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AN ADMISSIBILITY FOR TOPOLOGICAL DEGREE OF HERZ-TYPE BESOV AND TRIEBEL–LIZORKIN SPACES

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Abstract. In this paper, an admissibility for topological degree of Herz spaces, Herz-type Besov and Triebel–Lizorkin spaces is given

1. Introduction

It is well known that the theory of function spaces is an important tool for the study of ordinary and partial differential equations, in special the theory of Besov and Triebel–Lizorkin spaces. These two scales of function spaces include many classical spaces as special cases, for example, the Hölder spaces, the Sobolev spaces, the Bessel potential spaces, the Zygmund spaces, the local Hardy spaces and the space $\mathrm{bmo}(\mathbb{R}^n)$. All the above mentioned spaces have been studied intensively and applied in many fields; see for examples, [1], [6]–[10].

Recently, similar to classical Besov and Triebel–Lizorkin spaces, the Herz-type Besov and Triebel–Lizorkin spaces, $\dot{K}_q^{\alpha,p}B_\beta^s$, $K_q^{\alpha,p}B_\beta^s$, $\dot{K}_q^{\alpha,p}F_\beta^s$ and $K_q^{\alpha,p}F_\beta^s$ on $\mathbb{R}^n$ for $s \in \mathbb{R}$, $0 < \beta \leq \infty$, $0 < q,p < \infty$ and $-n/q < \alpha$, were introduced in [14] and [12] by Dachun Yang and the author. These spaces are the generalizations of the inhomogeneous Besov and Triebel–Lizorkin spaces; see [7]. Some

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basic properties on these spaces were also given in [14], which include embedding properties, maximal inequality, Fourier multiplier and lifting properties. Moreover, in [13], they established the connections between Herz-type Triebel–Lizorkin spaces with Herz spaces and Herz-type Bessel potential spaces; see[2] and [4]. The boundedness of a class of pesudo-differential operator on these spaces was obtained in [11].

It was known that the existence and multiplicity results for solutions of semilinear elliptic boundary valued problems can be obtained by topological methods, topological degree of maps acting in Banach spaces. However, in practice, there are many spaces are only quasi-Banach spaces. To extend the degree theory to larger classes of topological vector spaces, Klee in [3] introduced the notion of so-called admissible topological vector spaces. It is known that the classical Besov and Triebel–Lizorkin spaces are admissible topological vector spaces, see e.g. [6]. In this paper we will consider the admissibility of Herz-type Besov and Triebel–Lizorkin spaces. Our result will state in the next section. In the remainder of this section, we recall some definitions we will need.

Let $Q_k = \{x \in \mathbb{R}^n : |x| \le 2^k\}$ and $D_k = Q_k \setminus Q_{k-1}$ for $k \in \mathbb{Z}$. If E is a subset of $\mathbb{R}^n$, we let χ_E denote the characteristic function of the set E. For convenience, we denote χ_{D_k} by χ_k.

Definition 1.1 (see [2]). Let $\alpha \in \mathbb{R}$ and $0 < p$, $q \le \infty$.

(a) The homogeneous Herz space $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ is defined in terms of

$$\|f\|_{\dot{K}_q^{\alpha,p}} = \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|f\chi_k\|_{L^q}^p \right\}^{1/p}$$

by letting

$$\dot{K}_q^{\alpha,p}(\mathbb{R}^n) = \{f \in L^q_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}_q^{\alpha,p}} < \infty\}.$$

(b) The non-homogeneous Herz space $K_q^{\alpha,p}(\mathbb{R}^n)$ is defined in terms of

$$\|f\|_{K_q^{\alpha,p}} = \left\{ \|f\chi_{Q_0}\|_{L^q}^p + \sum_{k=1}^{\infty} 2^{k\alpha p} \|f\chi_k\|_{L^q}^p \right\}^{1/p}$$

by letting

$$K_q^{\alpha,p}(\mathbb{R}^n) = \{f \in L^q_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{K_q^{\alpha,p}} < \infty\}.$$

Here the usual modifications are made when $p = \infty$ or $q = \infty$.

