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FUNCTION BASES FOR TOPOLOGICAL VECTOR SPACES

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Abstract. Our main interest in this work is to characterize certain operator spaces acting on some important vector-valued function spaces such as $(V_a)_{c_0}^{a\in\mathbb{A}}$, by introducing a new kind basis notion for general Topological vector spaces. Where $\mathbb{A}$ is an infinite set, each V_a is a Banach space and $(V_a)_{c_0}^{a\in\mathbb{A}}$ is the linear space of all functions $x\colon\mathbb{A}\to\bigcup V_a$ such that, for each $\varepsilon>0$, the set $\{a\in\mathbb{A}:\|x_a\|>\varepsilon\}$ is finite or empty. This is especially important for the vector-valued sequence spaces $(V_i)_{c_0}^{i\in\mathbb{N}}$ because of its fundamental place in the theory of the operator spaces (see, for example, [12]).

1. Introduction

The absence of a basis notion in vector-valued function spaces causes many deficiencies in the structural and geometric characterizations of the spaces. In those spaces having infinite dimension some important topics such as the approximation property, separability, representation of operators and every topics in which the bases are used cannot be investigated sufficiently. Especially, in the representations of the spaces of continuous linear operators on some Banach spaces bases play a crucial role. For example, if Y is an arbitrary Banach space, then the operator space $\mathcal{L}(\ell_1,Y)$ is equivalent, by the mapping $u\to\{u(e_n)\}$,

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to the space

$$l_\infty(Y) = \left\{ \{y_n\} \subset Y : \sup_{1 \le n < \infty} \|y_n\| < \infty \right\},$$

endowed with the norm $\|\{y_n\}\| = \sup_{1 \le n < \infty} \|y_n\|$ where $\{e_n\}$ is the unit vector basis of ℓ_1. Further, for example, a continuous linear operator T from ℓ_∞ into ℓ_1 is equivalent, by the mapping $T \to \{T^*(f_n)\}$, to a sequence $\{y_n\} \subset ba(\mathbb{N}, \sum) = \ell_\infty^*$ such that $\sum_n y_n$ is weakly* unconditionally Cauchy, [13, pp. 167], where $\{f_n\}$ is the sequence of associate coordinate functionals to the unit vector basis $\{e_n\}$ of ℓ_1 and T^* is the adjoint operator of T. A reasonable questions to ask at this point is what is the extension of this results to the vector-valued function spaces $\ell_1(\mathbb{A}, X)$ and $c_0(\mathbb{A}, X)$? More generally, what is the characterizations of the operators between the Banach function spaces $\lambda(\mathbb{A}, X)$? Where λ is the any of the spaces ℓ_∞, c_0 or $\ell_p, p \ge 1$, $\mathbb{A}$ is an infinite set, X is a Banach space and the definitions of the spaces $\lambda(\mathbb{A}, X)$ are given in the next section. We shall seek an answer to this questions for more general cases. The answer contains many difficulties since the vector-valued function spaces such as $\ell_p(\mathbb{A}, X)$ and $c_0(\mathbb{A}, X)$ doesn't have a basis in general. Such questions on the structural and geometric properties of these spaces lead us to seek an extended definition of the classical basis notion.

Geometric and other structural properties of some vector-valued function spaces has been studied intensively in the last twenty years. Especially, the space of all bounded X-valued functions has many literature. Some important references on $\ell_\infty(\mathbb{A}, X)$ are [2]–[5], [7]. In general, these spaces have derived properties from X. For example, there are many examples, [3], [5], of normed spaces of $\ell_\infty(\mathbb{A}, X)$ which are barrelled whenever X is barrelled. Also, unordered Baire-like properties of these spaces and of its subspaces also depend on the space X (see [4] and [7]).

Further, Ferrando and Lüdkovsky, [6], have considered the function space $c_0(\mathbb{A}, X)$ which is an important extension of the classical Banach space c_0. They proved again that $c_0(\mathbb{A}, X)$ is either barreled, ultrabornological or unordered baire-like if and only if X is, respectively. An analogous result for the X-valued sequence space $c_0(X) = c_0(\mathbb{N}, X)$ was obtained earlier by J. Mendoza [11]. In [16], for a locally convex space X, we deal with the function spaces $\ell_1(\mathbb{A}, X)$ and $c_0(\mathbb{A}, X)$ and study some geometric and structural properties such as the separability and linear functionals on them. Furthermore, in [17], we characterized the continuous operators from an arbitrary Banach spaces to the any of these spaces by introducing a new notation as relative adjoint operators.

Our main interest in this work to give a systematic procedure on the structural investigation of some improved forms of mentioned function spaces and to characterize certain operators between them by means of the new basis notion.

2. Prerequisites

We use the notations $\mathbb{N}$, $\mathbb{C}$ and $\mathbb{R}$ for the sets of all positive integers, complex numbers and real numbers, respectively. For normed spaces X and Y over $\mathbb{C}$, $\mathcal{L}(X,Y)$ denotes the space of all continuous linear operators from X into Y while $\mathcal{L}(X) = \mathcal{L}(X,X)$ and X^* denotes the continuous dual of X. We shall write $\mathcal{L}_{\mathrm{SOT}}(X,Y)$ whenever we consider $\mathcal{L}(X,Y)$ with the strong operator topology instead of uniform (norm) topology. Further, we denote by B_X and S_X the closed unit ball and sphere of X, respectively.

Let us now introduce the notion of (unordered) infinite sums in a topological vector space (TVS, for short). Let $\mathbb{A}$ be an infinite set, $\{x_a : a \in \mathbb{A}\}$ be a family of vectors in a topological vector space (TVS) X and let $\mathcal{F}$ denote the family of all finite subsets of $\mathbb{A}$. $\mathcal{F}$ is directed by the inclusion relation $\subseteq$ and, for each $F \in \mathcal{F}$, we can form the finite sum $S_F = \sum_{a\in F} x_a$. If the net $(S_F : \mathcal{F})$ converges to some x in normed space X, then we say that the family $\{x_a : a \in \mathbb{A}\}$ is summable, or that the sum $\sum_{a\in\mathbb{A}} x_a$ exists, and we write $\sum_{a\in\mathbb{A}} x_a = x$ in X. We mean by the convergence of the net $(S_F : \mathcal{F})$ to x in X that, for each $\varepsilon > 0$, there exists an $F_0 = F_0(\varepsilon) \in \mathcal{F}$ such that $\|S_F - x\| < \varepsilon$ whenever $F_0 \subseteq F$. This also implies that the net $(S_F : \mathcal{F})$ is bounded. Note that, some convergent nets may not be bounded in general. The definition of summable family does not involve any ordering of the index set $\mathbb{A}$, and we may therefore say that the notion of a sum thus defined is commutative (unconditional). In case $\mathbb{A} = \mathbb{N}$, that the family $\{x_n : n \in \mathbb{N}\}$ is summable to x is equivalent to the series $\sum_{n\in\mathbb{N}} x_n$ is unconditionally convergent to x. Note that, a series $\sum_{n\in\mathbb{N}} x_n$ in a TVS is said to be unconditionally convergent if and only if for each permutation σ of $\mathbb{N}$, $\sum_{n\in\mathbb{N}} x_{\sigma(n)}$ is convergent. The definition of a convergent series, essentially, involves the order structure of $\mathbb{N}$. If the series $\sum_{n\in\mathbb{N}} x_n$ is convergent, and if σ is a permutation of $\mathbb{N}$, then the series $\sum_{n\in\mathbb{N}} x_{\sigma(n)}$ may not be convergent, that is, $\{x_n : n \in \mathbb{N}\}$ may not be summable [1, p. 270]. Hence summability of the family $\{x_n : n \in \mathbb{N}\}$ in X is stronger than the convergence of the series $\sum_{n\in\mathbb{N}} x_n$ in this sense.

