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ABELIANIZED OBSTRUCTION FOR FIXED POINTS OF FIBER-PRESERVING MAPS OF SURFACE BUNDLES

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Abstract. Let $f\colon M \to M$ be a fiber-preserving map where $S \to M \to B$ is a bundle and S is a closed surface. We study the abelianized obstruction, which is a cohomology class in dimension 2, to deform f to a fixed point free map by a fiber-preserving homotopy. The vanishing of this obstruction is only a necessary condition in order to have such deformation, but in some cases it is sufficient. We describe this obstruction and we prove that the vanishing of this class is equivalent to the existence of solution of a system of equations over a certain group ring with coefficients given by Fox derivatives.

1. Introduction

Fixed point theory of fiber-preserving maps of a bundle, has been studied for the past 40 years. One approach for this problem was developed by A. Dold in [3]. There, an index was defined as an element of a generalized cohomology theory. That index is a stable invariant (an element of the cohomotopy group of the disjoint union of the base with a point) and the dimension of the fiber plays no rôle. Another approach for the problem was developed by E. Fadell and S. Husseini in [5] using classical obstruction theory a la Steenrod. In [5] the fiber bundles are assumed to have as a fiber a manifold of dimension greater or equal to 3. This restriction leaves out the situation where the fiber is a closed surface

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or a circle. The case of S^1-fiber bundles has been considered in [9]. The purpose of this work is to provide some machinery to study fixed point of fiber-preserving maps of surface bundles. We will define a type of index for a fiber-preserving map f of a surface bundle which is going to be the *primary abelianized obstruction* to lift a map to a fibration. This type of index has already been considered in [6] for the usual case, i.e. the case where the base is a point. Then we will show how to decide when this obstruction vanishes or not, in terms of solution of a certain system of equations over a group ring $\mathbb{Z}[\pi]$ where the coefficients of the system are obtained using Fox calculus. This invariant is not expected in general to be a full invariant, in the sense that if it vanishes then one can deform the map by a fiber-preserving homotopy to a fixed point free map. Nevertheless, the works [10] and [11] show some cases where this invariant is indeed a full invariant.

Here we explain in more details how to define this index, which is a *primary abelianized obstruction*, for a fiber-preserving map of an arbitrary surface bundle. Also we will see how relevant it is for the fixed point problem of fiber-preserving maps on surface bundles, if the surface S is different from S^2 (the 2-sphere) and RP^2 (the projective plane). For some particular surface bundles there are simpler ways to obtain the index and we will describe them in Section 4, which is the section where we have a brief exposition of the recent applications of this index. Let $F \to M \to B$ be a bundle and $f\colon M \to M$ be a fiber-preserving map over B. When is f deformable over B to a fixed point free map g by a fiberwise homotopy over B? E. Fadell and S. Husseini in [5] considered this problem in the case where the fiber F, the base space B and the total space M are closed manifolds. They considered the fiber square $M \times_B M \to M$, i.e. the pullback fiber bundle of $p\colon M \to B$ by $p\colon M \to B$. Then the inclusion $M \times_B M - \Delta \to M \times_B M$, where Δ is the diagonal in $M \times_B M$, is replaced by the fiber bundle $q\colon E_B(M) \to M \times_B M$, whose fiber is denoted by $\mathcal{F}$. So we have the following diagram:

(1.1)

$$\begin{array}{ccccc} \mathcal{F} & & \mathcal{F} & & \\ \downarrow & & \downarrow & & \\ E_B(f) & \longrightarrow & E_B(M) & & \\ \overset{\sigma}{\nearrow} \downarrow q_f & & \downarrow q & & \\ M \xrightarrow{1} M & \xrightarrow[(1,f)]{} & M \times_B M & & \end{array}$$

where $q_f\colon E_B(f) \to M$ is the induced fiber bundle from q by $(1, f)$. From [5] we have

Theorem 1.1. *The map f is deformable to a fixed point free map g over B if and only if there exists a lift σ in diagram* (1.1).

Observe that the existence of σ is equivalent to the existence of a lift $M \to E_B(M)$ of the map $(1, f)$, since $E_B(f)$ is the pullback.

Let F be a closed surface different from S^2 (the 2-sphere) and RP^2 (the projective plane). As result of the Remarks 1.1 from [10, Section 1], we can conclude:

Remark 1.2.

