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NONTRIVIAL SOLUTIONS FOR SUPERQUADRATIC NONAUTONOMOUS PERIODIC SYSTEMS

Shouchuan Hu — Nikolas S. Papageorgiou

Abstract. We consider a nonautonomous second order periodic system with an indefinite linear part. We assume that the potential function is superquadratic, but it may not satisfy the Ambrosetti–Rabinowitz condition. Using an existence result for C^1-functionals having a local linking at the origin, we show that the system has at least one nontrivial solution.

1. Introduction

We consider the following second order periodic system

$$\begin{cases} -x''(t) - A(t)x(t) = \nabla F(t, x(t)) & \text{a.e. on } T = [0, b], \\ x(0) = x(b), \quad x'(0) = x'(b). \end{cases} \tag{1.1}$$

Here $A: T \to \mathbb{R}^{N \times N}$ is a continuous map such that for every $t \in T$, $A(t)$ is a symmetric $N \times N$ matrix; $F: T \times \mathbb{R}^N \to \mathbb{R}$ is measurable, it is C^1 in the $x \in \mathbb{R}^N$ variable and $\nabla F(t, x)$ denotes the gradient of the map $x \to F(t, x)$.

Problem (1.1) was studied by Rabinowitz [14] under the assumption that for each $t \in T$, the $N \times N$ matrix $A(t)$ is negative definite and the potential function $x \to F(t, x) = F(x)$ is strictly superquadratic. More precisely, it satisfies the

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so-called Ambrosetti–Rabinowitz condition (AR-condition), namely, there exist $\vartheta > 2$ and $M > 0$ such that

$$0 < \vartheta F(x) \leq (\nabla F(x), x)_{\mathbb{R}^N} \quad \text{for all } \|x\| \geq M. \tag{1.2}$$

The approach of Rabinowitz is variational, based on purely minimax methods. Since then, condition (1.2) has been used extensively in the study of superquadratic periodic systems. We mention the works of Ekeland–Ghoussoub [3], Girardi–Matzeu [7], Li–Willem [8], Long [9], Xu [16] and the references therein. An existence theorem for superquadratic periodic systems with a nonsmooth potential was proved by Motreanu–Motreanu–Papageorgiou [12], who employed a nonsmooth analog of condition (1.2). In all the aforementioned works, the approach is variational, based on the critical point theory. Superquadratic systems with a convex potential function were studied by Ekeland [2], Mawhin [10] and Mawhin–Willem [11], using the dual action principle. Recently, Fei [5] considered superquadratic Hamiltonian systems, without assuming the AR-condition and with a Hamiltonian function $H(t,x)$ which is nonnegative and C^1 on $T\times\mathbb{R}^N$ and proved an existence result using the linking theorem. Related are also the works of Faraci–Kristaly [4] and Motreanu–Motreanu–Papageorgiou [13].

In this paper we consider system (1.1), when the Carathéodory potential function F exhibits a superquadratic growth near infinity and near zero, but may not satisfy the AR-condition. Also, we do not impose any global sign condition on either F or A. Under these general conditions and using a slight variant of a result of Li–Willem [8], we are able to show that problem (1.1) admits at least one nontrivial solution.

In the next section, we present the mathematical tools to be used in this work. In Section 3, we have all the auxiliary results leading to the existence theorem.

2. Mathematical background

The hypotheses on the matrix valued function $t \to A(t)$, are the following:

(A) $A: T \to \mathbb{R}^{N\times N}$ is continuous such that $A(t)$ is symmetric for every $t \in T$.

In the analysis of problem (1.1), we will use the Sobolev space

$$W^{1,2}_{\mathrm{per}}((0,b),\mathbb{R}^N) = \{x \in W^{1,2}((0,b),\mathbb{R}^N) : x(0) = x(b)\}.$$

