Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 33, 2009, 51–64

WECKEN PROPERTY FOR PERIODIC POINTS ON THE KLEIN BOTTLE

Jerzy Jezierski — Edward Keppelmann — Wacław Marzantowicz

Abstract. Suppose $f: M \to M$ on a compact manifold. Let m be a natural number. One of the most important questions in the topological theory of periodic points is whether the Nielsen–Jiang periodic number $NF_m(f)$ is a sharp lower bound on $\#\mathrm{Fix}(g^m)$ over all $g \sim f$. This question has a positive answer if $\dim M \geq 3$ but in general a negative answer for self maps of compact surfaces. However, we show the answer to be positive when $M = \mathbb{K}$ is the Klein bottle. As a consequence, we reconfirm a result of Llibre and compute the set $\mathrm{HPer}(f)$ of homotopy minimal periods on the Klein bottle.

1. Introduction

The Wecken theorem (see [21], and also [16] for more details) confirmed, in dimension ≥ 3, an old conjecture of Nielsen which said that for a self-map f of a compact manifold M there exists a map g homotopic to f with $\#\mathrm{Fix}(g) = N(f)$ where $N(f)$ is the Nielsen number of f (see [17], [16], for a definition of $N(f)$). In a series of articles [8]–[11] the first author showed that a corresponding Wecken theorem for periodic points holds for any self-map $f: M \to M$ of a compact manifold M of dimension ≥ 3 with the Nielsen number $N(f)$ replaced by the full Nielsen-Jiang periodic number $NF_n(f)$ (cf. [17], and also [16] for a definition), where n is the specific period in consideration. For self-maps of compact surfaces

2000 *Mathematics Subject Classification.* 55M20, 57N99.
Key words and phrases. Periodic point, homotopy minimal period, Nielsen number, Nielsen–Jiang periodic number, Klein bottle.
The first and the third named authors supported by KBN grant no 2/PO3A/03929.

such theorem do not hold in general. There are examples (cf. [18]) of maps of compact oriented surfaces of genus ≥ 2 for which Wecken theorem does not hold.

In this note we prove the Wecken theorem for periodic points for self-maps of the Klein bottle $\mathbb{K}$. We use the fact that $\mathbb{K}$ is an Eilenberg–MacLane space of a solvable group G. $\mathbb{K}$ fibres over the circle S^1 with circles as fibres. Moreover, each self-map of $\mathbb{K}$ is homotopic to a fibre-map.

For each natural number n and self map f there are two Nielsen type periodic numbers $NP_n(f)$ and $NF_n(f)$. Both are f homotopy invariants. The first is a lower bound on the number of periodic points of f which have period exactly n and the second is a lower bound on the size of $\mathrm{Fix}(f^n)$. As a consequence of our results we get a formula for the prime Nielsen–Jiang periodic number $NP_n(f)$ used by Halpern in [4]. From it we derive a description of the set $\mathrm{HPer}(f)$ of homotopy minimal periods given by Llibre in [20]. It is worth emphasizing that our work includes a complete geometrical proof of the Wecken theorem for periodic points for self-maps of the Klein bottle, which is in fact slightly stronger than the main result of the unpublished work [4]. Moreover, this lets us put the calculation of the set $\mathrm{HPer}(f)$ in terms of the general scheme introduced in [19] and used in other papers of the authors (see [16, Chapter VI] for a detailed discussion). Finally we must add, that a description of the possible sets of homotopy minimal periods of self-maps of Klein bottle, together with the previously described cases of the circle [2], the two dimensional torus [1], the three dimensional torus [19], three dimensional nilmanifolds [15], three dimensional NR-solvmanifolds [14], and all other three dimensional solvamanifolds [13] completes a study of the feasible sets of homotopy minimal periods for all self-maps of compact solvmanifolds of dimension ≤ 3.

2. Klein bottle

Self-maps of the Klein bottle. We follow the notation of [4]. Let $A, B\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by $A(x,y) = (x+1,y)$, $B(x,y) = (-x, y+1)$ and let G be the group of self-homeomorphisms of $\mathbb{R}^2$ generated by A and B. We define the *Klein bottle* as the quotient space: $\mathbb{K} = G \setminus \mathbb{R}^2$.

Let $\pi\colon \mathbb{R}^2 \to \mathbb{K}$ denote the natural projection. The formula $p\colon \mathbb{K} \to \mathbb{R}/\mathbb{Z}$, $p[x,y] = [y]$ defines a locally trivial fibre bundle over the circle $\mathbb{R}/\mathbb{Z} = S^1$, with the circle as the fibre. Let $* = [0,0] \in \mathbb{K}$ be the chosen base point.

Since the natural projection $\pi\colon \mathbb{R}^2 \to \mathbb{K}$ is a covering of the contractible space $\mathbb{R}^2$, the Klein bottle is an Eilenberg–MacLane space. $\pi_1(\mathbb{K}, *) \equiv G$ is generated by the loops $a(t) = [t,0]$, $b(t) = [0,t]$ with subject to the single relation $ba = a^{-1}b$. In particular any element of $\pi_1(\mathbb{K})$ can be uniquely represented as $a^i b^j$, $(i,j \in \mathbb{Z})$.