(a) The fiber $\mathcal{F}$ has homotopy groups

$$\pi_{j-1}(\mathcal{F}) \cong \pi_j(M \times_B M, M \times_B M - \Delta) \cong \pi_j(F, F - x)$$

where x is a point in F. So these groups are trivial for $j > 2$.

(b) To construct a cross section of the bundle above, it suffices to construct a cross section over the 2-skeleton.

The remarks above show in our case how relevant is to decide when there is a cross section over the 2-skeleton. It is worth to mention that this question in many cases is equivalent to an algebraic question, as a consequence of Theorem 4.3.1, p. 265 in [1], (see also [10, Proposition 1.6]). More specifically, let $V \to W \to Y$ be a bundle where V is connected and $f\colon X \to Y$ a map. Suppose that $\pi_1(V) \to \pi_1(W)$ is injective. Then the map f can be lifted to W over the 2-skeleton of X if and only if there is a lift Γ of f_π:

$$\begin{array}{ccc} & & \pi_1(V) \\ & & \downarrow \\ & & \pi_1(W) \\ & \overset{\Gamma}{\nearrow} & \downarrow q \\ \pi_1(X) & \xrightarrow[f_\pi]{} & \pi_1(Y) \end{array}$$

where f_π is the homomorphism induced by f. To solve this algebraic problem is not an easy task, by all means. So a cohomology class with local coefficients system in $H^2(X; \{H_1(V)\})$, which is also called *the abelianized obstruction to lift f over the 2-skeleton*, is going to be defined. In general to decide if this obstruction vanish or not, is simpler than to decide if the lift Γ on the above diagram exists or not. The existence of a lift implies the vanishing of the abelianized obstruction, but the converse does not hold in general.

The manuscript is divided into 3 sections besides the introduction. In Section 2 we provide the general definition of the abelianized obstruction to deform a map into a subspace. In Section 3 we show how to decide if the abelianized obstruction vanishes in terms of the existence of solution of a system of equations. This is our main result. In Section 4 we apply and comment how to perform such calculation in some cases.

2. Generalities about obstruction (abelianized)

Let X be a connect CW-complex and Y a path connected space. Suppose that $Y_0 \subset Y$ is a path connected subspace such that $\pi_1(Y, Y_0)$ has cardinality one.

In this section we will describe the abelianized obstruction to deform a map $f\colon X \to Y$ to $Y_0 \subset Y$. The two main references are [13] and [18].

We assume that $f(X_1) \subset Y_0$. We will see that a cochain which represents the abelianized obstruction is given by

$$H_2(\overline{X_2}, \overline{X_1}) \xrightarrow{(\rho^{ab})^{-1}} {\pi_2}^{ab}(\overline{X_2}, \overline{X_1}) \xrightarrow{p_\pi^{ab}} \pi_2^{ab}(X_2, X_1) \xrightarrow{f_\pi} \pi_2^{ab}(Y, Y_0) = H_1(\mathcal{F}) \ .$$

Moreover, this cochain representing the obstruction can be interpreted as the following homomorphism C: we represent an element of $H_2(\overline{X_2}, \overline{X_1})$ as the class of a $\sigma\colon (I^2, \partial I^2) \to (\overline{X_2}, \overline{X_1})$ and let $C([\sigma]) = [fp(\sigma)] \in \pi_2^{ab}(Y, Y_0) = H_1(\mathcal{F})$.

We begin replacing the inclusion $Y_0 \to Y$ by a fibration and we obtain $\mathcal{F}(Y_0) \to \mathcal{E}(Y_0) \to Y$, where the fiber is connected. To simplify the notation we denote $\mathcal{E} = \mathcal{E}(Y_0)$ and $\mathcal{F} = \mathcal{F}(Y_0)$. It is well known that the fundamental group of the base of the fibration, $\pi_1(Y)$, acts on the abelian group $H_1(\mathcal{F})$. By means of the homomorphism f_π, we obtain that $H_1(\mathcal{F})$ becomes a $\mathbb{Z}[\pi_1(X)]$-module. Let X_n be the n-skeleton of X. Since X_1 and the fiber $\mathcal{F}$ are connected there is a lifting $g_1\colon X_1 \to \mathcal{E}$ so $q \circ g_1 = f_{|X_1}$, where $q\colon \mathcal{E} \to Y$ is the fibration. Following [13], we define a 2-cocycle as follows: given a 2-cell e_{β_j}, the attaching map from the boundary of the 2-cell into X_1 composite with the lifting g_1 is a loop in $\mathcal{E}$ which projects to a loop homotopically trivial in the base Y. So it can be deformed to a loop in the fiber and then we consider the element of $H_1(\mathcal{F})$ defined by such element. This cocycle depends on many choices but it determines a well defined 2-dimensional cohomology class $\mathcal{A}(f) \in H^2(X; \{H_1(\mathcal{F})\})$.