Recall that $W^{1,2}((0,b),\mathbb{R}^N)$ is compactly embedded in $C(T,\mathbb{R}^N)$. Hence, in the above definition, the evaluation of $W^{1,2}((0,b),\mathbb{R}^N)$ functions at $t=0$ and $t=b$ makes sense. We consider the Nemytskiĭ operator $\widehat{A} \in \mathcal{L}(C(T,\mathbb{R}^N), C(T,\mathbb{R}^N))$ corresponding to the map $t \to A(t)$, namely,

$$(\widehat{A}x)(t) = A(t)x(t).$$

From Mawhin–Willem [11, p. 89] and Showalter [15, p. 78], applying the spectral theorem for compact self-adjoint operators on a Hilbert space on the differential operator $x \to -x'' - \widehat{A}x$, we infer that there exists a sequence of eigenfunctions for the operator, which constitute an orthonormal basis for $L^2(T, \mathbb{R}^N)$ and an orthogonal basis for $W^{1,2}_{\mathrm{per}}((0,b), \mathbb{R}^N)$. Hence, we have the following orthogonal direct sum decomposition (spectral resolution):

$$W^{1,2}_{\mathrm{per}}((0,b), \mathbb{R}^N) = H_- \oplus H_0 \oplus H_+,$$

where

$$\begin{aligned} H_- &= \operatorname{span}\{x \in W^{1,2}_{\mathrm{per}}((0,b), \mathbb{R}^N) : -x'' - \widehat{A}x = \lambda x \text{ for some } \lambda < 0\}, \\ H_0 &= \ker(-x'' - \widehat{A}x), \\ H_+ &= \operatorname{span}\{x \in W^{1,2}_{\mathrm{per}}((0,b), \mathbb{R}^N) : -x'' - \widehat{A}x = \lambda x \text{ for some } \lambda > 0\}. \end{aligned}$$

Note that both H_- and H_0 are finite dimensional spaces.

In what follows, $\|\cdot\|_2$ denotes the norm of $L^2(T, \mathbb{R}^N)$, and $\|\cdot\|$ denotes both the norm of $\mathbb{R}^N$ and the norm of the Sobolev space $W^{1,2}_{\mathrm{per}}((0,b), \mathbb{R}^N)$. It will be clear from the context which one is used at each time.

The next lemma can be found in [12] and gives useful information about the component spaces H_- and H_+.

LEMMA 2.1.

(a) *There exists $\xi_0 > 0$ such that*

$$\|x'\|_2^2 - \int_0^b (A(t)x(t), x(t))_{\mathbb{R}^N}\, dt \geq \xi_0 \|x\|^2 \quad \text{for all } x \in H_+.$$

(b) *There exists $\xi_1 > 0$ such that*

$$\|x'\|_2^2 - \int_0^b (A(t)x(t), x(t))_{\mathbb{R}^N}\, dt \leq -\xi_1 \|x\|^2 \quad \text{for all } x \in H_-.$$

Now, let X be a Banach space that admits a direct sum decomposition $X = Y \oplus V$ and $\varphi \in C^1(X)$. We say that φ has a local linking at the origin (with respect to (Y, V)), if there exists $r > 0$ such that

$$\begin{cases} \varphi(y) \leq 0 & \text{for all } y \in Y \text{ with } \|y\| \leq r, \\ \varphi(v) \geq 0 & \text{for all } v \in V \text{ with } \|v\| \leq r. \end{cases}$$

Evidently, the origin is a critical point of φ, if φ has a local linking at 0.

Recall that $\varphi \in C^1(X)$ satisfies the "C-condition", if every sequence $\{x_n\} \subseteq X$ such that for all $n \geq 1$ and some $M_1 > 0$,

$$|\varphi(x_n)| \leq M_1 \quad \text{and} \quad (1 + \|x_n\|)\varphi'(x_n) \to 0 \quad \text{as } n \to \infty,$$

contains a strongly convergent subsequence.

The next result is essentially due to Li–Willem [8]. In their formulation, they use a graded version of the well-known PS-condition. Recalling that the deformation lemma is also valid if the functional satisfies the C-condition (see Bartolo–Benci–Fortunato [1] and Gasinski–Papageorgiou [6]), we can state the following slight variant of Theorem 2 in Li–Willem [8].

PROPOSITION 2.2. *If X is a Banach space with a direct decomposition $X = Y \oplus V$, $\dim Y < \infty$ and $\varphi \in C^1(X)$ satisfies the following conditions:*

(a) *φ has a local linking at 0 with respect to (Y, V);*
(b) *φ satisfies the C-condition;*
(c) *φ maps bounded sets into bounded sets;*
(d) *for every $E \subseteq V$ finite dimensional subspace, we have $\varphi(u) \to -\infty$ as $\|u\| \to \infty$, $u \in Y \oplus E$*

then φ has at least one nontrivial critical point.

