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NIELSEN COINCIDENCE THEORY OF FIBRE-PRESERVING MAPS AND DOLD'S FIXED POINT INDEX

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Dedicated to Albrecht Dold on the occasion of his 80th birthday

Abstract. Let $M \to B$, $N \to B$ be fibrations and $f_1, f_2 \colon M \to N$ be a pair of fibre-preserving maps. Using normal bordism techniques we define an invariant which is an obstruction to deforming the pair f_1, f_2 over B to a coincidence free pair of maps. In the special case where the two fibrations are the same and one of the maps is the identity, a weak version of our ω-invariant turns out to equal Dold's fixed point index of fibre-preserving maps. The concepts of Reidemeister classes and Nielsen coincidence classes over B are developed. As an illustration we compute e.g. the minimal number of coincidence components for all homotopy classes of maps between S^1-bundles over S^1 as well as their Nielsen and Reidemeister numbers.

1. Introduction and outline of results

Throughout this paper we consider the following situation:

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$$\begin{tikzcd} & F_M \arrow[d, phantom, "\bigcap"] & & F_N \arrow[d, phantom, "\bigcap"] \\ f, f_1, f_2, f', f_1', f_2', \dots \colon & M^m \arrow[rr] \arrow[dr, "p_M"'] & & N^n \arrow[dl, "p_N"] \\ & & B^b & \end{tikzcd} \tag{1.1}$$

Here B^b, M^m and N^n are smooth connected manifolds of the indicated dimensions, without boundary, B^b and M^m being compact. Moreover, p_M and p_N are smooth fibre maps with fibres F_M and F_N, respectively. The (continuous) maps $f_1, f_2, f, \dots$ as well as homotopies between them are always assumed to be fibre-preserving (so that e.g. $p_N \circ f = p_M$); we also call them *maps and homotopies over* B and write $f \sim_B f'$ if f, f' are homotopic in this sense. From now on we will drop the superscript which denotes the dimension of the manifold, unless this simplification is going to cause some confusion.

Question 1.1. Can the coincidence locus

$$C(f_1, f_2) := \{x \in M \mid f_1(x) = f_2(x)\}$$

be made empty by suitable homotopies of f_1 and f_2 over B? (If f_1 and f_2 can be deformed away from one another in this way we say that the pair (f_1, f_2) is *loose over* B or, shortly, B*-loose*).

More generally, we would like to estimate the *minimum number of pathcomponents*

$$\mathrm{MCC}_B(f_1, f_2) := \min\{\#\pi_0(C(f_1', f_2')) \mid f_i \sim_B f_i',\ i = 1, 2\} \tag{1.2}$$

of coincidence subspaces in M, achieved by suitable deformations of f_1 and f_2 over B.

For this purpose we study the geometry of the map

$$\begin{array}{llll} (1.3) & (f_1, f_2)\colon M \longrightarrow & N \times_B N & := \{(y_1, y_2) \in N \times N \mid p_N(y_1) = p_N(y_2)\} \\ & & \bigcup & \\ & & \Delta & := \{(y_1, y_2) \in N \times N \mid y_1 = y_2\}. \end{array}$$

After small deformations of f_1 and f_2 over B this map is smooth and transverse to the diagonal Δ so that the coincidence locus

$$C = C(f_1, f_2) = (f_1, f_2)^{-1}(\Delta) = \{x \in M \mid f_1(x) = f_2(x)\} \tag{1.4}$$

is an $(m - n + b)$-dimensional smooth submanifold of M.

Moreover, the tangent map of (f_1, f_2) induces an isomorphism of the normal bundles

$$\overline{g}_B^{\#} : \nu(C, M) \cong ((f_1, f_2)|C)^*(\nu(\Delta, N \times_B N)) \cong f_1^*(\mathrm{TF}(p_N))|C \tag{1.5}$$

here $\mathrm{TF}(p_N)$ denotes the tangent bundle along the fibres of p_N.

A third important coincidence datum is the lifting

$$\begin{tikzcd} & E_B(f_1,f_2) \arrow[d, "\mathrm{pr}"] \\ C \arrow[ur, "\widetilde{g}_B"] \arrow[r, "g=\mathrm{incl}"'] & M \end{tikzcd} \qquad \begin{aligned} E_B(f_1,f_2) := \{(x,\theta) \in M \times P(N) \mid p_N \circ \theta \equiv p_M(x), \\ \theta(0) = f_1(x),\ \theta(1) = f_2(x)\} \end{aligned} \tag{1.6}$$

defined by $\widetilde{g}_B(x) = (x, \text{constant path at } f_1(x) = f_2(x))$. Here $P(N)$ (and pr, respectively) denote the space of all continuous paths $\theta\colon [0,1] \to N$, with the compact-open topology (and the obvious projection, respectively; compare [10, diagram (5)]). The bordism class

$$\omega_B^{\#}(f_1,f_2) = [(C \subset M, \widetilde{g}_B, \overline{g}_B^{\#})]$$

of the resulting triple of the coincidence data (1.4)–(1.6) (which keeps track of the embedding of C in M and of its nonstabilized normal bundle) is independent of our choice of small deformations. It is our strongest obstruction to making the pair (f_1,f_2) loose over B. In certain settings (e.g. if $N = B \times S^{n-b}$) it yields a complete homotopy classification for maps over B. However, this ("strong") ω-invariant is often hard to compute.

The *stabilized* version

$$\widetilde{\omega}_B(f_1,f_2) = [(C, \widetilde{g}_B, \overline{g}_B)] \in \Omega_{m-n+b}(E_B(f_1,f_2); \widetilde{\varphi})$$

is much more manageable. It forgets about the map $g := pr \circ \widetilde{g}_B$ (cf. (1.6)) being an embedding, retains only the *stable* vector bundle isomorphism

$$\overline{g}_B\colon TC \oplus g^*(f_1^*(\mathrm{TF}(p_N))) \oplus \mathbb{R}^k \cong g^*(TM) \oplus \mathbb{R}^k, \quad k >> 0, \tag{1.7}$$

(compare (1.5)) and lies in the normal bordism group of singular $(m-n+b)$-manifolds in $E_B(f_1,f_2)$, with coefficient bundle

$$\widetilde{\varphi} := pr^*(f_1^*(\mathrm{TF}(p_N)) - TM) = pr^*(f_1^*(TN) - p_M^*(TB) - TM) \tag{1.8}$$

(compare e.g. [7, (2.1)]). The path space $E_B(f_1,f_2)$ and the resulting normal bordism group depend on the maps f_1, f_2, but homotopies induce group isomorphisms which preserve the $\widetilde{\omega}_B$-invariants (compare [9, (3.3)]). Therefore $\widetilde{\omega}_B(f_1,f_2)$ vanishes if f_1 and f_2 can be deformed to become coincidence free. In a suitable "stable dimension range" the converse holds.