Let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz space of all complex-valued rapidly decreasing infinitely differentiable functions on $\mathbb{R}^n$. Let $\mathcal{S}'(\mathbb{R}^n)$ be the set of all the tempered distribution on $\mathbb{R}^n$. If $\varphi \in \mathcal{S}(\mathbb{R}^n)$, then $\widehat{\varphi}$ denotes the Fourier transform of φ, and $\varphi^{\vee}$ denotes the inverse Fourier transform φ. Let function $\Phi \in \mathcal{S}(\mathbb{R}^n)$ satisfy the following conditions:

$$\operatorname{supp} \widehat{\Phi} \subset B(0,1) \quad \text{and} \quad \operatorname{supp} \widehat{\Phi} = 1 \quad \text{on } B(0,1/2).$$

Set $\Phi_j(x) = 2^{nj}\Phi(2^j x)$, $x \in \mathbb{R}^n$, for $j \in \mathbb{Z}$. We also put

$$\theta_j = \Phi_j(x) - \Phi_{j-1}(x).$$

Denote $\theta_0 = \Phi$. It follows that

$$\sum_{j=0}^{\infty} \widehat{\theta}_j(\xi) \equiv 1.$$

Definition 1.2. (a) Let $s, \alpha \in \mathbb{R}$, $0 < \beta \le \infty$, $0 < q$, $p \le \infty$. Then the Herz type Besov spaces as

$$K_q^{\alpha,p}B_\beta^s(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{K_q^{\alpha,p}B_\beta^s} = \|\{2^{sj}\theta_j * f\}_0^\infty\|_{\ell_\beta(K_q^{\alpha,p})} < \infty\},$$
$$\dot{K}_q^{\alpha,p}B_\beta^s(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{\dot{K}_q^{\alpha,p}B_\beta^s} = \|\{2^{sj}\theta_j * f\}_0^\infty\|_{\ell_\beta(\dot{K}_q^{\alpha,p})} < \infty\}.$$

(b) Let $s, \alpha \in \mathbb{R}$, $0 < \beta \le \infty$, $0 < q$, $p < \infty$. Then the Herz type Triebel–Lizorkin spaces as

$$K_q^{\alpha,p}F_\beta^s(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{K_q^{\alpha,p}F_\beta^s} = \|\{2^{sj}\theta_j * f\}_0^\infty\|_{K_q^{\alpha,p}(\ell_\beta)} < \infty\},$$
$$\dot{K}_q^{\alpha,p}F_\beta^s(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{\dot{K}_q^{\alpha,p}F_\beta^s} = \|\{2^{sj}\theta_j * f\}_0^\infty\|_{\dot{K}_q^{\alpha,p}(\ell_\beta)} < \infty\},$$

where $K_q^{\alpha,p}(\ell_\beta)$, $\ell_\beta(K_q^{\alpha,p})$ are the spaces of all sequences $\{g_j\}$ of measurable functions on $\mathbb{R}^n$ with finite quasi-norms

$$\|\{g_j\}\|_{K_q^{\alpha,p}(\ell_q)} = \|\|\{g_j\}\|_{\ell_q}\|_{K_q^{\alpha,p}} = \left\|\left(\sum_{j=0}^{\infty} |g_j(x)|^q\right)^{1/q} dx\right\|_{K_q^{\alpha,p}}$$

and

$$\|\{g_j\}\|_{\ell_q(K_q^{\alpha,p})} = \left(\sum_{j=0}^{\infty} \|g_j\|_{K_q^{\alpha,p}}^q\right)^{1/q},$$

respectively. $\dot{K}_q^{\alpha,p}(\ell_\beta)$ and $\ell_\beta(\dot{K}_q^{\alpha,p})$ are similar.

In what follows, letter C denote positive constant but it may change line to line. Constants may in general depend on all fixed parameters, and sometimes we show this dependence explicitly by writing, e.g. C_N.

2. The admissibility of Herz-type spaces

In this section we discuss the admissibility for topological degree of Herz-type Besov and Triebel–Lizorkin spaces. Firstly, we recall the definition of admissibility of quasi-normed space.

Definition 2.1. A quasi-normed space X is called to be *admissible*, if for every compact subset $E \subset X$ and for every $\varepsilon > 0$ there exists a continuous map $T\colon E \to X$ such that $T(E)$ is contained in a finite dimensional subset of X and $x \in E$ implies

$$\|Tx - x\|_X \leq \varepsilon.$$

It is known that every normed space is admissible.

