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A FORMULA FOR THE COINCIDENCE REIDEMEISTER TRACE OF SELFMAPS ON BOUQUETS OF CIRCLES

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Abstract. We give a formula for the coincidence Reidemeister trace of selfmaps on bouquets of circles in terms of the Fox calculus. Our formula reduces the problem of computing the coincidence Reidemeister trace to the problem of distinguishing doubly twisted conjugacy classes in free groups.

1. Introduction

Fadell and Husseini, in [3] proved the following:

Theorem 1.1 (Fadell, Husseini, 1983). *Let X be a bouquet of circles, $G = \pi_1(X)$, and let G_0 be the set of generators of G. If $f\colon X \to X$ induces the map $\varphi\colon G \to G$, then there is some lift $\widetilde{f}\colon \widetilde{X} \to \widetilde{X}$ to the universal covering space with*

$$RT(f,\widetilde{f}) = \rho\biggl(1 - \sum_{a \in G_0} \frac{\partial}{\partial a}\varphi(a)\biggr),$$

where $\rho\colon \mathbb{Z}G \to \mathbb{Z}\mathcal{R}(f)$ is the linearization of the projection into twisted conjugacy classes, and ∂ denotes the Fox derivative.

Theorem 1.1 reduces the calculation of the Reidemeister trace (and thus of the Nielsen number) in fixed point theory to the computation of twisted conjugacy classes. Our goal for this paper is to obtain a similar result in

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coincidence theory of selfmaps — a formula for the coincidence Reidemeister trace $RT(f, \widetilde{f}, g, \widetilde{g})$ in terms of Fox derivatives which reduces the computation to twisted conjugacy decisions.

The proof of Theorem 1.1 given in [3] is brief, thanks to a natural trace-like formula for the Reidemeister trace in fixed point theory. No such formula exists for the coincidence Reidemeister trace, and this will complicate our derivation. Our argument is based on first specifying a particular regular form for maps in Section 3 and for pairs of maps in Section 4. In Section 5 we give our main result.

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2. Preliminaries

Throughout the paper, let X be a bouquet of circles meeting at the base point x_0. Let $G = \pi_1(X)$, a free group, and let G_0 be the set of generators of G. Let $\widetilde{X}$ be the universal covering space of X with projection $p_X: \widetilde{X} \to X$, and choose once and for all a base point $\widetilde{x}_0 \in \widetilde{X}$ with $p_X(\widetilde{x}_0) = x_0$.

Given maps $f, g: X \to X$ and their induced homomorphisms $\varphi, \psi: G \to G$, we define an equivalence relation on G as follows: two elements $\alpha, \beta \in G$ are (*doubly*) *twisted conjugate* if

$$\alpha = \varphi(\gamma)\beta\psi(\gamma)^{-1}.$$

The equivalence classes with respect to this relation are the *Reidemeister classes*, and we denote the set of such classes as $\mathcal{R}(\varphi, \psi)$. Let $\rho: G \to \mathcal{R}(\varphi, \psi)$ be the projection into Reidemeister classes.

For any pair of maps $f, g: X \to X$, denote their coincidence set by

$$\mathrm{Coin}(f, g) = \{x \in X \mid f(x) = g(x)\}.$$

The set of coincidence points are partitioned into *coincidence classes* of the form $p_X(\mathrm{Coin}(\alpha^{-1}\widetilde{f}, \widetilde{g}))$, where $\alpha \in G$ and $\widetilde{f}, \widetilde{g}: \widetilde{X} \to \widetilde{X}$ are specified lifts of f and g. Lemma 2.3 of [2] shows that $\mathrm{Coin}(\alpha^{-1}\widetilde{f}, \widetilde{g}) = \mathrm{Coin}(\beta^{-1}\widetilde{f}, \widetilde{g})$ if and only if $\rho(\alpha) = \rho(\beta)$, and that $\mathrm{Coin}(\alpha^{-1}\widetilde{f}, \widetilde{g})$ and $\mathrm{Coin}(\beta^{-1}\widetilde{f}, \widetilde{g})$ are disjoint if $\rho(\alpha) \neq \rho(\beta)$. Thus the coincidence classes are represented by Reidemeister classes in G, and so each particular coincidence point has an associated Reidemeister class. For $x \in \mathrm{Coin}(f, g)$, let $[x_{\widetilde{f},\widetilde{g}}] \in \mathcal{R}(\varphi, \psi)$ denote the Reidemeister class $\rho(\alpha)$ for which $x \in p_X(\mathrm{Coin}(\alpha^{-1}\widetilde{f}, \widetilde{g}))$.

