{\alpha+1}\operatorname{sign} u_0$.

By Lemmas 5.2 and 5.3 and by the existence of a subsequence of u_ε convergent to u_0 in C^0-sense (see Theorem 4.9), we get that the only vanishing point of u_0 in $]0,\pi[$ is $\pi/2$. Hence for any $\varphi\in C_0^\infty(]0,\pi/2[)$ we get

$$\int_0^{\pi/2} u_\varepsilon'\varphi' = \int_0^{\pi/2} \frac{1}{(\varepsilon+u_\varepsilon)^{\alpha+1}}\,\varphi.$$

When $\varepsilon\to 0$, by the existence of a subsequence of u_ε convergent to u_0 in C^0-sense and in $H_0^{1,2}(\Omega)$ we get

$$\int_0^{\pi/2} u_0'\varphi' = \int_0^{\pi/2} \frac{1}{u_0^{\alpha+1}}\varphi, \quad \text{for all } \varphi\in H_0^{1,2}(\Omega).$$

Hence u_0 is a weak solution of $-u_0'' = 1/u_0^{\alpha+1}$ in the interval $[\delta,\pi/2-\delta]$ for all $\delta>0$. Thus, by a regularity argument we have that u_0 is of class C^2 in $]0,\pi/2[$. Hence, the claim.

Step 2. The function u_0 is the maximum point of the functional $E^{[0,\pi]}$ restricted to the path $\widetilde{\gamma}$, where $\widetilde{\gamma}(t)$ represents a function made by gluing together the positive minimum point of $E^{[0,\pi/2(1+t)]}$ with the negative minimum point of $E^{[\pi(1+t)/2,\pi]}$.

Indeed if we consider

$$u_{0,t} = \begin{cases} (1+t)^{2/(\alpha+2)}u_0\left(\dfrac{x}{1+t}\right) & 0\le x\le (1+t)\dfrac{\pi}{2}, \\ -(1-t)^{2/(\alpha+2)}u_0\left(\dfrac{\pi}{2}+\dfrac{x-(1+t)\frac{\pi}{2}}{1-t}\right) & (1+t)\dfrac{\pi}{2}\le x\le\pi. \end{cases}$$

we obtain that $\widetilde{\gamma}(t) = u_{0,t}$ and

$$E^{[0,\pi]}(u_{0,t}) = [(1+t)^{(2-\alpha)/(2+\alpha)} + (1-t)^{(2-\alpha)/(2+\alpha)}]E^{[0,\pi/2]}(u_0).$$

Then, 0 is a maximum point for the map $\{t\mapsto E^{[0,\pi]}(u_{0,t})\}$.

Step 3. $E^{[0,\pi]}(u_0) = \inf_{\gamma\in\mathcal{A}}\max_\gamma E^{[0,\pi]}$.

If $L = \inf_{\gamma\in\mathcal{A}} \max_\gamma E \lneqq E(u_0)$, then there exists $\widehat{\gamma} \in \mathcal{A}$ such that $\max_{\widehat{\gamma}} E < E(u_0)$. Now if we consider the path $\widehat{\gamma}_\varepsilon = [u_+^\varepsilon, u_+] \cup \widehat{\gamma} \cup [u_-, u_-^\varepsilon]$. By the convexity of E_ε on $[u_+^\varepsilon, u_+]$ and $[u_-, u_-^\varepsilon]$ we get

$$E_\varepsilon(u_\varepsilon) \leq \max_{\widehat{\gamma}_\varepsilon} E_\varepsilon = \max_{\widehat{\gamma}} E_\varepsilon \leq \max_{\widehat{\gamma}} E = E(u_0).$$

Hence $\sup_\varepsilon E_\varepsilon(u_\varepsilon) < E(u_0)$. Arguing as in the Step 3 of Theorem 4.9, by the fact that $\max_{\widetilde{\gamma}} E = E(u_0)$, we have

$$E(u_0) \leq \sup_\varepsilon E_\varepsilon(u_\varepsilon) \leq \max_{\widetilde{\gamma}} E = E(u_0).$$

And this is a contradiction. □

6. Saddle points of E_ε in the onedimensional case

If we divide the interval $[0,\pi]$ is equal parts, I_i, we prove that the function, made by gluing together the minimum points of $E_\varepsilon^{I_i}$, with alternate sign, is a saddle point of $E_\varepsilon^{[0,\pi]}$.

