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CONSTANTS OF MOTION FOR NON-DIFFERENTIABLE QUANTUM VARIATIONAL PROBLEMS

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Abstract. We extend the DuBois–Reymond necessary optimality condition and Noether's symmetry theorem to the scale relativity theory setting. Both Lagrangian and Hamiltonian versions of Noether's theorem are proved, covering problems of the calculus of variations with functionals defined on sets of non-differentiable functions, as well as more general non-differentiable problems of optimal control. As an application we obtain constants of motion for some linear and nonlinear variants of the Schrödinger equation.

1. Introduction

The notion of *symmetry* play an important role both in physics and mathematics. Symmetries are defined as transformations of a certain system, which result in the same object after the transformation is carried out. They are mathematically described by parameter groups of transformations. Their importance range from fundamental and theoretical aspects to concrete applications, having profound implications in the dynamical behavior of the systems, and in their basic qualitative properties (see [17] and references therein).

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Constants of motion are another fundamental notion of physics and mathematics. Typically, they are used in the calculus of variations and optimal control to reduce the number of degrees of freedom, thus reducing the problems to a lower dimension and facilitating the integration of the equations given by the necessary optimality conditions (see [6], [15] and references therein).

Emmy Noether was the first to prove, in 1918, that these two notions are connected: when a system exhibits a symmetry, then a constant of motion exists. The celebrated *Noether's theorem* provide an explicit formula for such constants of motion. Since the pioneer work of Emmy Noether, many extensions of the classical results were done both in the calculus of variations setting as well as in more general setting of optimal control (see [5], [7]–[9], [16], [18], [19] and references therein). All available versions of Noether's theorem are, however, proved for problems whose admissible functions are differentiable.

In 1992 L. Nottale introduced the theory of scale-relativity without the hypothesis of space-time differentiability [12], [13]. A rigorous foundation to Nottale's scale-relativity theory was recently given by J. Cresson [3], [4]. The calculus of variations developed in [3] cover sets of non differentiable curves, by substituting the classical derivative by a new complex operator, known as the *scale derivative*.

In this work we use the scale Euler–Lagrange equations and respective scale extremals [3], to prove an extension of Noether's theorem for problems of the calculus of variations and optimal control whose admissible functions are non-differentiable (Theorems 4.9 and 4.20). The results are proved by first extending the classical DuBois–Reymond necessary optimality condition to the scale calculus of variations (Theorem 4.8). Illustrative examples are given to Schrödinger equations in the scale framework [1]–[3].

2. Quantum calculus

In this section we briefly review the quantum calculus of [3], which extends the classical differential calculus to non-differentiable functions.

We denote by C^0 the set of real-valued continuous functions defined on $\mathbb{R}$.

Definition 2.1. Let $f \in C^0$. For all $\varepsilon > 0$, the ε left- and right-quantum derivatives of f, denoted respectively by $\Delta_\varepsilon^+ f(t)$ and $\Delta_\varepsilon^- f(t)$, are defined by

$$\Delta_\varepsilon^+ f(t) = \frac{f(t+\varepsilon) - f(t)}{\varepsilon}$$

and

$$\Delta_\varepsilon^- f(t) = \frac{f(t) - f(t-\varepsilon)}{\varepsilon}.$$

Remark 2.2. The ε left- and right-quantum derivative of a continuous function f correspond to the classical derivative of the ε-mean function f_ε^σ defined by

$$f_\varepsilon^\sigma(t) = \frac{\sigma}{\varepsilon}\int_t^{t+\sigma\varepsilon} f(s)\,ds,$$

with $\sigma = \pm$.

Next we define an operator which generalize the classical derivative.

DEFINITION 2.3. Let $f \in C^0$. For all $\varepsilon > 0$, the ε scale derivative of f at point t, denoted by $(\Box_\varepsilon f/\Box t)(t)$, is defined by

$$\frac{\Box_\varepsilon f}{\Box t}(t) = \frac{1}{2}[(\Delta_\varepsilon^+ f(t) + \Delta_\varepsilon^- f(t)) - i(\Delta_\varepsilon^+ f(t) - \Delta_\varepsilon^- f(t))].$$

REMARK 2.4. If f is differentiable, we can take the limit of the scale derivative when ε goes to zero. We then obtain the classical derivative $(df/dt)(t)$ of f at t.

We also need to extend the scale derivative to complex valued functions.

DEFINITION 2.5. Let f be a continuous complex valued function. For all $\varepsilon > 0$, the ε scale derivative of f, denoted by $\Box_\varepsilon f/\Box t$, is defined by

$$\frac{\Box_\varepsilon f}{\Box t}(t) = \frac{\Box_\varepsilon \mathrm{Re}(f)}{\Box t} + i\frac{\Box_\varepsilon \mathrm{Im}(f)}{\Box t},$$

where $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ denote the real and imaginary part of f, respectively.

In what follows, we will frequently use $\Box_\varepsilon$ to denote the scale derivative operator $\Box_\varepsilon/\Box t$.