Let a family $\{V_a : a \in \mathbb{A}\}$ of topological spaces be given. The product $\prod_{a\in\mathbb{A}} V_a$ is the set of all functions $x\colon \mathbb{A} \to \bigcup V_a$ such that $x(a) \in V_a$, for each $a \in \mathbb{A}$. Usually, we use the notation x_a for $x(a)$ since it is more convenient. By $V^{\mathbb{A}}$ we mean $\prod_{a\in A} V_a$ with $V_a = V$ for each $a \in \mathbb{A}$. Let us define $P_a\colon \prod_{a\in\mathbb{A}} V_a \to V_a$ by $P_a(x) = x_a$. This is called the projection on the a-th factor, or briefly, is called a-th-projection.

Let each V_a be a normed space over $\mathbb{C}$. Some important linear subspaces of $\prod_{a\in\mathbb{A}} V_a$ is given as follows; The space $(V_a)_{\ell_\infty}^{a\in\mathbb{A}}$ is the linear space of all functions $x\colon \mathbb{A} \to \bigcup V_a$ such that

$$\sup\{\|x_a\| : a \in \mathbb{A}\} < \infty.$$

Moreover, $(V_a)_{\ell_\infty}^{a\in\mathbb{A}}$ is a normed space with the sup norm

$$\|x\|_\infty = \sup\{\|x_a\| : a \in \mathbb{A}\}.$$

We denote by $(V_a)_{\ell_p}^{a\in\mathbb{A}}$, $1 \le p < \infty$ the set of all functions $x\colon \mathbb{A} \to \bigcup V_a$ such that $\{\|x_a\|^p : a \in \mathbb{A}\}$ is summable, i.e.

$$\sum_{a\in\mathbb{A}} \|x_a\|^p < \infty,$$

and it is a normed space with the norm

$$\|x\|_p = \left(\sum_{a\in\mathbb{A}} \|x_a\|^p\right)^{1/p}.$$

Also, the function space $(V_a)_{c_0}^{a\in\mathbb{A}}$ is the linear space of all functions $x\colon \mathbb{A} \to \bigcup V_a$ such that, for each $\varepsilon > 0$, the set

$$\{a \in \mathbb{A} : \|x_a\| > \varepsilon\}$$

is finite or empty. It is a normed space with the sup norm. A special case of $(V_a)_\lambda^{a\in\mathbb{A}}$ ($\lambda = \ell_\infty$, c_0 or ℓ_p) is $\lambda(\mathbb{A}, X)$ corresponding $V_a = X$ for each $a \in \mathbb{A}$. These function spaces are Banach spaces if and only if each V_a is a Banach space. Moreover, if $\mathbb{A}$ is a directed set then $(V_a)_\lambda^{a\in\mathbb{A}}$ is a linear space of all nets with corresponding property. They are usually denoted by $\lambda(X)$ in the case $\mathbb{A} = \mathbb{N}$ and $V_a = X$ for each $a \in \mathbb{A}$ and are called X-valued sequence spaces.

3. Function bases

One of the our main results is the following definition. It brings a new point of view to the basis notion.

DEFINITION 3.1. Let (X, T) be a TVS, $\mathbb{A}$ be a set and $\{V_a : a \in \mathbb{A}\}$ be a family of TVSs. A family $\{\eta_a : a \in \mathbb{A}\}$ of continuous linear functions $\eta_a\colon V_a \to X$ is called *an unordered* (*or unconditional*) *function basis for X with respect to the family* $\{V_a\}$, or briefly, is called *a uf-basis for X* (whenever $\{V_a\}$ and $\mathbb{A}$ are clear from context) if the following condition is satisfied. There exists a unique family $\{R_a : a \in \mathbb{A}\}$ of linear functions R_a from X onto V_a such that for each $x \in X$ the family $\{(\eta_a \circ R_a)(x) : a \in \mathbb{A}\}$ is summable, that is, the associate finite sums net $(\pi_F(x) : \mathcal{F})$ converges to x in X. Where, for each $F \in \mathcal{F}$,

$$\pi_F(x) = \sum_{a\in F} (\eta_a \circ R_a)(x),$$

and $\mathcal{F}$ is the directed family (by the inclusion relation $\subseteq$) of all finite subsets of the index set $\mathbb{A}$. Furthermore, $\{\eta_a\}$ is called *a uf-Schauder basis for X* whenever each R_a is continuous.

REMARK 3.2. In the special case $\mathbb{A} = \mathbb{N}$, convergence of the net $(\pi_F(x) : \mathcal{F})$ to x corresponds to the unconditional convergence of the series $\sum_{n=1}^{\infty}(\eta_n \circ R_n)(x)$ to x. In this case, summability of $\{(\eta_n \circ R_n)(x) : n \in \mathbb{N}\}$ is independent of the usual ordered structure of $\mathbb{N}$. If there exists a unique sequence $\{R_n\}$ such that the series $\sum_{n=1}^{\infty}(\eta_n \circ R_n)(x)$ converges to x (in the usual manner) then, by throwing the letter u, we say that $\{\eta_n\}$ is *an f-basis*. In this case the convergence of the series may not be unconditional.

REMARK 3.3. In Definition 3.1, the word *uf-basis for X* is replaced by the word *up-basis* whenever X is a vector subspace of the product space $\prod_{a\in\mathbb{A}} V_a$. Also, we use the word *up-Schauder basis* whenever each R_a is continuous.

DEFINITION 3.4. The family $\{R_a : a \in \mathbb{A}\}$ is called *associate family of functions* (A.F.F.) to the uf-basis $\{\eta_a : a \in \mathbb{A}\}$.

Let $\{\eta_a : a \in \mathbb{A}\}$ be a uf-basis for X with respect to the family $\{V_a\}$. Clearly, the finite summation $\pi_F(x)$ defines an operator π_F on X. This operator is called *F-projection* on X and it is continuous whenever $\{\eta_a\}$ is a uf-Schauder basis.