We will follow [18] in order to describe a cocycle which determines the abelianized obstruction $\mathcal{A}(\varphi)$ defined above. This description is suitable for our purpose and has been used in [17]. For this we consider:

(a) Let $p\colon \widetilde{X} \to X$ be the universal covering of X and denote $\overline{X}_n = p^{-1}(X_n)$.
(b) Let $g_1\colon X_1 \to \mathcal{E}$ be a lifting on the 1-skeleton X_1.
(c) Let G be a group and G^{ab} the abelianized group of G.
(d) Let $\rho^{ab}\colon {\pi_2}^{ab}(\overline{X_2}, \overline{X_1}) \to H_2(\overline{X_2}, \overline{X_1})$ be the induced homomorphism of the Hurewicz homomorphism $\rho\colon \pi_2(\overline{X_2}, \overline{X_1}) \to H_2(\overline{X_2}, \overline{X_1})$. Note that ρ^{ab} is an isomorphism.
(e) Let $p_\pi^{ab}\colon {\pi_2}^{ab}(\overline{X_2}, \overline{X_1}) \to \pi_2^{ab}(X_2, X_1)$ be the abelianized homomorphism induced by p_π.

According to Lemma 5.3, page 293 and in light of the Theorem 4.9, page 288, Chapter VI of [18], a cocycle which represents the primary abelianized obstruction $\mathcal{A}(f) \in H^2(X; \{H_1(\mathcal{F})\})$ is given by the following composition:

$$H_2(\overline{X_2}, \overline{X_1}) \xrightarrow{(\rho^{ab})^{-1}} {\pi_2}^{ab}(\overline{X_2}, \overline{X_1}) \xrightarrow{p_\pi^{ab}} \pi_2^{ab}(X_2, X_1) \xrightarrow{f_\pi} \pi_2^{ab}(Y, Y_0) = H_1(\mathcal{F})$$

We finish this section with some comments.

(a) The description of the abelianized obstruction that we are going to give below it is the same given for some particular fibrations in [17]. In [17] the description was given for spaces which arises from the study of fixed points of maps on certain T(Torus)-principal fibrations.

(b) The isomorphism $\pi_1(\mathcal{F}) \simeq \pi_2(Y, Y_0)$ induces an isomorphism $H_1(\mathcal{F}) \simeq \pi_2^{ab}(Y, Y_0)$ as $\pi_1(Y)$-module. We recall that the action of $\pi_1(Y)$ on $H_1(\mathcal{F})$ cames from the fibration and the action of $\pi_1(Y)$ on $\pi_2^{ab}(Y, Y_0)$ came from the action of $\pi_1(Y_0)$ and because $\pi_1(Y_0) \to \pi_1(Y)$ is surjective. So the primary abelianized obstruction $\mathcal{A}(f)$ lies in $H^2(X; \{\pi_2^{ab}(Y, Y_0)\})$.

(c) From this, when we look at the diagram (1.1), it is convenient to separate into two cases. The first case is when the homotopy group $\pi_2(Y, Y_0)$ is abelian. The second case is when it is not.

The first case happens when we consider a surface bundle where the fiber is the sphere S^2 or the projective space RP^2. In these cases the local coefficient system becomes, $\mathbb{Z}$ or $\mathbb{Z}+\mathbb{Z}$, respectively. Then the vanishing of this obstruction class implies that we have a lift over the 2-skeleton. Nevertheless, potentially there are higher obstructions in order to find a lift σ.

The second case is when $\pi_2(Y, Y_0)$ is not abelian. This is the case where the fiber is a closed surface different from S^2 and from RP^2.