Lemma 2.2. *Let A and B be quasi-normed spaces over either real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$. Let $T_A\colon A \to B$ and $T_B\colon B \to A$ be continuous maps. Suppose T_B be uniformly continuous on every bounded set of B, and $T_B \circ T_A = I_A$, where I_A is the identity on A. Then the admissibility of B implies that A is also admissible.*

Before discussing the admissibility of Herz-type Besov and Triebel–Lizorkin spaces, we first consider that of Herz spaces.

Theorem 2.3. *Let $\alpha \in \mathbb{R}$ and $0 < p,\ q \leq \infty$. Then the spaces $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ and $K_q^{\alpha,p}(\mathbb{R}^n)$ are admissible.*

Proof. The idea of the proof is similar to that of Lebesgue spaces. Since the result is new, we give the detail here. It is known that when $1 \leq p,\ q \leq \infty$ the space $K_q^{\alpha,p}(\mathbb{R}^n)$ is a Banach space. Therefore if $1 \leq p,\ q \leq \infty$ then the space $K_q^{\alpha,p}(\mathbb{R}^n)$ is admissible. Now we prove the theorem for all $0 \leq p,\ q \leq \infty$.

Let $0 < r < \infty$. Denote the operator $T_r\colon K_q^{\alpha,p}(\mathbb{R}^n) \to K_{q/r}^{r\alpha,p/r}(\mathbb{R}^n)$ by

$$T_r(f)(x) = \max(\mathcal{R}\, f(x), 0)^r,$$

where and follows $\mathcal{R}\, f$ is the real part of f.

It is easy to see that T_r is a bounded operator from $K_q^{\alpha,p}(\mathbb{R}^n)$ to $K_{q/r}^{r\alpha,p/r}$. We show it is uniformly continuous on bounded sets of $K_q^{\alpha,p}(\mathbb{R}^n)$. Let $M > 0$ and $\varepsilon > 0$ be fixed. Then for every $\delta > 0$ there exists a positive number C_δ such that

$$|x^r - y^r| \leq C_\delta |x - y|^r + \delta(x^r + y^r) \tag{2.1}$$

holds for all nonnegative numbers x, y, see e.g. [5]. Let $f, g \in K_q^{\alpha,p}(\mathbb{R}^n)$ with $\|f\|_{K_q^{\alpha,p}}, \|g\|_{K_q^{\alpha,p}} \leq M$. Then we have

$$\begin{aligned}
\|T_r f - T_r g\|_{K_{q/r}^{r\alpha,p/r}} \leq{}& C\Big(C_\delta \|| \max(\mathcal{R}\, f(\,\cdot\,), 0) - \max(\mathcal{R}\, g(\,\cdot\,), 0)|^r\|_{K_{q/r}^{r\alpha,p/r}} \\
&+ \delta \|| \max(\mathcal{R}\, f(\,\cdot\,), 0)|^r\|_{K_{q/r}^{r\alpha,p/r}} \\
&+ \delta \|| \max(\mathcal{R}\, g(\,\cdot\,), 0)|^r\|_{K_{q/r}^{r\alpha,p/r}}\Big) \\
\leq{}& C\Big(C_\delta \||f - g\|_{K_{q/r}^{r\alpha,p/r}}^r + \delta\|f\|_{K_{q/r}^{r\alpha,p/r}}^r + \delta\|g\|_{K_{q/r}^{r\alpha,p/r}}^r\Big).
\end{aligned}$$

Now it follows that

$$\|T_r f - T_r g\|_{K_{q/r}^{r\alpha,p/r}} \le \varepsilon$$

for sufficiently small $\delta > 0$ if $\|f\|_{K_q^{\alpha,p}}, \|g\|_{K_q^{\alpha,p}} \le M$ and

$$\|f - g\|_{K_q^{\alpha,p}} \le \delta_\varepsilon.$$

This means the map T_r is uniformly continuous on bounded subsets of $K_q^{\alpha,p}(\mathbb{R}^n)$. Let $0 < p,\ q \le \infty$, pick $0 < r < \min(p,q)$. Making use of Lemma 2.2, where $A = K_q^{\alpha,p}(\mathbb{R}^n)$, $B = K_{q/r}^{r\alpha,p/r}(\mathbb{R}^n)$,

$$T_A(f) = T_r(f) - T_r(-f) - iT_r(if) + iT_r(-if)$$

and

$$T_B(f) = T_{1/r}(f) - T_{1/r}(-f) - iT_{1/r}(if) + iT_{1/r}(-if),$$

we obtain that the space $K_q^{\alpha,p}(\mathbb{R}^n)$ is admissible. □

Now the main result is the following.