THEOREM 3.5. *Let $\mathbb{A}$ be a set and each V_a be a normed space. Then $(V_a)_{c_0}^{a\in\mathbb{A}}$ and $(V_a)_{\ell_p}^{a\in\mathbb{A}}$ has up-Schauder bases for every choice of the family $\{V_a : a \in \mathbb{A}\}$.*

PROOF. Define

$$I_a \colon V_a \to (V_a)_{\rho}^{a\in\mathbb{A}}, \quad \rho = c_0 \quad \text{or} \quad \ell_p,$$

by $I_a(t) = y$ such that $y(\delta) = t$ if $\delta = a$ otherwise $y(\delta) = 0$. Then the family $\{I_a : a \in \mathbb{A}\}$ is an up-Schauder basis for $(V_a)_{\rho}^{a\in\mathbb{A}}$. We prove the assertion only for $(V_a)_{c_0}^{a\in\mathbb{A}}$ since the proof for $(V_a)_{\ell_p}^{a\in\mathbb{A}}$ almost is the same. Choose the family $\{R_a : a \in \mathbb{A}\}$ as

$$R_a \colon (V_a)_{c_0}^{a\in\mathbb{A}} \to V_a; \quad R_a(x) = P_a(x) = x_a.$$

Then, we must show that the finite sums net $(\pi_F(x) : \mathcal{F})$ converges to x in $(V_a)_{c_0}^{a\in\mathbb{A}}$ where

$$\pi_F(x) = \sum_{a\in F}(I_a \circ P_a)(x) = \sum_{a\in F} I_a(x_a).$$

But, for some $x \in (V_a)_{c_0}^{a\in\mathbb{A}}$, $\pi_F(x)$ is the function x_F such that $x_F(a) = x_a$ if $a \in F$ and $x_F(a) = 0$ if $a \notin F$. Now, consider an arbitrary $\varepsilon > 0$. We must find a finite subset $F_0 = F_0(\varepsilon) \in \mathcal{F}$ such that, for each finite $F \supseteq F_0$,

$$\|x - x_F\|_{\infty} \leq \varepsilon.$$

Since the set $\{a \in \mathbb{A} : \|x_a\| > \varepsilon\}$ is finite or empty, taking F_0 as

$$F_0 = \{a \in \mathbb{A} : \|x_a\| > \varepsilon\},$$

we have

$$\|x - x_F\|_\infty = \|\{x_a : a \in \mathbb{A} - F\}\|_\infty \leq \varepsilon,$$

for each finite $F \supseteq F_0$.

For the uniqueness of the family $\{P_a\}$ suppose

$$\sum_{a\in\mathbb{A}} (I_a \circ P_a)(x) = \sum_{a\in\mathbb{A}} (I_a \circ R'_a)(x)$$

and

$$\pi^\circ_F(x) = \sum_{a\in F} (I_a \circ (P_a - R'_a))(x), \quad F \in \mathcal{F}.$$

Remember that $\|\pi^\circ_F(x)\|_\infty = \sup_{a\in F} \|(P_a - R'_a)(x)\|$ and $\|\pi^\circ_F(x)\|_\infty \leq \|\pi^\circ_G(x)\|_\infty$ for $F \subseteq G$. Since

$$\lim_{F\in\mathcal{F}} \|\pi^\circ_F(x)\|_\infty = 0,$$

we have that $(P_a - R'_a)(x) = 0$ for each a and for every $x \in (V_a)^{a\in\mathbb{A}}_{c_0}$. This implies, $P_a = R'_a$ for each a.

Further, each P_a is continuous because $\|x_a\| \leq \|x\|_\infty$. □

REMARK 3.6. Let X be a TVS over the field $\mathbb{C}$ possessing a basis $\{x_n\}$ (in the classical manner). Then the sequence $\{\eta_n\}$ of the functions

$$\eta_n : \mathbb{C} \to X : \eta_n(t) = t x_n$$

is an f-basis for X with respect to (the family) $\mathbb{C}$. Indeed; taking $\{R_n\}$ as the sequence of associate coordinate functionals (f_n) to the basis $\{x_n\}$, we obtain that

$$\sum_{n=1}^{\infty} (\eta_n \circ R_n)(x) = \sum_{n=1}^{\infty} f_n(x) x_n,$$

and so, it converges to x for each $x \in X$. Moreover, $\{\eta_n\}$ is a uf-basis for X if and only if $\{x_n\}$ is an unconditional basis of X.

It is classical that a TVS having a basis is separable. An evolution of this result is:

THEOREM 3.7. *Let X be a* TVS *for which a family $\{\eta_a : a \in \mathbb{A}\}$ be an uf-basis with respect to the family $\{V_a\}$ of separable* TVS*s. Then, X is separable if $\mathbb{A}$ is countable.*

PROOF. Let $\mathbb{A}$ be countable and W_a be countable dense subspace of V_a for each a. Then $\bigcup_{a\in\mathbb{A}} \eta_a(W_a)$ is countable dense subspace of X since the image of countable sets under a function is also countable, and a union of a countable family of countable sets is countable [8]. Furthermore, for some $x \in X$, there exists a net $\{x^a_\delta\}_\delta$ in W_a such that $x^a_\delta \to R_a(x)$ in V_a. So

$$\eta_a(x^a_\delta) \to (\eta_a \circ R_a)(x)$$

in X for each a. This implies

$$\sum_{a\in F}\eta_a(x^a_\delta)\to\sum_{a\in F}(\eta_a\circ R_a)(x)=\pi_F(x)\quad\text{for each } F\in\mathcal{F}.$$

Hence each neighbourhood $U_{\pi_F(x)}$ of $\pi_F(x)$ includes an element $\sum_{a\in F}\eta_a(x^a_\delta)$ of $\bigcup_{a\in\mathbb{A}}\eta_a(W_a)$. On the other hand, each neighbourhood U_x of x includes a neighbourhood $U_{\pi_{F_0}(x)}$ of some $\pi_{F_0}(x)$ since the net $(\pi_F(x),\mathcal{F})$ converges to x in X. Consequently, U_x includes an element of $\bigcup_{a\in\mathbb{A}}\eta_a(W_a)$. This shows that $\bigcup_{a\in\mathbb{A}}\eta_a(W_a)$ is dense in X. □

Another important attribute of classical bases is the fact that a TVS having a basis is considered as a sequence spaces. The analogue of this conjecture in the new definition is that if $\{\eta_a : a\in\mathbb{A}\}$ is a uf-basis for X with respect to the family $\{V_a\}$ then we consider X to be a subspace of $\prod_{a\in\mathbb{A}}V_a$. This is the content of the following theorem.

Theorem 3.8. *Let X be a* TVS *and $\{\eta_a : a\in\mathbb{A}\}$ be a uf-basis for X with respect to the family $\{V_a\}$ of* TVSs*. Then, X is linearly homeomorphic with a subspace Y of $\prod_{a\in\mathbb{A}}V_a$ such that $\{I_a\}$ is a up-basis for Y (with the family $\{P_a\}$ of a-th-projections as the* A.F.F. *to $\{I_a\}$). Where I_a is defined by $I_a(t)=\omega$ such that*

$$\omega(b)=\begin{cases}R_a(x) & \textit{for some } x\in X,\ \textit{if } a=b,\\ 0 & \textit{otherwise,}\end{cases}$$

and $\{R_a\}$ is the A.F.F. *to the uf-basis $\{\eta_a\}$. Further $\{\eta_a\}$ is Schauder basis if and only if $\{I_a\}$ is.*