3. Existence of nontrivial solutions

In this section we shall establish the existence of at least one nontrivial solution for problem (1.1). To do this, we employ hypothesis (A). Note that this hypothesis does not impose any sign condition on the matrix-valued map $t \to A(t)$, which means that in the spectral resolution of the linear differential operator $x \to -x'' - \widehat{A}x$, the negative, zero and positive parts, can all be non-trivial (i.e. $H_- \neq 0, H_0 \neq 0$ and $H_+ \neq 0$). So, our problem may have indefinite linear part. This is in sharp contrast to the work of Rabinowitz [14], where $H_- = H_0 = \{0\}$.

The hypotheses on the potential function F are given below, where

$$L^1(T)_+ = \{y \in L^1(T) : y(t) \geq 0 \quad \text{a.e. on } T\}.$$

(F) $F: T \times \mathbb{R}^N \to \mathbb{R}$ is a function such that:
(a) for all $x \in \mathbb{R}^N$, $t \to F(t,x)$ is measurable;
(b) for almost every $t \in T$, $x \to F(t,x)$ is continuously differentiable;
(c) for almost all $t \in T$ and all $x \in \mathbb{R}^N$ we have

$$\|\nabla F(t,x)\| \leq a(t) + c\|x\|^r,$$

where $a \in L^1(T)_+, c > 0$ and $1 < r < \infty$;
(d) there exists η, with either $\eta \geq 2$ or $\eta > r - 1$, such that

$$\liminf_{\|x\| \to \infty} \frac{(\nabla F(t,x), x)_{\mathbb{R}^N} - 2F(t,x)}{\|x\|^\eta} > 0 \quad \text{and} \quad \lim_{\|x\| \to \infty} \frac{F(t,x)}{\|x\|^2} = \infty$$

uniformly for almost every $t \in T$;

(e) $\lim_{x\to 0} F(t,x)/\|x\|^2 = 0$ uniformly for almost every $t \in T$ and if $\dim H_0 \neq 0$, then there exists $\delta > 0$ such that

$$\begin{aligned} &F(t,x) \le 0 \quad \text{for a.e. } t \in T \text{ and } \|x\| \le \delta, \\ \text{or}\quad &F(t,x) \ge 0 \quad \text{for a.e. } t \in T \text{ and } \|x\| \le \delta. \end{aligned}$$

REMARK 3.1. Hypotheses (F)(d) and (e) imply that the potential function $x \to F(t,x)$ is superquadratic both near infinity and near zero. Note that we do not require that $F \ge 0$ as in Fei [5]. The following functions satisfy hypotheses (F), but fail to satisfy the AR-condition (see (1.2)). For the sake of simplicity, we drop the t-variable. With $\alpha > 0$, consider

$$F_1(x) = \frac{1}{2}\|x\|^2 \ln(1+\|x\|^\alpha)$$

and with some appropriate choice of $\xi\colon \mathbb{R}_+ \to \mathbb{R}$, we may also consider

$$F_2(x) = \begin{cases} \xi(\|x\|) & \text{if } \|x\| \le 1, \\ \|x\|^2 \ln\|x\| + \xi(1) & \text{if } \|x\| > 1. \end{cases}$$

Let $\varphi\colon W^{1,2}_{\mathrm{per}}((0,b),\mathbb{R}^N) \to \mathbb{R}$ be the Euler functional for problem (1.1), defined by

$$\varphi(x) = \frac{1}{2}\|x'\|_2^2 - \frac{1}{2}\int_0^b (A(t)x(t),x(t))_{\mathbb{R}^N}\,dt - \int_0^b F(t,x(t))\,dt$$

for all $x \in W^{1,2}_{\mathrm{per}}((0,b),\mathbb{R}^N)$. Evidently, $\varphi \in C^1(W^{1,2}_{\mathrm{per}}((0,b),\mathbb{R}^N),\mathbb{R})$ and we have

$$\varphi'(x) = V(x) - \widehat{A}(x) - N(x), \tag{3.1}$$

where $V \in \mathcal{L}(W^{1,2}_{\mathrm{per}}((0,b),\mathbb{R}^N), W^{1,2}_{\mathrm{per}}((0,b),\mathbb{R}^N)^*)$ is defined by

$$\langle V(x),y\rangle = \int_0^b (x'(t),y'(t))_{\mathbb{R}^N}\,dt, \tag{3.2}$$

for all $x,y \in W^{1,2}_{\mathrm{per}}((0,b),\mathbb{R}^N)$. Here and in what follows, $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the duality brackets for the pair $(W^{1,2}_{\mathrm{per}}((0,b),\mathbb{R}^N), W^{1,2}_{\mathrm{per}}((0,b),\mathbb{R}^N)^*)$. Also,

$$N(x)(\,\cdot\,) = \nabla F(\,\cdot\,,x(\,\cdot\,)) \quad \text{for all } x \in W^{1,2}_{\mathrm{per}}((0,b),\mathbb{R}^N).$$

