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MULTIPLICITY OF MULTI-BUMP TYPE NODAL SOLUTIONS FOR A CLASS OF ELLIPTIC PROBLEMS IN $\mathbb{R}^N$

Claudianor O. Alves

Abstract. In this paper, we establish existence and multiplicity of multi-bump type nodal solutions for the following class of problems

$$-\Delta u + (\lambda V(x) + 1)u = f(u), \quad u > 0 \quad \text{in } \mathbb{R}^N,$$

where $N \geq 1$, $\lambda \in (0, \infty)$, f is a continuous function with subcritical growth and $V\colon \mathbb{R}^N \to \mathbb{R}$ is a continuous function verifying some hypotheses.

1. Introduction

In the present paper, we are concerned with existence and multiplicity of multi-bump type nodal solutions for the following class of problems

$$(\mathrm{P})_\lambda \qquad \begin{cases} -\Delta u + (\lambda V(x) + 1)u = f(u) & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases}$$

where $N \geq 1$, $\lambda \in (0, \infty)$, f is a continuous function with subcritical growth and $V\colon \mathbb{R}^N \to \mathbb{R}$ is a continuous function with $\inf_{\mathbb{R}^N} V(x) \geq 0$.

There exist a lot of papers concerning with existence and multiplicity of positive solutions to $(\mathrm{P})_\lambda$, where the behavior of function V is an important point to make a careful study about the behavior of the solutions, see for example, the

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papers of T. Bartsch and Z. Q. Wang [6], [7], M. Clapp and Y. H. Ding [13], C. Gui [18], Y. H. Ding and K. Tanaka [17], C. O. Alves [1], C. O. Alves, D. C. de Morais Filho and M. A. S. Souto [3], C. O. Alves and M. A. S. Souto [5] and references therein. The existence and multiplicity of nodal solutions have been considered also in some works, we would like to cite the papers of T. Bartsch, Z. Liu and T. Weth [9], T. Bartsch and T. Weth [8], T. Bartsch, T. Weth and M. Willem [10], A. Castro and M. Clapp [12], M. Clapp and Y. H. Ding [14], Z. Liu and Z.-Q. Wang [19], C. O. Alves and G. M. Figueiredo [2], C. O. Alves and S. H. M. Soares [4] and references therein.

In [14], M. Clapp and Y. H. Ding have considered the existence of nodal solution for a class of problems of the type

$$-\Delta u + \lambda V(x)u = \mu u + |u|^{2^*-2}u, \quad \text{in } \mathbb{R}^N.$$

Assuming that V is τ-invariant and $\inf_{\mathbb{R}^N} V(x) \geq 0$, they proved that there exists a family $\{u_\lambda\}$ of nodal solution, which has the following property: For each $\lambda_n \to \infty$, the sequence $\{u_{\lambda_n}\}$ converges in $H^1(\mathbb{R}^N)$ to a nontrivial solution u of the Dirichlet problem

$$\begin{cases} -\Delta u = \mu u + |u|^{2^*-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega = \operatorname{int} V^{-1}(\{0\})$. Moreover, it is proved also that u changes sign exactly once.

In [17], Y. H. Ding and K. Tanaka have considered the existence of multi-bump positive solutions to $(\mathrm{P})_\lambda$, by assuming that $f(u) = |u|^{q-1}u$ with $1 < q < (N+2)/(N-2)$, $\inf_{\mathbb{R}^N} V(x) \geq 0$ and the following conditions on the set $\Omega := \operatorname{int} V^{-1}(\{0\})$:

(H_1) Ω is non-empty, bounded, $\partial\Omega$ is smooth and $V^{-1}(\{0\}) = \overline{\Omega}$.

(H_2) Ω has k connected components denoted by Ω_j, that is, $\Omega = \Omega_1 \cup \ldots \cup \Omega_k$.

In that paper, Y. H. Ding and K. Tanaka used variational methods to establish the existence of $2^k - 1$ multi-bump positive solutions for λ large enough. More precisely, for each $\Gamma \subset \{1, \ldots, k\}$, there exists a family of positive solution $\{u_\lambda\}$ satisfying the following property: For each $\lambda_n \to \infty$, the sequence $\{u_{\lambda_n}\}$ converges in $H^1(\mathbb{R}^N)$ to a function u, which is a positive solution of the Dirichlet problem:

$$\begin{cases} -\Delta u + u = u^q & \text{in } \Omega_\Gamma, \\ u(x) > 0 & \text{in } \Omega_\Gamma, \\ u = 0 & \text{on } \partial\Omega_\Gamma, \end{cases}$$

where $\Omega_\Gamma = \bigcup_{j\in\Gamma} \Omega_j$.

In the recent papers [3] and [5], C. O. Alves et al motivated by [17] considered the existence of multi-bump positive solutions for $(\mathrm{P})_\lambda$, by assuming that the nonlinearity has a critical growth for the cases $N \geq 3$ and $N = 2$, respectively. In [18], C. Gui showed the existence of multi-bump positive solutions for a different class of elliptic problems from what considered in [17]. In [11], A. Cao and E. S. Noussair considered also the existence of multi-bump solution for the same class of problems studied in [18] but with critical frequency, that is, $\inf_{\mathbb{R}^N} V(x) = 0$.

Motivated by [14] and [17], we investigate in this paper the existence and multiplicity of multi-bump type nodal solutions to $(\mathrm{P})_\lambda$ by exploiting the number of connected components of $\Omega = \operatorname{int} V^{-1}(0)$. Our main result completes the studies made in [14] and [17] in the following points:

- In [17], the nonlinearity is homogeneous and the solutions found are positives.
- In [14], in some results it is assumed that V is τ-invariant. Moreover, the nodal solutions found are not of the type multi-bump.

Here, we use a result related to the existence of nodal solution with least energy on bounded domain due to T. Bartsch, T. Weth and M. Willem [10] (see also T. Bartsch and T. Weth [8]). Moreover, we modify all the sets that appear in the minimax arguments found in [17] to get the nodal solutions. The nodal solutions obtained are concentrated near of nodal solutions with least energy on the connected components Ω_j of Ω, when λ is sufficiently large.

The main result proved in this paper also can be seen as a complement of the studies made in [8], [9], [10] and [19], because we are working with a class of nodal solutions which was not considered in those papers.

In order to state our main result, we require the following assumptions on f:

$$\lim_{s \to 0} \frac{f(s)}{s} = 0. \tag{f_1}$$

There is $p \in (1, (N+2)/(N-2))$ if $N \geq 3$ and $p \in (1, \infty)$ if $N = 1, 2$ such that

$$\lim_{|s| \to \infty} \frac{f(s)}{|s|^p} = 0. \tag{f_2}$$

There is $\theta > 2$ verifying

$$0 < \theta F(s) \leq s f(s), \quad \text{for all } s \in \mathbb{R} \setminus \{0\}. \tag{f_3}$$

Moreover, we also assume

$$f(s)s - f'(s)s^2 < 0, \quad \text{for all } s \in \mathbb{R} \setminus \{0\}. \tag{f_4}$$

Our main result is the following

Theorem 1.1. *Assume that* (f_1)–(f_4) *and* (H_1)–(H_2) *hold. Then, for any non-empty subset* Γ *of* $\{1,\ldots,k\}$, *there exists* $\lambda^* > 0$ *such that, for* $\lambda \geq \lambda^*$, *problem* $(\mathrm{P})_\lambda$ *has a nodal solution* u_λ. *Moreover, the family* $\{u_\lambda\}_{\lambda\geq\lambda^*}$ *has the following property: For any sequence* $\lambda_n \to \infty$, *we can extract a subsequence* λ_{n_i} *such that* $u_{\lambda_{n_i}}$ *converges strongly in* $H^1(\mathbb{R}^N)$ *to a function* u *which satisfies* $u(x) = 0$ *for* $x \notin \Omega_\Gamma = \bigcup_{j\in\Gamma}\Omega_j$, *and the restriction* $u|_{\Omega_j}$ *is a nodal solution with least energy of*

$$-\Delta u + u = f(u), \quad u|_{\partial\Omega_j} = 0 \quad \text{for } j \in \Gamma.$$

2. Preliminaries

In this section, we fix some notations and recall some results related to existence of nodal solutions to $(\mathrm{P})_\lambda$ on the connected components Ω_j of Ω.