3. The vanishing of the abelianized obstruction

As before let X be a connected CW-complex. Therefore, we can assume that the CW structure of X has one 0-cell and consequently the 1-skeleton is a wedge of circles.

In the previous section we defined cocyles which represent the abelianized obstruction to factorize a map $f\colon X \to Y$ into $Y_0 \subset Y$ up to the 2-skeleton of X. We will obtain an equivalent condition for the class determined by this cocycle to be trivial. This condition is given in terms of solution of a system of equations over a group ring of the fundamental group of X where the entries are obtained using Fox calculus.

As in the previous section let $p\colon \widetilde{X} \to X$ be the universal covering, denote by X_n the n-skeleton of X and by $\overline{X}_n = p^{-1}(X_n)$.

Consider the following commutative diagram where the maps have been already defined in the previous section or they are self-explanatory.

$$
\begin{array}{ccccccc}
H_2(\overline{X_2},\overline{X_1}) & \xleftarrow[\simeq]{\rho^{ab}} & {\pi_2}^{ab}(\overline{X_2},\overline{X_1}) & \xrightarrow[\simeq]{p_\pi^{ab}} & \pi_2^{ab}(X_2,X_1) & \xrightarrow{f_\pi} & H_1\mathcal{F} \\
\downarrow{\scriptstyle \partial_2} & & & & & \nearrow_{\Psi} & \\
H_1(\overline{X_1}) & & & & & & \\
\downarrow{\scriptstyle j} & & & & & & \\
H_1(\overline{X_1},\overline{X_0}) & & & & & &
\end{array}
\tag{3.1}
$$

Observe that the composite of the maps in the horizontal line is a cocycle which represents exactly the abelianized obstruction $\mathcal{A}(f)$ as we have seen in the previous section. It is zero if and only if there is an equivariant homomorphism Ψ as in the diagram (3.1) which makes the entire diagram commutative.

The homomorphism

$$C = f_\pi \circ p_\pi^{ab} \circ (\rho^{ab})^{-1} \in \mathrm{Hom}^{\pi_1(X)}(\Gamma_*(\widetilde{X}); H_1(\mathcal{F}))$$

is a cocycle which is an equivariant homomorphism on the equivariant cellular chain complex denoted by $\Gamma_*(\widetilde{X})$.

Let $\langle \alpha_1, \dots, \alpha_n \mid \beta_1, \dots, \beta_m \rangle$ be a presentation of the group $\pi_1(X)$. Because X_1 is a wedge of S^1 and for each 2-cell e_{β_j} of X_2 it has the attaching map given by the relation β_j, we choose the generators of $H_1(\overline{X_1},\overline{X_0})$ and $H_2(\overline{X_2},\overline{X_1})$ which are bases as free $\pi_1(X)$-module and we choose the specific lifting g_1. Explicitly we have the following considerations:

(A) We note that $(g_1)_\pi(\beta_j) \in \pi_1(\mathcal{E})$ is mapped by the homomorphism $\pi_1(\mathcal{E}) \to \pi_1(Y)$ to zero; so we have $(g_1)_\pi(\beta_j) \in \pi_1(\mathcal{F})$. We denote this element in $H_1(\mathcal{F})$ by $(g_1)_\pi^{ab}(\beta_j) \in H_1(\mathcal{F})$.

(B) Let $c(\alpha_i)\colon (\Delta_1, \partial\Delta_1) \to (\overline{X}_1, \overline{X}_0)$ be a lifting of the generator $\alpha_i \in \pi_1(X)$ starting in $\overline{x}_0$ and let $\overline{x}_{\alpha_i}$ be its endpoints. These paths define a set of elements in $H_1(\overline{X}_1, \overline{X}_0)$ which form a basis of $H_1(\overline{X}_1, \overline{X}_0)$ as a free $\pi_1(X)$-module. By abuse of notation we denote the elements also by $c(\alpha_i)$.

(C) Let $\delta\colon \pi_2(X_2, X_1) \to \pi_1(X_1)$ be the connecting homomorphism. Take $\overline{e}_{\beta_j} \in H_2(\overline{X}_2, \overline{X}_1)$ generators such that $\delta \circ p_\pi \circ \rho^{-1}(\overline{e}_{\beta_j}) = \beta_j \in \pi_1(X_1)$, the relations in $\pi_1(X)$. These generators define a basis of $H_2(\overline{X}_2, \overline{X}_1)$ as a free $\pi_1(X)$-module.