By virtue of hypothesis (F)(c), we have

$$N(x) \in L^1(T,\mathbb{R}) \quad \text{for all } x \in W^{1,2}_{\mathrm{per}}((0,b),\mathbb{R}^N).$$

PROPOSITION 3.2. *If hypotheses* (A) *and* (F) *hold, then* φ *satisfies the* C-*condition.*

PROOF. Let $\{x_n\} \subseteq W^{1,2}_{\mathrm{per}}((0,b),\mathbb{R}^N)$ be a sequence such that for some $M_1 > 0$ and all $n \geq 1$,

$$|\varphi(x_n)| \leq M_1 \quad \text{and} \quad (1+\|x_n\|)\varphi'(x_n) \to 0 \quad \text{as } n \to \infty. \tag{3.3}$$

By virtue of hypothesis (F)(d), we can find $\beta, M_2 > 0$ such that

$$(\nabla F(t,x),x)_{\mathbb{R}^N} - 2F(t,x) \geq \beta\|x\|^\eta \tag{3.4}$$

for a.e. $t \in T$ and all $\|x\| \geq M_2$. On the other hand, (F)(c) implies that

$$|(\nabla F(t,x),x)_{\mathbb{R}^N} - 2F(t,x)| \leq a_1(t) \tag{3.5}$$

for almost every $t \in T$ and all $\|x\| \leq M_2$, with $a_1 \in L^1(T)_+$. From (3.4) and (3.5), it follows that

$$(\nabla F(t,x),x)_{\mathbb{R}^N} - 2F(t,x) \geq \beta\|x\|^\eta - a_2(t)$$

for almost every $t \in T$ and all $x \in \mathbb{R}$, with $a_2(t) = a_1(t) + \beta M_2^\eta$.

From (3.3), we have

$$|\langle \varphi'(x_n), u\rangle| \leq \frac{\varepsilon_n}{(1+\|x_n\|)}\|u\| \tag{3.6}$$

for all $u \in W^{1,2}_{\mathrm{per}}((0,b),\mathbb{R}^N)$, with $\varepsilon_n \downarrow 0$.

Let $u = x_n \in W^{1,2}_{\mathrm{per}}((0,b),\mathbb{R}^N)$, then (see (3.1) and (3.2))

$$-\|x_n'\|_2^2 + \int_0^b (A(t)x_n(t),x_n(t))_{\mathbb{R}^N}\,dt + \int_0^b (\nabla F(t,x_n(t)),x_n(t))_{\mathbb{R}^N}\,dt \leq \varepsilon_n. \tag{3.7}$$

Also, once again from (3.3), we have for all $n \geq 1$,

$$2\varphi(x_n) = \|x_n'\|_2^2 - \int_0^b (A(t)x_n(t),x_n(t))_{\mathbb{R}^N}\,dt - \int_0^b 2F(t,x_n(t))_{\mathbb{R}^N}\,dt \leq 2M_1. \tag{3.8}$$

Adding (3.7) and (3.8), we obtain

$$\int_0^b [(\nabla F(t,x_n(t)),x_n(t))_{\mathbb{R}^N} - 2F(t,x_n(t))]\,dt \leq M_3,$$

for some $M_3 > 0$ and all $n \geq 1$. Hence, for some $M_4 > 0$ and all $n \geq 1$ we have

$$\beta\|x_n\|_\eta^\eta \leq M_4. \tag{3.9}$$

Therefore, $\{x_n\}_{n\geq 1} \subseteq L^\eta(T,\mathbb{R}^N)$ is bounded.

Recall that we can write in a unique way $x_n = \overline{x}_n + x_n^0 + \widehat{x}_n$, with $\overline{x}_n \in H_-$, $x_n^0 \in H_0$ and $\widehat{x}_n \in H_+$.