Theorem 2.4. (a) *Let $0 < \beta \le \infty$, $0 < q,\ p \le \infty$, $\alpha > -n/q$, $a > n/q$. Then $K_q^{\alpha,p}B_\beta^s(\mathbb{R}^n)$ and $\dot{K}_q^{\alpha,p}B_\beta^s(\mathbb{R}^n)$ are admissible.*

(b) *Let $0 < \beta \le \infty$, $0 < q,\ p < \infty$, $\alpha > -n/q$, $a > n/\min(q,\beta)$. Then $K_q^{\alpha,p}F_\beta^s(\mathbb{R}^n)$ and $\dot{K}_q^{\alpha,p}F_\beta^s(\mathbb{R}^n)$ are admissible.*

We will use the method used in [6] to prove Theorem 2.4. To do this, we need preliminaries.

Firstly, let $\{f_k\}_0^\infty$ be a sequence of continuous and bounded functions defined on $\mathbb{R}^n$. Then for $k \in \mathbb{N}_0$ and $a > 0$, we denote

$$f_k^{*,a}(x) = \sup_{y\in\mathbb{R}^n} \frac{|f_k(y)|}{(1+2^k|x-y|)^a}, \quad x \in \mathbb{R}^n.$$

Then we denote

$$K_q^{\alpha,p}(\ell^\beta)_a = \{\{f_k\}_0^\infty : \|\{f_k\}_0^\infty\|_{K_q^{\alpha,p}(\ell^\beta)_a} = \|\{f_k^{*,a}\}_0^\infty\|_{K_q^{\alpha,p}(\ell^\beta)} < \infty\}$$

and

$$\ell^\beta(K_q^{\alpha,p})_a = \{\{f_k\}_0^\infty : \|\{f_k\}_0^\infty\|_{\ell^\beta(K_q^{\alpha,p})_a} = \|\{f_k^{*,a}\}_0^\infty\|_{\ell^\beta(K_q^{\alpha,p})} < \infty\}.$$

Similar for $\dot{K}_q^{\alpha,p}(\ell^\beta)_a$ and $\ell^\beta(\dot{K}_q^{\alpha,p})_a$.

Lemma 2.5. (a) *Let $0 < \beta \le \infty$, $0 < q,\ p \le \infty$, $\alpha \in \mathbb{R}$, $a > 0$. Then spaces $K_q^{\alpha,p}(\ell^\beta)_a$ and $\dot{K}_q^{\alpha,p}(\ell^\beta)_a$ are admissible.*

(b) *Let $0 < \beta \le \infty$, $0 < q,\ p < \infty$, $\alpha \in \mathbb{R}$, $a > 0$. Then spaces $\ell^\beta(K_q^{\alpha,p})_a$ and $\ell^\beta(\dot{K}_q^{\alpha,p})_a$ are admissible.*

Proof. Let $f = \{f_k\}_{k=0}^\infty$ be a sequence of continuous and bounded functions defined on $\mathbb{R}^n$. Let $0 < r < \infty$, we define

$$T_r(f_k)(x) = \max(\mathcal{R}\, f_k(x), 0)^r \quad \text{and} \quad T_r(f)(x) = \{T_r(f_k)(x)\}_{k=0}^\infty.$$

It is easy to see that T_r maps bounded sets of $K_q^{\alpha,p}(\ell^\beta)_a$ into bounded sets of $K_{q/r}^{r\alpha,p/r}(\ell^{\beta/r})_{ra}$. We show that T_r is uniformly continuous on bounded sets of $K_q^{\alpha,p}(\ell^\beta)_a$. By (2.1), we have

$$(T_r(f_k) - T_r(g_k))^{*,ra}(x) \le C_\delta[(f_k - g_k)^{*,a}(x)]^r + \delta[f_k^{*,a}(x)^r + g_k^{*,a}(x)^r]$$

for $\delta > 0$ and C_δ is a constant independent of functions f_k, g_k. Thus it follows

