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UNIFORM NONSQUARENESS
OF DIRECT SUMS OF BANACH SPACES

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Abstract. An inequality between James constants of Banach spaces X_s and the James constant of their direct sum is obtained. This gives a characterization of uniform nonsquareness of sums of Banach spaces.

1. Introduction

The class of uniformly nonsquare Banach spaces was introduced by James in [7]. He proved that those spaces are super-reflexive, i.e. they admit equivalent uniformly convex norms. In [4] Gao and Lau introduced a coefficient related to uniform nonsquareness, which is called the James constant of a Banach space. Uniform nonsquareness turned out to be useful in metric fixed point theory. The James constant and its modifications give sufficient conditions for normal structure (see [5], [2] and [8]) and even estimates for the normal structure coefficient (see [12]). In [6] it was proved that all uniformly nonsquare Banach spaces have the fixed point property for nonexpansive mappings on bounded closed convex sets. In this paper we deal with the problem of uniform nonsquareness of direct sums of Banach spaces. The basic result in this area was obtained by Kato, Saito and Tamura [9]. They found an equivalent condition for uniform nonsquareness of a direct sum of two Banach spaces. In their paper they asked if it is possible to generalize their result to direct sums of more than two spaces. A sufficient

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condition for uniform nonsquareness of direct sums of finite families of Banach spaces can be found in [3]. In this paper we give an answer to the problem posed in [9]. Following [1, p. 5] we consider a general product $X = (\sum_{s\in S} X_s)_Z$ of a family $\{X_s\}_{s\in S}$ of Banach spaces. We will show an inequality between the James constant $J(X)$ of X and James constants of the spaces X_s and Z. As a corollary we see that $X = (\sum_{s\in S} X_s)_Z$ is uniformly nonsquare if and only if the space Z is uniformly nonsquare and $\sup_{s\in S} J(X_s) < 2$. Our results can be applied in particular to the case when $Z = (Z, \|\cdot\|_Z)$ is a real Banach space with a 1-unconditional basis and (X_n) is a sequence of Banach spaces.

2. Preliminaries

Definition 2.1. A Banach space X is called uniformly nonsquare if there exists $\delta \in (0,1)$ such that for any $x, y \in X$ with $\|x\| = \|y\| = 1$ we have $\|x+y\|/2 \le 1-\delta$ or $\|x-y\|/2 \le 1-\delta$.

In [9] Kato, Saito and Tamura showed a theorem which can be formulated in the following way.

Theorem 2.2. *Let X and Y be Banach spaces and $\|\cdot\|_Z$ be a norm on $\mathbb{R}^2$ such that $\|(1,0)\|_Z = \|(0,1)\|_Z = 1$ and $\|\cdot\|_Z$ is monotone i.e. if $|x| \le |y|$, then $\|x\| \le \|y\|$. Then the following conditions are equivalent.*

(a) *$(X \oplus Y)_Z$ is uniformly nonsquare.*
(b) *X and Y are uniformly nonsquare and $\|\cdot\|_Z$ is different from l_1 and l_∞ norm.*

In their paper they asked about characterization of uniform nonsquareness of direct sums of more than two spaces. In [3] Dhompongsa, Kaewcharoen, Kaewkhao showed that if $X_1, \dots, X_n$ are uniformly nonsquare Banach spaces and $\|\cdot\|_Z$ is a monotone uniformly convex norm on $\mathbb{R}^n$ then $(X_1 \oplus \dots \oplus X_n)_Z$ is uniformly nonsquare.

The James constant, or the nonsquare constant of a Banach space X was defined by Gao and Lau in [4] as

$$J(X) = \sup\{\min\{\|x+y\|, \|x-y\|\} : x, y \in X, \|x\| = \|y\| = 1\}.$$

In this definition condition that $\|x\| = \|y\| = 1$ can be replaced by: x, y belong to the unit ball of X. Clearly, X is uniformly nonsquare if and only if $J(X) < 2$. In [5], Gao and Lau proved that, in general, $\sqrt{2} \le J(X) \le 2$ and X has uniform normal structure provided that $J(X) < 3/2$. Next, in [2] it was shown that $3/2$ can be replaced by $(1+\sqrt{5})/2$ in this theorem. The following refinement of the triangle inequality will be the main tool for the proof of our theorem.