In (3.6), let $u = \widehat{x}_n$. Exploiting the orthogonality of the component spaces, we have

$$\begin{aligned}(3.10)\quad \langle \varphi'(x_n), \widehat{x}_n\rangle = \|\widehat{x}_n'\|_2^2 - \int_0^b (A(t)\widehat{x}_n(t), \widehat{x}_n(t))_{\mathbb{R}^N}\,dt \\ - \int_0^b (\nabla F(t, x_n(t)), \widehat{x}_n(t))_{\mathbb{R}^N}\,dt \le \varepsilon_n.\end{aligned}$$

From Lemma 2.1(a), we have for all $n \ge 1$

$$(3.11)\qquad \xi_0\|\widehat{x}_n\|^2 \le \|\widehat{x}_n'\|_2^2 - \int_0^b (A(t)\widehat{x}_n(t), \widehat{x}_n(t))_{\mathbb{R}^N}\,dt.$$

Also, we have, for some $c_1, c_2, c_3 > 0$,

$$\begin{aligned}\int_0^b (\nabla F(t, x_n(t)), \widehat{x}_n(t))_{\mathbb{R}^N}\,dt &\le \int_0^b \|\nabla F(t, x_n(t))\| \cdot \|\widehat{x}_n(t)\|\,dt \\ \le c_1\|\widehat{x}_n\| \int_0^b \|\nabla F(t, x_n(t))\|\,dt &\le c_1\|\widehat{x}_n\| \int_0^b (a(t) + c\|x_n(t)\|^r)\,dt \quad \text{(see (F)(c))} \\ &\le c_2\|\widehat{x}_n\| + c_2\,\|\widehat{x}_n\| \max\{\|x_n\|_\eta^r, \|x_n\|_r^r\},\end{aligned}$$

where the last inequality is based on either $\eta \ge 2$ or $\eta > r - 1$, respectively.

In the case when $\eta \ge 2$, we have

$$\int_0^b (\nabla f(t, x_n(t)), \widehat{x}_n(t))_{\mathbb{R}^N}\,dt \le c_2\|\widehat{x}_n\| + c_3\|\widehat{x}_n\|\,\|x_n\|_\eta^r.$$

Thus,

$$\xi_0\|\widehat{x}_n\|^2 \le \varepsilon_n + c_2\|\widehat{x}_n\| + c_3\|\widehat{x}_n\|\,\|x_n\|_\eta^r.$$

Hence, since $\|\widehat{x}_n\| \le \|x_n\|$,

$$\xi_0\frac{\|\widehat{x}_n\|^2}{\|x_n\|^2} \le \frac{\varepsilon_n}{\|x_n\|^2} + c_2\frac{\|\widehat{x}_n\|}{\|x_n\|^2} + c_3\frac{\|\widehat{x}_n\|}{\|x_n\|^2}\,\|x_n\|_\eta^r \le \frac{\varepsilon_n}{\|x_n\|^2} + \frac{c_2}{\|x_n\|} + \frac{c_3}{\|x_n\|}||x_n\|_\eta^r.$$

Note that the sequence $||x_n\|_\eta^r$ is bounded, due to (3.9), the following procedure showing the boundedness of the sequence $\{x_n\} \subseteq W^{1,2}_{\mathrm{per}}((0,b), \mathbb{R}^N)$ works also in the current situation. So, we are going to go through details only for the second case.

To consider the second case, we have $\eta > r-1$. Clearly we can always assume that $\eta \le r$. Using the interpolation inequality (see Gasiński–Papageorgiou [6, p. 905]), we have

$$\begin{aligned}\|x_n\|_r \le \|x_n\|_\eta^{1-t}\|x_n\|_\infty^t \quad \text{where } t \in [0,1),\ \frac{1-t}{\eta} = \frac{1}{r}, \\ \Rightarrow \|x_n\|_r^r \le \widehat{M}\,\|x_n\|^{tr} \quad \text{for some } \widehat{M} > 0,\ \text{all } n \ge 1.\end{aligned}$$

Therefore

$$(3.12)\qquad \int_0^b \big(\nabla f(t, x_n(t)\big), \widehat{x}_n(t)\big)_{\mathbb{R}^N}\,dt \le c_2\|\widehat{x}_n\| + c_3\|\widehat{x}_n\|\,\|x_n\|^{tr}.$$