$$\begin{aligned}
\|\{T_r(f_k)\ &- T_r(g_k)\}_{k=0}^\infty\|_{L^{p(\cdot)/r}(\ell^{q/r})_{ra}} \\
&\le C(C_\delta\|\{|(f_k-g_k)^{*,a}|^r\}_{k=0}^\infty\|_{K_{q/r}^{r\alpha,p/r}(\ell^{\beta/r})_{ra}} \\
&\quad + \delta[\|\{|f_k^{*,a}|^r\}_{k=0}^\infty\|_{K_{q/r}^{r\alpha,p/r}(\ell^{\beta/r})_{ra}} + \|\{|g_k^{*,a}|^r\}_{k=0}^\infty\|_{K_{q/r}^{r\alpha,p/r}(\ell^{\beta/r})_{ra}}]) \\
&\le C(C_\delta\|\{(f_k-g_k)\}_{k=0}^\infty\|_{K_q^{\alpha,p}(\ell^q)_a} \\
&\quad + \delta[\|\{f_k\}_{k=0}^\infty\|_{K_q^{\alpha,p}(\ell^q)_a} + \|\{g_k\}_{k=0}^\infty\|_{K_q^{\alpha,p}(\ell^q)_a}]).
\end{aligned}$$

Similar to Theorem 2.3, we arrive at that T_r is uniformly continuous on bounded sets of $K_q^{\alpha,p}(\ell^\beta)_a$.

Let $0 < \beta \le \infty$, $0 < q$, $p \le \infty$, $\alpha \in \mathbb{R}$, and $a > 0$ be fixed, we pick $0 < r < \min\{\beta, q, p\}$. Now we set

$$A = K_q^{\alpha,p}(\ell^q)_a, \quad \text{and} \quad B = K_{q/r}^{r\alpha,p/r}(\ell^{\beta/r})_{ra}.$$

Then we let

$$T_A(\{f_k\}_{k=0}^\infty) = \{T_r(f_k) - T_r(-f_k) - iT_r(if_k) + iT_r(-if_k)\}_{k=0}^\infty$$

for $f = \{f_k\}_{k=0}^\infty \in K_q^{\alpha,p}(\ell^\beta)_a$ and

$$T_B(\{f_k\}_{k=0}^\infty) = \{T_{1/r}(f_k) - T_{1/r}(-f_k) - iT_{1/r}(if_k) + iT_{1/r}(-if_k)\}_{k=0}^\infty$$

for $f = \{f_k\}_{k=0}^\infty \in K_{q/r}^{r\alpha,p/r}(\ell^{\beta/r})_{ra}$. Therefore, the admissibility of $K_q^{\alpha,p}(\ell^\beta)_a$ follows from Lemma 2.2, since B is admissible because it is a normed space. $\square$

To finish the proof of Theorem 2.4, we also need maximal inequalities for Herz-type Besov and Triebel–Lizorkin spaces, see [12].

Let $\Psi, \psi \in \mathcal{S}(\mathbb{R}^n)$, $\varepsilon > 0$, integer $S \ge -1$ be such that

$$\begin{aligned}
|\widehat{\Psi}(\xi)| &> 0 \quad \text{on } \{|\xi| < 2\varepsilon\}, \\
|\widehat{\psi}(\xi)| &> 0 \quad \text{on } \{\varepsilon/2 < |\xi| < 2\varepsilon\},
\end{aligned}$$

and

$$D^\tau \widehat{\psi}(0) = 0 \quad \text{for all } |\tau| \le S.$$

For any $a > 0$, $f \in \mathcal{S}'(\mathbb{R}^n)$, and $x \in \mathbb{R}^n$, denote Maximal functions,

$$\Psi_a^* f(x) = \sup_{y\in\mathbb{R}^n} \frac{|\Psi * f(y)|}{(1+|x-y|)^a} \quad \text{and} \quad \psi_{j,a}^* f(x) = \sup_{y\in\mathbb{R}^n} \frac{|\psi_j * f(y)|}{(1+2^j|x-y|)^a}.$$

Lemma 2.6. (a) *Let* $s < S+1$, $0 < \beta \le \infty$, $0 < q$, $p \le \infty$, $\alpha > -n/q$, $a > n/q$. *Then for all* $f \in \mathcal{S}'(\mathbb{R}^n)$

$$\|\Psi_a^* f\|_{K_q^{\alpha,p}} + \|\{2^{sj}\psi_{j,a}^* f\}_1^\infty\|_{\ell_\beta(K_q^{\alpha,p})} \le C\|f\|_{K_q^{\alpha,p}B_\beta^s},$$
$$\|\Psi_a^* f\|_{\dot K_q^{\alpha,p}} + \|\{2^{sj}\psi_{j,a}^* f\}_1^\infty\|_{\ell_\beta(\dot K_q^{\alpha,p})} \le C|f\|_{\dot K_q^{\alpha,p}B_\beta^s}.$$