Throughout this paper we will use the following notations:

- If h is a measurable function, we denote by $\int_{\mathbb{R}^N} h$ the following integral $\int_{\mathbb{R}^N} h\,dx$.
- The symbols $\|u\|$, $|u|_r$ $(r > 1)$ and $|u|_\infty$ denote the usual norms in the spaces $H^1(\mathbb{R}^N)$, $L^r(\mathbb{R}^N)$ and $L^\infty(\mathbb{R}^N)$, respectively.
- For an open set $\Theta \subset \mathbb{R}^N$, the symbols $\|u\|_\Theta$, $|u|_{r,\Theta}$ $(r > 1)$ and $|u|_{\infty,\Theta}$ denote the usual norms in the spaces $H^1(\Theta)$, $L^r(\Theta)$ and $L^\infty(\Theta)$, respectively.
- For a measurable function u, we denote by u^+ and u^- the positive and negative part of u respectively, given by

$$u^+(x) = \max\{u(x), 0\} \quad \text{and} \quad u^-(x) = \min\{u(x), 0\}.$$

Hereafter, we will work with the space $\mathcal{H}_\lambda$ defined by

$$\mathcal{H}_\lambda = \left\{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)|u|^2 < \infty\right\}$$

endowed with the norm

$$\|u\|_\lambda = \left(\int_{\mathbb{R}^N} |\nabla u|^2 + (\lambda V(x) + 1)|u|^2\right)^{1/2}.$$

It easy to see that $(\mathcal{H}_\lambda, \|\cdot\|_\lambda)$ is a Hilbert space for $\lambda > 0$.

For an open set $\Theta \subset \mathbb{R}^N$, we also write

$$\mathcal{H}_\lambda(\Theta) = \left\{u \in H^1(\Theta) : \int_\Theta V(x)|u|^2 < \infty\right\}$$

and

$$\|u\|_{\lambda,\Theta} = \left(\int_\Theta |\nabla u|^2 + (\lambda V(x) + 1)|u|^2\right)^{1/2}.$$

As a consequence of the above considerations, if $\nu_0 > 0$ is sufficiently small we have that

$$\frac{1}{2}\|u\|_{\lambda,\Theta}^2 \le \|u\|_{\lambda,\Theta}^2 - \nu_0|u|_{2,\Theta}^2 \quad \text{for all } u \in \mathcal{H}_\lambda(\Theta) \text{ and } \lambda > 0. \tag{2.1}$$

For each $j \in \{1, \dots, k\}$, we fix a bounded open subset Ω'_j with smooth boundary such that

(i) $\overline{\Omega_j} \subset \Omega'_j$,
(ii) $\overline{\Omega'_j} \cap \overline{\Omega'_l} = \emptyset$ for all $j \neq l$,

and let us define the functionals I_j and $\Phi_{\lambda,j}$ on $H_0^1(\Omega_j)$ and $H^1(\Omega'_j)$, respectively by

$$I_j(u) = \frac{1}{2}\int_{\Omega_j}(|\nabla u|^2 + |u|^2) - \int_{\Omega_j} F(u)$$

and

$$\Phi_{\lambda,j}(u) = \frac{1}{2}\int_{\Omega'_j}(|\nabla u|^2 + (\lambda V(x)+1)|u|^2) - \int_{\Omega'_j} F(u).$$

It is well known that I_j and $\Phi_{\lambda,j}$ are C^1 and their critical points are weak solutions of the problems

$$\begin{cases} -\Delta u + u = f(u) & \text{in } \Omega_j, \\ u = 0 & \text{on } \partial\Omega_j, \end{cases} \tag{2.2}$$

and

$$\begin{cases} -\Delta u + (\lambda V(x)+1)u = f(u) & \text{in } \Omega'_j, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega'_j, \end{cases} \tag{2.3}$$

respectively. Hereafter, c_j, d_j, $c_{\lambda,j}$ and $d_{\lambda,j}$ denote the real numbers given by

$$\begin{aligned} c_j &= \min\{I_j(u) : u \in H_0^1(\Omega_j) \setminus \{0\}, I'_j(u)(u) = 0\}, \\ d_j &= \min\{I_j(u) : u^\pm \in H_0^1(\Omega_j) \setminus \{0\}, I'_j(u^\pm)(u^\pm) = 0\}, \\ c_{\lambda,j} &= \min\{\Phi_{\lambda,j}(u) : u \in H^1(\Omega'_j) \setminus \{0\}, \Phi'_{\lambda,j}(u)(u) = 0\}, \\ d_{\lambda,j} &= \min\{\Phi_{\lambda,j}(u) : u^\pm \in H^1(\Omega'_j) \setminus \{0\}, \Phi'_{\lambda,j}(u^\pm)(u^\pm) = 0\}. \end{aligned}$$

From results due to T. Bartsch, T. Weth and M. Willem [10] and T. Bartsch and T. Weth [8], there exist w_j and $w_{\lambda,j}$ nodal solutions of (2.2) and (2.3), respectively, such that

$$I_j(w_j) = d_j \quad \text{and} \quad \Phi_{\lambda,j}(w_{\lambda,j}) = d_{\lambda,j}.$$

In [17], it is proved that the numbers $c_{\lambda,j}$ and c_j verifying the following limit

$$c_{\lambda_n,j} \to c_j \quad \text{as } \lambda_n \to \infty$$

which will be used later on.

3. Localization of the concentration

In this section, as in M. del Pino and P. L. Felmer [16], C. Gui [18] and Y. H. Ding and K. Tanaka [17], we modify conveniently the function f.

Let $\nu_0 > 0$ be the constant given in (2.1), $a > 0$ verifying $\max\{f(a)/a, f(-a)/-a\} < \nu_0$ and $\widetilde{f}, \widetilde{F}: \mathbb{R} \to \mathbb{R}$ the following functions

$$\widetilde{f}(s) = \begin{cases} \dfrac{-f(-a)}{a} s & \text{if } s < -a, \\ f(s) & \text{if } |s| \le a, \\ \dfrac{f(a)}{a} s & \text{if } s > a, \end{cases} \qquad \text{and} \qquad \widetilde{F}(s) = \int_0^s \widetilde{f}(\tau)\, d\tau.$$

Using the above notations, we consider the functions

$$g(x,s) = \chi_\Gamma(x) f(s) + (1 - \chi_\Gamma(x)) \widetilde{f}(s)$$

and

$$G(x,s) = \int_0^s g(x,t)\, dt = \chi_\Gamma(x) F(s) + (1 - \chi_\Gamma(x)) \widetilde{F}(s)$$

where $\Gamma \subset \{1, \ldots, k\}$ is a non-empty set fixed and χ_Γ denotes the characteristic function of the set $\Omega'_\Gamma = \bigcup_{j \in \Gamma} \Omega'_j$.

Under the conditions (f_1)–(f_2), we can prove that functional $\Phi_\lambda: \mathcal{H}_\lambda \to \mathbb{R}$ given by

$$\Phi_\lambda(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + (\lambda V(x) + 1)|u|^2) - \int_{\mathbb{R}^N} G(x,u)$$

belongs to $C^1(\mathcal{H}_\lambda, \mathbb{R})$ and its critical points are weak solutions of

$$(\mathrm{A})_\lambda \qquad -\Delta u + (\lambda V(x) + 1) u = g(x,u) \quad \text{in } \mathbb{R}^N.$$

An immediate result related to nodal solutions of $(\mathrm{A})_\lambda$ is the following

Lemma 3.1. *If u_λ is a nodal solution of $(\mathrm{A})_\lambda$ verifying $|u(x)| \le a$ in $\mathbb{R}^N \setminus \Omega'_\Gamma$, then it is a nodal solution to $(\mathrm{P})_\lambda$.*

In the sequel, we study the convergence of Palais–Smale sequences related to Φ_λ, that is, of sequences $\{u_n\} \subset \mathcal{H}_\lambda$ verifying

$$\Phi_\lambda(u_n) \to c \quad \text{and} \quad \Phi'_\lambda(u_n) \to 0 \tag{3.1}$$

for some $c \in \mathbb{R}$ (shortly $\{u_n\}$ is a $(\mathrm{PS})_c$ sequence).