THEOREM 2.6 (cf. [3]). *Let f and g be two C^0 functions. For all $\varepsilon > 0$ one has*

$$\begin{aligned}\Box_\varepsilon(f\cdot g) &= \Box_\varepsilon f\cdot g + f\cdot\Box_\varepsilon g \\ &\quad + \varepsilon i(\Box_\varepsilon f \boxminus_\varepsilon g - \boxminus_\varepsilon f \Box_\varepsilon g - \Box_\varepsilon f \Box_\varepsilon g - \boxminus_\varepsilon f \boxminus_\varepsilon g)\end{aligned} \tag{2.1}$$

where $\boxminus f$ denotes the complex conjugate of $\Box f$.

REMARK 2.7. For two differentiable functions f and g, one obtains the classical Leibniz rule $(f\cdot g)' = f'\cdot g + f\cdot g'$ by taking the limit of (2.1) when ε goes to zero.

REMARK 2.8. It is not difficult to prove the following equality:

$$\int_a^b \Box_\varepsilon f(t)\,dt = \frac{1}{2}[(f_\varepsilon^+(t) + f_\varepsilon^-(t)) - i(f_\varepsilon^+(t) - f_\varepsilon^-(t))]\Big|_a^b. \tag{2.2}$$

When ε goes to zero, (2.2) reduces to

$$\int_a^b \frac{d}{dt}f(t)\,dt = f(t)|_a^b.$$

Definition 2.9 (α-Hölderian functions). A continuous real valued function f is said to be α-Hölderian, $0 < \alpha < 1$, if for all $\varepsilon > 0$ and all t, $t' \in \mathbb{R}$ there exists a constant c such that $|t - t'| \le \varepsilon$ implies $|f(t) - f(t')| \le c\varepsilon^{\alpha}$.

We denote by H^{α} the set of continuous functions which are α-Hölderian.

Theorem 2.10 (cf. [3]). *Let $f(t,x)$ be a C^{n+1} real valued function and $x(t) \in H^{1/n}$, $n \ge 1$. For all $\varepsilon > 0$ sufficiently small one has*

$$\frac{\Box_\varepsilon f}{\Box t}(t, x(t)) = \frac{\partial f}{\partial t}(t, x(t)) + \sum_{j=1}^{n} \frac{1}{j!} \frac{\partial^j f}{\partial x^j}(t, x(t))\varepsilon^{j-1} a_{\varepsilon,j}(t) + o(\varepsilon^{1/n})$$

where

$$a_{\varepsilon,j}(t) = \frac{1}{2}[((\Delta_+^\varepsilon x)^j - (-1)^j(\Delta_-^\varepsilon x)^j) - i((\Delta_+^\varepsilon x)^j + (-1)^j(\Delta_-^\varepsilon x)^j)].$$

Lemma 2.11 is crucial for our purposes (see proof of Theorem 4.9).

Lemma 2.11 (cf. [3]). *Let $h \in H^{\beta}$, $\beta \ge \alpha 1_{[1/2,1]} + (1-\alpha)1_{]0,1/2[}$, satisfy $h(a) = h(b) = 0$ for some $a, b \in \mathbb{R}$. If $f_\varepsilon \colon \mathbb{R} \mapsto \mathbb{C}$, $\varepsilon > 0$, is such that for all $t \in [a,b]$ one has*

$$\sup_{s \in \{t, t+\sigma\varepsilon\}} |f_\varepsilon(s)| \le C\varepsilon^{\alpha-1},$$

then

$$\int_a^b \frac{\Box_\varepsilon}{\Box t}(f_\varepsilon(t) h(t))\, dt = o(\varepsilon^{\alpha+\beta-1})$$

and

$$\varepsilon \int_a^b Op_\varepsilon(f_\varepsilon) Op'_\varepsilon(h)\, dt = o(\varepsilon^{\alpha+\beta})$$

where Op_ε and Op'_ε are either $\Box_\varepsilon$ or $\boxminus_\varepsilon$.

3. Review of the classical Noether's theorem

There are several ways to prove the classical Noether's theorem. In this section we review one of those proofs.

We begin by formulating the fundamental problem of the calculus of variations: to minimize

$$I[q(\cdot)] = \int_a^b L(t, q(t), \dot{q}(t))\, dt \tag{3.1}$$

under given boundary conditions $q(a) = q_a$ and $q(b) = q_b$, and where $\dot{q} = dq/dt$. The Lagrangian $L \colon [a,b] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is assumed to be a C^1-function with respect to all its arguments, and admissible functions $q(\cdot)$ are assumed to be C^2-smooth.

DEFINITION 3.1 (Invariance of (3.1)). The functional (3.1) is said to be invariant under the s-parameter group of infinitesimal transformations

$$\begin{cases} \bar{t} = t + s\tau(t,q) + o(s), \\ \bar{q}(t) = q(t) + s\xi(t,q) + o(s), \end{cases} \tag{3.2}$$

if

$$\int_{t_a}^{t_b} L(t, q(t), \dot{q}(t))\, dt = \int_{\bar{t}(t_a)}^{\bar{t}(t_b)} L(\bar{t}, \bar{q}(\bar{t}), \dot{\bar{q}}(\bar{t}))\, d\bar{t} \tag{3.3}$$

for any subinterval $[t_a, t_b] \subseteq [a, b]$.

We will denote by $\partial_i L$ the partial derivative of L with respect to its i-th argument, $i = 1, 2, 3$.