Proof. Let

$$Y=\left\{y\in\prod_{a\in\mathbb{A}}V_a : y_a=R_a(x)\text{ for some } x\in X,\right\}$$

and define $T\colon X\to Y$ by $Tx=y$. Clearly T is a linear isomorphism. Further, by imposing on Y the quotient topology $\mathcal{QT}$ by the operator T, we identified X with Y as topologically since T is continuous in $\mathcal{QT}$. Moreover,

$$\begin{aligned}y=Tx&=\sum_{a\in\mathbb{A}}(T\circ\eta_a\circ R_a)(x)=\sum_{a\in\mathbb{A}}(T\circ\eta_a)(R_a(x))\\&=\sum_{a\in\mathbb{A}}(T\circ\eta_a)(y_a)=\sum_{a\in\mathbb{A}}(T\circ\eta_a\circ P_a)(y)=\sum_{a\in\mathbb{A}}(I_a\circ P_a)(y)\end{aligned}$$

where P_a is the a-th-projection on the product space $\prod_{a\in\mathbb{A}}V_a$ and $I_a=T\circ\eta_a$. Hence $\{I_a\}$ is a up-basis. Further, P_a is continuous from $(Y,\mathcal{QT})$ to V_a if and only if R_a is continuous from X to V_a. □

An outstanding contribution of the classical bases to the structural investigation of Banach spaces is the approximation property (AP, for short). Note that, a Banach space X is said to have AP if, for every compact set K in X and every $\varepsilon > 0$, there exists an operator $T\colon X \to X$ of finite rank (i.e. $T(x) = \sum_{i=1}^{n} x_i^*(x)x_i$, for some $\{x_i\}_{i=1}^n \subset X$ and $\{x_i^*\}_{i=1}^n \subset X^*$) so that $\|T(x) - x\| \le \varepsilon$, for every $x \in K$. It is classical that a Banach space having a basis also has the AP. However, this is directly depend on the family $\{V_a\}$ here.

THEOREM 3.9. *Let X be a Banach space for which $\{\eta_a : a \in \mathbb{A}\}$ be a uf-Schauder basis with respect to the family $\{V_a\}$ of Banach spaces. Then X has the approximation property if and only if each V_a has.*

PROOF. It follows from the above theorem that we may consider X as to be a subspace of $\prod_{a\in\mathbb{A}} V_a$ with the up-basis $\{I_a\}$ and with the A.F.F. $\{P_a\}$. Let X has the AP and fix some $a \in \mathbb{A}$. Suppose some $K \subset V_a$ is compact and let $\varepsilon > 0$. Since $I_a(K)$ is also compact in X, there exists a finite rank operator T on X such that

$$\|T(I_a(y)) - I_a(y)\| \le \varepsilon$$

for every $y \in K$. This implies

$$\begin{aligned} \|(P_a \circ T \circ I_a)(y) - y\| &= \|(P_a \circ T \circ I_a)(y) - (P_a \circ I_a)(y)\| \\ &\le \|P_a\|\|T(I_a(y)) - I_a(y)\| \le \varepsilon. \end{aligned}$$

Hence $P_a \circ T \circ I_a$ is the desired finite rank operator (corresponding K and ε).

Conversely, let each V_a has the AP. It isn't hard to see that any finite direct sum of Banach spaces has the AP if and only if each component space has this property. Let $K \subset X$ be compact and $\varepsilon > 0$. There exists an $F_0(\varepsilon) \in \mathcal{F}$, such that

$$\|\pi_F(x) - x\| \le \varepsilon/2, \quad \text{for each finite } F \supseteq F_0,$$

since the net $(\pi_F(x), \mathcal{F})$ converges to x in X. Further, for each $F \in \mathcal{F}$, $\pi_F(K)$ is a compact subset of $\bigoplus_{a\in F} V_a$ and so there exists a finite rank operator T on $\bigoplus_{a\in F} V_a$ such that

$$\|T(\pi_F(x)) - \pi_F(x)\| \le \varepsilon/2,$$

for every $x \in K$. Thus, for each finite $F \supseteq F_0$,

$$\begin{aligned} \|(T \circ \pi_F)(x) - x\| &= \|T(\pi_F(x)) - \pi_F(x) + \pi_F(x) - x\| \\ &\le \|T(\pi_F(x)) - \pi_F(x)\| + \|\pi_F(x) - x\| \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

This implies, each $T \circ \pi_F$ such that $F \supseteq F_0$ is the desired finite rank operator.□

Now, let us introduce the notion f-biorthogonal systems in Banach spaces.

DEFINITION 3.10. Let $\{V_a : a \in \mathbb{A}\}$ be a family of Banach spaces, X be an arbitrary Banach space, $\{\eta_a : a \in \mathbb{A}\} \in \prod_{a\in\mathbb{A}} \mathcal{L}(V_a, X)$ and $\{R_a : a \in \mathbb{A}\} \in$

$\prod_{a\in\mathbb{A}}\mathcal{L}(X,V_a)$. If $R_a\circ\eta_b=I_{V_a}$ (identity of V_a) whenever $a=b$, and $R_a\circ\eta_b=0$ (zero operator from V_b to V_a) whenever $a\neq b$, then the family

$$\{(\eta_a,R_a):a\in\mathbb{A}\}$$

is called *an f-biorthogonal system for X with respect to the family $\{V_a\}$*, or briefly, is called *an f-biorthogonal system for X*. Further, we use the term *p-biorthogonal system for X* whenever X is a subspace of $\prod_{a\in\mathbb{A}}V_a$.

PROPOSITION 3.11. *Let $\{\eta_a:a\in\mathbb{A}\}$ be a uf-basis for X. Then, the family*

$$\{(\eta_a,R_a):a\in\mathbb{A}\}$$

is an f-biorthogonal system for X where $\{R_a\}$ is the A.F.F. *to the uf-basis.*

PROOF. For an arbitrary $v\in V_b$, $\eta_b(v)\in X$, and so

$$\eta_b(v)=\sum_{a\in\mathbb{A}}(\eta_a\circ R_a)(\eta_b(v))=\sum_{a\in\mathbb{A}}\eta_a[(R_a\circ\eta_b)(v)].$$

Hence, by the uniqueness of the family $\{R_a\}$, this equality holds if and only if $R_a\circ\eta_b=I_{V_a}$ for $a=b$, and $R_a\circ\eta_b=0$ for $a\neq b$. □

THEOREM 3.12. *Let X be a Banach space for which $\{\eta_a\}$ be a uf-basis for X with respect to the family $\{V_a\}$ of Banach spaces. Then*

$$\|x\|^\diamond=\sup_{F\in\mathcal{F}}\{\|\pi_F(x)\|\}$$

defines a norm on X, equivalent to the initial norm of X, and

$$\|x\|\leq\|x\|^\diamond\leq K\|x\|,\quad x\in X,$$

for some $K>0$.

The proof is quite similar to the its classical analogue (see [9] and [13]).

DEFINITION 3.13. The number $\sup_{F\in\mathcal{F}}\|\pi_F\|$ is called the *basis constant* of the uf-basis $\{\eta_a:a\in\mathbb{A}\}$ and a uf-basis whose basis constant is 1 is called *monotone.*

EXAMPLE 3.14. Up-bases of $(V_a)^{a\in\mathbb{A}}_{c_0}$ and $(V_a)^{a\in\mathbb{A}}_{\ell_p}$ given in Theorem 3.5 are monotone. Let us prove the assertion only for $(V_a)^{a\in\mathbb{A}}_{c_0}$. Fix some $v_a\in S_{V_a}$ and define $x^a\colon\mathbb{A}\to V_a$, for some $a\in\mathbb{A}$, by

$$x^a(b)=\begin{cases}v_a & \text{if } b=a,\\ 0 & \text{if } b\neq a.\end{cases}$$

Then, for some $F\in\mathcal{F}$ containing a, we have

$$\pi_F(x^a)=\sum_{b\in F}(I_b\circ P_b)(x^a)=x^a.$$

Thus, for some $x^a \in B_{(V_a)_{c_0}^{a\in\mathbb{A}}}$, $\|\pi_F(x^a)\|_\infty = \|x^a\|_\infty$. Also, for every $x \in B_{(V_a)_{c_0}^{a\in\mathbb{A}}}$,

$$\|\pi_F(x)\|_\infty = \left\|\sum_{a\in F}(I_a \circ P_a)(x)\right\|_\infty \le \left\|\sum_{a\in\mathbb{A}}(I_a \circ P_a)(x)\right\|_\infty = \|x\|_\infty,$$

whence, we have $\|\pi_F\| = 1$ for each $F \in \mathcal{F}$.