Proposition 3.2. *The functional Φ_λ satisfies $(\mathrm{PS})_c$ condition for all $c \in \mathbb{R}$. More precisely, any $(\mathrm{PS})_c$ sequence $\{u_n\} \subset \mathcal{H}_\lambda$ has a strongly convergent subsequence in $\mathcal{H}_\lambda$.*

Proof. Let $\{u_n\} \subset \mathcal{H}_\lambda$ be a Palais–Smale sequence. Using assumption (f_3) and the inequality

$$\Phi_\lambda(u_n) - \frac{1}{\theta} \Phi'_\lambda(u_n)(u_n) \le c + \|u_n\|_\lambda,$$

which holds for n sufficiently large, it follows that $\{u_n\}$ is bounded. This way, for some subsequence, still denoted by $\{u_n\}$, there exists $u \in \mathcal{H}_\lambda$ such that

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{weakly in } \mathcal{H}_\lambda \text{ and } H^1(\mathbb{R}^N), \\ u_n &\to u \quad \text{in } L^q_{\text{loc}}(\mathbb{R}^N) \text{ for all } q \in [1, 2^*) \end{aligned} \tag{3.2}$$

and

$$\Phi'_\lambda(u) = 0. \tag{3.3}$$

These limits combined with the growth of g give

$$g(x, u_n)u_n \to g(x, u)u \quad \text{in } L^1_{\text{loc}}(\mathbb{R}^N). \tag{3.4}$$

Once for any bounded sequence $(\varphi_n) \subset \mathcal{H}_\lambda$, we can easily see that $\Phi'_\lambda(u_n)\varphi_n \to 0$, by fixing the sequence

$$\varphi_n(x) = \sigma(x)u_n(x)$$

where $\sigma \in C^\infty(\mathbb{R}^N)$ is given by

$$\begin{aligned} \sigma(x) &= 1 && \text{for all } x \in B^c_R(0), \\ \sigma(x) &= 0 && \text{for all } x \in B^c_{R/2}(0), \\ \sigma(x) &\in [0, 1] && \text{with } \Omega'_\Gamma \subset B_R(0), \end{aligned}$$

by a argument found in M. Del Pino and P. L. Felmer [16, Lemma 1.1], it is possible to prove that for each $\varepsilon > 0$ fixed, there exists $R > 0$ such that

$$\int_{\{x \in \mathbb{R}^N : |x| \geq R\}} |\nabla u_n|^2 + (\lambda V(x) + 1)|u_n|^2 \leq \varepsilon \quad \text{for } n \in \mathbb{N}. \tag{3.5}$$

Combining (3.5) with Sobolev embeddings and using the fact that g has subcritical growth, for each $\varepsilon > 0$ fixed, there exists $R > 0$ such that

$$\int_{B^c_R(0)} g(x, u_n)u_n, \ \int_{B^c_R(0)} g(x, u)u < \frac{\varepsilon}{3}. \tag{3.6}$$

From (3.4) and (3.6), it follows that

$$\int_{\mathbb{R}^N} g(x, u_n)u_n \to \int_{\mathbb{R}^N} g(x, u)u \quad \text{as } n \to \infty. \tag{3.7}$$

Now, from (3.1)–(3.3) we derive the equality

$$\|u_n - u\|^2_\lambda = \int_{\mathbb{R}^N} g(x, u_n)u_n - \int_{\mathbb{R}^N} g(x, u)u + o_n(1)$$

which together with (3.7) yields $u_n \to u$ in $\mathcal{H}_\lambda$. □

Our next goal is to study the behavior of a generalized Palais–Smale sequence corresponding to a sequence of functionals. From now on, we say that a sequence

$\{u_n\} \subset H^1(\mathbb{R}^N)$ is $(\mathrm{PS})_{\infty,c}$ sequence, if there exist $\lambda_n \to \infty$ such that $u_n \in \mathcal{H}_{\lambda_n}$ and

$$(\mathrm{PS})_{\infty,c} \qquad \Phi_{\lambda_n}(u_n) \to c \quad \text{and} \quad \|\Phi'_{\lambda_n}(u_n)\|^*_{\lambda_n} \to 0.$$

Proposition 3.3. *Let $\{u_n\}$ be a $(\mathrm{PS})_{\infty,c}$ sequence. Then, for some subsequence, still denoted by $\{u_n\}$, there exists $u \in H^1(\mathbb{R}^N)$ such that*

$$u_n \to u \quad \text{in } H^1(\mathbb{R}^N).$$

Moreover,

(a) *For $\Omega_\Gamma = \bigcup_{j\in\Gamma} \Omega_j$, we have that $u \equiv 0$ in $\mathbb{R}^N \setminus \Omega_\Gamma$ and u is a solution of*

$$(\mathrm{P})_j \qquad \begin{cases} -\Delta u + u = f(u) & \text{in } \Omega_j, \\ u = 0 & \text{on } \partial\Omega_j, \end{cases}$$

for each $j \in \Gamma$.

(b) $\|u_n - u\|_{\lambda_n} \to 0$.

(c) *u_n also satisfies*

$$\lambda_n \int_{\mathbb{R}^N} V(x)|u_n|^2 \to 0, \qquad \|u_n\|^2_{\lambda_n \mathbb{R}^N \setminus \Omega_\Gamma} \to 0$$

and

$$\|u_n\|^2_{\lambda_n,\Omega'_j} \to \int_{\Omega_j} (|\nabla u|^2 + |u|^2) \quad \text{for all } j \in \Gamma.$$

Proof. As in the proof of Proposition 3.2, it is easy to check that $\{\|u_n\|_{\lambda_n}\}$ is bounded in $\mathbb{R}$. Thus, we can assume that, for some $u \in H^1(\mathbb{R}^N)$,

$$u_n \rightharpoonup u \quad \text{weakly in } H^1(\mathbb{R}^N) \tag{3.8}$$

and $u_n(x) \to u(x)$ almost everywhere in $\mathbb{R}^N$. In the following, for each $m \in \mathbb{N}$, we denote by C_m the set given by

$$C_m = \left\{ x \in \mathbb{R}^N : V(x) \geq \frac{1}{m} \right\}.$$

Then,

$$\int_{C_m} |u_n|^2 \leq \frac{m}{\lambda_n} \int_{\mathbb{R}^N} \lambda_n V(x)|u_n|^2 \leq \frac{m}{\lambda_n} \|u_n\|^2_{\lambda_n}.$$

This combined with Fatou's Lemma leads to

$$\int_{C_m} |u|^2 = 0, \quad \text{for all } m \in \mathbb{N}.$$

Thus $u(x) = 0$ on $\bigcup_{m=1}^{\infty} C_m = \mathbb{R}^N \setminus \overline{\Omega}$ and we can assert that

$$u|_{\Omega_j} \in H^1_0(\Omega_j) \quad \text{for all } j \in \{1, \ldots, k\}.$$

Once $\Phi'_{\lambda_n}(u_n)\varphi \to 0$ as $n \to \infty$, for each $\varphi \in C_0^\infty(\Omega_j)$ (and hence for each $\varphi \in H_0^1(\Omega_j)$), it follows from (3.8)

$$\int_{\Omega_j} \nabla u \nabla \varphi + u\varphi - \int_{\Omega_j} g(x,u)\varphi = 0, \tag{3.9}$$

which gives $u|_{\Omega_j}$ is a solution of $(\mathrm{P})_j$ for each $j \in \{1, \ldots, k\}$. Moreover, for each $j \in \{1, \ldots, k\} \setminus \Gamma$, setting $\varphi = u|_{\Omega_j}$ in (3.9), we have

$$\int_{\Omega_j} |\nabla u|^2 + |u|^2 - \int_{\Omega_j} \widetilde{f}(u)u = 0$$

that is,

$$\|u\|^2_{\lambda,\Omega_j} - \int_{\Omega_j} \widetilde{f}(u)u = 0.$$

Since $\widetilde{f}(s)s \leq \nu_0 |s|^2$ for all $s \in \mathbb{R}$, combining this inequality with (2.1) we get

$$\delta_0 ||u||^2_{2,\Omega_j} \leq \|u\|^2_{\lambda,\Omega_j} - \nu_0 |u|^2_{2,\Omega_j} \leq \|u\|^2_{\lambda,\Omega_j} - \int_{\Omega_j} \widetilde{f}(u)u = 0.$$

Thus, $u = 0$ in Ω_j, for $j \in \{1, \ldots, k\} \setminus \Gamma$, and the proof of (a) is complete.