(D) If A is a matrix with entries $a_{ij} \in R$, and $\varphi\colon R \to S$ is a homomorphism of rings then A^φ means the matrix with entries $\varphi(a_{ij})$.

With these generators and notations we have:

THEOREM 3.1.

(a) $C(\overline{e}_{\beta_j}) = (g_1)^{ab}_{\pi}(\beta_j)$, $j = 1, \dots, m$.

(b) $j \circ \partial_2(\overline{e}_{\beta_j}) = \sum_{i=1}^{n} [\partial\beta_j/\partial\alpha_i] \cdot c(\alpha_i)$, $j = 1, \dots, m$ *where* $\partial/\partial\alpha_i$ *are the free derivatives with respect to* α_i *in the sense of* [8] *and the symbol* "$\cdot$" *denotes the product using the* $\pi_1(X)$*-module structure on* $H_1(\overline{X}_1, \overline{X}_0)$.

(c) *There is an equivariant homomorphism* Ψ *in diagram* (3.1), *i.e.* $\mathcal{A}(f) = 0$ *if and only if there exists a solution for the following matricial equation on* $\mathbb{Z}(\pi_1(X))$*:*

$$\begin{pmatrix} \dfrac{\partial\beta_1}{\partial\alpha_1} & \dfrac{\partial\beta_1}{\partial\alpha_2} & \cdots & \dfrac{\partial\beta_1}{\partial\alpha_n} \\ \dfrac{\partial\beta_2}{\partial\alpha_1} & \dfrac{\partial\beta_2}{\partial\alpha_2} & \cdots & \dfrac{\partial\beta_2}{\partial\alpha_n} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial\beta_m}{\partial\alpha_1} & \dfrac{\partial\beta_m}{\partial\alpha_2} & \cdots & \dfrac{\partial\beta_m}{\partial\alpha_n} \end{pmatrix}^{\mathbb{Z}[(g_1)_\pi]} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} (g_1)^{ab}_{\pi}(\beta_1) \\ (g_1)^{ab}_{\pi}(\beta_2) \\ \vdots \\ (g_1)^{ab}_{\pi}(\beta_m). \end{pmatrix}$$

PROOF. (a) It follows from the definition of the abelianized obstruction and the consideration (A).

(b) This is classical. Recall that $H_1(\overline{X}, \overline{X}_0)$ is the homology of the complexes $H_i(\overline{X}_i, \overline{X}_{i-1})$ and for our purpose it suffices to consider $\overline{X}$ of dimension 2. The calculation of the boundary homomorphism in question is exactly described in [2, Chapter 9, Section A], for the purpose of to show Proposition 9.2. As result of section B in [2, Chapter 9], we have that the derivations in part A in [2, Chapter 9] are the Fox derivations. So the result follows.

(c) Suppose that the system has a solution $(x_1, \dots, x_n)$. Define

$$\Psi\colon H_1(\overline{X_1}, \overline{X_0}) \to H_1\mathcal{F} \quad \text{by} \quad \Psi(c(\alpha_i)) = x_i.$$

This defines an equivariant homomorphism and using (b) one can see that we have a commutative diagram. Finally, suppose that we have an equivariant homomorphism Ψ which makes the diagram commutative. Let $x_i = \Psi(c(\alpha_i))$. From the commutative of the diagram and using (b) we obtain that

$$(g_1)^{ab}_{\pi}(\beta_j) = \sum_{i=1}^{n} \left[\frac{\partial\beta_j}{\partial\alpha_i}\right] \cdot \Psi(c(\alpha_i)) = \sum_{i=1}^{n} \mathbb{Z}[(g_1)_\pi]\left(\frac{\partial\beta_j}{\partial\alpha_i}\right) \cdot x_i.$$

But these equations as j runs from 1 to n is the system given in the matricial form in (c) and the result follows. □

4. Special cases and the general case

In this section we consider the following examples: (1) An arbitrary surface bundle over a space B; (2) Torus bundles over the circle; (3) T-principal bundles where T is the torus.