Lemma 2.3 ([11]). *For any nonzero elements x, y in a Banach spaces X we have*

$$\|x+y\| + \left(2 - \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|\right)\min\{\|x\|,\|y\|\} \\ \leq \|x\| + \|y\| \leq \|x+y\| + \left(2 - \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|\right)\max\{\|x\|,\|y\|\}.$$

3. Results

Let $S \neq \emptyset$ be an arbitrary set. Consider the space $\mathrm{Map}\,(S,\mathbb{R})$ of all functions from S to $\mathbb{R}$ with the standard operations and order. Let $(Z, \|\cdot\|_Z)$ be a real Banach space such that Z is a linear subspace of $\mathrm{Map}\,(S,\mathbb{R})$ and if $x \in Z$, $y \in \mathrm{Map}\,(S,\mathbb{R})$ and $|y| \leq |x|$ then $y \in Z$ and $\|y\|_Z \leq \|x\|_Z$. Given a family $\{X_s\}_{s\in S}$ of Banach spaces, we can therefore define the direct sum (see [1, p. 5]) $X = (\sum_{s\in S} X_s)_Z$ as the space of all functions $x = \{x(s)\}_{s\in S}$ where $x(s) \in X_s$ for every $s \in S$ such that $\{\|x(s)\|\}_{s\in S} \in Z$. We endow X with the norm given by the formula

$$\|x\| = \|\{\|x(s)\|\}_{s\in S}\|_Z.$$

Theorem 3.1. *Let $X = (\sum_{s\in S} X_s)_Z$ be as above. Then*

$$J(X) \leq 2 - \frac{1}{2}\left(2 - \sup_{s\in S} J(X_s)\right)(2 - J(Z)).$$

Proof. Observe that

$$\lambda = \frac{1}{2}\left(2 - \sup_{s\in S} J(X_s)\right)$$

belongs to the interval $[0,1]$. For every $\varepsilon > 0$ there exist $u = \{u(s)\}_{s\in S}$, $v = \{v(s)\}_{s\in S}$ in the unit sphere of X such that

$$J(X)(1-\varepsilon) \leq \min\{\|u-v\|, \|u+v\|\}.$$

Let S_1 be the set of all $s \in S$ such that $u(s) = 0$ or $v(s) = 0$ or $u(s)$, $v(s)$ are not zero and

$$\left\|\frac{u(s)}{\|u(s)\|} + \frac{v(s)}{\|v(s)\|}\right\| = \min_{k=\pm 1}\left\|\frac{u(s)}{\|u(s)\|} + k\frac{v(s)}{\|v(s)\|}\right\|.$$

If $s \in S_1$ and $u(s) \neq 0$, $v(s) \neq 0$, then from Lemma 2.3 we obtain

$$\begin{aligned}\|u(s)+v(s)\| &\leq \|u(s)\| + \|v(s)\| \\ &\qquad - \left(2 - \left\|\frac{u(s)}{\|u(s)\|} + \frac{v(s)}{\|v(s)\|}\right\|\right)\min\{\|u(s)\|,\|v(s)\|\} \\ &= \max\{\|u(s)\|,\|v(s)\|\} \\ &\qquad + \left(\left\|\frac{u(s)}{\|u(s)\|} + \frac{v(s)}{\|v(s)\|}\right\| - 1\right)\min\{\|u(s)\|,\|v(s)\|\}\end{aligned}$$

$$
\begin{aligned}
&\le \max\{\|u(s)\|,\|v(s)\|\} + (J(X_s)-1)\min\{\|u(s)\|,\|v(s)\|\} \\
&\le \max\{\|u(s)\|,\|v(s)\|\} + (1-2\lambda)\min\{\|u(s)\|,\|v(s)\|\} \\
&= (1-\lambda)(\|u(s)\|+\|v(s)\|) + \lambda|\|u(s)\|-\|v(s)\||.
\end{aligned}
$$