Definition 6.1. Let $I_i = [(i-1)\pi/(n+1), i\pi/(n+1)]$, $i = 1, \ldots, n+1$, $n \in \mathbb{N}$, be the equal subintervals of $[0,\pi]$. We define the functions $u_\varepsilon^{(n)}$ such that

$$u_\varepsilon^{(n)}|_{I_i} := (-1)^{(i+1)} u_+^{\varepsilon,i} \quad \text{for all } n \in \mathbb{N}$$

where $u_+^{\varepsilon,i}$ is the positive minimum point of $E_\varepsilon^{I_i}$.

To simplify the notation we consider the case $n = 2$.

Remark 6.2. By (2.7) we can verify that $u_\varepsilon^{(2)}$ is a weak critical point of E_ε. By the following inequality we get that $\|u_\varepsilon^{(2)}\|$ is bounded:

$$E_\varepsilon^{[0,\pi]}(u_\varepsilon^{(2)}) \leq \sum_{i=1}^{2} E^{I_i}(u_+^{I_i}). \tag{6.1}$$

Now using Definition 6.1 and Remark 2.13 we get that $u_\varepsilon^{(2)}$ converges to $u^{(2)}$ weakly in $H_0^{1,2}([0,\pi])$ as $\varepsilon \to 0$, and $u^{(2)}|_{I_i} = (-1)^{i+1} u_+^{I_i}$.

At this point we define $\widetilde{\Gamma}_\varepsilon(t,\tau)$ as in (5.13)

$$\widetilde{\Gamma}_\varepsilon(t,\tau) = \begin{cases} (1+t)^{2/(\alpha+2)} u_\varepsilon^{(2)}\left(\dfrac{x}{1+t}\right) & 0 \leq x \leq (1+t)B, \\ -(1-t+\tau)^{2/(\alpha+2)} u_\varepsilon^{(2)}\left(\dfrac{(2+\tau)B - x}{1-t+\tau}\right), & \\ & (1+t)B \leq x \leq (2+\tau)B, \\ (1-\tau)^{2/(\alpha+2)} u_\varepsilon^{(2)}\left(\dfrac{3B - x}{1-\tau}\right), & (2+\tau)B \leq x \leq 3B, \end{cases} \tag{6.2}$$

where $u_\varepsilon^{(2)}$ takes the place of u_ε. Here $B = \pi/3$. Since $u_\varepsilon^{(2)} \in C^1([0,\pi]) \cap H^{2,2}([0,\pi])$ and $u_\varepsilon^{(2)}$ is a weak critical point of E_ε, we get that $\widetilde{\Gamma}_\varepsilon\colon [-1,1] \times [-1,1] \to H_0^{1,2}([0,\pi])$ is of C^1 class. Hence the following functions

$$v_1^\varepsilon := \lim_{t\to 0} \frac{\widetilde{\Gamma}_\varepsilon(t,0) - u_\varepsilon^{(2)}}{t}, \quad v_2^\varepsilon := \lim_{\tau\to 0} \frac{\widetilde{\Gamma}_\varepsilon(0,\tau) - u_\varepsilon^{(2)}}{\tau},$$

are well defined and we get

$$v_1^\varepsilon = \begin{cases} \dfrac{2}{2+\alpha} u_\varepsilon^{(2)}(x) - x(u_\varepsilon^{(2)})' & 0 \le x \le \dfrac{\pi}{3}, \\ -\dfrac{2}{2+\alpha} u_\varepsilon^{(2)}\left(\dfrac{2\pi}{3} - x\right) & \\ \qquad -\left(\dfrac{2\pi}{3} - x\right)(u_\varepsilon^{(2)})'\left(\dfrac{2\pi}{3} - x\right) & \dfrac{\pi}{3} \le x \le \dfrac{2\pi}{3}, \\ 0 & \dfrac{2\pi}{3} \le x \le \pi, \end{cases} \tag{6.3}$$