Theorem 1.2. *Assume that $m < 2(n-b) - 2$. Then a pair (f_1,f_2) is loose over B if and only if $\widetilde{\omega}_B(f_1,f_2) = 0$.*

In the proof (outlined in the Section 2 below) the path-space $E_B(f_1,f_2)$ plays a significant role: the lifting $\widetilde{g}_B$ (cf. (1.6)) allows us to construct the homotopies which deform f_1, f_2 away from one another. Quite generally $E_B(f_1,f_2)$ is a very interesting space with a rich topology. Already its decomposition into pathcomponents leads to the fibre theoretical analogue of the (algebraic) Reidemeister equivalence relation (on $\pi_1(F_N)$) and to the corresponding notion of the *Nielsen numbers*

$$N_B(f_1,f_2) \le N_B^{\#}(f_1,f_2) \le \mathrm{MCC}_B(f_1,f_2).$$

These are nonnegative integers counting the path-components of $E_B(f_1,f_2)$ which contribute non-trivially to $\widetilde{\omega}_B(f_1,f_2)$ and $\omega_B^{\#}(f_1,f_2)$, respectively (for details see Section 4 below). Clearly these Nielsen numbers form lower bounds for the minimum number $\mathrm{MCC}_B(f_1,f_2)$ (cf. (1.2)); in particular, they are simple numerical looseness obstructions. Moreover, the Nielsen numbers are obviously smaller or equal to the *geometric Reidemeister number*

$$\#R_B(f_1,f_2) := \#\pi_0(E_B(f_1,f_2)) \tag{1.9}$$

(i.e. the number of path-components of the space $E_B(f_1,f_2)$, cf. (1.6); its relation to the classical (algebraic) Reidemeister number will be explained in Section 3). Another simplification of our $\widetilde{\omega}_B$-invariant forgets about the path-space $E_B(f_1,f_2)$ and the lifting $\widetilde{g}_B$ altogether and keeps track only of the inclusion $g\colon C \subset M$ (as a continuous map) and of the description of the stable normal bundle of C given by (1.7). We obtain the normal bordism class

$$\omega_B(f_1,f_2) = [(C,g,\overline{g}_B)] \in \Omega_{m-n+b}(M;\varphi) \tag{1.10}$$

where

$$\varphi := f_1^*(\mathrm{TF}(p_N)) - TM = f_1^*(TN) - p_M^*(TB) - TM \tag{1.11}$$

(compare (1.8)). Homotopies of f_2 yield bordant triples of coincidence data $(C,g,\overline{g}_B)$ and hence the same ω_B-invariants.

Special Case 1.3 (trivial base space). If the base space B consists of a single point we drop the subscript B from our notations and obtain the invariants $\omega^{\#}$, $N^{\#}$, $\widetilde{\omega}$, N and ω discussed in [8]–[10]. (For further literature concerning this special case see e.g. [2]–[5], [11] and [12] as well as the references listed there).

Special Case 1.4 (trivial target fibration). If the target fibration is a product, $N = B \times F_N$, we may write $f_i =: (p_M, f_i')$, $i = 1,2$. Then the $\omega_B^{\#}$-, $\widetilde{\omega}_B$- and ω_B-invariants of (f_1,f_2) are related to the corresponding (unfibered) invariants of (f_1',f_2') via bijections (which preserve 0); in particular

$$N_B^{\#}(f_1,f_2) = N^{\#}(f_1',f_2') \quad \text{and} \quad N_B(f_1,f_2) = N(f_1',f_2').$$

Special Case 1.5 (fixed points). If the two fibrations coincide and f_1 is the identity map id, then $C(\mathrm{id},f)$ is the fixed point locus of f, the coefficient bundles

$\widetilde{\varphi}$ and φ are the pullbacks of the virtual vector bundle $-TB$ under projections, and our ω-invariant can be weakened further to yield the bordism class

$$p_{M*}(\omega_B(\mathrm{id}, f)) = \left[\left(C \xrightarrow{p_M|} B,\, \overline{g}_B\colon TC \xrightarrow[\text{stably}]{\cong} (p_M|)^*(TB)\right)\right] \in \Omega_b(B; -TB)$$

This procedure neglects the "vertical" aspects of our fixed point data.

On the other hand A. Dold [1] has defined a fixed point index $I^h(f)$ of f for every multiplicative generalised cohomology theory h with unit. In view of the universality property of stable cohomotopy theory the strongest ("universal") version of Dold's index takes the form

$$I(f) \in \pi^0_{\text{stable}}(B^+) = \varinjlim [\Sigma^k B^+, S^k]; \tag{1.12}$$

and actually classifies certain "horizontal" fixed point phenomena (cf. [1, Theorem 4.3]); here B^+ denotes the space B with a disjointly added point.

Note that the Pontrjagin–Thom procedure yields a canonical isomorphism

$$\mathrm{PT}\colon \pi^0_{\text{stable}}(B^+) \xrightarrow{\cong} \Omega_b(B; -TB) \tag{1.13}$$

(which will be described in Section 5 below).

Theorem 1.6. *For every map $f\colon M \to M$ over B*

$$I(f) = (\mathrm{PT})^{-1}(p_{M*}(\omega_B(\mathrm{id}, f))).$$

The proof will be given in Section 5 below.

As in illustration of our notions and methods we calculate the minimum number MCC_B (as well as the Reidemeister and the Nielsen numbers) and the ω_B-invariant for all pairs of B-maps involving the torus and/or the Klein bottle over $B = S^1$. Note that this is way outside of the stable dimension range discussed in Theorem 1.2.

Example 1.7 (S^1-bundles over S^1). Let M, N be (possibly different) fibre spaces over S^1 with fibre S^1. Thus M (and also N) is either the torus

$$T = S^1 \times S^1 = I \times S^1/(0, z) \sim (1, z), \quad z \in S^1 \tag{1.14}$$

or the Klein bottle

$$K = I \times S^1/(0, z) \sim (1, \overline{z}), z \in S^1 \tag{1.15}$$

with the standard projection to $I/0 \sim 1 = S^1$. We define two sections s_ε, $\varepsilon = \pm 1$, by

$$s_\varepsilon([t]) = [(t, \varepsilon)]. \tag{1.16}$$

Given a map $f\colon M \to N$ over S^1 we have two well defined numerical invariants:

$$q(f) := (\text{degree of } f| : F_M \to F_N) \in \mathbb{Z} \tag{1.17}$$

(this vanishes if $M \neq N$); and

$$r(f) := \text{degree of } \left(B = S^1 \xrightarrow{f \circ s_{+1}} N = [0,1] \times S^1/\!\sim \longrightarrow S^1 \right); \tag{1.18}$$

this lies in $\mathbb{Z}$ (and in $\mathbb{Z}_2$, respectively) if $N = T$ (and if $N = K$ *and* f preserves the base point $[(0,1)]$, respectively); this number measures roughly how often the section $f \circ s_{+1}$ (assumed to be base point preserving if $N = K$) "winds around the fiber in N". A base point free description of $r(f)$ in the case $N = K$ is as follows: $r(f)$ equals the mod 2 integer 0 (and 1, respectively) if $f \circ s_{+1}$ is homotopic (through sections in K) to s_{+1} (and to s_{-1}, respectively).