This gives us the inequality

$$
\|u(s)+v(s)\| \le (1-\lambda)(\|u(s)\|+\|v(s)\|) + \lambda|\|u(s)\|-\|v(s)\|| \tag{3.1}
$$

which holds also if $u(s)=0$ or $v(s)=0$. Consequently, it holds for all $s\in S_1$. Put

$$
z_0(s) = \|u(s)\|+\|v(s)\|, \quad s\in S
$$

and

$$
z_1(s) = \begin{cases} |\|u(s)\|-\|v(s)\|| & \text{for } s\in S_1, \\ \|u(s)\|+\|v(s)\| & \text{for } s\in S_2, \end{cases}
$$

where $S_2 = S\setminus S_1$. We have

$$
\begin{aligned}
&(1-\lambda)z_0(s)+\lambda z_1(s) \\
&\qquad = \begin{cases} (1-\lambda)(\|u(s)\|+\|v(s)\|)+\lambda|\|u(s)\|-\|v(s)\|| & \text{for } s\in S_1, \\ \|u(s)\|+\|v(s)\| & \text{for } s\in S_2, \end{cases}
\end{aligned}
$$

so, by (3.1), we get

$$
\begin{aligned}
J(X)(1-\varepsilon) &\le \|u+v\| \le \|(1-\lambda)z_0+\lambda z_1\|_Z \\
&\le (1-\lambda)(\|u\|+\|v\|)+\lambda\|z_1\|_Z = 2(1-\lambda)+\lambda\|z_1\|_Z.
\end{aligned}
$$

This gives us the inequality

$$
J(X)(1-\varepsilon) - 2(1-\lambda) \le \lambda\|z_1\|_Z. \tag{3.2}
$$

If $s\in S_2$, then $u(s)\neq 0$, $v(s)\neq 0$ and

$$
\left\|\frac{u(s)}{\|u(s)\|}-\frac{v(s)}{\|v(s)\|}\right\| = \min_{k=\pm 1}\left\|\frac{u(s)}{\|u(s)\|}+k\frac{v(s)}{\|v(s)\|}\right\|.
$$

A reasoning similar to the previous one shows that

$$
J(X)(1-\varepsilon) - 2(1-\lambda) \le \lambda\|z_2\|_Z \tag{3.3}
$$

where

$$
z_2(s) = \begin{cases} \|u(s)\|+\|v(s)\| & \text{for } s\in S_1, \\ |\|u(s)\|-\|v(s)\|| & \text{for } s\in S_2. \end{cases}
$$

Put

$$
y_1(s) = \|u(s)\|, \quad s\in S, \qquad y_2(s) = \begin{cases} \|v(s)\| & \text{for } s\in S_1, \\ -\|v(s)\| & \text{for } s\in S_2. \end{cases}
$$

Then $\|y_1\|_Z = \|u\| = 1$ and $\|y_2\|_Z = \|v\| = 1$. Moreover,

$$
|y_1-y_2|(s) = \begin{cases} |\|u(s)\|-\|v(s)\|| & \text{for } s\in S_1, \\ \|u(s)\|+\|v(s)\| & \text{for } s\in S_2, \end{cases}
$$

which shows that $\|y_1 - y_2\|_Z = \|z_1\|_Z$ and

$$|y_1 + y_2|(s) = \begin{cases} \|u(s)\| + \|v(s)\| & \text{for } s \in S_1, \\ |\|u(s)\| - \|v(s)\|| & \text{for } s \in S_2, \end{cases}$$

which implies that $\|y_1 + y_2\|_Z = \|z_2\|_Z$. From (3.2) and (3.3) we therefore see that

$$\lambda J(Z) \geq \lambda \min\{\|y_1 - y_2\|_Z, \|y_1 + y\|_Z\} \geq J(X)(1 - \varepsilon) - 2(1 - \lambda).$$

Passing to the limit with $\varepsilon \to 0$, we obtain $\lambda J(Z) \geq J(X) - 2(1 - \lambda)$ which means that

$$J(X) \leq \lambda J(Z) + 2(1 - \lambda) = 2 - \frac{1}{2}\Big(2 - \sup_{s \in S} J(X_k)\Big)(2 - J(Z)). \qquad \square$$

It is easy to see that all spaces X_s and Z are isometric to subspaces of $(\sum_{s \in S} X_s)_Z$. In view of Theorem 3.1 this gives us the following result.