Since $\mathbb{K}$ is an Eilenberg–MacLane space,

$$[(\mathbb{K},*),(\mathbb{K},*)] = \mathrm{Hom}(G,G)$$

where $[(\mathbb{K},*),(\mathbb{K},*)]$ denotes the set of homotopy classes of self-maps of $\mathbb{K}$ preserving the chosen point. Now each $[f] \in [(\mathbb{K},*),(\mathbb{K},*)]$ gives the homomorphism $f_\#\colon \pi_1(\mathbb{K}) \to \pi_1(\mathbb{K})$ and

$$f_\#(a) = a^u b^z, \quad f_\#(b) = a^v b^w$$

for some integers u, v, w, z. Halpern [4] showed that then always $z = 0$ and moreover, w even implies $u = 0$. See also [3].

We will give explicit formulae for a map representing each self-homomorphism of the fundamental group, i.e. an explicit formulae for a representative of each homotopy class of self-maps of $\mathbb{K}$.

This is accomplished by picking a specific representative in each homotopy class of self maps. A deformation of this cannonical representative then suitably realizes the appropriate lower bound for $NF_n(f)$. Suppose $f\colon M \to M$ on a compact manifold. Let m be a natural number. One of the most important questions in the topological theory of periodic points is whether the Nielsen–Jiang periodic number $NF_m(f)$ is a sharp lower bound on $|\mathrm{Fix}(g^m)|$ over all $g \sim f$. This question has a positive answer if $\dim M \geq 3$ but in general a negative answer for self maps of compact surfaces. However, we show the answer to be positive when $M = K$ is the Klein bottle.

As a consequence, we reconfirm a result of Llibre and compute the set $\mathrm{Hper}(f)$ of homotopy minimal periods on the Klein bottle.

- For w odd, and $0 \leq x, y \leq 1$, we define

$$f[x,y] = [ux + vy, wy] \quad \text{for } x \in \mathbb{R},\ 0 \leq y \leq 1.$$

- For w even, and $0 \leq x, y \leq 1$, we define

$$f[x,y] = [vy, wy] \quad \text{where } x, y \in \mathbb{R}.$$

We will say that a self-map of the Klein bottle is of *a standard form* if it is given by one of the above formulae.

Let us notice that any map in the standard form is a fibre map

$$\begin{array}{ccc} \mathbb{K} & \xrightarrow{f} & \mathbb{K} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ S^1 & \xrightarrow[\bar{f}]{} & S^1 \end{array}$$

where $p([x,y]) = [y]$ with $y \sim y + q$ for any integer q and $[y]$ denotes the equivalence class on $\mathbb{R}$ which forms S^1.

Let us notice that the induced map of the base space is given by $\overline{f}[y] = [wy] \in \mathbb{R}/\mathbb{Z} = S^1$ so its degree $\deg(\overline{f}) = w$.

The following Lemma is a consequence of [22]. However, to make the paper more self-contained we give an easy proof in this special case.

Lemma 2.1. *Let $f: \mathbb{K} \to \mathbb{K}$ be a fibre map (not necessarily in standard form) of the Klein bottle withw $\neq 1$. Then for some $b = [y] \in S^1$ we have that fixed points x, y lying in a fibre $\mathbb{K}_b = S^1$ are Nielsen related as $x, y \in \mathrm{Fix}(f_b)$ if and only if they are Nielsen related as $x, y \in \mathrm{Fix}(f)$.*

Proof. Since $\Rightarrow$ is obvious, we prove $\Leftarrow$. Let $\omega: I \to \mathbb{K}$ be a path satisfying $\omega(0) = x$, $\omega(1) = y$ (rel. end points), $\omega \sim f\omega$ (rel. endpoints) in $\mathbb{K}$.

Since x and y are in the same fibre, $p\omega: [0,1] \to S^1$ is a loop based at b. Moreover, $\overline{f}p\omega = pf\omega$ and $p\omega$ represent the same element in $\pi_1(S^1) = \mathbb{Z}$. Now $\overline{f}_\#(p\omega) = p\omega$ as loop classes $p\omega$ represents the trivial element in $\pi_1(S^1) = \mathbb{Z}$. This implies that there is a homotopy contracting $p\omega$ to the constant loop at b. This null homotopy lifts to a homotopy in $\mathbb{K}$ from ω to a path ω' joining x and y entirely within the fibre $\mathbb{K}_b$. Now

$$f\omega' \sim f\omega \sim \omega \sim \omega'$$

in $\mathbb{K}$. Thus $f\omega' \sim \omega'$ in $\mathbb{K}$. In remains to show that $f\omega' \sim \omega'$ in $\mathbb{K}_b$. This follows from the fact that the inclusion $\mathbb{K}_b \subset \mathbb{K}$ induces a monomorphism of fundamental groups. This is a consequence of the exact homotopy sequence

$$\cdots \to 0 = \pi_2(S^1) \to \pi_1(\mathbb{K}_b) \to \pi_1(\mathbb{K}) \to \cdots \qquad \square$$

Lemma 2.2. *Let $f: K \to K$ be a fibre map of the Klein bottle in standard form with $w = 1$ and $|u| \neq 1$. Thenf is naturally fibre-preserving and the restriction of the projection $p_|: \mathrm{Fix}(f) \to S^1$ is a finite covering map.*

Proof. If $u = 0$ then $p_|$ is a homeomorphism and the conclusion is immediate.