Let $f, g: X \to X$ be mappings with isolated coincidence points, and for each coincidence point x let $U_x \subset X$ be a neighborhood of x containing no other coincidence points. Then we wish to define the coincidence Reidemeister trace as a sum of the classes $[x_{\widetilde{f},\widetilde{g}}]$ with each coefficient given by the coincidence index at x.

The coincidence index is in general not defined for nonmanifolds such as our bouquet of circles X, but we can construct an appropriate index without too much difficulty. It will be shown in the next section that we may assume that g is homotopic to the identity in a neighborhood of the base point x_0, and thus that the coincidence index of f and g on a neighborhood of x_0 is simply the fixed point index of f at x_0 (which is defined for any ANR). Near any point x other than x_0, the space X is an orientable differentiable manifold, and we define the coincidence index as usual for that setting (see [5]):

$$\mathrm{ind}(f, g, U_x) = \mathrm{sign}(\det(dg_x - df_x)),$$

where df_x and dg_x denote the derivatives of f and g at x. (That this determinant can be assumed to be nonzero is a consequence of our construction below.)

Thus we define the *coincidence Reidemeister trace* as:

$$RT(f, \widetilde{f}, g, \widetilde{g}) = \iota(f, U_{x_0})[x_{0\,\widetilde{f},\widetilde{g}}] + \sum_{x \in \mathrm{Coin}(f,g) - x_0} \mathrm{ind}(f, g, U_x)[x_{\widetilde{f},\widetilde{g}}],$$

where ι denotes the fixed point index. Indeed we can define a local Reidemeister trace: for any open set U not cointaining x_0, define

$$RT(f, \widetilde{f}, g, \widetilde{g}, U) = \sum_{x \in \mathrm{Coin}(f,g,U)} \mathrm{ind}(f, g, U_x)[x_{\widetilde{f},\widetilde{g}}],$$

where $\mathrm{Coin}(f, g, U) = \mathrm{Coin}(f, g) \cap U$. If U does contain x_0, then we add $\iota(f, U_{x_0})[x_{0\,\widetilde{f},\widetilde{g}}]$ into the sum in the obvious way. Clearly this local Reidemeister trace is equal to the nonlocal version if U is taken to be X, and has the following additivity property: if V and W are disjoint subsets of U with $\mathrm{Coin}(f, g, U) \subset V \cup W$, then

$$RT(f, \widetilde{f}, g, \widetilde{g}, U) = RT(f, \widetilde{f}, g, \widetilde{g}, V) + RT(f, \widetilde{f}, g, \widetilde{g}, W).$$

As for defining the Reidemeister trace for any pair of maps $f, g\colon X \to X$ (perhaps having nonisolated coincidence points) with lifts $\widetilde{f}, \widetilde{g}\colon \widetilde{X} \to \widetilde{X}$, we first change f and g by a homotopy to f', g' so that they have isolated coincidence points (that this is possible will be a consequence of our construction in Theorem 4.1). Now the homotopies of f, g to f', g' can be lifted to a homotopy of $\widetilde{f}, \widetilde{g}$ to some lifts $\widetilde{f}', \widetilde{g}'$. We then define

$$RT(f, \widetilde{f}, g, \widetilde{g}) = RT(f', \widetilde{f}', g', \widetilde{g}').$$

That this is well defined will be a consequence of the homotopy invariance of the coincidence index and the homotopy-relatedness of coincidence classes (see Lemma 5.1 of [6]).

We will now review the necessary properties of the Fox calculus (see e.g. [1]). If $\{x_i\}$ are the generators of a free group G, then the operators

$$\frac{\partial}{\partial x_i}: G \to \mathbb{Z}G$$

are defined by:

$$\frac{\partial}{\partial x_i} 1 = 0, \qquad \frac{\partial}{\partial x_i} x_j = \delta_{ij}, \qquad \frac{\partial}{\partial x_i}(uv) = \frac{\partial}{\partial x_i} u + u \frac{\partial}{\partial x_i} v,$$

where δ_{ij} is the Kronecker delta, and $u, v \in G$ are any words. Two important formulas can be obtained from the above:

$$\frac{\partial}{\partial x_i} x_i^{-1} = -x_i^{-1}, \qquad \frac{\partial}{\partial x_i}(h_1 \dots h_n) = \sum_{k=1}^{n} h_1 \dots h_{k-1} \frac{\partial}{\partial x_i} h_k.$$

3. A regular form for mappings

In this section we will describe a standard form for selfmaps of X. Each circle of X is represented by some generator of the fundamental group. For each generator $a \in G_0$, let $|a| \subset X$ be the circle represented by a (including the base point x_0).