To show (b), we begin observing that arguing as in the proof of Proposition 3.2, for each $\varepsilon > 0$ fixed, there exists $R > 0$ such that

$$\int_{\{x \in \mathbb{R}^N : |x| \geq R\}} |\nabla u_n|^2 + (\lambda_n V(x) + 1)|u_n|^2 \leq \varepsilon \quad \text{for } n \in \mathbb{N}.$$

This inequality implies that

$$\int_{\mathbb{R}^N} g(x, u_n)u_n \to \int_{\mathbb{R}^N} g(x,u)u \quad \text{as } n \to \infty.$$

Using the limit $\|\Phi'_{\lambda_n}\|^*_{\lambda_n} \to 0$ together with the fact that $u \in H_0^1(\Omega_\Gamma)$, we get the equality

$$\|u_n - u\|^2_{\lambda_n} = \int_{\mathbb{R}^N} g(x,u_n)u_n - \int_{\mathbb{R}^N} g(x,u)u + o_n(1)$$

which yields

$$\|u_n - u\|^2_{\lambda_n} \to 0 \tag{3.10}$$

and (b) follows. To prove (c), notice that

$$\int_{\mathbb{R}^N} \lambda_n V(x)|u_n|^2 = \int_{\mathbb{R}^N} \lambda_n V(x)|u_n - u|^2 \leq C\|u_n - u\|^2_{\lambda_n}$$

so,

$$\int_{\mathbb{R}^N} \lambda_n V(x)|u_n|^2 \to 0 \quad \text{as } n \to \infty.$$

The other limits also follow immediately from (3.10). $\square$

Proposition 3.4. *Let $\{u_\lambda\}$ be a family of nodal solution of* (A)$_\lambda$ *with $u_\lambda \to 0$ in $H^1(\mathbb{R}^N \setminus \Omega_\Gamma)$ as $\lambda \to \infty$. Then, there exists $\lambda^* > 0$ such that u_λ is a nodal solution of* (P)$_\lambda$ *for all $\lambda \geq \lambda^*$.*

Proof. In this proof, we will use the Moser iteration technique [20] and the same arguments found in [1, Proposition 3.2]. The basic idea is the following: Fixing $\Omega'_j \subset \widetilde{\Omega}_j$ and $\sigma \in C^\infty(\mathbb{R}^N)$ verifying

$$\begin{aligned} 0 \leq \sigma(x) &\leq 1 \quad \text{for all } x \in \mathbb{R}^N, \\ \sigma(x) &= 0 \quad \text{for all } x \in \bigcup_{j\in\Gamma} \Omega'_j, \\ \sigma(x) &= 1 \quad \text{for all } x \in \mathbb{R}^N \setminus \bigcup_{j\in\Gamma} \widetilde{\Omega}_j, \end{aligned}$$

let us define for each $\lambda, L, \beta > 1$ the functions

$$u^+_{L,\lambda} = \begin{cases} u^+_\lambda & \text{if } u_\lambda \leq L, \\ L & \text{if } u_\lambda \geq L, \end{cases}$$

$$z^+_{L,\lambda} = \sigma^2 |u^+_{L,\lambda}|^{2(\beta-1)} u_\lambda, \qquad w^+_{L,\lambda} = \sigma u_\lambda |u^+_{L,\lambda}|^{\beta-1}.$$

Since u_λ is a solution of (A)$_\lambda$, using $z^+_{L,\lambda}$ as a test function and the fact that $|g(x,s)| \leq \nu_0 |s|^2$ for all $x \in \mathbb{R}^N \setminus \Omega'_\Gamma$, we get

$$|w^+_{L,\lambda}|^2_{2^*} \leq C \int_{\mathbb{R}^N} |\nabla w^+_{L,\lambda}|^2 \leq C\beta^2 \int_{\mathbb{R}^N} |\nabla \sigma|^2 |u_\lambda|^2 |u^+_{L,\lambda}|^{2(\beta-1)}. \tag{3.11}$$

The estimate (3.11) yields

$$|w^+_{L,\lambda}|^2_{2^*,\mathcal{B}} \leq C_1 \beta^2 \bigg(\int_\Upsilon |u_\lambda|^2 |u^+_{L,\lambda}|^{2(\beta-1)} \bigg)$$

where $\Upsilon = \bigcup_{j\in\Gamma} (\widetilde{\Omega}_j \setminus \Omega'_j)$ and $\mathcal{B} = \mathbb{R}^N \setminus \bigcup_{j\in\Gamma} \Omega'_j$.

Now, the last inequality together with the Moser iteration lead to

$$|u^+_\lambda|_{\infty,\mathcal{B}} \leq C_3 |u^+_\lambda|_{2^*,\Upsilon}$$

for some positive constant C_3. On the other hand, by hypothesis

$$u_\lambda \to 0 \quad \text{in } H^1(\mathbb{R}^N \setminus \Omega_\Gamma) \text{ as } \lambda \to \infty,$$

then, this limit combined with (3.12) implies that

$$|u^+_\lambda|_{\infty,\mathbb{R}^N\setminus\Omega'_\Gamma} \leq a \quad \text{for all } \lambda \geq \lambda^*$$

for some $\lambda^* > 0$. A similar argument can be use to prove that

$$|u^-_\lambda|_{\infty,\mathbb{R}^N\setminus\Omega'_\Gamma} \leq a \quad \text{for all } \lambda \geq \lambda^*$$

for some $\lambda^* > 0$. Therefore,

$$|u_\lambda|_{\infty,\mathbb{R}^N\setminus\Omega'_\Gamma} \leq a \quad \text{for all } \lambda \geq \lambda^*.$$

This, together with Lemma 3.1, yields u_λ is a solution of $(\mathrm{P})_\lambda$ for all $\lambda \geq \lambda^*$. $\square$

4. A special class of functions

In what follows, let us fix $R > 0$ verifying

$$I_j(R^{-1}w_j^{\pm}), I_j(Rw_j^{\pm}) < \frac{I_j(w_j^{\pm})}{2} \quad \text{for all } j \in \Gamma. \tag{4.1}$$

Moreover, without loss of generality, we assume $\Gamma = \{1, \dots, l\} (l \leq k)$, and define $\gamma_0 \colon [1/R^2, 1]^{2l} \to \mathcal{H}_\lambda$ by

$$\gamma_0(s_1, \dots, s_l, t_1, \dots, t_l)(x) = \sum_{j=1}^{l} s_j R w_j^+(x) + \sum_{j=1}^{l} t_j R w_j^-(x) \tag{4.2}$$

and

$$S_{\lambda,\Gamma} = \inf_{\gamma \in \Sigma_\lambda} \max_{(\overrightarrow{s}, \overrightarrow{t}) \in [1/R^2,1]^{2l}} \Phi_\lambda(\gamma(\overrightarrow{s}, \overrightarrow{t}))$$

where $(\overrightarrow{s}, \overrightarrow{t}) = (s_1, \dots, s_l, t_1, \dots, t_l)$ and

$$\Sigma_\lambda = \{\gamma \in C([1/R^2, 1]^{2l}, \mathcal{H}_\lambda) : \gamma^{\pm}|_{\Omega'_j} \neq 0 \text{ for all } j \in \Gamma \\ \text{and } (\overrightarrow{s}, \overrightarrow{t}) \in [1/R^2, 1]^{2l},\ \gamma = \gamma_o \text{ on } \partial([1/R^2, 1]^{2l})\}.$$

We remark that $\gamma_o \in \Sigma_\lambda$, so $\Sigma_\lambda \neq \emptyset$ and $S_{\lambda,\Gamma}$ is well defined.