4. Function bases and operators spaces

In this section we deal with Banach spaces and continuous operators between them. The object is to give a general conjecture characterizing certain operator spaces acting on some important vector-valued function spaces such as $(V_a)_{c_0}^{a\in\mathbb{A}}$ and $(V_a)_{\ell_p}^{a\in\mathbb{A}}$. This is especially important for the vector-valued sequence spaces $(V_i)_{c_0}^{i\in\mathbb{N}}$ because of its fundamental place in the theory of the operator spaces (see, for example, [12]).

Following theorem which is a generalization of the result given in [13, p. 162] is one of the main yields of this work.

THEOREM 4.1. *Let X be a Banach space for which $\{\eta_a : a \in \mathbb{A}\}$ be a uf-Schauder basis with respect to the family $\{V_a\}$ of Banach spaces and Y be an arbitrary Banach space. Then $\mathcal{L}(X,Y)$ is topologically isomorphic by the mapping*

$$T \to \{T \circ \eta_a : a \in \mathbb{A}\}$$

to the subspace

$$\Lambda = \left\{\chi = \{\chi_a\} : \sup_{F\in\mathcal{F}} \sup_{x\in B_X} \left\|\sum_{a\in F}(\chi_a \circ R_a)(x)\right\| \le \infty\right\}$$

of the product space $\prod_{a\in\mathbb{A}} \mathcal{L}(V_a, Y)$, endowed with the norm

$$\|\chi\| = \sup_{F\in\mathcal{F}} \sup_{x\in B_X} \left\|\sum_{a\in F}(\chi_a \circ R_a)(x)\right\|, \tag{4.1}$$

and there exists a constant K such that $\|T\| \le \|\chi\| \le K\|T\|$.

REMARK 4.2. By a topological isomorphism between Banach spaces X and Y we mean a linear mapping Ψ from X onto Y such that there exists some $K > 0$ with the property

$$\|x\| \le \|\Psi(x)\| \le K\|x\|, \quad \text{for every } x \in X.$$

Note that a topological isomorphism is automatically one to one and, sometimes, it is called a topological isomorphism into if it isn't onto. Further, a topological isomorphism is called an equivalence or isometric isomorphism whenever $K = 1$.

PROOF OF THEOREM 4.1. Define the linear operator

$$\Psi\colon \mathcal{L}(X,Y) \to \Lambda \quad \text{by } \Psi(T) = \chi \text{ such that } \chi_a = T \circ \eta_a.$$

Let us first show that the function $\|\cdot\|$ in (4.1) is well-defined. Pick some $\chi \in \Lambda$ and say

$$\pi'_F(x) = \sum_{a\in F} (\chi_a \circ R_a)(x), \quad \text{for } F \in \mathcal{F}.$$

It is continuous linear operator on X, whence,

$$\sup_{F\in\mathcal{F}} \sup_{x\in B_X} \left\| \sum_{a\in F} (\chi_a \circ R_a)(x) \right\| = \sup_{F\in\mathcal{F}} \|\pi'_F\| < \infty$$

by the uniform boundedness principle.

Let $\|\chi\| = 0$, then $\sup_{x\in B_X} \|\pi'_F(x)\| = 0$ for each $F \in \mathcal{F}$, in particular, for $F = \{a\}$ $(a \in \mathbb{A})$. This implies $\|(\chi_a \circ R_a)(x)\| = 0$ for each a and $x \in B_X$. This implies $\chi_a = 0$ for $R_a \neq 0$, and so $\chi = 0$. Other properties of norm can easily be verified.

Let $T \in \mathcal{L}(X,Y)$ and say $\chi_a = T \circ \eta_a$. Then

$$\begin{aligned} Tx &= T\Bigg(\sum_{a\in\mathbb{A}} (\eta_a \circ R_a)(x)\Bigg) = \sum_{a\in\mathbb{A}} (T \circ \eta_a \circ R_a)(x) \\ &= \sum_{a\in\mathbb{A}} (\chi_a \circ R_a)(x) = \lim_{F\in\mathcal{F}} \pi'_F(x) \end{aligned}$$

for $x \in X$. Thus, $\sup_{F\in\mathcal{F}} \|\pi'_F(x)\| < \infty$ so that $\sup_{F\in\mathcal{F}} \|\pi'_F\| < \infty$ by the uniform boundedness principle. This means $\chi \in \Lambda$, whence, Ψ is well-defined. Further

$$\|T\| \le \sup_{F\in\mathcal{F}} \|\pi'_F\| = \sup_{F\in\mathcal{F}} \sup_{x\in B_X} \|\sum_{a\in F} (\chi_a \circ R_a)(x)\| = \|\chi\|.$$

On the other hand, for each $F \in \mathcal{F}$,

$$\begin{aligned} \left\| \sum_{a\in F} (\chi_a \circ R_a)(x) \right\| &= \left\| T\Bigg(\sum_{a\in F} (\eta_a \circ R_a)(x)\Bigg) \right\| \le \|T\| \left\| \sum_{a\in F} (\eta_a \circ R_a)(x) \right\| \\ &= \|T\| \|\pi_F(x)\| \le \|T\| \|x\|^\diamond \le K \|T\| \|x\| \end{aligned}$$

by the Theorem 3.12, whence, $\|\chi\| \le K\|T\|$. This shows that Ψ is a topological isomorphism into. Now let us prove that Ψ is also surjective. Let $\chi \in \Lambda$ be arbitrary and write

$$Tx = \sum_{a\in\mathbb{A}} (\chi_a \circ R_a)(x), \quad \text{for } x \in X,$$

We shall show that this formula defines a bounded linear operator from X into Y. Let us first show that it is well-defined. This, obviously, equivalent to the summability of the family $\{(\chi_a \circ R_a)(x) : a \in \mathbb{A}\}$ to Tx, and so is equivalent to the convergence of the net $(\pi'_F(x) : \mathcal{F})$ to Tx for each $x \in X$. From the fact that the family $\{(\eta_a, R_a) : a \in \mathbb{A}\}$ is an f-biorthogonal system, we have

$$\pi'_{F\cup F_1}(\pi_F(x)) = \pi'_F(\pi_F(x)), \quad F, F_1 \in \mathcal{F}.$$

This implies $(\pi'_F(\pi_F(x)) : \mathcal{F})$ is a Cauchy net in X, and so it has a limit. For the set of all $\pi_F(x)$ is dense in X, and Y is complete we have $(\pi'_F(x) : \mathcal{F})$ converges to Tx, whence, T is well-defined. Linearity is obvious and T is continuous since $\sup_{x\in B_X} \|Tx\| \le \|\chi\|$. □

REMARK 4.3. It is clear from the proofs of Theorem 4.1 and the following Theorem 4.11 that the topological isomorphisms are equivalences whenever the uf-bases are monotone.