Therefore we can assume that $|u| \geq 2$. Since $w = 1$, $\overline{f} = \mathrm{id}_{S^1}$ and f maps each fibre is into itself.

Suppose $z \in \mathrm{Fix}(f)$ and let $p(z) = b$. Since $|u| \geq 2$, the restrictions of f to the fibres are expanding so in particular the derivative $D(f_b)$ is not the identity at the point z. Thus the Implicit Function Theorem implies that $p: \mathrm{Fix}(f) \to S^1$ is a local homeomorphism. Since $\mathrm{Fix}(f)$ is compact, $p: \mathrm{Fix}(f) \to S^1$ is a covering map. $\square$

3. Wecken Theorem

In [17] Boju Jiang defined a homotopy invariant lower bound $NF_n(f)$ on the number of periodic points of period n for a self-map of a compact polyhedron.

Later it was shown [8]–[11] that for manifolds of dimension ≥ 3, this lower bound is optimal, i.e. there always exists a map g is homotopic to f with $\#\mathrm{Fix}(g^n) = NF_n(f)$.

The aim of this paper is to prove that $NF_n(f)$ is also optimal for self-maps of the Klein bottle.

THEOREM 3.1. *Any map of the Klein bottle $f\colon \mathbb{K} \to \mathbb{K}$ is homotopic to a map realizing the number $NF_n(f)$, i.e. there is a homotopy from f to a map g satisfying $\#\mathrm{Fix}(g^n) = NF_n(f)$.*

The rest of this section is devoted to a proof of Theorem 3.1.

The definition of $NF_n(f)$, in general, is a bit complicated ([17], [6], [7] and also [15] for longer exposition). However, for the sake of our paper it is enough to use the following

LEMMA 3.2. *Let $f\colon X \to X$ be a self-map of an compact polyhedron. Then*

$$\#\mathrm{Fix}(f^n) \geq NF_n(f) \geq \sum_{k|n} NP_k(f) = \sum_{k|n} (\#\mathcal{IEOR}(f^k) \times k) \geq \sum_{k|n} \#\mathcal{IER}(f^k)$$

(where $\mathcal{IER}(f^k)$ $(\mathcal{IEOR}(f^k))$ denote the sets of irreducible essential (orbits of) Reidemeister class of f^k and $NP_k(f) = \#\mathcal{IEOR}(f^k) \times k$ (by definition) is a Nielsen type homotopy invariantlower bound on the number of periodic points of f of the minimal period k, see [6].

PROOF. The first inequality is the basic property of $NF_n(f)$. The second inequality is a straight consequence of of the fact that when $n|k$ then it means that each essential irreducible Reidemeister orbit of f^n makes a unique contribution to the points of period k.

The equality is the definition of $NP_n(f)$ given above. To prove the last inequality it is enough to notice that the length of each orbit in $\mathcal{IER}(f^k)$ is at most k. This means that

$$\#\mathcal{IEOR}(f^k) \times k \geq \#\mathcal{IER}(f^k). \qquad \square$$

Thus to prove Theorem 3.1 it is enough to produce a map g homotopic to f satisfying $\#\mathrm{Fix}(g^n) = \sum_{k|n} \#\mathcal{IER}(g^k)$. In fact this equality and Lemma 3.2 along with the basic property of NF_n imply $\#\mathrm{Fix}(g^n) \geq NF_n(g) \geq \#\mathrm{Fix}(g^n)$ which gives the desired equality $\#\mathrm{Fix}(g^n) = NF_n(f)$.

LEMMA 3.3. *Let $f\colon X \to X$ be a self-map of a finite polyhedron and n a natural number. If for each divisor l of n, and each point $x \in \mathrm{Fix}(f^n)$ whose orbit $\langle x \rangle = \{x, fx, f^2x, \dots\}$ has length l, the subset $\{x\} \subset \mathrm{Fix}(f^l)$ is an irreducible essential Nielsen class of f^l then $\#\mathrm{Fix}(f^n) = NF_n(f)$.*

Proof. First we notice that a self-map $f: X \to X$ defines a map

$$\alpha: \mathrm{Fix}(f^n) \to \bigcup_{k|n} \mathcal{R}(f^k)$$

(disjoint union) sendinga periodic point x to the class in $\mathcal{R}(f^l)$ represented by x where l denotes the length of the orbit $\langle x \rangle = \{x, fx, f^2x, \dots, f^{l-1}(x)\}$. In our case, by assumption, each class $\alpha(x)$ is essential. Moreover, $\alpha(x)$ must be also irreducible, since if $\alpha(x) \in \mathcal{R}(f^l)$ would reduce to $\mathcal{R}(f^h)$, for some $h < l$, then the length of the orbit $\langle \alpha(x) \rangle = \{\alpha(x), \alpha(fx), \dots\}$ would be at most $h < l$. This gives the map

$$\alpha': \mathrm{Fix}(f^n) \to \bigcup_{k|n} \mathcal{IER}(f^k).$$

We will show that α' is a bijection. Let $A \in \mathcal{IER}(f^k)$. Since A is essential, it contains a point in $\mathrm{Fix}(f^s) \subseteq \mathrm{Fix}(f^n)$ whose orbit has length $s|l$. Since A is irreducible, this must mean that $s = k$. This proves that α' is onto.