Returning to (3.10) and using (3.11) and (3.12), we obtain

$$\xi_0 \|\widehat{x}_n\|^2 \le \varepsilon_n + c_2 \|\widehat{x}_n\| + c_3 \|\widehat{x}_n\| \, \|x_n\|^{tr}.$$

Hence, since $\|\widehat{x}_n\| \le \|x_n\|$,

$$\begin{aligned} \xi_0 \frac{\|\widehat{x}_n\|^2}{\|x_n\|^2} &\le \frac{\varepsilon_n}{\|x_n\|^2} + c_2 \frac{\|\widehat{x}_n\|}{\|x_n\|^2} + c_3 \frac{\|\widehat{x}_n\|}{\|x_n\|^2} \, \|x_n\|^{tr} \\ &\le \frac{\varepsilon_n}{\|x_n\|^2} + \frac{c_2}{\|x_n\|} + \frac{c_3}{\|x_n\|^{1-tr}}. \end{aligned} \tag{3.13}$$

Suppose that $\{x_n\} \subseteq W^{1,2}_{\rm per}((0,b),\mathbb{R}^N)$ is unbounded. By passing to a suitable subsequence if necessary, we may assume that $\|x_n\| \to \infty$. Note that $\eta > r-1$ is equivalent to $tr < 1$. So from (3.13) it follows that

$$\frac{\|\widehat{x}_n\|}{\|x_n\|} \to 0. \tag{3.14}$$

In a similar way, using in (3.6) $u = \overline{x}_n$ and invoking this time Lemma 2.1(b), we obtain that

$$\frac{\|\overline{x}_n\|}{\|x_n\|} \to 0. \tag{3.15}$$

Let $y_n = x_n/\|x_n\|$. Since $\|y_n\| = 1$ for all $n \ge 1$, we may assume that

$$y_n \xrightarrow{w} y \quad \text{in } W^{1,2}_{\rm per}((0,b),\mathbb{R}^N) \quad \text{and} \quad y_n \to y \quad \text{in } C(T,\mathbb{R}^N).$$

By virtue of (3.14) and (3.15), we see that $y_0 \in H_0$. Also due to (3.9), $y = 0$. Moreover, because H_0 is finite dimensional, we have

$$y_n^0 = \frac{x_n^0}{\|x_n\|} \to 0 \quad \text{in } W^{1,2}_{\rm per}((0,b),\mathbb{R}^N). \tag{3.16}$$

Combining (3.14)–(3.16), we have

$$1 = \|y_n\| \le \frac{\|\overline{x}_n\| + \|x_n^0\| + \|\widehat{x}_n\|}{\|x_n\|} \to 0 \quad \text{as } n \to \infty,$$

which is a contradiction. This proves that $\{x_n\}_{n\ge1} \subseteq W^{1,2}_{\rm per}((0,b),\mathbb{R}^N)$ is bounded and so, we may assume that

$$x_n \xrightarrow{w} x \quad \text{in } W^{1,2}_{\rm per}((0,b),\mathbb{R}^N) \quad \text{and} \quad x_n \to x \quad \text{in } C(T,\mathbb{R}^N). \tag{3.17}$$

From (3.6) with $u = x_n - x$, we have

$$\begin{aligned} \bigg| \langle V(x_n), x_n - x\rangle &- \int_0^b (A(t)x_n(t), x_n(t) - x(t))_{\mathbb{R}^N}\, dt \\ &- \int_0^b (\nabla F(t, x_n(t)), x_n(t) - x(t))_{\mathbb{R}^N}\, dt \bigg| \le \varepsilon_n. \end{aligned} \tag{3.18}$$

Due to (3.17) and the hypotheses (A) and (F)(c), we have

$$\int_0^b (A(t)x_n(t), x_n(t) - x(t))_{\mathbb{R}^N}\, dt \to 0,$$
$$\int_0^b (\nabla F(t, x_n(t)), x_n(t) - x(t))_{\mathbb{R}^N}\, dt \to 0.$$

So, by (3.18) it follows that

$$\lim_{n\to\infty} \langle V(x_n), x_n - x\rangle = 0. \tag{3.19}$$

Note that $V(x_n) \xrightarrow{w} V(x)$ in $W^{1,2}_{\mathrm{per}}((0,b),\mathbb{R}^N)^*$. Hence from (3.17) and (3.19) we obtain $\|x_n'\|_2^2 \to \|x'\|_2^2$.