THEOREM 3.2 (Necessary and sufficient condition of invariance). *If functional* (3.1) *is invariant under transformations* (3.2)*, then*

$$\partial_1 L(t, q, \dot{q})\tau + \partial_2 L(t, q, \dot{q}) \cdot \xi + \partial_3 L(t, q, \dot{q}) \cdot (\dot{\xi} - \dot{q}\dot{\tau}) + L(t, q, \dot{q})\dot{\tau} = 0. \tag{3.4}$$

PROOF. Since (3.3) is to be satisfied for any subinterval $[t_a, t_b]$ of $[a, b]$, one can get rid off of the integral sign in (3.3) and write the equivalent equality

$$L(t, q, \dot{q}) = \left[L\left(t + s\tau + o(s), q + s\xi + o(s), \frac{\dot{q} + s\dot{\xi} + o(s)}{1 + s\dot{\tau} + o(s)} \right) \right] \frac{d\bar{t}}{dt}. \tag{3.5}$$

Equation (3.4) is obtained differentiating both sides of condition (3.5) with respect to s and then putting $s = 0$. □

DEFINITION 3.3 (Constant of motion). A quantity $C(t, q(t), \dot{q}(t))$, $t \in [a, b]$, is said to be a *constant of motion* if $(d/dt)C(t, q(t), \dot{q}(t)) = 0$ for all the solutions q of the Euler–Lagrange equation

$$\frac{d}{dt}\partial_3 L(t, q(t), \dot{q}(t)) = \partial_2 L(t, q(t), \dot{q}(t)). \tag{3.6}$$

THEOREM 3.4 (DuBois–Reymond necessary optimality condition). *If function q is a minimizer or maximizer of functional* (3.1)*, then*

$$\partial_1 L(t, q(t), \dot{q}(t)) = \frac{d}{dt}\{L(t, q(t), \dot{q}(t)) - \partial_3 L(t, q(t), \dot{q}(t)) \cdot \dot{q}(t)\}. \tag{3.7}$$

PROOF. The conclusion follows by direct calculations using the Euler–Lagrange equation (3.6):

$$\begin{aligned} \frac{d}{dt}\{L(t, q, \dot{q}) - \partial_3 L(t, q, \dot{q}) \cdot \dot{q}\} &= \partial_1 L(t, q, \dot{q}) + \partial_2 L(t, q, \dot{q}) \cdot \dot{q} \\ &\quad + \partial_3 L(t, q, \dot{q}) \cdot \ddot{q} - \frac{d}{dt}\partial_3 L(t, q, \dot{q}) \cdot \dot{q} - \partial_3 L(t, q, \dot{q}) \cdot \ddot{q} \\ &= \partial_1 L(t, q, \dot{q}) + \dot{q} \cdot \left(\partial_2 L(t, q, \dot{q}) - \frac{d}{dt}\partial_3 L(t, q, \dot{q})\right) = \partial_1 L(t, q, \dot{q}). \end{aligned}$$ □

Theorem 3.5 (Noether's theorem). *If (3.1) is invariant under (3.2), then*

$$C(t,q,\dot q) = \partial_3 L(t,q,\dot q)\cdot \xi(t,q) + (L(t,q,\dot q) - \partial_3 L(t,q,\dot q)\cdot \dot q)\tau(t,q) \tag{3.8}$$

is a constant of motion.

Proof. To prove Noether's theorem we use the Euler–Lagrange equations (3.6) and the DuBois–Reymond necessary optimality condition (3.7) into the necessary and sufficient condition of invariance (3.4):

$$\begin{aligned}
0 &= \partial_1 L(t,q,\dot q)\tau + \partial_2 L(t,q,\dot q)\cdot\xi + \partial_3 L(t,q,\dot q)\cdot(\dot\xi - \dot q\dot\tau) + L(t,q,\dot q)\dot\tau \\
&= \partial_2 L(t,q,\dot q)\cdot\xi + \partial_3 L(t,q,\dot q)\cdot\dot\xi + \partial_1 L(t,q,\dot q)\tau + \dot\tau(L(t,q,\dot q) - \partial_3 L(t,q,\dot q)\cdot\dot q) \\
&= \frac{d}{dt}\partial_3 L(t,q,\dot q)\cdot\xi + \partial_3 L(t,q,\dot q)\cdot\dot\xi \\
&\quad + \frac{d}{dt}\{L(t,q,\dot q) - \partial_3 L(t,q,\dot q)\cdot\dot q\}\tau + \dot\tau(L(t,q,\dot q) - \partial_3 L(t,q,\dot q)\cdot\dot q) \\
&= \frac{d}{dt}\{\partial_3 L(t,q,\dot q)\cdot\xi + (L(t,q,\dot q) - \partial_3 L(t,q,\dot q)\cdot\dot q)\tau\}. \qquad \square
\end{aligned}$$

4. Main results: Non-differentiable Noether-type theorems

The classical Noether's theorem is valid for extremals $q(\,\cdot\,)$ which are C^2 differentiable, as considered in Section 3. The biggest class where a Noether-type theorem has been proved is the class of Lipschitz functions [18]. In this work we prove a more general Noether-type theorem, valid for non-differentiable scale extremals.

4.1. Calculus of variations with scale derivatives. In [3] the calculus of variations with scale derivatives is introduced and respective Euler–Lagrange equations derived. In this section we obtain a formulation of Noether's theorem for the scale calculus of variations. The proof of our Noether's theorem is done in two steps: first we extend the DuBois–Reymond condition to problems with scale derivatives (Theorem 4.8); then, using this result, we obtain the scale/quantum Noether's theorem (Theorem 4.9).