On the other hand α' is also injective, since two points $x \neq x' \in \mathrm{Fix}(f^l)$ represent different Nielsen (and hence also Reidemeister) classes. □

Remark 3.4. The map $f: S^1 = \mathbb{R}/\mathbb{Z} \to S^1$ given by $f[t] = [kt]$, $k \in \mathbb{Z}$, $k \neq \pm 1$ satisfies the assumptions of Lemma 3.3.

Now we come back to self-maps of the Klein bottle. We start with a lemma showing that in many cases the map given by the standard formula already realizes the least number of periodic points.

Lemma 3.5. *Let $f: \mathbb{K} \to \mathbb{K}$ be given by the formula $f[x, y] = [ux + vy, wy]$ with $|u| \neq 1$ and $|w| \neq 1$. Then f satisfies the assumptions of Lemma 3.3. In particular,*

$$\#\mathrm{Fix}(f^n) = NF_n(f).$$

Proof. Let $x \neq y \in \mathrm{Fix}(f^k)$, $k|n$. It will suffice to that x, y represent different essential singleton Nielsen classes in $\mathrm{Fix}(f^k)$.

Suppose to the contrary that x and y are Nielsen equivalent under f^k. Then px and py are Nielsen related under f^k. Since $|w| \neq 1$, Remark 3.4 implies that $px = py$, so the points x, y belong to the same fibre $x, y \in p^{-1}(b_0)$. Now the assumption that x, y are Nielsen related in $\mathrm{Fix}(f^k)$ and Lemma 2.1 imply that x and y are Nielsen related in $\mathrm{Fix}((f_{b_0})^k)$ and hence since $|u| \neq 1$ Remark 3.4 implies $x = y$.

The class $\{x\} \subset \mathrm{Fix}(f^k)$ is essential by the Fixed Point Index Product Formula. □

Remark 3.6. The above lemma also holds if $w = -1$, $|u| \neq 1$ and n is odd. We may follow the same proof using the fact that then all divisors k of n will be odd and so $w^k = -1 \neq 1$. This tells us that all relevant Nielsen classes are essential.

REMARK 3.7. If $w = -1$, $|u| \neq 1$ and n is even, then it follows from Lemma 3.5 that the assumptions of the Lemma 3.3 hold for orbits of odd length. More precisely, for each point $x \in \mathrm{Fix}(f^n)$ whose orbit $x, fx, f^2x, \ldots$ has odd length l, the subset $\{x\} \subset \mathrm{Fix}(f^l)$ is an irreducible essential Nielsen class.

Lemmas 3.3, 3.5 and Remarks 3.6, 3.7 give Theorem 3.1 for $|u| \neq 1$ and $|w| \neq 1$. Now we consider the remaining cases in the table and discussion below.

The entries within the table indicate the number in the discussion where this is considered.

$w \setminus u$	$u = -1$	$u = 0$	$u = +1$	$\|u\| \geq 2$
$w = -1$	2	5	2	6
$w = 0$	4	4	4	4
$w = +1$	1	1	1	1
$\|w\| \geq 2$	3	5	3	Lemma 3.5

Case 1. $w = 1$. Then the base map $\overline{f}: S^1 \to S^1$ is the identity, hence after a small twist of $\overline{f}$, by an angle which is an irrational multiple of 2π, has no periodic points. This homotopy of $\overline{f}$ lifts to a homotopy $f_t: \mathbb{K} \to \mathbb{K}$ with no periodic points.

Case 2. $u = \pm 1$ and $w = -1$. Since $\deg(\overline{f}) = -1$, after a small homotopy we may assume that the map $\overline{f}$ has only two periodic points $[0]$ and $[1/2]$. Moreover, the fibre map over these points has degree -1 and $+1$ respectively. Now after a homotopy near these fibres we get exactly two periodic points over $[0]$ and no periodic points, over $[1/2]$. It remains to notice that the fixed point index in each of the two fixed points over $[0]$ is $(+1)(+1) \neq 0$ (from Fixed Point Index Product Formula) so from Lemma 2.1 they represent different essential Nielsen classes in $\mathrm{Fix}(f)$.

Case 3. $u = \pm 1$ and $|w| \geq 2$. Consider the set $\mathrm{Fix}(\overline{f}^n)$ and an orbit of points of length l. Let $b \in\in \mathrm{Fix}(\overline{f})$ denote a point in this orbit. If $\deg(f^l_b) = +1$ then we can deform f, near a fibre over a point from the considered orbit, to make $f^l_b: p^{-1}(b) \to p^{-1}(b)$ a map without periodic points. If $\deg(f^l_b) = -1$ then we can deform f as above to assure that $f^l_b: p^{-1}(b) \to p^{-1}(b)$ is a mapwith exactly two periodic points.