We know that $x_n' \xrightarrow{w} x'$ in $L^2(T,\mathbb{R}^N)$. Therefore, by the Kadec–Klee property of Hilbert spaces, we have $x_n' \to x'$ in $L^2(T,\mathbb{R}^N)$, which implies that $x_n \to x$ in $W^{1,2}_{\mathrm{per}}((0,b),\mathbb{R}^N)$. $\square$

Proposition 3.3. *If* (A) *and* (F) *hold, then* φ *has a local linking at* 0.

Proof. First, we assume that $\dim H_0 \neq 0$ and the first option in (F)(e) holds, namely, $F(t,x) \le 0$ for almost every $t \in T$ and all $\|x\| \le \delta$.

We consider the orthogonal direct sum decomposition:

$$W^{1,2}_{\mathrm{per}}((0,b),\mathbb{R}^N) = H_- \oplus V, \quad \text{with } V = H_0 \oplus H_+.$$

By (F)(e), given any $\varepsilon > 0$ we can find $\delta_1 = \delta_1(\varepsilon) > 0$ such that

$$|F(t,x)| \le \frac{\varepsilon}{2}\|x\|^2 \quad \text{for a.e. } t \in T \text{ and all } \|x\| \le \delta_1. \tag{3.20}$$

Since $W^{1,2}_{\mathrm{per}}((0,b),\mathbb{R}^N)$ is embedded compactly into $C(T,\mathbb{R}^N)$, we can find $\rho > 0$ so small, that $\|x\| \le \rho$ implies that $\|x\|_\infty \le \delta_1$. Therefore, by (3.20) we have that

$$|F(t,x(t))| \le \frac{\varepsilon}{2}\|x(t)\|^2$$

for almost every $t \in T$ and all $x \in W^{1,2}_{\mathrm{per}}((0,b),\mathbb{R}^N)$ with $\|x\| \le \rho$.

Let $x \in H_-$ with $\|x\| \le \rho$. Then,

$$\begin{aligned}\varphi(x) = \frac{1}{2}\|x'\|_2^2 - \frac{1}{2}\int_0^b (A(t)x(t), x(t))_{\mathbb{R}^N}\, dt - \int_0^b F(t,x(t))\, dt \\ \le -\frac{\xi_1}{2}\|x\|^2 + \frac{\varepsilon}{2}\|x\|^2.\end{aligned}$$

So, if we choose $\varepsilon < \xi_1$, then

$$\varphi(x) \le 0 \quad \text{for all } x \in H_- \text{ with } \|x\| \le \rho. \tag{3.21}$$

Next, let $x \in V = H_0 \oplus H_+$. Then, $x = x^0 + \widehat{x}$, with $x^0 \in H_0$ and $\widehat{x} \in H_+$. Since $W^{1,2}_{\mathrm{per}}((0,b),\mathbb{R}^N)$ is embedded compactly into $C(T,\mathbb{R}^N)$, we can find $\xi_2 > 0$

such that $\|u\|_\infty \leq \xi_2\|u\|$ for all $u \in V$. In (3.20), we let $\rho \in (0, \delta/\xi_2]$, with $\delta > 0$ as in (F)(e). Hence, for $x \in V$ with $\|x\| \leq \rho$, we have

$$\|x(t)\| \leq \|x\|_\infty \leq \xi_2\|x\| \leq \delta \quad \text{for all } t \in T.$$

By (F)(e) (first option), this implies that $F(t, x(t)) \leq 0$ almost everywhere on T. Thus,

$$-\int_0^b F(t, x(t))\, dt \geq 0. \tag{3.22}$$

Therefore, for $x \in V$ with $\|x\| \leq \rho$, we have

$$\begin{aligned}\varphi(x) &= \frac{1}{2}\|x'\|_2^2 - \frac{1}{2}\int_0^b (A(t)x(t), x(t))_{\mathbb{R}^N}\, dt - \int_0^b F(t, x(t))\, dt \\ &\geq \frac{\xi_0}{2}\|x\|^2 \quad \text{(see Lemma 2.1(a) and (3.22)).}\end{aligned}$$

Thus,

$$\varphi(x) \geq 0 \quad \text{for all } x \in V \text{ with } \|x\| \leq \rho. \tag{3.23}$$

From (3.21) and (3.23) it follows that φ has a local linking at 0.