The problem of the calculus of variations with scale derivatives is defined as

$$I[q(\,\cdot\,)] = \int_{a-\varepsilon}^{b+\varepsilon} L(t,q(t),\Box_\varepsilon q(t))\,dt \to \min \tag{4.1}$$

under given boundary conditions $q(a-\varepsilon) = q_{a_\varepsilon}$ and $q(b+\varepsilon) = q_{b_\varepsilon}$, $0 < \varepsilon \ll 1$, $q(\,\cdot\,) \in H^\alpha$, $0 < \alpha < 1$. The Lagrangian $L\colon [a-\varepsilon, b+\varepsilon] \times \mathbb{R}^n \times \mathbb{C}^n \to \mathbb{C}$ is assumed to be a C^1-function with respect to all its arguments satisfying

$$||DL(t,q(t),\Box_\varepsilon q(t))|| \le K, \tag{4.2}$$

where K is a non-negative constant, D denotes the differential and $||\cdot||$ is a norm for matrices (see [3]).

Remark 4.1. In the case of admissible differentiable functions $q(\cdot)$, problem (4.1) tends to problem (3.1) when ε tends to zero.

Remark 4.2. We need to assume $a-\varepsilon \leq t \leq b+\varepsilon$ in order to avoid problems with the definition of scale derivative in the boundaries of the interval.

Theorem 4.3 (Scale Euler–Lagrange equation, cf. [3]). *If q is a minimizer of problem* (4.1)*, then q satisfy the following scale Euler–Lagrange equation:*

$$\partial_2 L(t, q(t), \Box_\varepsilon q(t)) - \Box_\varepsilon \partial_3 L(t, q(t), \Box_\varepsilon q(t)) = 0. \tag{4.3}$$

Definition 4.4 (Scale extremals). The solutions $q(t)$ of the scale Euler–Lagrange equation (4.3) are called *scale extremals*.

Definition 4.5 (cf. Definition 3.1). The functional (4.1) is said to be invariant under a s-parameter group of infinitesimal transformations

$$\begin{cases} \bar{t} = t + s\tau(t,q) + o(s), \\ \overline{q}(t) = q(t) + s\xi(t,q) + o(s), \end{cases} \tag{4.4}$$

τ, $\xi \in H^\beta$, $\beta \geq \alpha 1_{[1/2,1]} + (1-\alpha)1_{]0,1/2[}$, if

$$\int_{t_a}^{t_b} L(t, q(t), \Box_\varepsilon q(t))\, dt = \int_{\bar{t}(t_a)}^{\bar{t}(t_b)} L(\bar{t}, \overline{q}(\bar{t}), \Box_\varepsilon \overline{q}(\bar{t}))\, dt \tag{4.5}$$

for any subinterval $[t_a, t_b] \subseteq [a-\varepsilon, b+\varepsilon]$.

Theorem 4.6 establish a necessary and sufficient condition of invariance for (4.1). Condition (4.6) will be used in the proof of our Noether-type theorem.

Theorem 4.6 (cf. Theorem 3.2). *If functional* (4.1) *is invariant under the one-parameter group of transformations* (4.4)*, then*

$$\begin{aligned} \int_{t_a}^{t_b} [\partial_1 L(t, q(t), \Box_\varepsilon q(t))\tau + \partial_2 L(t, q(t), \Box_\varepsilon q(t)) \cdot \xi \\ + \partial_3 L(t, q(t), \Box_\varepsilon q(t)) \cdot (\Box_\varepsilon \xi - \Box_\varepsilon q(t) \Box_\varepsilon \tau)]\, dt = 0 \end{aligned} \tag{4.6}$$

for any subinterval $[t_a, t_b] \subseteq [a-\varepsilon, b+\varepsilon]$.

Proof. Equation (4.5) is equivalent to

$$\begin{aligned} &\int_{t_a}^{t_b} L(t, q(t), \Box_\varepsilon q(t))\, dt \\ &\quad = \int_{t_a+s\tau}^{t_b+s\tau} L\left(t + s\tau + o(s), q + s\xi + o(s), \frac{\Box_\varepsilon q + s\Box_\varepsilon \xi + o(s)}{1 + s\Box_\varepsilon \tau + o(s)}\right) dt. \end{aligned} \tag{4.7}$$

Differentiating both sides of equation (4.7) with respect to s, then putting $s=0$, we obtain equality (4.6). □

DEFINITION 4.7 (Scale constants of motion). We say that quantity $C(t, q(t), \Box_\varepsilon q(t))$ is a *scale constant of motion* if, and only if, $C(t, q(t), \Box_\varepsilon q(t)) = \text{constant}$ along all the scale extremals $q(\,\cdot\,)$ (cf. Definition 4.4).

Theorem 4.8 generalizes the DuBois–Reymond necessary optimality condition (cf. Theorem 3.4) for problems of the calculus of variations with scale derivatives.