Returning to the base point free setting we obtain:

Proposition 1.8. *Two maps $f, \widehat{f}\colon M \to N$ over S^1 are homotopic over S^1 if and only if $q(f) = q(\widehat{f})$ and $r(f) = r(\widehat{f})$.*

Thus each homotopy class can be represented by a map in a rather natural standard form (enjoying constant angular velocities both along each fibre and for $f \circ s_{+1}$). This is very helpful when we analyse coincidence data.

Now consider any two maps $f_1, f_2\colon M \to N$ over S^1 and put

$$q := q(f_1) - q(f_2) \quad \text{and} \quad r := r(f_1) - r(f_2) \tag{1.19}$$

(compare (1.17) and (1.18)).

Theorem 1.9. *The minimum number* $\mathrm{MCC}_B(f_1, f_2)$ *is equal to the Nielsen numbers* $N_B(f_1, f_2)$ *and* $N_B^{\#}(f_1, f_2)$ (*and also to* $\#R_B(f_1, f_2)$ *whenever this Reidemeister number is finite*). *More precisely:*

(a) *Assume $N = S^1 \times S^1$. Then $(q, r) \in \mathbb{Z} \times \mathbb{Z}$ and we have:*

$$\begin{aligned} \gcd(q,r) &= \mathrm{MCC}_B(f_1,f_2) = N_B(f_1,f_2) = \#R_B(f_1,f_2) && \textit{if } (q,r) \neq (0,0); \\ 0 &= \mathrm{MCC}_B(f_1,f_2) = N_B(f_1,f_2) \neq \#R_B(f_1,f_2) = \infty && \textit{if } (q,r) = (0,0). \end{aligned}$$

In particular, the pair (f_1, f_2) is loose over B if and only if $f_1 \sim_B f_2$.

(b) *Assume $N = K$. Then $(q, r) \in \mathbb{Z} \times \mathbb{Z}_2$ and we have: if $q \neq 0$:*

$$\mathrm{MCC}_B(f_1,f_2) = N_B(f_1,f_2) = \#R_B(f_1,f_2) = \begin{cases} |q|/2 & \textit{if } q \textit{ even, } r = 1, \\ [|q|/2] + 1 & \textit{else}; \end{cases}$$

if $q = 0$:

$$\mathrm{MCC}_B(f_1,f_2) = N_B(f_1,f_2) = \begin{cases} 0 & \textit{if } r \neq 0, \\ 1 & \textit{if } r = 0, \end{cases} \neq \#R_B(f_1,f_2) = \infty.$$

In particular, the pair (f_1, f_2) *is loose over* B *if and only if it consists of two "antipodal" maps, i.e.* $-f_1 \sim_B f_2$.

Note that here the value of the Nielsen number is always 0 or 1 or the Reidemeister number. A similar result in an entirely different setting was proved in [12, Theorem 1.31].

Clearly, in all of Example 1.7 the $\widetilde{\omega}_B$-invariant is a complete looseness obstruction. Actually, already its weaker version $\omega_B(f_1, f_2) \in \Omega_1(M; \varphi)$ (cf. (1.10)) allows us to distinguish maps up to homotopy over S^1.

Theorem 1.10. *Let* (M, N) *be any of the four combinations of* S^1*-bundles over* S^1 *and let* $f_1, f_2 \colon M \to N$ *be maps over* S^1. *Then there are canonical isomorphisms which describe* $\Omega_1(M; \varphi)$ (*and correspondingly* $\omega_B(f_1, f_2)$) *as* (*an element of*) *a direct sum of three groups, as follows* (*compare Proposition* 1.8 *and Theorem* 1.9)

<table>
<tr><th>(M, N)</th><th>$\Omega_1(M; \varphi)$</th><th colspan="3">$\omega_B(f_1, f_2)$</th></tr>
<tr><td>(T, T)</td><td>$Z \oplus \mathbb{Z} \oplus \mathbb{Z}_2$</td><td>$q$</td><td>$r$</td><td rowspan="4">$\rho_2(N_B(f_1, f_2))$
(*comp. Theorem* 1.9)</td></tr>
<tr><td>(K, K)</td><td>$Z \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$</td><td>$q$</td><td>$r + 1 + \rho_2(q)$</td></tr>
<tr><td>(K, T)</td><td>$0 \oplus \mathbb{Z} \oplus \mathbb{Z}_2$</td><td>$q$</td><td>$r$</td></tr>
<tr><td>(T, K)</td><td>$0 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$</td><td>$q$</td><td>$r + 1$</td></tr>
</table>

Here $\rho_2 \colon \mathbb{Z} \to \mathbb{Z}_2$ *denotes reduction* mod 2. *In particular, for every map* $f \colon M \to N$ *over* S^1 *the "fibred degree"* (*or "root invariant"*) $\omega_B(f, s_{+1} \circ p_M)$ *determines* $q(f)$ *and* $r(f)$ *and hence the homotopy class* (*over* S^1) *of* f.

Remark 1.11. In view of Proposition 1.8 the homotopy class of f is already determined by the first two components of $\omega_B(f, s_{+1} \circ p_M)$ or, equivalently, by

$$\mu(\omega_B(f, s_+ \circ p_M)) \in H_1(M; \widetilde{\mathbb{Z}}_\varphi)$$

where μ denotes the Hurewicz homomorphism into the first homology group of M with integer coefficients (which are twisted like the orientation line bundle of φ). This is a very special phenomenon, related to the fact that both the torus and the Klein bottle are $K(\pi, 1)'s$. For general M and N the methods of singular homology theory are often far too weak, and the full power of our approach (based on normal bordism theory and the pathspace $E_B(f_1, f_2)$) yields better results.

Remark 1.12. Consider a selfmap of the torus T or the Klein bottle K over S^1. Dold's fixed point index [1] in its strongest form lies in

$$\pi^0_{\text{stable}}((S^1)^+) \cong \Omega_1^{\text{fr}}(S^1) \cong \mathbb{Z} \oplus \mathbb{Z}_2$$

and captures precisely the first and third components of $\omega_B(\mathrm{id}, f)$ or, equivalently, $\pm q = \pm(\deg(f|F) - 1)$ as well as the Nielsen number $N_B(f, \mathrm{id})$, taken mod 2. However, it looses all information about the characteristic winding number r which together with q-determines f and which measures the "vertical" aspect of the generic fixed point circles.

2. The ω-invariants

Given maps $f_1, f_2\colon M \to N$ over B, the definition of $\omega_B^{\#}(f_1, f_2)$, $\widetilde{\omega}_B(f_1, f_2)$ and $\omega_B(f_1, f_2)$ (as outlined in the introduction) is completely analoguous to the definition (given in [10] and [9]) of the corresponding invariants for ordinary maps between manifolds (or, equivalently, for maps over $B = \{\text{point}\}$). Therefore many of the notions, methods and results of the ordinary (fibration free) coincidence theory allow a straightforward generalization to the setting of fibre preserving maps.