Lemma 4.1. *For any $\gamma \in \Sigma_\lambda$ there exists $(\overrightarrow{s_*}, \overrightarrow{t_*}) \in [1/R^2, 1]^{2l}$ such that*

$$\Phi'_{\lambda,j}(\gamma^{\pm}(\overrightarrow{s_*}, \overrightarrow{t_*}))(\gamma^{\pm}(\overrightarrow{s_*}, \overrightarrow{t_*})) = 0 \quad \textit{for all } j \in \{1, \dots, l\}.$$

Proof. For each $\gamma \in \Sigma_\lambda$, let us define the function $H \colon [1/R^2, 1]^{2l} \to \mathbb{R}$ given by

$$H(\overrightarrow{s}, \overrightarrow{t}) = \big(\Phi'_{\lambda,1}(\gamma^+).(\gamma^+), \dots, \Phi'_{\lambda,l}(\gamma^+).(\gamma^+), \\ \Phi'_{\lambda,1}(\gamma^-).(\gamma^-), \dots, \Phi'_{\lambda,l}(\gamma^-).(\gamma^-))$$

where

$$\Phi'_{\lambda,j}(\gamma^{\pm}).(\gamma^{\pm}) = \Phi'_{\lambda,j}(\gamma^{\pm}(\overrightarrow{s}, \overrightarrow{t})).(\gamma^{\pm}(\overrightarrow{s}, \overrightarrow{t})) \quad \text{for } j \in \{1, \dots, l\}.$$

Since

$$H(\overrightarrow{s}, \overrightarrow{t}) = H_0(\overrightarrow{s}, \overrightarrow{t}) \quad \text{for all } (\overrightarrow{s}, \overrightarrow{t}) \in \partial([1/R^2, 1]^{2l})$$

where

$$H_0(\overrightarrow{s}, \overrightarrow{t}) = (\Phi'_{\lambda,1}(\gamma_o^+).(\gamma_o^+), \dots, \Phi'_{\lambda,l}(\gamma_o^+).(\gamma_o^+), \\ \Phi'_{\lambda,1}(\gamma_o^-).(\gamma_o^-), \dots, \Phi'_{\lambda,l}(\gamma_o^-).(\gamma_o^-))$$

and, by (f_4), $d(H_0, (1/R^2, 1)^{2l}, 0) = 1$, (topological degree). Using topological degree, we derive $d(H, (1/R^2, 1)^{2l}, 0) = 1$.

The last equality implies that there exists $(\overrightarrow{s_*}, \overrightarrow{t_*}) \in [1/R^2, 1]^{2l}$ such that $H(\overrightarrow{s_*}, \overrightarrow{t_*}) = 0$, which proves the lemma. □

In the sequel, we denote by D_Γ the number $D_\Gamma = \sum_{j=1}^{l} d_j$.

Proposition 4.2. *The numbers D_Γ and $S_{\lambda,\Gamma}$ verify the following relations*

(a) $\sum_{j=1}^{l} d_{\lambda,j} \le S_{\lambda,\Gamma} \le D_\Gamma$ *for all* $\lambda \ge 1$.
(b) $S_{\lambda,\Gamma} \to D_\Gamma$ *as* $\lambda \to \infty$.

Proof. (a) Since γ_o defined in (4.2) belongs to Σ_λ, we have

$$\begin{aligned} S_{\lambda,\Gamma} &\le \max_{(\overrightarrow{s}, \overrightarrow{t}) \in [1/R^2,1]^{2l}} \Phi_\lambda(\gamma_o(\overrightarrow{s}, \overrightarrow{t})) \\ &= \max_{(s_1,\dots,s_l)\in[1/R^2,1]^l} \sum_{j=1}^{l} I_j(s_j R w_j^+) + \max_{(t_1,\dots,t_l)\in[1/R^2,1]^l} \sum_{j=1}^{l} I_j(t_j R w_j^-). \end{aligned}$$

From definition of w_j, it is standard the equality

$$\max_{z\in[1/R^2,1]} I_j(zRw_j^\pm) = I_j(w_j^\pm) \quad \text{for all } j \in \Gamma \tag{4.3}$$

and, thus,

$$S_{\lambda,\Gamma} \le \sum_{j=1}^{l} d_j = D_\Gamma.$$

Taking $(\overrightarrow{s_*}, \overrightarrow{t_*}) \in [1/R^2, 1]^{2l}$ given by Lemma 4.1, it follows that

$$\Phi_{\lambda,j}(\gamma(\overrightarrow{s_*}, \overrightarrow{t_*})) \ge d_{\lambda,j} \quad \text{for all } j \in \Gamma.$$

On the other hand, recalling that $\Phi_{\lambda,\mathbb{R}^N\setminus\Omega'_\Gamma}(u) \ge 0$ for all $u \in H^1(\mathbb{R}^N \setminus \Omega'_\Gamma)$, we get the inequality

$$\Phi_\lambda(\gamma(\overrightarrow{s_*}, \overrightarrow{t_*})) \ge \sum_{j=1}^{l} \Phi_{\lambda,j}(\gamma(\overrightarrow{s_*}, \overrightarrow{t_*}))$$

which yields

$$\max_{(\overrightarrow{s}, \overrightarrow{t})\in[1/R^2,1]^{2l}} \Phi_\lambda(\gamma(\overrightarrow{s}, \overrightarrow{t})) \ge \Phi_\lambda(\gamma(\overrightarrow{s_*}, \overrightarrow{t_*})) \ge \sum_{j=1}^{l} d_{\lambda,j}.$$

From definition of $S_{\lambda,\Gamma}$, we can conclude

$$S_{\lambda,\Gamma} \ge \sum_{j=1}^{l} d_{\lambda,j}$$

and the proof of (a) is complete.

(b) The same arguments used in proof of Proposition 3.3 work to prove that for each $j \in \Gamma$ fixed, $d_{\lambda,j} \to d_j$ as $\lambda \to \infty$, and, therefore,

$$\sum_{j=1}^{l} d_{\lambda,j} \to D_\Gamma.$$

The last limit together with (a) implies that (b) holds. □

5. A special family of nodal solutions to $(\mathrm{A})_\lambda$

In this section, we show the existence of a special family of nodal solutions to $(\mathrm{A})_\lambda$ for λ large enough. These nodal solutions are exactly the nodal solutions given in Theorem 1.1.

Hereafter, $E^+_{\lambda,j}$ and $E^-_{\lambda,j}$ denote the cone of nonnegative and nonpositive functions belongs to $\mathcal{H}_\lambda(\Omega'_j)$, respectively, that is

$$\begin{aligned} E^+_{\lambda,j} &= \{u \in \mathcal{H}_\lambda(\Omega'_j) : u(x) \geq 0 \text{ a.e. in } \Omega'_j\}, \\ E^-_{\lambda,j} &= \{u \in \mathcal{H}_\lambda(\Omega'_j) : u(x) \leq 0 \text{ a.e. in } \Omega'_j\}. \end{aligned}$$

From definition of γ_o, there exist positive constants τ and $\lambda^* > 0$ such that

$$\mathrm{dist}_{\lambda,j}(\gamma_o(\overrightarrow{s}, \overrightarrow{t}), E^\pm_{\lambda,j}) > \tau \quad \text{for all } (\overrightarrow{s}, \overrightarrow{t}) \in [1/R^2, 1]^{2l},\ j \in \Gamma \text{ and } \lambda \geq \lambda^*,$$

where $\mathrm{dist}_{\lambda,j}(K, F)$ denotes the distance between sets of $\mathcal{H}_\lambda(\Omega'_j)$. Taking the number τ obtained in the last inequality, we define

$$\Theta = \{u \in \mathcal{H}_\lambda : \mathrm{dist}_{\lambda,j}(u, E^\pm_{\lambda,j}) \geq \tau \text{ for all } j \in \Gamma\}.$$

Moreover, for any $c, \mu > 0$ and $0 < \delta < \tau/2$, we consider the sets

$$\Phi^c_\lambda = \{u \in \mathcal{H}_\lambda : \Phi_\lambda(u) \leq c\} \quad \text{and} \quad B_{\lambda,\mu} = \{u \in \Theta_{2\delta} : |\Phi_\lambda(u) - S_{\lambda,\Gamma}| \leq \mu\}$$

where Θ_r, for $r > 0$, denotes the set $\Theta_r = \{u \in \mathcal{H}_\lambda : \mathrm{dist}(u, \Theta) \leq r\}$.