THEOREM 4.8 (Scale DuBois–Reymond necessary condition). *If $q(\,\cdot\,)$ is a minimizer of problem* (4.1)*, then it satisfy the following condition:*

$$\begin{aligned}\frac{\Box_\varepsilon}{\Box t}\bigg\{L\bigg(t, q, \frac{\Box_\varepsilon q}{\Box t}\bigg) - \partial_3 L\bigg(t, q, \frac{\Box_\varepsilon q}{\Box t}\bigg) \cdot \frac{\Box_\varepsilon q}{\Box t}\bigg\}& \\ = \partial_1 L\bigg(t, q, \frac{\Box_\varepsilon q}{\Box t}\bigg) - \varepsilon i(\Box_\varepsilon f \boxminus_\varepsilon g - \boxminus_\varepsilon f \Box_\varepsilon g - \Box_\varepsilon f \Box_\varepsilon g - \boxminus_\varepsilon f \boxminus_\varepsilon g)&\end{aligned} \tag{4.8}$$

where $f = \partial_3 L(t, q, \Box_\varepsilon q/\Box t)$, $g = \Box_\varepsilon q/\Box t$, $\boxminus f$ and $\boxminus g$ are the complex conjugate of $\Box f$ and $\Box g$, respectively.

PROOF. The scale DuBois–Reymond necessary condition (4.8) follows from the linearity of the scale derivative operator, Theorems 2.6 and 2.10, and the scale Euler–Lagrange equations (4.3):

$$\begin{aligned}&\Box_\varepsilon\{L(t,q,\Box_\varepsilon q) - \partial_3 L(t, q, \Box_\varepsilon q) \cdot \Box_\varepsilon q\} \\ &\quad = \partial_1 L(t, q, \Box_\varepsilon q) + \partial_2 L(t, q, \Box_\varepsilon q) \cdot \Box_\varepsilon q + \partial_3 L(t, q, \Box_\varepsilon q) \cdot \Box_\varepsilon \Box_\varepsilon q \\ &\qquad - \Box_\varepsilon \partial_3 L(t, q, \Box_\varepsilon q) \cdot \Box_\varepsilon q - \partial_3 L(t, q, \Box_\varepsilon q) \cdot \Box_\varepsilon \Box_\varepsilon q \\ &\qquad - \varepsilon i(\Box_\varepsilon f \boxminus_\varepsilon g - \boxminus_\varepsilon f \Box_\varepsilon g - \Box_\varepsilon f \Box_\varepsilon g - \boxminus_\varepsilon f \boxminus_\varepsilon g) \\ &\quad = \partial_1 L(t, q, \Box_\varepsilon q) + \Box_\varepsilon q \cdot (\partial_2 L(t, q, \Box_\varepsilon q) - \Box_\varepsilon \partial_3 L(t, q, \Box_\varepsilon q)) \\ &\qquad - \varepsilon i(\Box_\varepsilon f \boxminus_\varepsilon g - \boxminus_\varepsilon f \Box_\varepsilon g - \Box_\varepsilon f \Box_\varepsilon g - \boxminus_\varepsilon f \boxminus_\varepsilon g) \\ &\quad = \partial_1 L(t, q, \Box_\varepsilon q) - \varepsilon i(\Box_\varepsilon f \boxminus_\varepsilon g - \boxminus_\varepsilon f \Box_\varepsilon g - \Box_\varepsilon f \Box_\varepsilon g - \boxminus_\varepsilon f \boxminus_\varepsilon g). \quad \Box\end{aligned}$$

Theorem 4.9 establish an extension of Noether's theorem for problems of the calculus of variations with scale derivatives.

THEOREM 4.9 (Scale Noether's theorem in Lagrangian form). *If functional* (4.1) *is invariant in the sense of Definition* 4.5*, then*

$$\begin{aligned}C(t, q(t), \Box_\varepsilon q(t)) = \partial_3 L(t, q, \Box_\varepsilon q)) \cdot \xi(t, q)& \\ + (L(t, q, \Box_\varepsilon q) - \partial_3 L(t, q, \Box_\varepsilon q) \cdot \Box_\varepsilon q)\tau(t, q)&\end{aligned} \tag{4.9}$$

is a scale constant of motion (*cf. Definition* 4.7)*.*

REMARK 4.10. If the admissible functions q are differentiable, the scale constant of motion (4.9) tends to (3.8) when we take the limit $\varepsilon \to 0$.

PROOF. Noether's scale constant of motion (4.9) follows by using the scale DuBois–Raymond condition (4.8), the scale Euler–Lagrange equation (4.3) and Theorem 2.6 into the necessary and sufficient condition of invariance (4.6):

$$\begin{aligned}
(4.10)\quad 0 = & \int_{t_a}^{t_b} [\partial_1 L(t,q(t),\Box_\varepsilon q(t))\tau + \partial_2 L(t,q(t),\Box_\varepsilon q(t)) \cdot \xi \\
& + \partial_3 L(t,q,\Box_\varepsilon q) \cdot (\Box_\varepsilon \xi - \Box_\varepsilon q \Box_\varepsilon \tau) + L\Box_\varepsilon \tau - L\Box_\varepsilon \tau]\, dt \\
= & \int_{t_a}^{t_b} [\tau \Box_\varepsilon (L(t,q,\Box_\varepsilon q) - \partial_3 L(t,q,\Box_\varepsilon q) \cdot \Box_\varepsilon q) \\
& + (L(t,q,\Box_\varepsilon q) - \partial_3 L(t,q,\Box_\varepsilon q) \cdot \Box_\varepsilon q)\Box_\varepsilon \tau \\
& + \xi \cdot \Box_\varepsilon \partial_3 L(t,q,\Box_\varepsilon q) + \partial_3 L(t,q,\Box_\varepsilon q) \cdot \Box_\varepsilon \xi]\, dt + R(\varepsilon) \\
= & \int_{t_a}^{t_b} \frac{\Box_\varepsilon}{\Box t} \{\partial_3 L(t,q,\Box_\varepsilon q) \cdot \xi + (L(t,q,\Box_\varepsilon q) - \partial_3 L(t,q,\Box_\varepsilon q) \cdot \Box_\varepsilon q)\tau\}\, dt \\
& + R(\varepsilon) + R'(\varepsilon),
\end{aligned}$$