Next, assume that $\dim H_0 \neq 0$ and the second option in (F)(e) holds, namely,

$$F(t, x) \geq 0 \quad \text{for a.e. } t \in T \text{ and all } \|x\| \leq \delta.$$

In this case, we simply set $W^{1,2}_{\text{per}}((0, b), \mathbb{R}^N) = Y \oplus H_+$ with $Y = H_- \oplus H_0$, and use Lemma 2.1(b) instead. Then the proof can easily go through as in the case with the first option.

Finally, the case when $\dim H_0 = 0$ becomes obvious. □

Set $Y = H_- \oplus H_0$. We have

PROPOSITION 3.4. *If* (A) *and* (F) *hold and $E \subseteq H_+$ is a finite dimensional subspace, then $\varphi(u) \to -\infty$ as $\|u\| \to \infty$ with $u \in Y \oplus E$.*

PROOF. By (F)(d), given any $\sigma > 0$, we can find $M_4 > 0$ such that

$$F(t, x) \geq \sigma\|x\|^2$$

for almost every $t \in T$ and all $\|x\| \geq M_4$.

(F)(c) implies that

$$|F(t, x)| \leq a_3(t) \tag{3.25}$$

for almost every $t \in T$, all $\|x\| < M_4$, with some $a_3 \in L^1(T)_+$.

From (3.24) and (3.25), we have

$$F(t, x) \geq \sigma\|x\|^2 - a_4(t)$$

for almost every $t \in T$ and all $x \in \mathbb{R}^N$, with $a_4 = a_3(t) + \sigma M_4^2$.

Let $u \in Y \oplus E$. Then, for some $c_5 > 0$,

$$\begin{aligned}(3.26)\qquad \varphi(u) &= \frac{1}{2}\|u'\|_2^2 - \frac{1}{2}\int_0^b (A(t)u(t), u(t))_{\mathbb{R}^N}\, dt - \int_0^b F(t, u(t))\, dt \\ &\leq \frac{1}{2}\|u'\|_2^2 - \frac{1}{2}\int_0^b (A(t)u(t), u(t))_{\mathbb{R}^N}\, dt - \sigma\|u\|_2^2 + c_5.\end{aligned}$$

Since the space $Y \oplus E$ is finite dimensional, all norms on it are equivalent. Therefore, we can find $\xi_3, \xi_4 > 0$ such that

$$\frac{1}{2}\|u'\|_2^2 - \frac{1}{2}\int_0^b (A(t)u(t), u(t))_{\mathbb{R}^N}\, dt \leq \xi_3\|u\|^2$$

and

$$\xi_4\|u\|^2 \leq \|u\|_2^2 \quad \text{for all } u \in Y \oplus E.$$

Using these in (3.26), we obtain

$$(3.27)\qquad \varphi(u) \leq (\xi_3 - \sigma\xi_4)\|u\|^2 + c_5.$$

Since $\sigma > 0$ is arbitrary, from (3.27) it follows that $\varphi(u) \to -\infty$ as $\|u\| \to \infty$ with $u \in Y \oplus E$. $\square$

Now we are ready to state our existence theorem for problem (1.1).

Theorem 3.6. *If* (A) *and* (F) *hold, then problem* (1.1) *has at least one nontrivial solution* $\widetilde{x} \in C^1(T, \mathbb{R}^N)$.

Proof. Clearly, due to (F)(c), φ maps bounded sets to bounded sets. Hence, because of Propositions 3.2–3.4, we can apply Proposition 2.2 and obtain some $\widetilde{x} \in W^{1,2}_{\rm per}((0,b), \mathbb{R}^N)$, with $\widetilde{x} \neq 0$, such that

$$(3.28)\qquad \varphi'(\widetilde{x}) = V(\widetilde{x}) - \widehat{A}\widetilde{x} - N(\widetilde{x}) = 0.$$

From (3.28) as in [12], using integration by parts, we have $\widetilde{x} \in C^1(T, \mathbb{R}^N)$ and solves problem (1.1). $\square$

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Shouchuan Hu
College of Mathematics
Shandong Normal University
Jinan, Shandong, P. R. CHINA
and
Department of Mathematics
Missouri State University
Springfield, MO 65897, USA

E-mail address: hu@math.smsu.edu

Nikolas S. Papageorgiou
Department of Mathematics
National Technical University
Zografou Campus
Athens 15780, GREECE

E-mail address: npapg@math.ntua.gr