COROLLARY 4.4. *Let Y be an arbitrary Banach space, $1 < p < \infty$ and $1/p + 1/q = 1$. Then the operator space $\mathcal{L}(\ell_p(\mathbb{A}, X), Y)$ is equivalent by the mapping*

$$T \to \{T \circ I_a : a \in \mathbb{A}\}$$

to the Banach space E_q; the space of all $\chi = \{\chi_a\} \in \mathcal{L}(X, Y)^{\mathbb{A}}$ such that

$$\|\chi\| = \sup_{g\in B_{Y^*}} \left(\sum_{a\in\mathbb{A}} \|g \circ \chi_a\|^q\right)^{1/q} < \infty.$$

PROOF. Take $X = \ell_p(\mathbb{A}, X)$ with the monotone up-basis $\{I_a : a \in \mathbb{A}\}$ in Theorem 4.1. Remember that $R_a = P_a$, whence,

$$\begin{aligned}
\|\chi\| &= \sup_{F\in\mathcal{F}} \sup_{x\in B_{\ell_p(\mathbb{A},X)}} \left\| \sum_{a\in F} (\chi_a \circ P_a)(x) \right\|_Y \\
&= \sup_{F\in\mathcal{F}} \sup_{x\in B_{\ell_p(\mathbb{A},X)}} \sup_{g\in B_{Y^*}} \left| g\left(\sum_{a\in F} (\chi_a \circ P_a)(x) \right) \right| \\
&= \sup_{F\in\mathcal{F}} \sup_{g\in B_{Y^*}} \sup_{x\in B_{\ell_p(\mathbb{A},X)}} \left| \sum_{a\in F} (g \circ \chi_a)(x_a) \right| \\
&= \sup_{F\in\mathcal{F}} \sup_{g\in B_{Y^*}} \left(\sum_{a\in F} \|g \circ \chi_a\|^q \right)^{1/q} = \sup_{g\in B_{Y^*}} \left(\sum_{a\in\mathbb{A}} \|g \circ \chi_a\|^q \right)^{1/q}.
\end{aligned}$$

We also use the fact that $\ell_p(\mathbb{A}, X)^* = \ell_q(\mathbb{A}, X^*)$ in the last two rows. □

It must be noted that $\ell_q(\mathbb{A}, \mathcal{L}(X, Y)) \subseteq E_q$. This is clear since

$$\sup_{g\in B_{Y^*}} \left(\sum_{a\in\mathbb{A}} \|g \circ \chi_a\|^q \right)^{1/q} \le \sup_{g\in B_{Y^*}} \|g\| \left(\sum_{a\in\mathbb{A}} \|\chi_a\|^q \right)^{1/q} = \left(\sum_{a\in\mathbb{A}} \|\chi_a\|^q \right)^{1/q}.$$

Let us show now that the above inclusion relation may be strict.

EXAMPLE 4.5. Let $X = c_0$, $Y = \ell_p$ and $\mathbb{A} = \mathbb{N}$. Take $\chi = (\chi_n)$ as $\chi_n = (a_{ij}^n)$: $a_{nn}^n = 1$, $a_{ij}^n = 0$ otherwise, that is,

$$\chi = \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \dots \right\}_{n=1}^{\infty}.$$

Previously, let us prove that $\chi \in E_q$. Recall that each continuous operator from c_0 into ℓ_p must have an infinite matrix form [10]. Obviously, each $\chi_n \in \mathcal{L}(c_0, \ell_p)$. Further, for $x = (x_n) \in \ell_p(\mathbb{N}, c_0) = \ell_p(c_0)$

$$\chi_n(x_n) = (0, \dots, 0, x_n^n, 0, \dots), \quad n \in \mathbb{N},$$

where each $x_n = (x_n^k)_{k=1}^{\infty} \in c_0$.

On the other hand, for each $g = (g_n) \in \ell_p^* = \ell_q$ and for $y \in B_{c_0}$,

$$|(g \circ \chi_n)(y)| = |g(0, \dots, 0, y_n, 0, \dots)| = |g_n y_n| \le |g_n|.$$

Also, for $y = e^k$,

$$|(g \circ \chi_n)(x_n)| = |g_n|.$$

Hence,

$$\|g \circ \chi_n\| = |g_n|.$$

This implies

$$\sup_{g \in B_{\ell_p^*}} \left(\sum_{n \in \mathbb{N}} \|g \circ \chi_n\|^q \right)^{1/q} = \sup_{g \in B_{\ell_q}} \left(\sum_{n \in \mathbb{N}} |g_n|^q \right)^{1/q} = 1,$$

that is, $\chi = (\chi_n) \in E_q$. However, $\sum_{n \in \mathbb{N}} \|\chi_n\|^q = \infty$ since each $\|\chi_n\| = 1$. This means $\chi \notin \ell_q(\mathcal{L}(c_0, \ell_p))$.

COROLLARY 4.6. *Let each V_a and Y be Banach spaces. Then $\mathcal{L}((V_a)_{\ell_1}^{a \in \mathbb{A}}, Y)$ is equivalent, by the mapping*

$$T \to \{T \circ I_a : a \in \mathbb{A}\},$$

to the space $(\mathcal{L}(V_a, Y))_{\ell_\infty}^{a \in \mathbb{A}}$, where (I_a) is the up-basis of $(V_a)_{\ell_1}^{a \in \mathbb{A}}$.

PROOF. Take $X = (V_a)_{\ell_1}^{a \in \mathbb{A}}$ in Theorem 4.1 with the monotone up-basis (I_a). Hence, we have

$$\sum_{a \in F} (\chi_a \circ R_a)(x) = \sum_{a \in F} \chi_a(x_a) = \sum_{a \in F} T(I_a(x_a)), \quad F \in \mathcal{F}$$

for $x \in (V_a)_{\ell_1}^{a \in \mathbb{A}}$. This implies

$$\|\chi\| = \sup_{F \in \mathcal{F}} \sup_{\sum_{a \in \mathbb{A}} \|x_a\| \le 1} \|\sum_{a \in F} T(I_a(x_a))\| = \sup_{a \in \mathbb{A}} \|T \circ I_a\| = \sup_{a \in \mathbb{A}} \|\chi_a\|. \qquad \square$$

However, $\mathcal{L}((V_a)_{c_0}^{a\in\mathbb{A}}, Y)$ is not equivalent to the space $(\mathcal{L}(V_a, Y))_{\ell_1}^{a\in\mathbb{A}}$. The former may contain strictly the latter.