It remains to show that each of the two points $z, z' \in p^{-1}(b) \cap \mathrm{Fix}(f^l)$ is an essential irreducible class in $\mathcal{R}(f^l)$. In fact,

- the two points are not Nielsen related by Lemma 2.1,

- the Fixed Point Index Product Formula implies that the fixed point index for each $b \in \mathrm{Fix}(\overline{f})$ we have that equals $\mathrm{sign}(1 - w^l) \neq 0$, hence these Nielsen classes are essential,
- each Nielsen $\{z\}$ class is irreducible, since otherwise has the same minimal period as the class $\{p(z)\}$ for $\overline{f}$ and hence the result follows from Lemma 3.3.

Case 4. $w = 0$. Then we may assume that $\overline{f}(S^1) = b_0$, Since this image is a singleton essential fixed point class of $\overline{f}$, we will have that $NF_n(f) = NF_n(f_b)$ for all n. We may assume that, after a deformation, each point in $\mathrm{Fix}(f_{b_0}^n)$ is an essential Nielsen class of $\overline{f}_{b_0}^n$. Then, from Lemma 2.1 and the Fixed Point Index Product Formula, each point in $\mathrm{Fix}(f^n) = \mathrm{Fix}(f_{b_0}^n)$ is an essential Nielsen class.

Case 5. $u = 0$. Then f_b is constant for each $b \in \mathrm{Fix}(\overline{f})$ hence, we have that $p\colon \mathrm{Fix}(f^n) \to \mathrm{Fix}(\overline{f}^n)$ is an index preserving homeomorphism.

Cases 1–5 all involved fibre preserving homotopies. This will not be possible in the last case.

Case 6. $w = -1$ and $|u| \geq 2$.

Now the map f is given by the formula

$$f[x, y] = [ux + vy, -y]$$

Since the case for n is even follows from Remark 3.6, it remains to consider the case where n is odd. Now essential classes occur only in $\mathrm{Fix}(f^k)$ for odd $k|n$. Moreover from Remark 3.6, each point $z \in \mathrm{Fix}(f^k)$ with k-odd forms a singleton essential irreducible class. This implies that

$$NF_n(f) = \# \bigcup_{k|n \text{ with } k \text{ odd}} \mathrm{Fix}(f^k).$$

The theorem will be shown once we remove from $\mathrm{Fix}(f^n)$ the points whose minimal periods are even.

As we have noticed (Lemma 2.2) the set of periodic points splits into circles

$$\mathrm{Fix}(f^n) = S_1 \cup \cdots \cup S_h$$

and f permutes these circles giving rise to the *orbits of circles*.

We fix neighbourhoods $U_i \supset S_i$ so thin that $\mathrm{cl}\, U_i \cap \mathrm{cl}\, U_j = \emptyset$ for $i \neq j$. Moreover, in each orbit we fix one circle and we call it as the orbit's *leading* circle. For each leading circle S_i we fix another neighbourhood U_i' satisfying: $S_i \subset U_i' \subset U_i$ and $f^k(S_i) \subset U_j$ implies $f^k(U_i') \subset U_j$ for $k = 1, \ldots, n$.

Let us fix an orbit of circles and let S_0 be its leading circle. Let k be the smallest number satisfying $f^k(S_0) \subset S_0$.

(1) If k is even then we will find a homotopy f_t with the carrier in U_i' so that $f_0 = f$ and

$$\operatorname{Fix}(f_1^n) = \operatorname{Fix}(f^n) \setminus (S_0 \cup f(S_0) \cup f^2(S_0) \cup \ldots \cup f^{k-1}(S_0)),$$

i.e. we remove precisely the orbit of circles represented by S_0.

(2) Now let k be odd. Now $\operatorname{Fix}(f^k) \cap S_0$ consists of two points z_1, z_2. We will find a homotopy carried by U_0' so that

$$\operatorname{Fix}(f_1^n) = \operatorname{Fix}(f^n) \setminus (S_0 \cup f(S_0) \cup f^2(S_0) \cup \ldots \cup f^{k-1}(S_0) \\ \cup \{z_1, f(z_1), \ldots, f^{k-1}(z_1); z_2, f(z_2), \ldots, f^{k-1}(z_2)\}$$

i.e. we remove the orbit of S_0 leaving only the orbits of the points z_1 and z_2 (the only points of odd period).

We may do these operations to all orbits in $\{S_1, \ldots, S_h\}$ simultaneously. Since the orbits of points are mutually disjoint, so are the orbits of their sufficiently small neighbourhoods, hence no new periodic points appear. This homotopy will produce the map g satisfying

$$\operatorname{Fix}(g^n) = \bigcup_i \{z_{1i}, f(z_{1i}), \ldots, f^{k_i-1}(z_{1i}); z_{2i}, f(z_{2i}), \ldots, f^{k_i-1}(z_{2i})\}$$

where the summation runs over the set of all $1 \le i \le h$ for which S_i is a leading circle and the number

$$k_i = \text{ the length of the orbit containing } S_i$$

is odd. Thus only the orbits of odd length remain and hence all the points of even period are removed as required.

It remains two prove the above two claims. This will be done in the next Lemma.