where $R(\varepsilon)$ and $R'(\varepsilon)$ are integrals with terms resulting from the application of formula (2.1) of Theorem 2.6. Taking into consideration condition (4.2) and Lemma 2.11, the integrals $R(\varepsilon)$ and $R'(\varepsilon)$ vanish. Thus, (4.10) simplify to

$$(4.11)\quad \int_{t_a}^{t_b} \frac{\Box_\varepsilon}{\Box t} \{\partial_3 L(t,q,\Box_\varepsilon q) \cdot \xi + (L(t,q,\Box_\varepsilon q) - \partial_3 L(t,q,\Box_\varepsilon q) \cdot \Box_\varepsilon q)\tau\}\, dt = 0.$$

Using formula (2.2) and having in mind that (4.11) holds for an arbitrary $[t_a, t_b] \subseteq [a-\varepsilon, b+\varepsilon]$, we conclude that

$$\partial_3 L(t,q,\Box_\varepsilon q) \cdot \xi + (L(t,q,\Box_\varepsilon q) - \partial_3 L(t,q,\Box_\varepsilon q) \cdot \Box_\varepsilon q)\tau = \text{constant}. \qquad \Box$$

4.2. Scale optimal control. Theorem 4.9 gives a Lagrangian formulation of Noether's principle to the non-differentiable scale setting. Now we give a scale Hamiltonian formulation of Noether's principle for more general scale problems of optimal control (Theorem 4.20). The result is obtained as a corollary of Theorem 4.9.

We define the scale optimal control problem as follows:

$$(4.12)\quad \begin{gathered} I[q(\cdot), u(\cdot)] = \int_{a-\varepsilon}^{b+\varepsilon} L(t,q(t),u(t))\, dt \to \min, \\ \Box_\varepsilon q(t) = \varphi(t,q(t),u(t)), \end{gathered}$$

under the given initial condition $q(a-\varepsilon) = q_{a_\varepsilon}$, $0 < \varepsilon \ll 1$, $q(\cdot)$, $u(\cdot) \in H^\alpha$, $0 < \alpha < 1$. The Lagrangian $L\colon [a-\varepsilon, b+\varepsilon] \times \mathbb{R}^n \times \mathbb{C}^m \to \mathbb{C}$ and the velocity vector $\varphi\colon [a-\varepsilon, b+\varepsilon] \times \mathbb{R}^n \times \mathbb{C}^m \to \mathbb{C}^n$ are assumed to be C^1-functions with respect to all its arguments. Similarly as before, we assume that

$$||DL(t,q(t),u(t))|| \leq K,$$

where K is a non-negative constant, D denotes the differential and $||\cdot||$ a classical norm of matrices.

Remark 4.11. In the particular case when $\varphi(t,q,u) = u$, (4.12) is reduced to the scale problem of the calculus of variations (4.1).

Remark 4.12. If functions q are differentiable, problem (4.12) tends to the classical problem of optimal control,

$$I[q(\,\cdot\,),u(\,\cdot\,)] = \int_a^b L(t,q(t),u(t))\,dt \longrightarrow \min,$$
$$\dot q(t) = \varphi(t,q(t),u(t)),$$

as $\varepsilon \to 0$.

Theorem 4.13. *If* $(q(\,\cdot\,),u(\,\cdot\,))$ *is a minimizer of* (4.12), *then there exists a co-vector function* $p(t) \in H^{\alpha}([a-\varepsilon,b+\varepsilon];\mathbb{R}^n)$ *such that the following conditions hold:*

(a) *the scale Hamiltonian system*

$$\begin{cases} \Box_\varepsilon q(t) = \partial_4 \mathcal{H}(t,q(t),u(t),p(t)), \\ \Box_\varepsilon p(t) = -\partial_2 \mathcal{H}(t,q(t),u(t),p(t)); \end{cases} \tag{4.13}$$

(b) *the stationary condition*

$$\partial_3 \mathcal{H}(t,q(t),u(t),p(t)) = 0; \tag{4.14}$$

where the Hamiltonian $\mathcal{H}$ *is defined by*

$$\mathcal{H}(t,q,u,p) = L(t,q,u) + p\cdot\varphi(t,q,u). \tag{4.15}$$

Remark 4.14. The first equation in the scale Hamiltonian system (4.13) is nothing more than the scale control system $\Box_\varepsilon q(t) = \varphi(t,q(t),u(t))$ given in the formulation of problem (4.12).