Notice that for each $\mu > 0$, there exists $\Lambda^* = \Lambda^*(\mu) > 0$ such that $w = \sum_{j=1}^l w_j \in B_{\lambda,\mu}$ for all $\lambda \geq \Lambda^*$, because $w \in \Theta$, $\Phi_\lambda(w) = D_\Gamma$ and $S_{\lambda,\Gamma} \to D_\Gamma$ as $\lambda \to \infty$. Therefore $B_{\lambda,\mu} \neq \emptyset$ for λ sufficiently large.

In the sequel, let us consider $\overline{B}_{M+1}(0) = \{u \in \mathcal{H}_\lambda : \|u\|_\lambda \leq M + 1\}$ where M is a constant large enough independent of λ verifying

$$\|\gamma(\overrightarrow{s}, \overrightarrow{t})\|_\lambda, \left\| \sum_{j=1}^{k} w_j \right\|_\lambda \leq \frac{M}{2} \quad \text{for all } (\overrightarrow{s}, \overrightarrow{t}) \in [1/R^2, 1]^{2l}.$$

Moreover, let us denote by $\mu^* > 0$ the real number

$$\mu^* = \min\left\{ \frac{I_j(w_j^\pm) + M + \delta}{4} : j \in \Gamma \right\}. \tag{5.1}$$

Proposition 5.1. *For each $\mu > 0$ fixed, there exist $\sigma_o = \sigma_o(\mu) > 0$ and $\Lambda_* = \Lambda(\mu) \geq 1$ independent of λ such that*

$$\|\Phi'_\lambda(u)\|^*_\lambda \geq \sigma_o \quad \text{for } \lambda \geq \Lambda_* \text{ and all } u \in (B_{\lambda,2\mu} \setminus B_{\lambda,\mu}) \cap \overline{B}_{M+1}(0) \cap \Phi_\lambda^{D_\Gamma}.$$

Proof. Arguing by contradiction, we assume that there exist $\lambda_n \to \infty$ and

$$u_n \in (B_{\lambda_n,2\mu} \setminus B_{\lambda_n,\mu}) \cap \overline{B}_{M+1}(0) \cap \Phi_{\lambda_n}^{D_\Gamma}$$

such that $\|\Phi'_{\lambda_n}(u_n)\|^*_{\lambda_n} \to 0$. Since $u_n \in B_{\lambda_n,2\mu}$ and $\{\|u_n\|_{\lambda_n}\}$ is a bounded sequence, it follows that $\{\Phi_{\lambda_n}(u_n)\}$ is also bounded. Thus we may assume

$$\Phi_{\lambda_n}(u_n) \to c \in (-\infty, D_\Gamma]$$

after extracting a subsequence if necessary. Applying Proposition 3.3, we can extract a subsequence $u_n \to u$ in $H^1(\mathbb{R}^N)$ where $u \in H^1_0(\Omega_\Gamma)$ is a solution of (P_j) with

$$\|u_n - u\|_{\lambda_n} \to 0, \quad \lambda_n \int_{\mathbb{R}^N} V(x)|u_n|^p \to 0 \quad \text{and} \quad \|u_n\|_{\lambda_n,\mathbb{R}^N\setminus\Omega_\Gamma} \to 0.$$

Once $u_n \in \Theta_{2\delta}$ for all $n \in \mathbb{N}$, we have that $\|u_n^\pm\|_{\lambda_n,\Omega'_j} \not\to 0$ for all $j \in \Gamma$, from where it follows that $\|u^\pm\|_{\Omega_j} \neq 0$ for all $j \in \Gamma$, so that u is a nodal solution of (P_j) for all $j \in \Gamma$ and

$$\sum_{j=1}^{l} d_j \leq \sum_{j=1}^{l} I_j(u|_{\Omega_j}) \leq D_\Gamma.$$

This fact leads to $I_j(u|_{\Omega_j}) = d_j$ for all $j \in \Gamma$, and hence $\Phi_{\lambda_n}(u_n) \to D_\Gamma$. On the other hand, since $S_{\lambda_n,\Gamma} \to D_\Gamma$, we can conclude that $u_n \in B_{\lambda_n,\mu} \cap \Phi_{\lambda_n}^{D_\Gamma}$ for n large enough, which is an absurd. □

Proposition 5.2. *For each $\mu \in (0, \mu^*)$, there exists $\Lambda^* = \Lambda^*(\mu) > 0$ such that for all $\lambda \geq \Lambda^*$ the functional Φ_λ has a critical point in $B_{\lambda,\mu} \cap \overline{B}_{M+1}(0) \cap \Phi_\lambda^{D_\Gamma}$.*

Proof. Arguing again by contradiction, we assume that there exists $\mu \in (0, \mu^*)$ and a sequence $\lambda_n \to \infty$, such that Φ_{λ_n} has not critical points in $B_{\lambda_n,\mu} \cap \overline{B}_{M+1}(0) \cap \Phi_\lambda^{D_\Gamma}$. Since the Palais–Smale condition holds for Φ_{λ_n} (see Proposition 3.2), there exists a constant $d_{\lambda_n} > 0$ such that

$$\|\Phi'_{\lambda_n}(u)\|^*_{\lambda_n} \geq d_{\lambda_n} \quad \text{for all } u \in B_{\lambda_n,\mu} \cap \overline{B}_{M+1}(0) \cap \Phi_{\lambda_n}^{D_\Gamma}.$$

Moreover, from Proposition 5.1, we also have

$$\|\Phi'_{\lambda_n}(u)\|^*_{\lambda_n} \geq \sigma_o \quad \text{for all } u \in (B_{\lambda_n,2\mu} \setminus B_{\lambda_n,\mu}) \cap \overline{B}_{M+1}(0) \cap \Phi_{\lambda_n}^{D_\Gamma}$$

where $\sigma_o > 0$ is independent of λ_n for n large enough. In what follows, $\Psi_n\colon \mathcal{H}_{\lambda_n} \to \mathbb{R}$ and $H_n\colon \Phi_{\lambda_n}^{c_\Gamma} \to \mathcal{H}_{\lambda_n}$ are continuous functions verifying

$$\begin{aligned} \Psi_n(u) &= 1 \quad \text{for } u \in B_{\lambda_n,3\mu/2} \cap \Theta_\delta \cap \overline{B}_M(0), \\ \Psi_n(u) &= 0 \quad \text{for } u \notin B_{\lambda_n,2\mu} \cap \overline{B}_{M+1}(0), \\ 0 \le \Psi_n(u) &\le 1 \quad \text{for } u \in \mathcal{H}_{\lambda_n}, \end{aligned}$$

and

$$H_n(u) = \begin{cases} -\Psi_n(u)\|Y_n(u)\|^{-1}Y_n(u) & \text{for } u \in B_{\lambda_n,2\mu} \cap \overline{B}_{M+1}(0), \\ 0 & \text{for } u \notin B_{\lambda_n,2\mu} \cap \overline{B}_{M+1}(0), \end{cases}$$

where Y_n is a pseudo-gradient vector field for Φ_{λ_n} on $\mathcal{M}_n = \{u \in \mathcal{H}_{\lambda_n} : \Phi'_{\lambda_n} \neq 0\}$. Hereafter, we denote by m_0^n the real number given by

$$m_0^n = \sup\{\Phi_{\lambda_n}(u) : u \in \gamma_0([1/R^2,1])^{2l} \setminus (B_{\lambda_n,\mu} \cap \overline{B}_M(0))\}$$

which verifies $\limsup_{n\to\infty} m_0^n < D_\Gamma$. Moreover, let us denote by $K_n > 0$ a constant verifying

$$|\Phi_{\lambda_n,j}(u) - \Phi_{\lambda_n,j}(v)| \le K_n\|u-v\|_{\lambda_n,\Omega'_j} \quad \text{for all } u,v \in \overline{B}_{M+1}(0) \text{ and all } j \in \Gamma.$$