COROLLARY 4.7. *Let each V_a and Y be Banach spaces. Then the operator space $\mathcal{L}((V_a)_{c_0}^{a\in\mathbb{A}}, Y)$ is equivalent by the mapping*

$$T \to \{T \circ I_a : a \in \mathbb{A}\}$$

to the Banach space E_1; the space of all $\chi = \{\chi_a\} \in \prod_{a\in\mathbb{A}} \mathcal{L}(V_a, Y)$ such that $\sum_{a\in\mathbb{A}} \|g \circ \chi_a\| < \infty$ for each $g \in B_{Y^*}$, endowed with the norm

$$\|\chi\| = \sup_{g\in B_{Y^*}} \sum_{a\in\mathbb{A}} \|g \circ \chi_a\| < \infty.$$

PROOF. Taking $X = (V_a)_{c_0}^{a\in\mathbb{A}}$ in Theorem 4.1 with the monotone up-basis (I_a), one can complete the proof by using the same technique in Corollary 4.4 with the fact $[(V_a)_{c_0}^{a\in\mathbb{A}}]^* = (V_a^*)_{\ell_1}^{a\in\mathbb{A}}$. □

EXAMPLE 4.8. Clearly, $(\mathcal{L}(V_a, Y))_{\ell_1}^{a\in\mathbb{A}} \subseteq E_1$ in Corollary 4.7 and the inclusion relation may be strict. let $\mathbb{A} = \mathbb{N}$, $V_n = \ell_{1+1/n}$ and $Y = c_0$. Take $\chi = (\chi_n)$ as is in Example 4.5, that is,

$$\chi = \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \dots \end{bmatrix}, \dots \right\}_{n=1}^{\infty}.$$

Clearly, each $\chi_n \in \mathcal{L}(\ell_{1+1/n}, c_0)$ and $\|\chi_n\| = 1$. This means $\chi \notin (\mathcal{L}(\ell_{1+1/n}, c_0))_{\ell_1}^{a\in\mathbb{A}}$. However, the same discussion as in Example 4.5 shows that

$$\sup_{g\in B_{c_0^*}} \sum_{n\in\mathbb{N}} \|g \circ \chi_n\| = \sup_{g\in B_{\ell_1}} \sum_{n\in\mathbb{N}} |g_n| = 1,$$

whence, $\chi \in E_1$.

Following notion was introduced in [17] and was used in the characterization of certain operator spaces. Here, we use it in Theorem 4.11 to give a general conjecture to characterization of operators from an arbitrary Banach space to the another one which has an uf-basis.

DEFINITION 4.9. Let X, Y, Z be an arbitrary Banach space and let $T \in \mathcal{L}(X, Y)$. Then we define *Z-adjoint operator of T* by

$$\begin{aligned} T_Z^* &: \mathcal{L}(Y, Z) \to \mathcal{L}(X, Z) \\ &: R \to T_Z^*(R) \end{aligned}$$

such that

$$(T_Z^*(R))(x) = R(Tx) \quad \text{for each } x \in X.$$

This is reduced to classical definition of adjoint operator whenever $Z = \mathbb{K}(\mathbb{R}$ or $\mathbb{C})$, the scalar field of both X and Y. Z-adjoint operators satisfies the usual attributes of the adjoint operators. For example T^*_Z is linear continuous and $\|T\| = \|T^*_Z\|$ [17]. Further, it can be proved also as in the classical case that

$$(\lambda T + \beta A)^*_Z = \lambda T^*_Z + \beta A^*_Z \quad \text{and} \quad (TA)^*_Z = A^*_Z T^*_Z$$

for all scalars λ, β and for all $T, A \in \mathcal{L}(X, Y)$.

However, there exists important structural differences between adjoint and Z-adjoint operators. Let us now present an example illustrating dimensional differences between these two adjoint types. This example is taken from [17].

Example 4.10. Take $X = Y = Z = c_0$ in Definition 4.9 and consider the difference operator Δ from c_0 into c_0 such that $\Delta x = (x_k - x_{k+1})_{k=1}^{\infty}$ for $x = (x_k)_{k=1}^{\infty} \in c_0$. Clearly, Δ is given by the infinite matrix

$$\Delta = \begin{bmatrix} 1 & -1 & 0 & 0 & \dots \\ 0 & 1 & -1 & 0 & \dots \\ 0 & 0 & 1 & -1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

It is clear that the adjoint Δ^* is given by

$$\Delta^* = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots \\ -1 & 1 & 0 & 0 & \dots \\ 0 & -1 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

However it is illustrated in [17] that the c_0-adjoint $\Delta^*_{c_0}$ of Δ is the following four-dimensional infinite matrix $(a^*_{ijkl})_{i,j,k,l \in \mathbb{N}}$, such that

$$a^*_{ijkl} = \begin{cases} 1 & \text{if } i = k \text{ and } j = l. \\ -1 & \text{if } i = k,\ j = l - 1 \text{ and } l \neq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Unfortunately, we cannot associate the self-adjointness to new adjoint type in the case $Z \neq \mathbb{K}$, even if X and Y are the same Hilbert space.

Now, we are ready to state another fundamental result of this work.

Theorem 4.11. *Let X be a Banach space for which $\{\eta_a : a \in \mathbb{A}\}$ be a uf-basis with respect to the family $\{V_a\}$ of Banach spaces and Y be another arbitrary Banach space. Then $\mathcal{L}(Y, X)$ is isomorphic by the mapping*

$$T \to \{T^*_{V_a}(R_a) : a \in \mathbb{A}\}$$

to the subspace

$$\Phi = \{\varphi = \Big\{\varphi_a\} \in \prod_{a\in\mathbb{A}} \mathcal{L}(Y, V_a) : \{(\eta_a \circ \varphi_a)(y) : a \in \mathbb{A}\}$$
$$\text{is summable in } X, \quad \text{for each } y \in Y\Big\},$$

endowed with the norm

$$\|\varphi\| = \sup_{F\in\mathcal{F}} \sup_{y\in B_Y} \Big\| \sum_{a\in F} (\eta_a \circ \varphi_a)(y) \Big\|, \tag{4.2}$$

and there exists a constant K such that

$$\|T\| \le \|\varphi\| \le K\|T\|.$$

PROOF. As is similar to the proof of Theorem 4.1, we have (4.2) is finite by the uniform boundedness principles, and satisfies the norm conditions.

Let $T \in \mathcal{L}(Y, X)$ and say $\varphi_a = T^*_{V_a}(R_a)$. Then, for $y \in Y$,

$$\begin{aligned} Ty &= \sum_{a\in\mathbb{A}} (\eta_a \circ R_a)(Ty) = \sum_{a\in\mathbb{A}} \eta_a[R_a(Ty)] \\ &= \sum_{a\in\mathbb{A}} [\eta_a \circ (T^*_{V_a}(R_a))](y) = \sum_{a\in\mathbb{A}} (\eta_a \circ \varphi_a)(y), \end{aligned}$$

whence, $\varphi = \{\varphi_a : a \in \mathbb{A}\} \in \Phi$ and

$$\|T\| = \sup_{y\in B_Y} \|\sum_{a\in\mathbb{A}} (\eta_a \circ \varphi_a)(y)\| \le \sup_{F\in\mathcal{F}} \sup_{y\in B_Y} \|\sum_{a\in F} (\eta_a \circ \varphi_a)(y)\| = \|\varphi\|.$$