In particular, the proof of Theorem 1.2 proceeds in direct analogy to the proof of Theorem 1.10 in ([9, pp. 213 and 223–224]): we just have to replace $N \times N$ by $N \times_B N$. Our ("stable") dimension condition means that the dimension of $C(f_1, f_2)$, augmented by 2, is strictly smaller than the codimension in M. Hence here $\widetilde{\omega}_B(f_1, f_2)$ is precisely as strong as $\omega_B^{\#}(f_1, f_2)$, and $N_B(f_1, f_2) = N_B^{\#}(f_1, f_2)$; nulbordism data can be realized by a suitably embedded manifold in $M \times I$ with a nonstabilized description of its normal bundle and, above all, without new coincidences occurring in its shadow (cf. [9, p. 224]).

Remark 2.1. The interested reader may check when the methods of [9, (1.10) and (4.7)], can be generalized to yields the full equality $\mathrm{MCC}_B(f_1, f_2) = N_B(f_1, f_2)$ ("Wecken theorem").

Next let us consider the special case where the target fibration is trivial. Given maps over B,

$$f_i = (p_M, f_i')\colon M \to N = B \times F_N, \quad i = 1, 2,$$

we see that $E_B(f_1, f_2)$ can be identified with the path-space $E(f_1', f_2')$ discussed in [9] and [10]. Thus the ω-invariants and Nielsen numbers *over* B of the pair (f_1, f_2) are equal to the corresponding *ordinary* (unfibred) invariants of (f_1', f_2').

3. The algebraic Reidemeister classes over B and the space $E_B(f_1, f_2)$

In this section we fix maps $f_1, f_2\colon M \to N$ over B. We will give an algebraic description of the (geometric) Reidemeister set $\pi_0(E_B(f_1, f_2))$ (compare (1.9)). This generalizes and refines the classical approach. As an application we will compute Reidemeister numbers for maps into the Klein bottle.

Choose a coincidence point $x_0 \in C(f_1, f_2)$ (if it does not exist, the pair (f_1, f_2) is loose and our initial Question 1.1 needs no further answer). Put $y_0 := f_1(x_0) = f_2(x_0)$ and let $F_N \subset N$ be the fibre over $b_0 := p_M(x_0)$.

Using homotopy lifting extension properties (compare [13, I.7.16]) of the (Serre) fibration p_N we construct a well defined operation

$$*_B \colon \pi_1(M, x_0) \times \pi_1(F_N, y_0) \longrightarrow \pi_1(F_N, y_0)$$

as follows. Given loops $c\colon (I, \partial I) \to (M, x_0)$ and $\theta\colon (I, \partial I) \to (F_N, y_0)$, lift the homotopy

$$h\colon I \times I \to B, \quad h(s,t) := p_M \circ c(s), \tag{3.1}$$

to a map $\widetilde{h}\colon I \times I \to N$ such that

$$\widetilde{h}(0,t) = \theta(t), \quad \widetilde{h}(s,0) = f_1 \circ c(s), \quad \widetilde{h}(s,1) = f_2 \circ c(s) \tag{3.2}$$

for all $s, t \in I$. Then the loop θ' defined by $\theta'(t) := \widetilde{h}(1,t)$ lies entirely in F_N. Due to the very special form of h (cf. (3.1)) the homotopy class $[\theta']$ of θ' in F_N (and *not* just in N) depends only on the homotopy classes of c and θ. We put

$$[c] *_B [\theta] := [\theta'].$$

DEFINITION 3.1. Two elements $[\theta]$, $[\theta'] \in \pi_1(F_N, y_0)$ are called *Reidemeister equivalent over B* if there exists $[c] \in \pi_1(M, x_0)$ such that $[c] *_B [\theta] = [\theta']$.

The *algebraic Reidemeister set* $R_B(f_1, f_2, x_0)$ is the resulting set of equivalence classes (i.e. of orbits of the group action $*_B$ of $\pi_1(M, x_0)$ on (the set) $\pi_1(F_N, y_0)$).

Its cardinality is called *Reidemeister number of f_1, f_2 over B*.

There is also the classical group action (without any reference to B)

$$*\colon \pi_1(M, x_0) \times \pi_1(N, y_0) \longrightarrow \pi_1(N, y_0)$$

determined by the induced homomorphisms $f_{j*}\colon \pi_1(M, x_0) \to \pi_1(N, y_0)$, $j = 1, 2$, i.e.

$$[c] * [\theta] := f_{1*}([c])^{-1} \cdot [\theta] \cdot f_{2*}([c]) \tag{3.3}$$

for $[c] \in \pi_1(M, x_0)$ and $[\theta] \in \pi_1(N, y_0)$ (compare e.g. [9, (2.1)]).

In view of the boundary conditions (3.2) of the lifting $\widetilde{h}$ (which takes its value in N and, in general, not already in F_N) we see that

$$[c] * i_*([\theta]) = i_*([c] *_B [\theta]) \tag{3.4}$$

for all $[c] \in \pi_1(M, x_0)$, $[\theta] \in \pi_1(F_N, y_0)$; here $i\colon F_N \to N$ denotes the inclusion. In particular, the standard action $*$ (cf. (3.3)) restricts to an action of $\pi_1(M, x_0)$ on $i_*(\pi_1(F_N, y_0))$. In general this yields a coarser equivalence relation than the one defined by our action $*_B$ (e.g. when $p_M = p_N\colon S^{2k+1} \to \mathbb{C}P(k)$, $k \geq 1$, is the Hopf fibration, then $R_B(f_1, f_2, x_0) = \pi_1(S^1) \cong \mathbb{Z}$, but $i_*(\pi_1(F_N)) = 0$). However, if i_* is injective (e.g. when $\pi_2(B) = 0$) then (3.4) can be used to compute

$$R_B(f_1, f_2, x_0) = \pi_1(F_N, y_0)/\sim *_B \approx i_*(\pi_1(F_N, y_0))/\sim *. \tag{3.5}$$

In particular, when $B = \{b_0\}$ and hence $F_N = N$, our definition of an algebraic Reidemeister set coincides with the usual one.