Case 1. Let $S \to E \to B$ be a S-bundle (S a closed surface) over a space B and $f\colon E \to E$ be a fiber-preserving map. In this case the fixed point free problem is equivalent to the problem of deforming the map $(1, f)\colon M \to M \times_B M$ into $M \times_B M - \Delta$ (see [5]). Then, we want to find a lift $\overline{g}$ for the diagram

$$\begin{array}{ccc} & & \mathcal{F}(M \times M - \Delta) \\ & & \downarrow \\ & & \mathcal{E}(M \times M - \Delta) \\ & \overset{\overline{g}}{\nearrow} & \downarrow q \\ M & \xrightarrow[(1,f)]{} & M \times_B M \end{array}$$

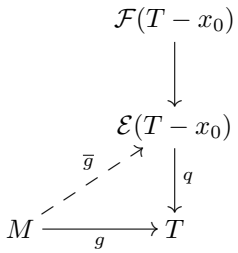
The primary abelianized obstruction can be studied using Theorem 3.1. In order to apply Theorem 3.1 we have that the local coefficient system here is given by the formula given in Corollary 3.4 in [5]. Further, we can use the formula given by Proposition 3.5 in [5] since in the fibration above the fundamental group of the fiber injects in the fundamental group of the total space. Although in our case the fiber dimension is smaller than 3, the proof given in [5] can be adapted to find the action of the fundamental group of the base on $H_1(F)$, by making the obvious adjustments. Compare also with [6]. For the specific case where the base is S^1 and the fiber is the Klein bottle the abelianized primary obstruction has been studied in detail in [11]. From the results one can read that in many cases the vanishing of abelianized obstruction suffices for the existence of the lift, therefore we are able to deform the map fiberwise to a fixed point free map. But it is not clear yet if this happens in all cases. See last section.

Case 2. Let $T \to MA \to S^1$ be a T-bundle over S^1 and $f\colon MA \to MA$ be a fiber-preserving map. The space MA is obtained from $[0, 1] \times S^1$ when we identify $(0, x) \sim (1, A(x))$ and A is a unimodular matrix with coefficients in $\mathbb{Z}$. In this case the fixed point free problem is equivalent to the problem of deforming a certain map $g\colon MA \to MA$ into $MA - S^1$ (see [10]). Then we want to find a lift $\overline{g}$ for the diagram

$$\begin{array}{ccc} & & \mathcal{F}(MA - S^1) \\ & & \downarrow \\ & & \mathcal{E}(MA - S^1) \\ & \overset{\overline{g}}{\nearrow} & \downarrow q \\ MA & \xrightarrow[g]{} & MA \end{array}$$

The primary abelianized obstruction can be studied using Theorem 3.1. It has been studied in detail in [10]. The action of $\pi_1(MA)$ on $\pi_1(\mathcal{F}(MA - S^1) \simeq \Pi_2(T, T-1)$ is given by Theorem 2.2, item 2 in [10]. After a long calculation solving all cases, one can read from the results that the cases where one can find a lift are exactly the ones where the abelianized obstruction vanish.

Case 3. Let $T \to M \to B$ be a T-principal bundle over a base B and $f\colon M \to M$ be a fiber-preserving map. In this case the fixed point free problem is equivalent to the problem of deforming a certain map $g\colon M \to T$ into $T - x_0$ (see [17]). Then we want to find a lift $\overline{g}$ for the diagram



The primary abelianized obstruction can be studied using Theorem 3.1. It has been studied in detail in [17]. The action of $\pi_1(T)$ on $H_1(\mathcal{F}) \simeq \mathbb{Z}[\pi_1(T)]$ is the left multiplication $\alpha\beta$ where $\alpha \in \pi_1(T)$, it is viewed in the base of the fibration and $\beta \in \pi_1(T) \subset H_1(\mathcal{F}) \simeq \mathbb{Z}[\pi_1(T)]$. By means of g it induces a structure of $\mathbb{Z}[\pi_1(M)]$-module on $\mathbb{Z}[\pi_1(T)]$. In [17] several results are obtained for fixed point of T-principal bundles. We conclude by remark that in [17] is given an example of a T-principal bundle where the abelianized obstruction is trivial but one cannot lift the map over the 2-skeleton. Nevertheless if the bundle is over S^1 then the bundle is trivial and this phenomenon does not happens. We reproduce the example.