Remark 4.15. In classical mechanics p is called the *generalized momentum*. In the language of optimal control [14], p is known as the adjoint variable.

Definition 4.16. A triplet $(q(\,\cdot\,),u(\,\cdot\,),p(\,\cdot\,))$ satisfying the conditions of Theorem 4.13 will be called a *scale Pontryagin extremal*.

Remark 4.17. In the particular case when $\varphi(t,q,u) = u$, Theorem 4.13 reduces to Theorem 4.3: the stationary condition (4.14) gives $p = -\partial_3 L$ and the second equation in the scale Hamiltonian system (4.13) gives $\Box_\varepsilon p(t) = -\partial_2 L$. Comparing both equalities, one obtains the scale Euler–Lagrange equation (4.3): $\Box_\varepsilon \partial_3 L = \partial_2 L$. In other words, the scale Pontryagin extremals (Definition 4.16) are a generalization of the scale Euler–Lagrange extremals (Definition 4.4).

PROOF OF THEOREM 4.13. Using the Lagrange multiplier rule, (4.12) is equivalent to the augmented problem

$$J[q(\cdot),u(\cdot),p(\cdot)] = \int_a^b [\mathcal{H}(t,q(t),u(t),p(t)) - p(t)\cdot\Box_\varepsilon q(t)]\,dt \to \min\,. \tag{4.16}$$

The necessary optimality condition (4.13)–(4.14) is obtained from the Euler–Lagrange equations (4.3) applied to problem (4.16):

$$\begin{cases} \dfrac{\Box_\varepsilon}{\Box t}\dfrac{\partial}{\partial\Box_\varepsilon q}(\mathcal{H}-p\cdot\Box_\varepsilon q) = \dfrac{\partial}{\partial q}(\mathcal{H}-p\cdot\Box_\varepsilon q),\\ \dfrac{\Box_\varepsilon}{\Box t}\dfrac{\partial}{\partial\Box_\varepsilon u}(\mathcal{H}-p\cdot\Box_\varepsilon q) = \dfrac{\partial}{\partial u}(\mathcal{H}-p\cdot\Box_\varepsilon q),\\ \dfrac{\Box_\varepsilon}{\Box t}\dfrac{\partial}{\partial\Box_\varepsilon p}(\mathcal{H}-p\cdot\Box_\varepsilon q) = \dfrac{\partial}{\partial p}(\mathcal{H}-p\cdot\Box_\varepsilon q) \end{cases} \Leftrightarrow \begin{cases} -\Box_\varepsilon p = \partial_2\mathcal{H},\\ 0=\partial_3\mathcal{H},\\ 0=\partial_4\mathcal{H}-\Box_\varepsilon q. \end{cases}$$

□

The notion of invariance for problem (4.12) is defined using the equivalent augmented problem (4.16).

DEFINITION 4.18 (cf. Definition 4.5). The functional (4.16) is said to be invariant under the s-parameter group of infinitesimal transformations

$$\begin{cases} \bar{t} = t + s\tau(t,q,u,p) + o(s),\\ \overline{q}(t) = q(t) + s\xi(t,q,u,p) + o(s),\\ \overline{u}(t) = u(t) + s\varrho(t,q,u,p) + o(s),\\ \overline{p}(t) = p(t) + s\varsigma(t,q,u,p) + o(s), \end{cases}$$

τ, ξ, ϱ, $\varsigma \in H^\beta$, $\beta \geq \alpha 1_{[1/2,1]} + (1-\alpha)1_{]0,1/2[}$, if

$$\int_{t_a}^{t_b} [\mathcal{H}(t,q(t),u(t),p(t)) - p(t)\cdot\Box_\varepsilon q(t)]\,dt = \int_{\bar{t}(t_a)}^{\bar{t}(t_b)} [\mathcal{H}(\bar{t},\overline{q}(\bar{t}),\overline{u}(\bar{t}),\overline{p}(\bar{t})) - \overline{p}(\bar{t})\cdot\Box_\varepsilon\overline{q}(\bar{t})]\,dt$$

for any subinterval $[t_a,t_b]\subseteq[a-\varepsilon,b+\varepsilon]$.

DEFINITION 4.19 (cf. Definition 4.7). A function $C(t,q(t),u(t),p(t))$ preserved along any scale Pontryagin extremal $(q(\cdot),u(\cdot),p(\cdot))$ of problem (4.12) is said to be a *scale constant of motion* for (4.12).

Theorem 4.20 gives a Noether-type theorem for scale optimal control problems (4.12).

Theorem 4.20 (Scale Noether's theorem in Hamiltonian form). *If we have invariance in the sense of Definition* 4.18*, then*

$$C(t, q(t), u(t), p(t)) = \mathcal{H}(t, q(t), u(t), p(t))\tau - p(t) \cdot \xi \tag{4.17}$$

is a scale constant of motion for (4.12).

Proof. The scale constant of motion (4.17) is obtained by applying Theorem 4.9 to problem (4.16). □

Remark 4.21. For the scale problem of the calculus of variations (4.1) the Hamiltonian (4.15) takes the form

$$\mathcal{H} = L + p \cdot u, \quad \text{with } u = \Box_\varepsilon q(t) \text{ and } p = -\partial_3 L$$

(cf. Remark 4.17). In this case the scale constant of motion (4.17) reduces to (4.9).