More general, injectivity criteria for i_* may be extracted from the exact sequence

$$\cdots \longrightarrow \pi_2(B, b_0) \longrightarrow \pi_1(F_N, y_0) \xrightarrow{i_*} \pi_1(N, y_0) \xrightarrow{p_{N*}} \pi_1(B, b_0).$$

Next let us compare our algebraic and geometric Reidemeister sets (cf. Definition 3.1 and (1.9)). By definition $E_B(f_1, f_2)$ is the space of pairs (x, θ) where x is a point in M and θ is a path in N from $f_1(x)$ to $f_2(x)$ which stays entirely in one fibre of p_N. In view of the very special form of the homotopy h (cf. (3.1)) its lifting $\widetilde{h}$ determines a path

$$s \in I \longrightarrow ((c(s), \widetilde{h}(s, -)) \in E_B(f_1, f_2)$$

joining (x_0, θ) to (x_0, θ'). Actually every other path in $E_B(f_1, f_2)$ which starts and ends in the fibre $\mathrm{pr}^{-1}(\{x_0\}) = \{x_0\} \times \Omega(F_N, y_0)$ of pr (cf. (1.6)) can be obtained in this way from some lifted homotopy $\widetilde{h}$ as in (3.1) and (3.2). In other words, two classes $[\theta], [\theta'] \in \pi_1(F_N, y_0)$ are Reidemeister equivalent over B if and only if (x_0, θ) and (x_0, θ') lie in the same path-component of $E_B(f_1, f_2)$. Thus the map

$$R_B(f_1, f_2, x_0) \longrightarrow \pi_0(E_B(f_1, f_2)),$$

which is induced by the fibre inclusion $\Omega(F_N, y_0) \approx \mathrm{pr}^{-1}(\{x_0\}) \subset E_B(f_1, f_2)$ and by the resulting map

$$\pi_1(F_N, y_0) = \pi_0(\Omega(F_N, y_0)) \longrightarrow \pi_0(E_B(f_1, f_2)),$$

is injective. It is also onto. Indeed, given any point (x, θ) of $E_B(f_1, f_2)$, we can pick a path in M from x to x_0 and lift it to a path in $E_B(f_1, f_2)$ which joins (x, θ) to some point in $\mathrm{pr}^{-1}(\{x_0\})$.

We have showed

Theorem 3.2. *For every pair $f_1, f_2 \colon M \to N$ of maps over B and for every choice $x_0 \in C(f_1, f_2)$ and $y_0 = f_1(x_0) = f_2(x_0)$ of base points there is a canonical bijection*

$$R_B(f_1, f_2, x_0) \approx \pi_0(E_B(f_1, f_2))$$

between the algebraic and geometric Reidemeister sets.

COROLLARY 3.3. *The Reidemeister number depends only on the* (*base point free*) *homotopy classes of f_1 and f_2 over B.*

PROOF. Indeed, any pair of homotopies $f_1 \sim f_1'$, $f_2 \sim f_2'$ over B induces a fibre homotopy equivalence $E_B(f_1, f_2) \sim E_B(f_1', f_2')$ over M (comp. [9, (3.2)]). $\square$

EXAMPLE 3.4 (Maps into the Klein bottle). We illustrate the previous discussion by a calculation which we will need in the proof of Theorem 1.9.

PROPOSITION 3.5. *Consider maps $f_1, f_2 \colon M \to K$ over S^1 where M is the torus T or the Klein bottle K* (*and use the notations* (1.17)–(1.19)). *If $M = T$ or $q = 0$, then the Reidemeister number $\#R_B(f_1, f_2)$ is infinite. If $M = K$ and $q \neq 0$, then*

$$\#R_B(f_1, f_2) = \begin{cases} |q|/2 & \textit{if } q \equiv 0(2),\ r \neq 0, \\ [|q|/2] + 1 & \textit{else.} \end{cases}$$

PROOF. In view of Corollary 3.3 f_2 map $x_0 = [(0,1)]$ to $y_0 = [(0,1)]$ (cf. (1.14) and (1.15)). Let us use these base points for computing the algebraic Reidemeister set. Then $\pi_1(M)$ (and $\pi_1(K)$, respectively) is generated by

$$a_M := i_{M*}(g) \quad \text{and} \quad b_M := s_{+1*}(g)$$

(and by $a := i_*(g)$ and $b := s_{+1*}(g)$, respectively) where i_M, i, s_{+1} denote fibre inclusions and the section defined in (1.16); g is the standard generator of $\pi_1(S^1)$.

Since $\pi_2(S^1)$ vanishes, i_* is injective and we have to evaluate only the standard action (3.3) of $\pi_1(M)$ on $\pi_1(F_N) \cong \mathbb{Z}$. Given $k \in \mathbb{Z}$, we obtain

$$\begin{aligned} a_M * a^k &= a^{k-(q(f_1)-q(f_2))} = a^{k-q}, \\ b_M * a^k &= f_{1*}(b_M)^{-1} \cdot a^k \cdot f_{2*}(b_M) \\ &= b^{-1} \cdot a^{-r(f_1)} \cdot a^k \cdot a^{r(f_2)} \cdot b = a^{r(f_1)-r(f_2)-k} \end{aligned}$$

where we consider $r(f_j) \in \{0,1\}$ as an *integer* so that $f_{j*}(b_M) = a^{r(f_j)} \cdot b$, $j = 1,2$ (compare (6.1)). Therefore we can interpret $\pi_1(F_N)/\sim *_B$ (cf. (3.5)) as the orbit set of the involution ι on $\mathbb{Z}/q\mathbb{Z}$ defined by

$$\iota([k]) := [r(f_1) - r(f_2) - k], \quad [k] \in \mathbb{Z}/q\mathbb{Z}.$$

In particular, its cardinality is infinite if $q = 0$. This is e.g. always the case when $M = T$, since the map $f_j| \colon F_M \to F_N = S^1$ is freely homotopic to its own complex conjugate and hence has degree $q(f_j) = 0$, $j = 1,2$.

For the remainder of the proof it suffices to consider the case where $M = K$ and $q > 0$. Then

$$\#R_B(f_1, f_2, x_0) = (q + \#\mathrm{Fix}(\iota))/2. \tag{3.6}$$

Clearly the fixed point set $\mathrm{Fix}(\iota)$ of ι consists just of the solutions of the linear equation

$$2[k] = [r(f_1) - r(f_2)]$$

in $\mathbb{Z}/q\mathbb{Z}$. Therefore it is easy to see that

$$\#\mathrm{Fix}(\iota) = \begin{cases} 1 & \text{if } q \text{ is odd,} \\ 2 & \text{if } q \equiv 0(2),\ r(f_1) = r(f_2), \\ 0 & \text{if } q \equiv 0(2),\ r(f_1) \neq r(f_2). \end{cases}$$

In view of (3.6) this completes the proof. □

4. Nielsen coincidence classes over B

In this section we extend J. Jezierski's notion of Nielsen classes over B (cf. [6]) in the obvious way from fixed points to coincidences of maps f_1, f_2 over B. The resulting decomposition of the coincidence set turns out to correspond precisely to the decomposition of the space

$$E_B(f_1, f_2) = \bigcup_{A \in \pi_0(E_B(f_1,f_2))} A$$

into path-components and yields the description of

$$\widetilde{\omega}_B(f_1, f_2) = \{(\widetilde{\omega}_B(f_1, f_2))_A\} \in \Omega_*(E_B(f_1, f_2); \widetilde{\varphi}) = \oplus_A \Omega_*(A; \widetilde{\varphi}|A) \tag{4.1}$$

as a direct sum. We will discuss the Nielsen number

$$N_B(f_1, f_2) := \#\{A \in \pi_0(E_B(f_1, f_2)) \mid (\widetilde{\omega}_B(f_1, f_2))_A \neq 0\} \tag{4.2}$$

(which counts the nontrivial direct summands of $\widetilde{\omega}_B(f_1, f_2)$) and its nonstabilized analogue

$$N_B^{\#}(f_1, f_2) := \#\{A \in \pi_0(E_B(f_1, f_2)) \mid (\omega_B^{\#}(f_1, f_2))_A \neq 0\}. \tag{4.3}$$

In classical fixed point theory (where B consists of a single point) both definitions (4.2) and (4.3) just yield the familiar notion of the Nielsen fixed point number.