Example 4.1. From [17] we will write an example of a map $f\colon M \to T$ such that the abelianized obstruction to deform the map into $T - \{x_0\}$ is trivial but the map cannot be deformed, where T is the torus. Let M be an orientable closed manifold with

$$\pi_1(M) = \langle \gamma_1, \gamma_2; [[\gamma_1, \gamma_2], \gamma_2[\gamma_1, \gamma_2]\gamma_2^{-1}] \rangle.$$

So we have only one relation and using previous notation for the presentation of a group we have $\beta_1 = [[\gamma_1, \gamma_2], \gamma_2[\gamma_1, \gamma_2]\gamma_2^{-1}]$.

Let $f\colon M \to T$ be a map such that $f_\pi(\gamma_1) = a$ and $f_\pi(\gamma_2) = b$ where a, b are generators of $\pi_1(T)$. From the definition of the homomorphism $(g_1)_\pi^{ab}$ follows that $(g_1)_\pi^{ab}(\beta_1)$ is trivial. Therefore from Theorem 3.1 the abelianized obstruction vanish. But we cannot find a lifting for f_π, consequently for f. A lifting for f_π would have image a free group of rank 1, which is a contradiction. It is

straighforward, using the map f, construct an example on the trivial principal bundles $M \times T$ in terms of the fixed point of fiber-preserving maps.

5. The Klein bottle case

In this last section we would like to state the results obtained from [11]. The details will appear elsewhere. Let ϕ be a homeomorphism of the Klein bottle and $M(\phi)$ be the quotient space $K \times [0,1]$, where we are identifying $(x,0)$ with $(\phi(x),1)$. Consider all maps $f\colon M(\phi) \to M(\phi)$ over S^1, i.e. $p \circ f = p$, where $p\colon M(\phi) \to S^1$ is the projection, such that when restricted to the fiber can be deformed to fixed point free map. There are four isotopy classes of homeomorphisms of K. The induced homomorphisms on the fundamental group of K by the four homeomorphisms ϕ above are $\phi_p(1,\eta)\colon \alpha \to \alpha$, $\beta \to \alpha^p\beta^\eta$ for $p \in \{0,1\}$, $\eta \in \{\pm 1\}$. Here α, β denote the generators of the Klein bottle under the relation $\alpha\beta\alpha\beta^{-1}$.

Case I	(I.1)	$f_s(r,1,0,2k)$:	$\alpha \to \alpha^r$,	
$\phi_0(1,1)$			$\beta \to \alpha^s\beta$,	
			$c_0 \to \beta^{2k}c_0$,	$r,s,k \in \mathbb{Z}$
	(I.2)	$f_s(0,1,s,2k+1)$:	$\alpha \to 1$,	
			$\beta \to \alpha^s\beta$,	
			$c_0 \to \alpha^s\beta^{2k+1}c_0$,	$s,k \in \mathbb{Z}$
Case II		$f_s(2r+1,1,r,2k)$:	$\alpha \to \alpha^{2r+1}$,	
$\phi_1(1,1)$			$\beta \to \alpha^s\beta$,	
			$c_0 \to \alpha^r\beta^{2k}c_0$,	$r,s,k \in \mathbb{Z}$
Case III	(III.1)	$f_s(r,1,0,2k)$:	$\alpha \to \alpha^r$,	
$\phi_0(1,-1)$			$\beta \to \alpha^s\beta$,	
			$c_0 \to \beta^{2k}c_0$,	$r,s,k \in \mathbb{Z}$
	(III.2)	$f_s(0,1,s,2k+1)$:	$\alpha \to 1$,	
			$\beta \to \alpha^s\beta$,	
			$c_0 \to \alpha^s\beta^{2k+1}c_0$,	$s,k \in \mathbb{Z}$
Case IV		$f_s(2r+1,1,r,2k)$:	$\alpha \to \alpha^{2r+1}$,	
$\phi_1(1,-1)$			$\beta \to \alpha^s\beta$,	
			$c_0 \to \alpha^r\beta^{2k}c_0$,	$r,s,k \in \mathbb{Z}$