Lemma 3.8. *Let f be a self-map of the Klein bottle given in the canonical form with $w = -1$ and $|u| \ge 2$. Let S be a leading circle in an orbit of length k. Then:*

(a) *If k is even then there is a homotopy f_t whose the carrier is in an arbitrarily prescribed neighbourhood $U' \supset S$ with $f_0 = f$ and*

$$\operatorname{Fix}(f_1^n) = \operatorname{Fix}(f^n) \setminus (S \cup f(S) \cup f^2(S) \cup \ldots \cup f^{k-1}(S)).$$

(b) *If k is odd then $\operatorname{Fix}(f^k) \cap S$ consists of two points z_1, z_2. There is a homotopy f_t whose carrier is in an arbitrarily prescribed neighbourhood of*

$U' \supset S$ that

$$\operatorname{Fix}(f_1^n) = \{\operatorname{Fix}(f^n) \setminus (S \cup f(S) \cup f^2(S) \cup \dots)\} \\ \cup \{z_1, f(z_1), \dots, f^{k-1}(z_1); z_2, f(z_2), \dots, f^{k-1}(z_2)\}.$$

Proof. Let us notice that we may assume that the given neighbourhood U' is so small that $U', f(U'), \dots, f^k(U')$ are disjoint from the other orbits of circles.

Case 1. Let k be even. Since $\overline{f}^k = \mathrm{id}$, the restriction $f^k\colon S \to S$ is a natural transformation of the covering $p\colon S \to S^1$. Therefore either $f^k(z) = z$ for all $z \in S$ or $f^k(z) \neq z$for all $z \in S$. In the first case there are no periodic points in S there is nothing to prove. Therefore, let us assume that $f^k(x) = x$ for all $x \in S$. We can define a smooth vector field v_z which is zero for $z \notin U'$ and is such that $p_* v_z$ is nonzero, for all $z \in U'$, and $p_* v_z$ shows an orientation of the circle S^1. Let $h\colon \mathbb{K} \to \mathbb{K}$ be a Poincare map induced by the vector field v_z. We will show that, for h sufficiently close to the identity, the map $g = fh$ satisfies

$$\operatorname{Fix}(g^n) = \operatorname{Fix}(f^n) \setminus \bigcup_r f^r(S)$$

where the summation runs over the set of all divisors of n. It is enough to show that $g^n(z) \neq z$ for all $z \in U'$. We notice that

$$pg^k(z) = p(f^{k-1} fh)(z) = pf^k h(z) = ph(z).$$

Here the first equality follows from the fact that $g = f$ except on U' and the following:

1. $g(z) = fh(z) \subset fh(U') = f(U')$ and successively $g^2(z) \in f^2(U'), \dots,$ $g^{k-1}(z) \in f^{k-1}(U')$,
2. the sets $U', f(U'), \dots, f^{k-1}(U')$ are mutually disjoint

The last equality follows from $pf^k = \overline{f}^k p = \mathrm{id}_{S^1} p$. Since $ph(z) = \exp(\phi)p(z)$ for a small positive angle ϕ, we have that $pg^k(z) = \exp(\phi)p(z)$. Using the same argument and substituting $z = g^k(z)$ we get $pg^{2k}(z) = \exp(\phi')p(z)$ and consequently $pg^n(z) = \exp(\phi'')p(z)$ where ϕ' and ϕ'' are small positive angles (which depend on n). The last equality implies that $g^n(z) \neq z$ for all $z \in U'$.

Case 2. Let k be odd. Then the restriction $f^k\colon S \to S$ covers the flip map of S^1, hence f is also the flip map and has exactly two fixed points z_1, z_2. Now $S \setminus \{z_1, z_2\}$ splits into two connected components. Let S_1, S_2 denote their closures. Notice that f^k permutes S_1 with S_2.

Let us fix a neighbourhood U'' satisfying

1. $U'' \cap S = S_1 \setminus \{z_1, z_2\}$,
2. $U'' \subset U'$,
3. $U'' \cap f^s(U'') \neq \emptyset$ if and only if $2k|s$.

We fix a vector field v_z on $\mathbb{K}$, as above, but now with the carrier is in U''. We denote by h a Poincare map and $g = fh$. We will show that $\mathrm{Fix}(g^n) \cap U'' = \emptyset$. In fact

$$pg^{2k}(z) = p(f^{2k-1}fh)(z) = pf^{2k}h(z) = ph(z)$$

since $g(U'), \dots, g^{k-1}(U')$ are mutually disjoint from U' and $pf^{2k} = \overline{f}^{2k}p = \mathrm{id}_{S^1}p$. But $ph(z) = \exp(\phi)p(z)$ for a small positive angle ϕ. Now we may argue as in the previous case where k is even to get that $g^n(z) \neq z$ for all $z \in U''$. □

4. Homotopy minimal periods

In this section,we derive the description of the set of possible homotopy minimal periods of a self-map $f\colon \mathbb{K} \to \mathbb{K}$, given in [20], using the methods of the proof the Wecken Theorem 3.1.

Remark 4.1. Suppose that every self-map of a finite polyhedron $f\colon K \to K$ is homotopic to a map satisfying the assumptions of Lemma 3.3. Then

$$n \in \mathrm{HPer}(f) \text{ if and only if } NP_n(f) \neq 0$$

i.e. n is a homotopy period if and only if there is an essential irreducible Nielsen class in $\mathrm{Fix}(f^n)$.