Definition 4.1. Let $f_1, f_2 \colon M \to N$ be maps over B. Two coincidence points $x, x' \in C(f_1, f_2)$ are called *Nielsen equivalent over B* if there exist a path $c \colon I \to M$ joining x to x', as well as a homotopy $\widetilde{h} \colon I \times I \to N$ from $f_1 \circ c$ to $f_2 \circ c$ which keeps the end points fixed and such that for each $s \in I$ the whole image $\widetilde{h}(\{s\} \times I)$ lies in the fibre of p_N over $p_M \circ c(s)$.

PROPOSITION 4.2. *The coincidence points x and x' are Nielsen equivalent over B if and only if the map*

$$\widetilde{g}_B \colon C(f_1, f_2) \longrightarrow E_B(f_1, f_2)$$

(defined by $\widetilde{g}_B(x) = (x$,constant path at $f_1(x) = f_2(x))$ takes them into the same path-component A of $E_B(f_1, f_2)$. Therefore the Nielsen classes of (f_1, f_2) over B are just those inverse images $\widetilde{g}^{-1}(A)$, $A \in \pi_0(E_B(f_1, f_2))$, which are nonempty.

PROOF. (Compare also the proof of Theorem 3.2). The data $(c, \widetilde{h})$ in the Definition 4.1 represent just another way of describing a path in $E_B(f_1, f_2)$ from $\widetilde{g}_B(x)$ to $\widetilde{g}_B(x')$. Indeed, for every $s \in I$ the pair $(c(s), \widetilde{h}(s, \cdot\,))$ lies in $E_B(f_1, f_2)$, since $\widetilde{h}(s, \cdot\,)$ is a path joining $f_1(c(s))$ to $f_2(c(s))$ in the fibre $p_N^{-1}(p_M(c(s)))$. $\square$

COROLLARY 4.3. *Each Nielsen class is open and closed in $C(f_1, f_2)$.*

Indeed, it is not hard to see that each path-component A is open and closed in $E_B(f_1, f_2)$.

We want to consider only those Nielses classes which survive (in some sense) all possible B-homotopies of f_1, f_2. We try to detect them with the help of our ω-invariants.

After a suitable approximation of f_1, f_2 the coincidence set C is a clossed manifold, and so is each Nielsen class $C_A := \widetilde{g}_B^{-1}(A)$, $A \in \pi_0(E_B(f_1, f_2))$. We call it *strongly essential*, and *essential*, respectively, if the corresponding triple $(C_A, \widetilde{g}_B|C_A, \overline{g}^{(\#)}|C_A)$ of restricted coincidence data is not nullbordant (in the nonstabilized, and stabilized sense, respectively). Define $N_B^{\#}(f_1, f_2)$ and $N_B(f_1, f_2)$ to be the resulting numbers of (strongly) essential Nielsen classes.

THEOREM 4.4. *For all maps $f_1, f_2 \colon M \to N$ over B we have:*

(a) *the Nielsen numbers $N_B^{\#}(f_1, f_2)$ and $N_B(f_1, f_2)$ depend only on the homotopy classes of f_1, f_2 over B;*
(b) $N_B^{\#}(f_1, f_2) = N_B^{\#}(f_2, f_1)$ *and* $N_B(f_1, f_2) = N_B(f_2, f_1)$;
(c) $0 \leq N_B(f_1, f_2) \leq N_B^{\#}(f_1, f_2) \leq \mathrm{MCC}_B(f_1, f_2) < \infty$ *and* $N_B^{\#}(f_1, f_2) \leq \# R_B(f_1, f_2)$;
(d) *in classical fixed point theory (over B=point) both versions of our Nielsen numbers coincide with the classical notion of the Nielsen numbers.*

The proof proceeds as in [9, (1.9)], and [10, (1.2)].

Unlike the ω_B-invariants which lie in (possibly very complicated) bordism sets (varying with f_1, f_2) our Nielsen numbers are simple numerical looseness obstructions. To what extend are they less powerful?

Proposition 4.5. *$N_B(f_1,f_2)=0$ if and only if $\widetilde{\omega}_B(f_1,f_2)=0$.*

This follows from the direct sum decomposition (4.1).

It is not clear whether the corresponding statement holds for $N_B^{\#}$ and $\omega_B^{\#}$. If $N_B^{\#}(f_1,f_2)=0$, then the Nielsen classes C_A allow individual embedded nullbordisms in $M\times I$. But these may not fit together disjointly to yield an *embedded* nullbordism for all of $C(f_1,f_2)$ (which is needed to show that $\omega^{\#}(f_1,f_2)=0$).

5. Relation to Dold's index

In this section we study the special case 1.5 where the two fibrations p_M and p_N coincide, f_1 is the identity map id and we are interested in the fixed point behaviour of a map $f_2=f$ over B.

We will see that our weakened normal bordism invariant $p_{M*}(\omega(\mathrm{id},f))$ determines the strongest version of Dold's fixed point index (which generalizes the Lefschetz index, cf. [1]).

First let us describe the Pontrjagin–Thom isomorphism PT (cf. (1.13)) which relates these invariants. Given a real number $R>0$, let $\overline{D}^k(R)$ (and $D^k(R)$, respectively) denote the compact (and open, respectively) ball of radius R in euclidian space $\mathbb{R}^k$ and identify the quotient space

$$\overline{D}^k(R)/\partial\overline{D}^k(R)=\mathbb{R}^k/(\mathbb{R}^k-D^k(R))$$

with the sphere $S^k=\mathbb{R}^k\cup\{\infty\}$ in the standard fashion. Moreover, define

$$E^k_R:=B\times\overline{D}^k(R)\subset E^k:=B\times\mathbb{R}^k. \tag{5.1}$$

Then we can interpret the suspension

$$\Sigma^k B^+=B\times S^k/(B\times\{\infty\})=E^k/(E^k-\mathring{E}^k_R)=E^k_R/\partial E^k_R \tag{5.2}$$

as a one point-compactification of $\mathring{E}^k_R=B\times D^k(R)$. Now, given a map

$$u\colon(\Sigma^k B^+,\infty)\to(S^k,\infty),\quad k>>0,$$

up to homotopy, we may assume that $u|\mathring{E}^k_R$ is smooth with regular value $0\in\mathbb{R}^k\subset S^k$. Thus its inverse image $u^{-1}(\{0\})$ is a smooth submanifold of $B\times\mathbb{R}^k$ whose normal bundle is trivialized via the tangent map of u. The resulting normal bordism class $[(u^{-1}(\{0\}),\text{first projection},\overline{g}_B)]$ is the value of $[u]\in\pi^0_{stable}(B^+)$ under the Pontrjagin–Thom isomorphism PT (cf. (1.13); compare (1.7) and (1.12)).