$$v_2^\varepsilon = \begin{cases} 0 & 0 \le x \le \dfrac{\pi}{3}, \\ -\dfrac{2}{2+\alpha} u_\varepsilon^{(2)}\left(\dfrac{2\pi}{3} - x\right) & \\ \qquad -\left(x - \dfrac{\pi}{3}\right)(u_\varepsilon^{(2)})'\left(\dfrac{2\pi}{3} - x\right) & \dfrac{\pi}{3} \le x \le \dfrac{2\pi}{3}, \\ -\dfrac{2}{2+\alpha} u_\varepsilon^{(2)}(\pi - x) + (\pi - x)(u_\varepsilon^{(2)})'(\pi - x) & \dfrac{2\pi}{3} \le x \le \pi. \end{cases} \tag{6.4}$$

Let us consider the subspace V^ε of $H_0^{1,2}([0,\pi])$ spanned by v_1^ε and v_2^ε. Then we have that $H_0^{1,2}([0,\pi]) = V^\varepsilon \oplus W$, where

$$W = \{w \in H_0^{1,2}([0,\pi]) : w(\pi/3) = w(2\pi/3) = 0\}. \tag{6.5}$$

Indeed for $u \in H_0^{1,2}([0,\pi])$ we have $u = c_1 v_1^\varepsilon + c_2 v_2^\varepsilon + w$ where $w \in W$ and

$$c_1 = \frac{u(\pi/3)}{v_1(\pi/3)}, \quad c_2 = \frac{u(2\pi/3)}{v_2(2\pi/3)}.$$

Lemma 6.3. *The function $u_\varepsilon^{(2)}$ is the unique 2-saddle point of the functional E_ε, i.e.*

$$E_\varepsilon(u_\varepsilon^{(2)}) = \inf_{\phi\in\mathcal{A}_\varepsilon} \sup_{|t|^2+|\tau|^2\le\rho^2} E_\varepsilon(\phi(\widetilde{\Gamma}_\varepsilon(t,\tau))) \tag{6.6}$$

for some $\rho > 0$, where

$$\mathcal{A}_\varepsilon = \{\phi\colon \widetilde{\Gamma}_\varepsilon(B_\rho(0)) \to H_0^{1,2}([0,\pi]) \mid \phi \text{ continuous}, \ \phi|_{\widetilde{\Gamma}_\varepsilon(\partial B_\rho(0))} = \mathrm{id}\}.$$

Here $B_\rho(0) = \{(t,\tau) \in \mathbb{R}\times\mathbb{R} : |t|^2 + |\tau|^2 \le \rho^2\}$.

Proof. By Definition 6.1 and by (6.5) we have

$$E_\varepsilon(u_\varepsilon^{(2)} + w) \ge E_\varepsilon(u_\varepsilon^{(2)}) \quad \text{for all } w \in W. \tag{6.7}$$

By formulas (6.2), (6.3) and (6.4) we get

$$\Gamma(t,\tau)=u_\varepsilon^{(2)}+tv_1+\tau v_2+o(t,\tau). \tag{6.8}$$

Analogously as in the proof of Lemma 5.3 we have that $u_\varepsilon^{(2)}$ is the unique maximum point of E_ε on $\widetilde{\Gamma}_\varepsilon(B_\rho(0))$ for ρ small enough. By a version of the Saddle Point Theorem for locally Lipschitz functionals we get that $u_\varepsilon^{(2)}$ is a saddle point for E_ε satisfying (6.6).

At this point we prove that $u_\varepsilon^{(2)}$ is the unique two-saddle point of E_ε, i.e. it is the unique saddle point of E_ε satisfying (6.6).