5. Application: scale constants of motion for Schrödinger equations

In [3, §5] some fractional variants of the Schrödinger equation, with particular interest in quantum mechanics, are studied. It is proved that under certain conditions, solutions of both linear and nonlinear Schrödinger's equations coincide with the extremals of certain functionals (4.1) of the scale calculus of variations. In this section we use our Noether theorem to find scale constants of motion for the problems studied in [3, §5]. In all the examples we use the computer program [10], [11] ([1]) to compute the symmetries (i.e. the invariance transformations (4.4)).

Example 5.1. Let us consider the following nonlinear Schrödinger equation:

$$2i\gamma m\bigg[-\frac{1}{\Psi}\bigg(\frac{\partial\Psi}{\partial q}\bigg)^2\bigg(i\gamma + \frac{a_\varepsilon(t)}{2}\bigg) + \frac{\partial\Psi}{\partial t} + \frac{a_\varepsilon(t)}{2}\frac{\partial^2\Psi}{\partial q^2}\bigg] = (U(q) + \alpha(q))\Psi \tag{5.1}$$

where $m > 0$, $\gamma \in \mathbb{R}$, $U\colon \mathbb{R} \mapsto \mathbb{R}$, $\alpha(q)$ is an arbitrary continuous function, $q(t) \in H^{1/2}$, $\Psi(t, q)$ satisfy the condition

$$\frac{\Box_\varepsilon q(t)}{\Box t} = -i2\gamma \frac{\partial \ln(\Psi(t, q))}{\partial q}$$

and $a_\varepsilon\colon \mathbb{R} \mapsto \mathbb{C}$ is given by

$$a_\varepsilon(t) = \frac{1}{2}[((\Delta_\varepsilon^+ q(t))^2 - (\Delta_\varepsilon^- q(t))^2) - i((\Delta_\varepsilon^+ q(t))^2 + (\Delta_\varepsilon^- q(t))^2)].$$

([1]) The software is available from the Maple Application Centre at http://www.maplesoft.com/applications/app_center_view.aspx?AID=1983.

It is shown in [3, Theorem 5.1] that the solutions $q(t)$ of (5.1) coincide with the Euler–Lagrange extremals of functional (4.1) with the Lagrangian

$$L(t, q(t), \Box_\varepsilon q(t)) = \frac{1}{2} m (\Box_\varepsilon q(t))^2 + U(q).$$

The functional

$$I[q(\,\cdot\,)] = \frac{1}{2} \int_a^b \left[m \left(-i2\gamma \frac{\partial \ln(\Psi(t,q))}{\partial q} \right)^2 + 2U(q) \right] dt$$

is invariant in the sense of Definition 4.5 under the symmetries $(\tau, \xi) = (c, 0)$, where c is an arbitrary constant. It follows from our Theorem 4.9 that

$$-2m \left(\gamma \frac{\partial \ln(\Psi(t,q))}{\partial q} \right)^2 + U(q) \tag{5.2}$$

is a scale constant of motion: (5.2) is preserved along all the solutions $q(t)$ of the nonlinear Schrödinger equation (5.1).

Example 5.2. We now consider the following linear Schrödinger's equation:

$$i\overline{h} \frac{\partial \Psi}{\partial t} + \frac{\overline{h}^2}{2m} \frac{\partial^2 \Psi}{\partial q^2} = U(q)\Psi \tag{5.3}$$

where $\overline{h} = h/(2\pi)$, $m > 0$, $U \colon \mathbb{R} \mapsto \mathbb{R}$, $\Psi(t,q)$ satisfy

$$\frac{\Box_\varepsilon q(t)}{\Box t} = -i \frac{\overline{h}}{m} \frac{\partial \ln(\Psi(t,q))}{\partial q}$$

and $q(t) \in H^{1/2}$ are such that

$$\frac{1}{2} [((\Delta_\varepsilon^+ q(t))^2 - (\Delta_\varepsilon^- q(t))^2) - i((\Delta_\varepsilon^+ q(t))^2 + (\Delta_\varepsilon^- q(t))^2)] = -i \frac{\overline{h}}{m}.$$

In [3, Theorem 5.2] it is proved that the solutions of (5.3) coincide with Euler–Lagrange extremals of functional (4.1) with Lagrangian

$$L(t, q(t), \Box_\varepsilon q(t)) = \frac{1}{2} m (\Box_\varepsilon q(t))^2 + U(q).$$

It happens that functional

$$I[q(\,\cdot\,)] = \frac{1}{2} \int_a^b \left[m \left(-i \frac{\overline{h}}{m} \frac{\partial \ln(\Psi(t,q))}{\partial q} \right)^2 + 2U(q) \right] dt$$

is invariant in the sense of Definition 4.5 under the symmetries $(\tau, \xi) = (c, 0)$, where c is an arbitrary constant. It follows from our Theorem 4.9 that

$$-\frac{1}{2m} \left(\overline{h} \frac{\partial \ln(\Psi(t,q))}{\partial q} \right)^2 + U(q) = -\frac{1}{8m} \left(\frac{h}{\pi} \frac{\partial \ln(\Psi(t,q))}{\partial q} \right)^2 + U(q) \tag{5.4}$$

is a scale constant of motion: expression (5.4) is constant along all the solutions $q(t)$ of the linear Schrödinger's equation (5.3).

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