Proof of Theorem 1.6. In view of the homotopy invariance of Dold's index $I(f)$ (cf. [1, (2.9)]) we may assume that the map $(\mathrm{id},f)\colon M\to M\times_B M$ is smooth and transverse to the diagonal Δ. Then the fixed point set

$$C=C(\mathrm{id},f)=(\mathrm{id},f)^{-1}(\Delta) \tag{5.3}$$

is a smooth submanifold of M with the description

$$\overline{g}_B^{\#}\colon \nu_1 := \nu(C, M) \cong \mathrm{TF}(p_M)|C \tag{5.4}$$

of its normal bundle as in (1.5).

For large k there exists a smooth embedding $M \subset B \times \mathbb{R}^k$ over B whose normal bundle $\nu_2 := \nu(M, B \times \mathbb{R}^k)$ can be identified with a "vertical" subbundle of $T(B \times \mathbb{R}^k)$ which, together with $\mathrm{TF}(p_M)$, spans the tangent bundle along the fibres of $B \times \mathbb{R}^k \to B$. Let $\overline{V}$ (and V, respectively) be a corresponding compact (and open, respectively) tubular neighbourhood of M in $B \times \mathbb{R}^k$ and consider the composite map

$$\widehat{f}\colon \overline{V} \xrightarrow{\text{projection}} M \xrightarrow{f} M \subset B \times \mathbb{R}^k \tag{5.5}$$

over B. Clearly its fixed point set is also equal to C (cf. (5.3)). Hence there exists a radius $R > 0$ such that $v - \widehat{f}(v) \notin B \times D^k(R)$ for every $v \in \partial\overline{V}$. Moreover, we can pick a radius $\rho > 0$ such that the space $\overline{V}$ lies in $B \times \overline{D}^k(\rho)$. Collapsing its complement and using (5.1) and (5.2) we obtain the composite map

$$\widehat{u}\colon \Sigma^k B^+ = E^k_\rho / \partial E^k_\rho \longrightarrow \overline{V}/\partial\overline{V} \xrightarrow{\mathrm{id}-\widehat{f}} E^k/(E^k - \overset{\circ}{E}{}^k_R) = \Sigma^k B^+.$$

Now, according to [1, (2.15), (2.3) and (2.1)], Dold's indices of f and of $\widehat{f}|V$ (cf. (5.5)) agree and are defined to be the value of $1 \in \pi^0_{\mathrm{stable}}(B^+)$ under the induced homomorphism of $\widehat{u}$. In other words, we can represent $I(f)$ by the obvious composite map

$$u\colon \Sigma^k B^+ \xrightarrow{\widehat{u}} \Sigma^k B^+ \longrightarrow \Sigma^k(\{\mathrm{point}\}^+) = S^k.$$

Let us apply the Pontrjagin–Thom procedure (as described above) to $I(f) = [u]$. Clearly $u^{-1}(\{0\})$ is just the fixed point set C of f (cf. (5.3)). The trivialization

$$\overline{g}_B\colon \nu(C, B \times \mathbb{R}^k) = \nu_1 \oplus \nu_2|C \xrightarrow{\cong} \mathrm{TF}(p_M) \oplus \nu_2|C = C \times \mathbb{R}^k$$

is induced by the tangent map of $\mathrm{id} - f$. On ν_1 it coincides with $\overline{g}_B^{\#}$ (cf. (5.4)) and on ν_2 it is given by the identity map (since f is constant along each normal ball in the tubular neighbourhood V of M in $B \times \mathbb{R}^k$). Thus the data $(C \subset B \times \mathbb{R}^k \to B, \overline{g}_B)$, $k >> 0$, which describe $\mathrm{PT}(I(f))$ are just the stabilized coincidence data of (id, f), projected down to B. This proves the identity claimed in Theorem 1.6. $\square$

Remark 5.1. A key point in the previous proof is the fact that Dold's index remains unchanged by the passage $f \rightsquigarrow \widehat{f}$ (cf. (5.5)). This parallels closely the stabilizing transition $\omega^{\#}(\mathrm{id}, f) \rightsquigarrow \widetilde{\omega}(\mathrm{id}, f)$.

6. S^1-bundles over S^1

In this section we study the Example 1.7 of the introduction in some detail. In particular, we prove Theorems 1.9 and 1.10.

Given possible values q and r of the numerical invariants discussed in Proposition 1.8, let us describe the corresponding map $f\colon M \to N$ in standard form: for any element $[(t, z)]$ in the domain (cf. (1.14) or (1.15)), the standard map is defined by

$$f([(t,z)]) = \begin{cases} [(t, e^{2\pi irt} z^q)] & \text{if } N = T, \\ [(t, (-1)^r z^q)] & \text{if } N = K. \end{cases} \tag{6.1}$$

Using the linear structures on the universal covering spaces of T and K we see that every map over S^1 can be deformed over S^1 into standard form. This proves Proposition 1.8.

Next we calculate the group $\Omega_1(M;\varphi)$ in which the (weakened) coincidence invariant $\omega_B(f_1, f_2)$ of maps $f_1, f_2\colon M \to N$ over S^1 lies. Here φ is trivial if $M = N$; φ is the pullback $p_M^*(\lambda)$ of the nontrivial line bundle over $B = S^1$ if $M \neq N$ (cf. (1.11)).

From [7, Theorem 9.3], we obtain the exact sequence

$$0 \to \Omega_1^{\mathrm{fr}} \xrightarrow{\ \delta\ } \Omega_1(M;\varphi) \xrightarrow{\ \gamma\ } \overline{\Omega}_1(M;\varphi) \longrightarrow 0 \tag{6.2}$$

where γ forgets about stable vector bundle isomorphisms and retains only the corresponding orientation information. If $M = N$, then γ maps the classical framed bordism group $\Omega_1^{\mathrm{fr}}(M)$ to the oriented bordism group $\Omega_1(M) \cong H_1(M;\mathbb{Z})$, and the obvious forgetful homomorphism $\Omega_1^{\mathrm{fr}}(M) \to \Omega_1^{\mathrm{fr}}$ yields a splitting of (6.2). If $M \neq N$ then a splitting can be extracted from the exact Gysin sequence

$$\Omega_1^{\mathrm{fr}}(M) \xrightarrow{\ d\ } \Omega_1^{\mathrm{fr}}(\widetilde{M}) \xrightarrow{\ \mathrm{proj}_*\ } \Omega_1(M;\varphi) \longrightarrow 0$$

where $\widetilde{M}$ is the double cover (or S^0-bundle) corresponding to the line bundle $\lambda_M := p_M^*(\lambda)$ over M, d takes double coverings and proj denotes the obvious projection. (This is essentially the exact sequence of the pair $(\lambda_M, \lambda_M - s_0(M))$ and uses the Thom isomorphism

$$\Omega_i(\lambda_M, \lambda_M - s_0(M); -\lambda_M) \cong \Omega_{i-1}^{\mathrm{fr}}(M)$$

obtained by intersecting transversely with the zero section s_0 of λ_M).