COROLLARY 3.2. *Let* $X = (\sum_{s \in S} X_s)_Z$ *be as in Theorem* 3.1. *Then* X *is uniformly nonsquare if and only if the space* Z *is uniformly nonsquare and* $\sup_{s \in S} J(X_s) < 2$.

We say that a Schauder basis (e_k) of a real space Z is unconditional if whenever the series $\sum_{k=1}^{\infty} a_k e_k$ converges, it converges unconditionally, i.e. $\sum_{k=1}^{\infty} a_{\pi(k)} \cdot e_{\pi(k)}$ converges for any permutation π of $\mathbb{N}$. In this case

$$\lambda = \sup\left\{\left\|\sum_{k=1}^{\infty} \varepsilon_k a_k e_k\right\| : \left\|\sum_{k=1}^{\infty} a_k e_k\right\| = 1,\ \varepsilon_k = \pm 1\right\} < \infty$$

and we say that (e_k) is λ-unconditional. In the same way we can treat finite bases. Let Z be a space with a 1-unconditional basis (e_k). We can identify a vector $x = \sum_{k=1}^{\infty} a_k e_k$ with the sequence (a_k). Thus Z can be seen as a subspace of the space $\mathrm{Map}\,(\mathbb{N}, \mathbb{R})$. Since the basis (e_k) is 1-unconditional, the norm of Z is monotone (see [10, p. 19]). We can therefore consider the direct sum $X = (\sum_{k=1}^{\infty} X_k)_Z$ of a sequence (X_k) of Banach spaces. In this case an element $x \in X$ is of the form $x = (x_k)$, where $x_k \in X_k$ for every $k \in \mathbb{N}$ and

$$\|x\| = \left\|\sum_{k=1}^{\infty} \|x_k\| e_k\right\|.$$

So we obtain the following particular case of Theorem 3.1.

THEOREM 3.3. *Let* $Z = (Z, \|\cdot\|_Z)$ *be a real Banach space with a* 1-*unconditional basis and* (X_k) *be a sequence of Banach spaces. Put* $X = (\sum_{k=1}^{\infty} X_k)_Z$. *Then*

$$J(X) \leq 2 - \frac{1}{2}\Big(2 - \sup_{k \in \mathbb{N}} J(X_k)\Big)(2 - J(Z)).$$

Considering the case $Z = \mathbb{R}^n$, we obtain the following result.

Corollary 3.4. *Let $\|\cdot\|_Z$ be a monotone norm in $\mathbb{R}^n$ and $X_1, \ldots, X_n$ be Banach spaces. Put $Z = (\mathbb{R}^n, \|\cdot\|_Z)$ and $X = (X_1 \oplus \ldots \oplus X_n)_Z$. Then*

$$J(X) \le 2 - \frac{1}{2}\Big(2 - \max_{1\le k\le n} J(X_k)\Big)(2 - J(Z)).$$

Consequently, X is uniformly nonsquare if and only if all spaces $X_1, \ldots, X_n, Z$ are uniformly nonsquare.

Let $\|\cdot\|$ be a monotone norm in $\mathbb{R}^2$ such that $\|(1,0)\| = \|(0,1)\| = 1$. Then $(\mathbb{R}^2, \|\cdot\|)$ is uniformly nonsquare if and only if $\|\cdot\|$ is different from l_1 and l_∞ norm. Theorem 2.2 can be therefore seen as a partial case of Corollary 3.4.

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