Further, since $\|\pi_F(x)\| \le K\|x\|$, for some $K > 0 (x \in X,\ F \in \mathcal{F})$,

$$\begin{aligned} \|\varphi\| &= \sup_{F\in\mathcal{F}} \sup_{y\in B_Y} \Big\| \sum_{a\in F} (\eta_a \circ \varphi_a)(y) \Big\| \\ &= \sup_{F\in\mathcal{F}} \sup_{y\in B_Y} \Big\| \sum_{a\in F} [\eta_a \circ (T^*_{V_a}(R_a))](y) \Big\| = \sup_{F\in\mathcal{F}} \sup_{y\in B_Y} \Big\| \sum_{a\in F} (\eta_a \circ R_a)(Ty) \Big\| \\ &= \sup_{F\in\mathcal{F}} \sup_{y\in B_Y} \|\pi_F(Ty)\| \le K \sup_{y\in B_Y} \|Ty\| \le K\|T\|. \end{aligned}$$

This shows that the mapping $T \to \{T^*_{V_a}(R_a) : a \in \mathbb{A}\}$ is a topological isomorphism into. Furthermore, it is not hard to see that for some $\varphi \in \Phi$, the formula

$$Ty = \sum_{a\in\mathbb{A}} (\eta_a \circ \varphi_a)(y)$$

defines an operator $T \in \mathcal{L}(Y, X)$. Thus, the mapping $T \to \{T^*_{V_a}(R_a)\}$ is a topological isomorphism from $\mathcal{L}(Y, X)$ to Φ. $\square$

COROLLARY 4.12. *Let each V_a and Y be Banach spaces. Then $\mathcal{L}(Y,(V_a)_{c_0}^{a\in\mathbb{A}})$ is equivalent by the mapping*

$$T \to \{T^*_{V_a}(P_a) : a \in \mathbb{A}\}$$

to

$$\mathrm{SOT} - (\mathcal{L}(Y,V_a))_{c_0}^{a\in\mathbb{A}};$$

the Banach space of all functions $\varphi = \{\varphi_a\} \in \prod_{a\in\mathbb{A}} \mathcal{L}(Y,V_a)$ such that

$$\{\varphi_a(y) : a \in \mathbb{A}\} \in (V_a)_{c_0}^{a\in\mathbb{A}},$$

for each $y \in Y$, where (P_a) is the associate projections to the up-basis $\{I_a : a \in \mathbb{A}\}$ of $(V_a)_{c_0}^{a\in\mathbb{A}}$; endowed with the norm $\|\varphi\| = \sup_{a\in\mathbb{A}} \|\varphi_a\|$.

PROOF. Take $Y = (V_a)_{c_0}^{a\in\mathbb{A}}$ in Theorem 4.11 with monotone up-basis (I_a). The summability of the family $\{(I_a \circ \varphi_a)(y) : a \in \mathbb{A}\}$ in $(V_a)_{c_0}^{a\in\mathbb{A}}$ is equivalent to

$$\sup_{F\in\mathcal{F}} \left\| \sum_{a\in F} (I_a \circ \varphi_a)(y) \right\| = \sup_{F\in\mathcal{F}} \sup_{a\in F} \|\varphi_a(y)\| = \sup_{a\in\mathbb{A}} \|\varphi_a(y)\| < \infty$$

for each $y \in Y$. Further,

$$\|\varphi\| = \sup_{F\in\mathcal{F}} \sup_{y\in B_Y} \left\| \sum_{a\in F} (I_a \circ \varphi_a)(y) \right\| = \sup_{F\in\mathcal{F}} \sup_{y\in B_Y} \sup_{a\in F} \|\varphi_a(y)\| = \sup_{a\in\mathbb{A}} \|\varphi_a\|. \quad \square$$

COROLLARY 4.13. *$\mathcal{L}(Y,\ell_p(\mathbb{A},X))$ is equivalent, by the mapping*

$$T \to \{T^*_{V_a}(P_a) : a \in \mathbb{A}\}$$

to

$$\ell_p(\mathbb{A},\mathcal{L}_{\mathrm{SOT}}(Y,X));$$

the Banach space of all functions $\varphi = (\varphi_a) \in \mathcal{L}(Y,X)^{\mathbb{A}}$ which are pointwise absolutely p-summable, that is, such that $(\varphi_a(y)) \in \ell_p(\mathbb{A},X)$ for each $y \in Y$, endowed with the norm

$$\|\varphi\| = \sup_{y\in B_Y} \left(\sum_{a\in\mathbb{A}} \|\varphi_a(y)\|^p \right)^{1/p} \quad \text{where } p \geq 1.$$

PROOF. Take $X = \ell_1(A,X)$ in Theorem 4.11 with monotone up-basis (I_a). The family $\{(I_a \circ \varphi_a)(y) : a \in \mathbb{A}\}$ is summable in $\ell_p(\mathbb{A},X)$ if and only if $\sum_{a\in\mathbb{A}} \|\varphi_a(y)\|^p < \infty$ for each $y \in Y$. Also the norm

$$\|\varphi\| = \sup_{F\in\mathcal{F}} \sup_{y\in B_Y} \left\| \sum_{a\in F} (I_a \circ \varphi_a)(y) \right\|$$

in Theorem 4.11 is reduced to the form

$$\|\varphi\| = \sup_{y\in B_Y} \left(\sum_{a\in\mathbb{A}} \|\varphi_a(y)\|^p \right)^{1/p}. \quad \square$$

EXAMPLE 4.14. It is clear that $\ell_p(\mathbb{A}, \mathcal{L}(Y, X)) \subseteq \ell_p(\mathbb{A}, \mathcal{L}_{\mathrm{SOT}}(Y, X))$ and the inclusion relation may be strict. Let us show this for $p > 1$. Take $\mathbb{A} = N$, $X = \mathbb{C}$, $Y = \ell_p$ and $\varphi_n = f_n$, $n \in \mathbb{N}$; the associate coordinate functionals to the unit vectors basis $\{e_n\}$ of ℓ_1. Remember that $f_n(x) = x_n$ and each $f_n \in \ell_p^* = \ell_q$ since $\|f_n\| = 1$. In this case $(I_n \circ f_n)(y) = y_n e_n$ for each $y \in \ell_p$. Further

$$\|\{f_n\}_{n=1}^{\infty}\| = \sup_{F \in \mathcal{F}} \sup_{y \in B_{\ell_1}} \sum_{n \in F} |f_n(y)| = \sup_{F \in \mathcal{F}} \sup_{\sum_{n \in \mathbb{N}} |y_n| \leq 1} \sum_{n \in F} |y_n| = 1,$$

whence, $\{f_n\}_{n=1}^{\infty} \in \ell_p(\mathbb{A}, \mathcal{L}_{\mathrm{SOT}}(Y, X)) = \ell_p(\ell_p^*, weak^*)$. However, $\{f_n\}_{n=1}^{\infty} \notin \ell_p(\mathbb{A}, \mathcal{L}(Y, X)) = \ell_p(\ell_q)$ since $\|f_n\| = 1$ for each n.

A similar example may be given for the case $p = 1$.

REMARK 4.15. The characterizations of $\mathcal{L}((V_a)_{\ell_p}^{a \in \mathbb{A}}, Y)$ and $\mathcal{L}(Y, (V_a)_{\ell_p}^{a \in \mathbb{A}})$, $p > 1$, also can be derived from Theorems 4.1 and 4.11. We don't give here all characterizations but only some instances for mentioned function spaces. These theorems provide a fundamental tool in the representations of the operators between some vector-valued function spaces.

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