(b) *Let* $s < S+1$, $0 < \beta \le \infty$, $0 < q$, $p < \infty$, $\alpha > -n/q$, $a > n/\min(q,\beta)$. *Then for all* $f \in \mathcal{S}'$

$$\|\Psi_a^* f\|_{K_q^{\alpha,p}} + \|\{2^{sj}\psi_{j,a}^* f\}_1^\infty\|_{K_q^{\alpha,p}(\ell_\beta)} \le C\|f\|_{K_q^{\alpha,p}F_\beta^s},$$
$$\|\Psi_a^* f\|_{\dot K_q^{\alpha,p}} + \|\{2^{sj}\psi_{j,a}^* f\}_1^\infty\|_{\dot K_q^{\alpha,p}(\ell_\beta)} \le C\|f\|_{\dot K_q^{\alpha,p}F_\beta^s}.$$

Here the constant C *is independent of* f.

Proof of Theorem 2.4. We only prove the admissibility of $K_q^{\alpha,p}F_\beta^s(\mathbb{R}^n)$, similar for spaces $\dot K_q^{\alpha,p}F_\beta^s(\mathbb{R}^n)$, $K_q^{\alpha,p}B_\beta^s(\mathbb{R}^n)$ and $\dot K_q^{\alpha,p}B_\beta^s(\mathbb{R}^n)$. By the lifting property (see e.g. [14]), without loss of generality, we may assume $s = 0$. Let $a > n/\min\{\beta, q\}$, θ_j, $j \in \mathbb{N}_0$ as in Section 1. Let $A = K_q^{\alpha,p}F_\beta^s(\mathbb{R}^n)$ and $B = K_q^{\alpha,p}(\ell^\beta)_a$. Then we define for $f \in K_q^{\alpha,p}F_\beta^s(\mathbb{R}^n)$

$$T_A f = \{(\widehat{\theta}_k \widehat{f})^\vee\}_{k=0}^\infty.$$

By Lemma 2.6, we conclude that T_A is bounded continuous linear map from A to B. For $\{f_k\}_{k=0}^\infty \in K_q^{\alpha,p}(\ell^\beta)_a$, we define

$$T_B(\{f_k\}_{k=0}^\infty) = \sum_{k=1}^\infty [(\widehat{\theta}_{k-1} + \widehat{\theta}_k + \widehat{\theta}_{k+1})\widehat{f}_k]^\vee + [(\widehat{\theta}_0 + \widehat{\theta}_1)\widehat{f}_0]^\vee.$$

Thus, by the definition of θ_j,

$$\widehat{\theta}_j \widehat{T_B(\{f_k\}_{k=0}^\infty)} = (\widehat{\theta}_{j-2} + \widehat{\theta}_{j-1} + \widehat{\theta}_j)\widehat{\theta}_j \widehat{f}_{j-1} + \widehat{\theta}_j \widehat{f}_j + (\widehat{\theta}_j + \widehat{\theta}_{j+1} + \widehat{\theta}_{j+2})\widehat{\theta}_j \widehat{f}_{j+1},$$

where $\theta_j \equiv 0$ if $j < 0$. Therefore

$$\begin{aligned}\theta_j * T_B(\{f_k\}_{k=0}^\infty) \\ = (\theta_{j-2} + \theta_{j-1} + \theta_j) * \theta_j * f_{j-1} + \theta_j * f_j + (\theta_j + \theta_{j+1} + \theta_{j+2}) * \theta_j * f_{j+1}.\end{aligned}$$

Then, we have

$$|\theta_j * T_B(\{f_k\}_{k=0}^\infty)(x)| \le C(f_{j-1}^{*,a} + f_j^{*,a} + f_{j+1}^{*,a}),$$

where $f_j^{*,a} \equiv 0$ for $j < 0$.

So, T_B is a bounded continuous linear map from $K_q^{\alpha,p}(\ell^\beta)_a$ to $K_q^{\alpha,p}F_\beta^s(\mathbb{R}^n)$. Therefore, by Lemmas 2.2 and 2.5, $K_q^{\alpha,p}F_\beta^s(\mathbb{R}^n)$ is admissible. This completes the proof. □

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