Recall that any connected closed smooth 1-manifold S can carry two distinct stable framings:

(i) the invariant framing obtained from a *nonstable* parallelization TS $\cong$ $S \times \mathbb{R}$ (which is essentially invariant under rotations along the circle $S \cong S^1$); and

(ii) the boundary framing induced from a disk D which bounds $S = \partial D$.

The corresponding bordism classes are 1 and 0, respectively, in $\Omega_1^{\mathrm{fr}} \cong \mathbb{Z}_2$.

Now we can describe the direct sum decomposition of $\Omega_1(M;\varphi)$ in Theorem 1.10. The projection to the first (and the second, respectively) component group is obtained via intersecting circles in M with the fibre F_M (and with the section $s_{-1}(B)$ at -1, respectively); the three direct summands are generated by the circles $s_{+1}(B)$ and F_M (both with the boundary framing) and by

$$\delta(1) := [(\text{invariantly framed } S^1, \text{constant map})].$$

In order to compute the summands of $\omega_B(f_1, f_2)$ (corresponding to this decomposition of $\Omega_1(M;\varphi)$) we may assume that f_1, f_2 are in standard form (cf. (6.1)). Then the pairs (f_1, f_2) and $(f := f_1 \circ f_2^{-1},\ f_0 := f_2 \circ f_2^{-1} = s_{+1} \circ p_M)$ have the same coincidence locus C which consists of "parallel" circles in M. (Here we use fibrewise complex multiplication of standard maps; it is compatible with the gluing diffeomorphisms of T and K, cf. (1.14) and (1.15)). The transverse intersections of C with F_M and $s_{-1}(B)$ determine q and r (as indicated in the first two columns concerning $\omega_B(f_1, f_2)$ in the table of Theorem 1.10; the correction terms 1 and $\rho_2(q)$ result from the fact that the sections s_{+1} and s_{-1} have each a self-intersection in K).

Furthermore each circle S in the coincidence locus C is invariantly framed and hence contributes nontrivially to the third component of $\omega_B(f_1, f_2)$; it constitutes a full Nielsen class which therefore must be essential (see also the following proof). This establishes Theorem 1.10.

Proof of Theorem 1.9. If $N = S^1 \times S^1$, we are in the special case of a product fibration (c.f. special case 1.4), and our coincidence theory of maps f_1, f_2 over B reduces to the classical coincidence theory of their projections f_1', f_2' to the fibre S^1. But this situation has been thoroughly discussed in [9, Theorem 1.13 and Section 6], where even the fibre homotopy type of $E(f_1', f_2')$ over M is described. In particular, the Reidemeister number is just the cardinality of the cokernel of the induced homomorphism

$$f_{1*}' - f_{2*}' \colon H_1(M, \mathbb{Z}) \longrightarrow H_1(S^1, \mathbb{Z}) \cong \mathbb{Z}$$

whose image is generated by the greatest common divisor of $(q(f_1) - q(f_2))$ and $(r(f_1) - r(f_2))$. The Reidemeister number equals $\mathrm{MCC}(f_1', f_2') = N(f_1', f_2')$ except in the selfcoincidence case $f_1 \sim_B f_2$ when (f_1, f_2) is loose (cf. [9], (1.9)).

If $N = K$ the only sections (up to homotopy) of p_N are s_ε, $\varepsilon = \pm 1$ (cf. (1.16)); each can be deformed away from itself until it has only one selfintersection point in K. Therefore, if maps $f_i \colon M \to K$ over S^1 are homotopic to $s_{\varepsilon_i} \circ p_M$, $i = 1, 2$, (e.g. if $M = T$), their coincidence data can be represented by a whole fibre (or by $\emptyset$, respectively) when $\varepsilon_1 = \varepsilon_2$ (or $\varepsilon_1 \neq \varepsilon_2$, respectively), and $\mathrm{MCC}_B(f_1, f_2) = N_B^{\#}(f_1, f_2) = N_B(f_1, f_2)$ equals 1 (or 0, respectively).

It remains to study the coincidence behaviour of maps $f_1, f_2\colon K \to K$ over S^1 in standard form or, equivalently, of maps $f_1 \circ f_2^{-1} =: f$ and $f_2 \circ f_2^{-1} = s_{+1} \circ p_K =: f_0$ (here we use fibrewise complex multiplication). In view of the previous paragraph we may assume $q \neq 0$. Then the locus $C(f_1, f_2)$ consists of "horizontal" circles which are "parallel" to the sections $s_{\pm 1}$ and intersect each fibre S^1 in $\eta, \eta z_1, \dots, \eta z_1^{|q|-1}$ where $z_1 = e^{2\pi i/|q|}$ and $\eta = e^{\pi i r/|q|}$ for $r = 0, 1$.

Given $0 \leq k < k' < |q|$, we need to know when the coincidence points ηz_1^k, $\eta z_1^{k'}$ are Nielsen equivalent over B. This happens precisely if there is a path $c = l \cdot \widehat{c}$ from ηz_1^k to $\eta z_1^{k'}$ in K (consisting of a loop l at ηz_1^k, followed by a path $\widehat{c}$ in the fibre) such that $f \circ c = (f \circ l) \cdot (f \circ \widehat{c})$ is homotopic to $f_0 \circ c \sim f_0 \circ l$ keeping end points fixed. In other words, the loop $f \circ \widehat{c}$ which winds $k' - k + jq$, $j \in \mathbb{Z}$, times around the fibre is Reidemeister equivalent to the trivial loop. Since $\pi_2(S^1) = 0$ (cf. (3.5)) this means that k' is equal to k or to $-k + ((-1)^r - 1)/2 \bmod |q|$, or, equivalently, that ηz_1^k and $\eta z_1^{k'}$ lie in the same coincidence circle (due to the glueing reflection of K). Thus each Reidemeister class corresponds to an essential Nielsen class which consist of a single "horizontal" circle with winding number ± 1 or ± 2 with respect to the base S^1.

Recall that the Reidemeister numbers were computed in Proposition 3.5. $\square$

Remark 6.1. It is intriguing to compare the roles of the involution ι (in the proof of Proposition 3.5) on the one side and of complex conjugation (in the proof above) on the other side.

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