From definition of H_n, we derive that

$$\|H_n(u)\| \le 1 \quad \text{for all } n \in \mathbb{N} \text{ and } u \in \Phi_{\lambda_n}^{D_\Gamma},$$

consequently there is a deformation flow $\eta_n\colon [0,\infty) \times \Phi_{\lambda_n}^{D_\Gamma} \to \Phi_{\lambda_n}^{D_\Gamma}$ defined by

$$\frac{d\eta}{dt} = H_n(\eta), \quad \eta_n(0,u) = u \in \Phi_{\lambda_n}^{D_\Gamma}.$$

This flow satisfies the following basic properties

$$\Phi_{\lambda_n}(\eta_n(t,u)) \le \Phi_{\lambda_n}(u) \quad \text{for all } t \ge 0 \text{ and } u \in \mathcal{H}_{\lambda_n}$$

and

$$\eta_n(t,u) = u \quad \text{for all } t \ge 0 \text{ and } u \notin B_{\lambda_n,2\mu} \cap \overline{B}_{M+1}(0)$$

Claim 5.3. *There exists $T_n = T(\lambda_n) > 0$ and $\varepsilon^* > 0$ independent of n such that*

$$\limsup_{n\to\infty}\Big[\max_{(\overrightarrow{s},\overrightarrow{t})\in[1/R^2,1]^{2l}} \Phi_{\lambda_n}(\eta_n(T_n,\gamma_0(\overrightarrow{s},\overrightarrow{t})))\Big] < D_\Gamma - \varepsilon^*$$

In fact, set $u = \gamma_0(\overrightarrow{s},\overrightarrow{t})$, $\widetilde{d}_{\lambda_n} = \min\{d_{\lambda_n},\sigma_0\}$, $T_n = \sigma_0\mu/2\widetilde{d}_{\lambda_n}$ and $\widetilde{\eta_n}(t) = \eta_n(t,u)$. if $u \notin B_{\lambda_n,\mu} \cap \overline{B}_M(0) \cap \Theta_\delta$, from definition of m_0^n we get

$$\Phi_{\lambda_n}(\eta_n(t,u)) \le \Phi_{\lambda_n}(u) \le m_0^n \quad \text{for all } t \ge 0.$$

On the other hand, if $u \in B_{\lambda_n,\mu} \cap \overline{B}_M(0) \cap \Theta_\delta$, we have to consider the following cases:

Case 1. $\widetilde{\eta_n}(t) \in B_{\lambda_n,3\mu/2} \cap \overline{B}_M(0) \cap \Theta_\delta$ for all $t \in [0,T_n]$.

Case 2. $\widetilde{\eta_n}(t_0) \notin B_{\lambda_n,3\mu/2} \cap \overline{B}_M(0) \cap \Theta_\delta$ for some $t_0 \in [0, T_n]$.

Following the same arguments found in Y. H. Ding and Tanaka [17], Case 1 implies that there exists $\varepsilon^* > 0$ independent of n such that

$$\Phi_{\lambda_n}(\widetilde{\eta_n}(T_n)) \le D_\Gamma - \varepsilon^*.$$

Related to Case 2, we have the following situations:

(a) There exists $t_2 \in [0, T_n]$ such that $\widetilde{\eta_n}(t_2) \notin \Theta_\delta$, and thus for $t_1 = 0$ it follows that

$$\|\widetilde{\eta_n}(t_2) - \widetilde{\eta_n}(t_1)\| \ge \delta > \mu$$

because $\widetilde{\eta_n}(t_1) = u \in \Theta$.

(b) There exists $t_2 \in [0, T_n]$ such that $\widetilde{\eta_n}(t_2) \notin \overline{B}_M(0)$, so that for $t_1 = 0$ we get

$$\|\widetilde{\eta_n}(t_2) - \widetilde{\eta_n}(t_1)\| \ge \frac{M}{2} > \mu$$

because $\widetilde{\eta_n}(t_1) = u \in \overline{B}_{M/2}(0)$.

(c) $\widetilde{\eta_n}(t) \in \Theta_\delta \cap \overline{B}_M(0)$ for all $t \in [0, T_n]$, and there are $0 \le t_1 \le t_2 \le T_n$ such that $\widetilde{\eta_n}(t) \in B_{\lambda_n,3\mu/2} \setminus B_{\lambda_n,\mu}$ for all $t \in [t_1, t_2]$ with

$$|\Phi_{\lambda_n}(\widetilde{\eta_n}(t_1)) - S_{\lambda_n,\Gamma}| = \mu \quad \text{and} \quad |\Phi_{\lambda_n}(\widetilde{\eta_n}(t_2)) - S_{\lambda_n,\Gamma}| = 3\mu/2.$$

Using the definition of K_n, we have that

$$\|\widetilde{\eta_n}(t_2) - \widetilde{\eta_n}(t_1)\| \ge \frac{\mu}{2K_n}.$$

The estimates showed in (a)–(c) yield, there exists $C > 0$ such that $t_2 - t_1 \ge C\mu$. This, combined with some arguments found in [17], gives that there exists $\varepsilon^* > 0$ independent of n such that

$$\limsup_{n\to\infty} \left[\max_{(\overrightarrow{s}, \overrightarrow{t}) \in [1/R^2, 1]^{2l}} \Phi_{\lambda_n}(\eta_n(T_n, \gamma_0(\overrightarrow{s}, \overrightarrow{t}))) \right] \le D_\Gamma - \varepsilon^*$$

and the proof of Claim 5.2 is complete.

Now, our goal is to prove that $(\overrightarrow{s}, \overrightarrow{t}) \to \eta_n(T_n, \gamma_o(\overrightarrow{s}, \overrightarrow{t}))$ belongs to Σ_{λ_n} for n large enough. To this end, we begin observing that $\eta_n(\gamma_o(\overrightarrow{s}, \overrightarrow{t}))$ is a continuous functions in $[1/R^2, 1]^{2l}$. Hence, we have to show that

$$\eta_n(T_n, \gamma_o(\overrightarrow{s}, \overrightarrow{t})) = \gamma_o(\overrightarrow{s}, \overrightarrow{t}) \quad \text{for all } (\overrightarrow{s}, \overrightarrow{t}) \in \partial([1/R^2, 1]^{2l})$$

and

$$(\eta_n(T_n, \gamma_o(\overrightarrow{s}, \overrightarrow{t})))^\pm \in H^1(\Omega'_j) \setminus \{0\},$$

for all $j \in \Gamma$ and all $(\overrightarrow{s}, \overrightarrow{t}) \in [1/R^2, 1]^{2l}$.