If w_ε is a saddle point satisfying (6.6), then it is a weak critical point for E_ε, hence by Lemma 5.1 and Remark 4.11 we have that the vanishing point of w_ε divide the interval $[0,\pi]$ in a finite number ν_ε of equal parts I_i, and

$$w_\varepsilon|_{I_i}=(-1)^{i+1}u_+^{\varepsilon,i}$$

where $u_+^{\varepsilon,i}$ is the positive minimum point of $E_\varepsilon^{I_i}$. If we argue as in Lemma 5.2 and 5.3 we can verify that the number of the vanishing points of w_ε is exactly 2. We use respectively for $\nu_\varepsilon\ge 4$ the function

$$\widetilde{\Gamma}(t,\tau,s)(x)=\begin{cases}(1+t)^{2/(2+\alpha)}u_\varepsilon^{(2)}\left(\dfrac{x}{1+t}\right) & 0\le x\le(1+t)B,\\ -(1-t+\tau)^{2/(2+\alpha)}u_\varepsilon^{(2)}\left(\dfrac{(2+\tau)B-x}{1-t+\tau}\right) & (1+t)B\le x\le(2+\tau)B,\\ (1-\tau)^{2/(2+\alpha)}u_\varepsilon^{(2)}\left(\dfrac{3B-x}{1-\tau}\right) & (2+\tau)B\le x\le 3B,\\ -(1+s)^{2/(2+\alpha)}u_\varepsilon^{(2)}\left(\dfrac{x-3B}{1+s}\right) & 3B\le x\le(4+s)B,\\ (1-s)^{2/(2+\alpha)}u_\varepsilon^{(2)}\left(\dfrac{5B-x}{1-s}\right) & (4+s)B\le x\le 5B,\\ u_\varepsilon(x) & x\ge 5B,\end{cases}$$

and for $\nu_\varepsilon=3$ the function

$$\widetilde{\Gamma}(t,\tau,s)(x)=\begin{cases}(1+t)^{2/(2+\alpha)}u_\varepsilon^{(2)}\left(\dfrac{x}{1+t}\right) & 0\le x\le(1+t)B,\\ -(1-t+\tau)^{2/(2+\alpha)}u_\varepsilon^{(2)}\left(\dfrac{(2+\tau)B-x}{1-t+\tau}\right) & (1+t)B\le x\le(2+\tau)B,\\ (1-\tau+s)^{2/(2+\alpha)}u_\varepsilon^{(2)}\left(\dfrac{(3+s)B-x}{1-\tau+s}\right) & (2+\tau)B\le x\le(3+s)B,\\ -(1-s)^{2/(2+\alpha)}u_\varepsilon^{(2)}\left(\dfrac{4B-x}{1-s}\right) & (3+s)B\le x\le 4B,\end{cases}$$

with $B = \pi/\nu_\varepsilon$. So the number of vanishing point of w_ε is 2, hence $w_\varepsilon = u_\varepsilon^{(2)}$.□

Now we get a property which characterizes the solutions of (1.1) found in [15] which are made by gluing together the minimum point of the functionals E^{I_i} where $I_i = [(i-1)\pi/(n+1), i\pi/(n+1)]$.

Theorem 6.4. *The function* $u_0^{(2)} \in H_0^{1,2}([0,\pi])$*, such that* $u^{(2)}|_{I_i} = u_+^{I_i}$*, with* $I_i = [(i-1)\pi/3, i\pi/3]$*,* $i = 1,2,3$*, can be characterized as the weak limit in* $H_0^{1,2}([0,\pi])$*, as* ε *tends to zero, of* $u_\varepsilon^{(2)}$*, which is the unique* 2*-saddle point of* E_ε*. Moreover,*

$$E^{[0,\pi]}(u_0^{(2)}) = \inf_{\phi\in\mathcal{A}_0} \max_{|t|^2+|\tau|^2\le\rho^2} E^{[0,\pi]}(\phi(\Gamma_0(t,\tau))) \tag{6.9}$$

for some $\rho > 0$*, where*

$$\mathcal{A}_0 = \{\phi\colon \Gamma_0(B_\rho(0)) \to \mathcal{E}^{[0,\pi]} \mid \phi \text{ continuous}, \ \phi|_{\Gamma_0(\partial B_\rho(0))} = \mathrm{id}\}.$$