Proof. $\Leftarrow$ is obvious. Now we assume that there is no essential irreducible Nielsen class in $\mathrm{Fix}(f^n)$. It remains to notice that the map g homotopic to f which satisfies the assumptions of Lemma 3.3 has no orbit of points of length n. In fact such an orbit of points in $\mathrm{Fix}(g^n)$ would determine an essential irreducible orbit of Reidemeister classes of g^n. □

Theorem 4.2. *Let $f\colon \mathbb{K} \to \mathbb{K}$ be a self-map of the Klein bottle inducing the homotopy group homomorphism $f_\#(a) = a^u b^v$, $f_\#(b) = b^w$. Then*

$$\mathrm{HPer}(f) = \begin{cases} \emptyset & \text{for } w = 1, \\ \{1\} & \text{for } (w = 0) \text{ or } (w = -1 \text{ and } |u| \leq 1), \\ \mathbb{N} \setminus \{2\} & \text{for } w = -2, \\ \mathbb{N} \setminus 2\mathbb{N} & \text{for } w = -1 \text{ and } |u| \geq 2, \\ \mathbb{N} & \text{for } w \leq -3 \text{ or } w \geq 2. \end{cases}$$

Proof. Let w be even. Then $u = 0$, hence every fibre map is homotopic to a constant and we have that$\mathrm{HPer}(f) = \mathrm{HPer}(\overline{f})$. Since $\overline{f}$ is a self-map of the circle of (even) degree w,

$$\mathrm{HPer}(f) = \begin{cases} \{1\} & \text{for } w = 0, \\ \mathbb{N} \setminus \{2\} & \text{for } w = -2, \\ \mathbb{N} & \text{for } w \leq -4 \text{ or } w \geq 2, \ (w \text{ is even}), \end{cases}$$

see [1].

Now we consider the case when w is odd. Let $w = 1$. Then the base map is homotopic, by an irrational twist, to a map with no periodic points. This implies $\mathrm{HPer}(f) = \emptyset$.

Let $|w| \geq 3$. We will show that in this case $\mathrm{HPer}(f) = \mathbb{N}$. First we recall that in the homotopy exact sequence

$$1 \to \pi_1(S^1) \to \pi_1(\mathbb{K}) \to \pi_1(S^1) \to 1$$

arising from natural fibration of the Klein bottle $p: \mathbb{K} \to S^1$, the action α of $\pi_1(S^1) = \mathbb{Z}$ on the fibre $\pi_1(S^1) = \mathbb{Z}$ is given by $\alpha[k] = (-1)^k$. Let f be a self-map of the Klein bottle in the standard form. Let us denote $\varepsilon_w^j = [j/|w-1|] \in \mathbb{R}/\mathbb{Z}$. Then since $\overline{f}$ is the standard map of degree w on S^1, $\mathrm{Fix}(\overline{f}) = \{\varepsilon_w^0, \varepsilon_w^1, \ldots, \varepsilon_w^{|w-2|}\}$. Now $\deg(f_{\varepsilon_w^0}) = u$ implies $\deg(f_{\varepsilon_w^j}) = (-1)^j \cdot u$.

Let k be a natural number. Let us notice that then the points $\varepsilon^1_{w^k}$, $\varepsilon^2_{w^k}$ are essential irreducible classes of $\overline{f}^k$ ($|w| \geq 3$). Moreover if $\deg f_{\varepsilon^1_{w^k}} = u$ then $\deg f_{\varepsilon^2_{w^k}} = -u$.

Now at least one of the numbers $N(f^k_{\varepsilon^1_{w^k}}) = |1-u|$ or $N(f^k_{\varepsilon^2_{w^k}}) = |1+u|$ is not zero, hence there are essential classes in, at least one, of the fibre maps $f^k_{\varepsilon^1_{w^k}}$ or $f^k_{\varepsilon^2_{w^k}}$. Since the points $\varepsilon^1_{w^k}$, $\varepsilon^2_{w^k}$ are essential irreducible classes of $\overline{f}^k$, the fibre over (at least one) of these points, contains also essential irreducible classes of f^k. This proves that $k \in \mathrm{HPer}(f)$.

It remains to consider the case $w = -1$. First we notice that when k is even, $N(\overline{f}^k) = 0$ so f has no essential classes. On the other hand, the base map $\overline{f}$ contains two essential Reidemeister classes and the fixed points representing these classes also represent unique essential classes of for all odd iterations $\overline{f}^k$ of f. Let $b_1, b_2 \in S^1$ be the points representing these classes.