Once $\mu \in (0, \mu^*)$, (4.1), (4.3) and (5.1) lead to

$$|\Phi_{\lambda_n}(\gamma_o(\overrightarrow{s}, \overrightarrow{t})) - D_\Gamma| \ge 2\mu^* \quad \text{for all } (\overrightarrow{s}, \overrightarrow{t}) \in \partial([1/R^2, 1]^{2l}) \text{ and } n \in \mathbb{N}.$$

Hence, by using again the fact that $S_{\lambda,\Gamma} \to D_\Gamma$ as $\lambda \to \infty$, there is $n_0 > 0$ such that

$$|\Phi_{\lambda_n}(\gamma_o(\overrightarrow{s}, \overrightarrow{t})) - S_{\lambda_n,\Gamma}| > 2\mu \quad \text{for all } (\overrightarrow{s}, \overrightarrow{t}) \in \partial([1/R^2, 1]^{2l}) \text{ and } n \geq n_0,$$

which implies that $\gamma_o(\overrightarrow{s}, \overrightarrow{t}) \notin B_{\lambda_n,2\mu}$ for all $(\overrightarrow{s}, \overrightarrow{t}) \in \partial([1/R^2, 1]^{2l})$ and $n \geq n_0$. From this,

$$\eta_n(T_n, \gamma_o(\overrightarrow{s}, \overrightarrow{t})) = \gamma_o(\overrightarrow{s}, \overrightarrow{t}) \quad \text{for all } (\overrightarrow{s}, \overrightarrow{t}) \in \partial([1/R^2, 1]^{2l}) \text{ and } n \geq n_0.$$

On the other hand, since $\eta_n(T_n, \gamma_o(\overrightarrow{s}, \overrightarrow{t})) \in \Theta_{2\delta}$ for all n, we reach that

$$\mathrm{dist}_{\lambda_n,j}(\eta_n(T_n, \gamma_o(\overrightarrow{s}, \overrightarrow{t})), E^{\pm}_{\lambda_n,j}) \geq \tau - 2\delta > 0.$$

Then, $(\eta_n(T_n, \gamma_o(\overrightarrow{s}, \overrightarrow{t})))^{\pm}|_{\Omega_j} \neq 0$ for all $j \in \Gamma$, and we can conclude that $\eta_n(T_n, \gamma_o(\overrightarrow{s}, \overrightarrow{t}))$ belongs to Σ_{λ_n} for n large enough. Combining the definition of $S_{\lambda,\Gamma}$ with Claim 5.3 and the fact that $\eta_n(T_n, \gamma_o(\overrightarrow{s}, \overrightarrow{t}))$ belongs to Σ_{λ_n} for n large enough, we get the inequality

$$\limsup_{n\to+\infty} S_{\lambda_n,\Gamma} \leq D_\Gamma - \varepsilon^*$$

which contradicts the Proposition 4.2. □

From the last proposition, we have the following result

Corollary 5.4. *For each $\mu \in (0, \mu^*)$ fixed, there exists $\Lambda^* = \Lambda^*(\mu) > 1$ such that $(\mathrm{A})_\lambda$ has a nodal solution $u_\lambda \in B_{\lambda,\mu}$ for all $\lambda \geq \Lambda^*$.*

6. Proof of Theorem 1.1

From Corollary 5.4, for each $\mu \in (0, \mu^*)$ fixed, there exists $\Lambda^* = \Lambda^*(\mu) > 1$ such that $(\mathrm{A})_\lambda$ has a nodal solution $u_\lambda \in B_{\lambda,\mu}$ for $\lambda \geq \Lambda^*$ with

$$\mathrm{dist}_{\lambda,j}(u_\lambda, E^{\pm}_{\lambda,j}) \geq \tau - 2\delta > 0 \quad \text{for all } j \in \Gamma. \tag{6.1}$$

Repeating the same arguments used in the proof of Proposition 3.3, we get

$$u_\lambda \to 0 \quad \text{in } H^1(\mathbb{R}^N \setminus \Omega_\Gamma) \text{ as } \lambda \to \infty.$$

This together with Proposition 3.4 gives u_λ is a nodal solution of $(\mathrm{P})_\lambda$ for λ large enough.

Fixing $\lambda_n \to \infty$ and $\mu_n \to 0$, the sequence $\{u_{\lambda_n}\}$ verifies

$$\Phi'_{\lambda_n}(u_{\lambda_n}) = 0 \quad \text{and} \quad \Phi_{\lambda_n}(u_{\lambda_n}) = S_{\lambda_n,\Gamma} + o_n(1),$$

that is,

$$\Phi'_{\lambda_n}(u_{\lambda_n}) = 0 \quad \text{and} \quad \Phi_{\lambda_n}(u_{\lambda_n}) = D_\Gamma + o_n(1)$$

and, therefore, $\{u_{\lambda_n}\}$ is a $(\mathrm{PS})_{\infty,D_\Gamma}$ sequence. By Proposition 3.3, for some subsequence, still denoted by $\{u_{\lambda_n}\}$, there exists $u \in H_0^1(\Omega_\Gamma)$ such that

$$u_{\lambda_n} \to u \quad \text{in } H^1(\mathbb{R}^N), \quad \lambda_n \int_{\mathbb{R}^N} V(x)|u_{\lambda_n}|^2 \to 0 \quad \text{and} \quad \|u_{\lambda_n}\|^2_{\lambda_n,\mathbb{R}^N\setminus\Omega_\Gamma} \to 0.$$

These facts imply that

$$I_j'(u) = 0 \quad \text{for all } j \in \Gamma \quad \text{and} \quad \sum_{j=1}^{l} I_j(u) = D_\Gamma. \tag{6.2}$$

Once $\{u_{\lambda_n}\}$ verifies (6.1), we derive that $\|u^\pm_{\lambda_n}\|_{\lambda_n,\Omega_j'} \not\to 0$ for all $j \in \Gamma$. Hence, from definition of g, it follows that there is $\tau_* > 0$ such that

$$\int_{\Omega_j'} |u^\pm_{\lambda_n}|^{p+1} \geq \tau_* \quad \text{for all } n \in \mathbb{N} \text{ and all } j \in \Gamma,$$

and thus

$$\int_{\Omega_j} |u^\pm|^{p+1} \geq \tau_* \quad \text{for all } j \in \Gamma.$$

Thereby, u changes signal on Ω_j for all $j \in \Gamma$, and therefore,

$$I_j(u) \geq d_j \quad \text{for all } j \in \Gamma. \tag{6.3}$$

From (6.2) and (6.3) $I_j(u) = d_j$ for all $j \in \Gamma$. This shows that $u|_{\Omega_j}$ is a nodal solution with least energy in Ω_j for each $j \in \Gamma$, and the proof of Theorem 1.1 is complete. □

7. Final remarks

The method used in the present paper can be used to show the existence of multi-bump type solutions joining positive, negative and nodal least energy solutions. The main modifications should be make in the Sections 4 and 5, for example, if you want to get a positive solution w_1 on Ω_1 and a negative solution w_2 on Ω_2, we must to change $w_1^\pm$ and $w_2^\pm$ by w_1 and w_2, respectively. Other modifications must be make in the definition of $S_{\lambda,\Gamma}$ and in the sets $B_{\lambda,\mu}$. Moreover, we need to replace d_1 and d_2 by mountain pass levels c_1 and c_2 associated with the energy functionals I_1 and I_2, respectively. From this, we have the following theorem

Theorem 7.1. *Assume that* (f_1)–(f_4) *and* (H_1)–(H_2) *hold. Then, for any non-empty subsets* Γ_1, Γ_2, Γ_3 *of* $\{1,\ldots,k\}$ *with* $\Gamma_s \cap \Gamma_t = \emptyset$ *for* $s \neq t$, *there exists* $\lambda^* > 0$ *such that, for* $\lambda \geq \lambda^*$, *problem* $(\mathrm{P})_\lambda$ *has a nontrivial solution* u_λ *that satisfies: For any sequence* $\lambda_n \to \infty$, *we can extract a subsequence* λ_{n_i} *such that* $u_{\lambda_{n_i}}$ *converges strongly in* $H^1(\mathbb{R}^N)$ *to a function* u *which satisfies* $u(x) = 0$ *for* $x \notin \Omega_\Gamma = \bigcup_{j\in\Gamma} \Omega_j$ *where* $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, *and the restriction* $u|_{\Omega_j}$ *is* *a positive solution if* $j \in \Gamma_1$, *a negative solution if* $j \in \Gamma_2$ *and a nodal solution if* $j \in \Gamma_3$ *with least energy of the problem*

$$-\Delta u + u = f(u), \quad u|_{\partial \Omega_j} = 0.$$

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Claudianor O. Alves
Universidade Federal de Campina Grande
Departamento de Matemática e Estatística
CEP:58109-970, Campina Grande-PB, BRAZIL

E-mail address: coalves@dme.ufcg.edu.br