Proof. By Remark 6.2 and Lemma 6.3 we get the first claim. Now we prove (6.9). Firstly we define

$$\Gamma_0(t,\tau) = \begin{cases} (1+t)^{2/(\alpha+2)}\, u_0^{(2)}\left(\dfrac{x}{1+t}\right) & 0 \le x \le (1+t)B, \\ -(1-t+\tau)^{2/(\alpha+2)} u_0^{(2)}\left(\dfrac{(2+\tau)B - x}{1-t+\tau}\right) & \\ & (1+t)B \le x \le (2+\tau)B, \\ (1-\tau)^{2/(\alpha+2)} u_0^{(2)}\left(\dfrac{3B-x}{1-\tau}\right) & (2+\tau)B \le x \le 3B, \end{cases} \tag{6.10}$$

with $|t| \le 1$, $|\tau| \le 1$ and $B = \pi/3$. We get

$$\begin{aligned} E^{[0,\pi]}(\Gamma_0(t,\tau)) = [(1+t)^{(2-\alpha)/(2+\alpha)} + (1-t+\tau)^{(2-\alpha)/(2+\alpha)} \\ + (1-t)^{(2-\alpha)/(2+\alpha)}] E^{[0,\pi/3]}(u_0^{(2)}). \end{aligned}$$

Then $(0,0)$ is the unique maximum point for the functional

$$(t,\tau) \mapsto E^{[0,\pi]}(\Gamma_0(t,\tau)) \quad \text{with } |t| \le 1 \text{ and } |\tau| \le 1.$$

So $\max_{|t|^2+|\tau|^2\le\rho^2} E^{[0,\pi]}(\Gamma_0(t,\tau)) = E^{[0,\pi]}(u_0^{(2)})$.

Moreover, given $\varepsilon > 0$, we show that it exists an homeomorphism between the sets $S_1\{tv_1^\varepsilon + \tau v_2^\varepsilon \in V^\varepsilon : |t|^2+|\tau|^2 \le \rho^2\}$ and $S_2 = \{\Gamma_0(t,\tau) : |t|^2+|\tau|^2 \le \rho^2\}$, for some $\rho > 0$. We set

$$P_{V^\varepsilon}(\Gamma_0(t,\tau)) := \alpha_\varepsilon(t,\tau)v_1^\varepsilon + \beta_\varepsilon(t,\tau)v_2^\varepsilon$$

where $P_{V^\varepsilon}\colon H_0^{1,2}([0,\pi]) \to V^\varepsilon$ is the projection onto V^ε. We have

$$\begin{aligned} \alpha_\varepsilon(t,\tau) &= \frac{\Gamma_0(t,\tau)(\pi/3)}{-(\pi/3)(u_0^{(2)})'(\pi/3)}, & \beta_\varepsilon(t,\tau) &= \frac{\Gamma_0(t,\tau)(2\pi/3)}{-(2\pi/3)(u_0^{(2)})'(2\pi/3)}, \\ \alpha_\varepsilon(0,0) &= 0, & \beta_\varepsilon(0,0) &= 0. \end{aligned}$$

Using (6.10) we get that the operator $(t,\tau)\mapsto(\alpha_\varepsilon(t,\tau),\beta_\varepsilon(t,\tau))$ is an homeomorphism between the sets S_1 and S_2, for t and τ such that $|t|^2+|\tau|^2\le\rho^2$, for some $\rho>0$. By Definition 6.1 and by the definition of the subspace W (see (6.5)) we have

$$E^{[0,\pi]}(u_0^{(2)}+w)=\sum_{i=1}^{3}E^{I_i}(u_0^{(2)}+w)\ge E^{[0,\pi]}(u_0^{(2)}).$$

By a well-known argument of the topological degree we have that

$$\phi(\Gamma_0(B_\rho(0)))\cap W\neq\emptyset$$

for any $\phi\colon\Gamma_0(B_\rho(0))\to H_0^{1,2}([0,\pi])$ continuous with $\phi|_{\Gamma_0(\partial B_\rho(0))}=\mathrm{id}$. Then

$$\max_{|t|^2+|\tau|^2\le\rho^2}E^{[0,\pi]}(\phi(\Gamma_0(t,\tau)))\ge E^{[0,\pi]}(u_0^{(2)}).$$

By the fact that $\max_{|t|^2+|\tau|^2\le\rho^2}E^{[0,\pi]}(\Gamma_0(t,\tau))=E^{[0,\pi]}(u_0^{(2)})$ we get the claim.□

Remark 6.5. For $u_\varepsilon^{(n)}$ with $n>2$, the generalization of Lemma 6.3 and Theorem 6.4 are straightforward. So we can characterize the saddle points of $E_\varepsilon^{[0,\pi]}$ by their nodal set. For n-saddle point of $E_\varepsilon^{[0,\pi]}$ we mean a saddle point of $E_\varepsilon^{[0,\pi]}$ with respect to the decomposition of $H_0^{1,2}([0,\pi])$ of the type: $H_0^{1,2}([0,\pi])=\widehat V\oplus\widehat W$, with $\dim\widehat V=n$.