Now if $|u| \geq 2$, then

$$\mathrm{HPer}(f_{b_i}) = \begin{cases} \mathbb{N} & \text{when } \deg(f_{b_i}) \neq -2, \\ \mathbb{N} \setminus \{2\} & \text{when } \deg(f_{b_i}) = -2. \end{cases}$$

Because the b_i are fixed points of $\overline{f}$ and we can assume that $\overline{f}$ has no other periodic points, we have that

$$\mathrm{Hper}(f) = [\mathrm{Hper}(f_{b_1}) \cup \mathrm{Hper}(f_{b_2})] - 2\mathbb{N}.$$

Furthermore, since the degrees of the fibre maps over b_1 and b_2 are opposite, $\mathrm{HPer}(f_{b_i}) = \mathbb{N}$ over at least one of these points. This implies $\mathrm{HPer}(f) = \mathbb{N} \setminus 2\mathbb{N}$

Now we assume that $w = -1$ and $|u| \leq 1$. We will show that then $\mathrm{HPer}(f) = \{1\}$. If $u = 0$ then $\mathrm{HPer}(f) = \mathrm{HPer}(\overline{f}) = \{1\}$, since $\deg(\overline{f}) = w = -1$. If $u = \pm 1$ then the degree over one class of $\overline{f}$ is $+1$ and over the other one is -1. The set of homotopy periods of the fibre map over the first class is empty while over the second class it is in $\{1\}$. This gives $\mathrm{HPer}(f) = \{1\}$. □

Remark 4.3. It is interesting to note that the parameter v plays no role on the homotopy minimal periods.

As a byproduct we get the following Sharkovski-type theorem:

Theorem 4.4. *Let $f\colon K \to K$ be a self-map of the Klein bottle. If 2 is a homotopy period of f then all natural numbers are homotopy periods.*

References

[1] L. Alsedá, S. Baldwin, J. Llibre, R. Swanson and W. Szlenk, *Minimal sets of periods for torus maps via Nielsen numbers*, Pacific J. Math. **169** (1995), 1–32.

[2] L. Block, J. Guckenheimer, M. Misiurewicz and L. S. Young, *Periodic points and topological entropy of one-dimensional maps*, Lectures Notes in Math. **819** (1983), Springer Verlag Berlin, Heidelberg-New York, 18–24.

[3] R. Dobrenko and J. Jezierski, *The coincidence Nielsen number on non-orientable manifolds*, Rocky Mountain J. Math. **23** (1003), 67–85.

[4] B. Halpern, *Periodic points on the Klein bottle*, manuscript (1978).

[5] Ph. Heath and E. Keppelmann, *Fibre techniques in Nielsen periodic point theory on nil and solvmanifolds* I, Topology Appl. **76** (1997), 217–247.

[6] P. Heath, R. Piccinini and C. You, *Nielsen type numbers for periodic points* I, Topological Fixed Point Theory and Applications (B. Jiang, ed.), Lecture Notes in Math., vol. 1411, Springer, Berlin, 1989.

[7] P. Heath and C. You, *Nielsen type numbers for periodic points* II, Topology Appl. **43** (1992), 219–236.

[8] J. Jezierski, *Cancelling periodic points*, Math. Ann. **321** (2001), 107–130.

[9] ———, *Wecken's Theorem for periodic points*, Topology **42** (2003), 1101–1124.

[10] ———, *Weak Wecken's theorem for periodic points in dimension* 3, Fund. Math. **180** (2003), 223–239.

[11] ———, *Wecken's theorem for periodic points in dimension at least* 3, Topology Appl. **153** (2006), 1825–1837.

[12] ———, *Homotopy periodic sets of selfmaps of real projective space*, Bull. Mexican Math. Soc. **11** (2005), 293–302.

[13] J. Jezierski, E. Keppelmann and W. Marzantowicz, *Homotopy minimalperiods for maps of three dimensional solvmanifolds*, Topology Appl. (to appear).

[14] J. Jezierski, J. Kędra and W. Marzantowicz, *Homotopy minimal periods for NR-solvmanifolds maps*, Topology and Appl. **144** (2004), no. 1–3, 29–49.

[15] J. Jezierski and W. Marzantowicz, *Homotopy minimal periods for maps of three dimensional nilmanifolds*, Pacific J. Math. **209** (2003), 85–101.

[16] ———, *Homotopy Methods in Topological Fixed and Periodic Points Theory*, Series: Topological Fixed Point Theory and Its Applications, vol. 3, Springer, 2005, pp. XI, 319.

[17] B. Jiang, *Lectures on Nielsen Fixed Point Theory*, Contemp. Math., vol. 14, Providence, 1983.

[18] ———, *Fixed points and braids*, Invent. Math. **75** (1984), 69–74.

[19] B. Jiang and J. Llibre, *Minimal sets of periods for torus maps*, Discr. Contin. Dynam. Systems **4** (1998), 301–320.

[20] J. Llibre, *A note on the set of periods for Klein bottle maps*, Pacific J. Math. **157** (1993), 87–93.

[21] F. Wecken, *Fixpunktklassen,* I, Math. Ann. **117** (1941), 659–671; II **118** (1942), 216–234; III **118** (1942), 544–577.

[22] C. Y. You, *Fixed point classes of a fiber map*, Pacific J. Math **100** (1981).

Manuscript received February 23, 2008

Jerzy Jezierski
Institute of Applied Mathematics
University of Agriculture
Nowoursynowska 166
02-787 Warszawa, POLAND

E-mail address: jezierski@alpha.sggw.waw.pl

Edward Keppelmann
Department of Mathematics
University of Nevada Reno
Reno NV, USA

E-mail address: edkeppelma@unr.edu

Wacław Marzantowicz
Faculty of Mathematics and Computer Science
Adam Mickiewicz University of Poznań
Umultowska 87
61-614 Poznań, POLAND

E-mail address: marzan@amu.edu.pl