Table 1

Case I	$f_s(r,1,0,0)$:	$\alpha \to \alpha^r$,	
$\phi_0(1,1)$		$\beta \to \alpha^s\beta$,	
		$c_0 \to \quad c_0$	$r,s \in \mathbb{Z}$
Case II	$f_s(2r+1,1,r,0)$:	$\alpha \to \alpha^{2r+1}$,	
$\phi_1(1,1)$		$\beta \to \alpha^s\beta$,	
		$c_0 \to \alpha^r c_0$,	$r,s \in \mathbb{Z}$
Case III	$f_{2s+1}(1,1,0,4k)$:	$\alpha \to \alpha$,	
$\phi_0(1,-1)$		$\beta \to \alpha^{2s+1}\beta$,	
		$c_0 \to \beta^{4k} c_0$,	$s,k \in \mathbb{Z}$
	$f_{2s+1}(-1,1,0,4k+2)$:	$\alpha \to \alpha^{-1}$,	
		$\beta \to \alpha^{2s+1}\beta$,	
		$c_0 \to \beta^{4k+2} c_0$,	$s,k \in \mathbb{Z}$
	$f_s(0,1,s,2k+1)$:	$\alpha \to 1$,	
		$\beta \to \alpha^s\beta$,	
		$c_0 \to \alpha^s\beta^{2k+1} c_0$,	$s,k \in \mathbb{Z}$
Case IV	$f_{2s}(4l+3,1,2l+1,4k)$:	$\alpha \to \alpha^{4l+3}$,	
$\phi_1(1,-1)$		$\beta \to \alpha^{2s}\beta$,	
		$c_0 \to \alpha^{2l+1}\beta^{4k} c_0$,	$s,k,l \in \mathbb{Z}$
	$f_{2s}(4l+1,1,2l,4k+2)$:	$\alpha \to \alpha^{4l+1}$,	
		$\beta \to \alpha^{2s}\beta$,	
		$c_0 \to \alpha^{2l}\beta^{4k+2} c_0$,	$s,k,l \in \mathbb{Z}$
	$f_{2s+1}(4l+3,1,2l+1,4k+2)$:	$\alpha \to \alpha^{4l+3}$,	
		$\beta \to \alpha^{2s+1}\beta$,	
		$c_0 \to \alpha^{2l+1}\beta^{4k+2} c_0$,	$s,k \in \mathbb{Z}$
	$f_{2s+1}(4l+1,1,2l,4k)$:	$\alpha \to \alpha^{4l+1}$,	
		$\beta \to \alpha^{2s+1}\beta$,	
		$c_0 \to \alpha^{2l}\beta^{4k} c_0$,	$s,k,l \in \mathbb{Z}$

TABLE 2

The fundamental group of $M(\phi_p(1,\eta))$, where by abuse of notation $\phi_p(1,\eta)$ also denotes the homeomorphism which induces homomorphism above, have a presentation

$$\pi_1(M(\phi_p(1,\eta)),\mathbf{0})) = \langle \alpha,\beta,c_0 \mid \alpha\beta\alpha\beta^{-1} = 1, c_0\alpha {c_0}^{-1} = \alpha, c_0\beta {c_0}^{-1} = \alpha^p\beta^\eta \rangle$$

For any homeomorphism $\phi\colon K \to K$, $M(\phi)$ is homeomorphic over S^1 to $M(\phi_p(1,\eta))$ where $\phi_p(1,\eta)$ is given above. From Theorem 2.4 in [11] we have

THEOREM 5.1. *Let $f\colon M(\phi_q(1,\eta)) \to M(\phi_q(1,\eta))$ be a map over S^1, where $q \in \{0,1\}$ and $\eta = \pm 1$. If the Nielsen number of f restricted to the fiber is zero then $f_\pi\colon \pi_1(M(\phi_q(1,\eta))) \to \pi_1(M(\phi_q(1,\eta)))$ is given by the Table* 1.

For all the maps above mentioned we can decide which ones can be deformed over S^1 to fixed point free. This is the main result in [11, Theorem 6.26]. From there we can say:

THEOREM 5.2. *Let ϕ be a homeomorphism of K, where K denotes the Klein bottle and let $M(\phi)$ be the quotient space $K \times [0,1]$ where we are identifying $(x,0)$ with $(\phi(x),1)$. Then $M(\phi)$ is a fiber bundle over the circle S^1, where the fiber is K, and let $f\colon M(\phi) \to M(\phi)$ be a fiber-preserving map over S^1. If f belongs to one of the cases of the previous theorem then it can be deformed to a fixed point free map g by a fiberwise homotopy over S^1 if and only if f belongs to the correspondent case on the Table* 2.

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