Theorem 6.6. *The function* $u_0^{(n)}\in H_0^{1,2}([0,\pi])$, *such that* $u_0^{(n)}|_{I_i}=u_+^{I_i}$, *with* $I_i=[(i-1)\pi/(n+1),i\pi/(n+1)]$, $i=1,\dots,n+1$, *can be characterized as the weak limit in* $H_0^{1,2}([0,\pi])$, *as* ε *tends to zero, of* $u_\varepsilon^{(n)}$ *which is the unique* n*-saddle point of* E_ε. *Moreover,*

$$E^{[0,\pi]}(u_0^{(n)})=\inf_{\phi\in\mathcal A_0}\max_{\sum_{i=1}^n|t_i|^2\le\rho}E^{[0,\pi]}(\phi(\Gamma_0(t_1,\dots,t_n))),$$

where $\mathcal A_0=\{\phi\colon\Gamma_0(B_\rho(0))\to\mathcal E^{[0,\pi]}\mid\phi\text{ continuous},\ \phi|_{\Gamma_0(\partial B_\rho(0))}=\mathrm{id}\}$. *Here* $B_\rho(0)=\{\mathbf t:=t_1,\dots,t_n:\sum_{i=1}^n|t_i|^2\le\rho^2\}$ *and*

$$\Gamma_0(\mathbf t)=\begin{cases}(-1)^2(1+t_1)^{2/(\alpha+2)}u_0^{(n)}\left(\dfrac{x}{1+t_1}\right) & 0\le x\le(1+t_1)B,\\ (-1)^3(1-t_1+t_2)^{2/(\alpha+2)}u_0^{(n)}\left(\dfrac{(2+t_2)B-x}{1-t_1+t_2}\right) & \\ & (1+t_1)B\le x\le(2+t_2)B,\\ (-1)^4(1-t_2+t_3)^{2/(\alpha+2)}u_0^{(n)}\left(\dfrac{(3+t_3)B-x}{1-t_2+t_3}\right) & \\ & (2+t_2)B\le x\le(3+t_3)B,\\ \dots\dots\dots\dots\dots\dots & \\ (-1)^{n+2}(1-t_n)^{2/(\alpha+2)}u_0^{(n)}\left(\dfrac{(n+1)B-x}{1-t_n}\right) & \\ & (n+t_n)B\le x\le(n+1)B.\end{cases}$$

Here $B = \pi/(n+1)$.

Remark 6.7. Using the definition of McKenna and Reichel introduced in [15], if we denote by $\mathcal{Z} = \{\pi/(n+1), 2\pi/(n+1), \dots, n\pi/(n+1)\}$, we have

$$\frac{d^2}{dt^2} u_0^{(n)}(t) + PV_{\mathcal{Z}} (u_0^{(n)})^{-(\alpha+1)}(t) = 0,$$
$$u_0^{(n)}(i\pi/(n+1)) = 0, \quad i = 1, \dots, n+1,$$

where $PV_{\mathcal{Z}}$ stands for the principal value centered at $\pi/(n+1), 2\pi/(n+1), \dots,$ $n\pi/(n+1)$, i.e.

$$\langle PV_{\mathcal{Z}} \varphi, \psi \rangle = \lim_{\rho \to 0} \int_0^{\pi/(n+1)-\rho} + \int_{\pi/(n+1)+\rho}^{2\pi/(n+1)-\rho} + \dots + \int_{n\pi/(n+1)+\rho}^{\pi} \varphi(t)\psi(t)\, dt$$

for all $\psi \in C_0^\infty([0,\pi])$.

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