For simplicity in our notation, we parameterize each circle by the interval $[0, 1]$ with endpoints identified. The circles of X will be parameterized so that the base point x_0 is identified with 0 (or equivalently, with 1). For any generator $a \in G$ and any $x \in [0, 1]$, let $[x]_a$ denote the point of $|a| \subset X$ which has coordinate x. Interval-like subsets of X will be denoted e.g. $(x_1, x_2)_a$ for the subset of points in $|a|$ parameterized by the interval (x_1, x_2).

Homotopy classes of mappings of X are characterized by their induced mappings on the fundamental groups. Consider the example where $G = \langle a, b \rangle$ and the mapping $f: X \to X$ induces $\varphi: G \to G$ with

$$\varphi(a) = ab^{-1}a^{-1}b^2.$$

Geometrically speaking, the above formula for φ indicates that f is homotopic to a map fixing the base point x_0 and mapping some interval $(0, x_1)_a$ bijectively onto $|a| - x_0$, maps some interval $(x_1, x_2)_a$ bijectively onto $|b| - x_0$ (in the "reverse direction"), and so on.

We can represent the action of this map on $|a|$ pictorially as in Figure 1, where in this case $x_i = [i/5]_a$, and the label on each interval indicates that the interval is being mapped bijectively onto the corresponding circle. Note that sliding the points x_i ($i \neq 0$) around the circle will not change the homotopy class of f, provided that no x_i ever moves across another, and the ordering of the labels is preserved.

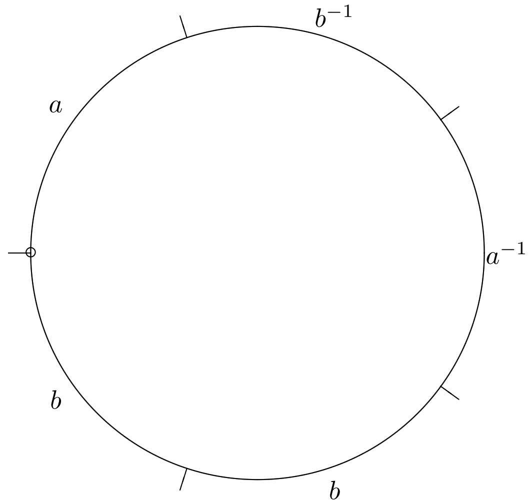

Figure 1. Diagram of the action on $|a|$ of a map with $\varphi(a) = ab^{-1}a^{-1}b^2$. The base point x_0 is circled at left

Thus any map $f\colon X \to X$ is homotopic to a map which is characterized as follows: for each generator $a \in G_0$, specify n_a intervals $I_1^a, \dots I_{n_a}^a$ together with labels $\{h_1^a, \dots, h_{n_a}^a\}$, where each of h_i^a are letters of G (a *letter* of G is an element which is either a generator or the inverse of a generator of G). These intervals will be called *intervals of* f, and the labels will be called the *labels of* f.

Since we are concerned only with homotopy classes of maps, we may assume that f maps $(x_i, x_{i+1})_a$ affine linearly onto $(0,1)_b$ for some generator $b \in G$. A map specified by intervals and labels which is linear on each interval in this way will be called *regular*.

Specifying a map f by intervals and labels gives precise information which can be used to compute the derivatives of f at any point. For any interval $I = (x_i, x_{i+1})_a$, define $w(I)$, the *width* of I, as the real number $x_{i+1} - x_i$. It is easy to verify that for any coincidence point $x \in I$, we have

$$df_x = \pm \frac{1}{w(I)},$$

where the sign above is $+$ when I is labeled by a generator, and $-$ when I is labeled by the inverse of a generator.

4. The Reidemeister trace of a pair of regular maps

Let $f, g\colon X \to X$ be a regular pair of maps, and choose lifts $\widetilde{f}$ and $\widetilde{g}$ of f and g respectively so that $\widetilde{f}(\widetilde{x}_0) = \widetilde{g}(\widetilde{x}_0)$. In this section we will describe a method for calculating $RT(f, \widetilde{f}, g, \widetilde{g})$.

For each generator a of G, write

$$f(a) = h_1^a \dots h_{n_a}^a, \qquad g(a) = l_1^a \dots l_{m_a}^a,$$

where all h_i^a and l_j^a are letters of G. Without loss of generality, we will assume the action of f and g on $|a|$ is specified by intervals and labels as pictured in Figure 2.

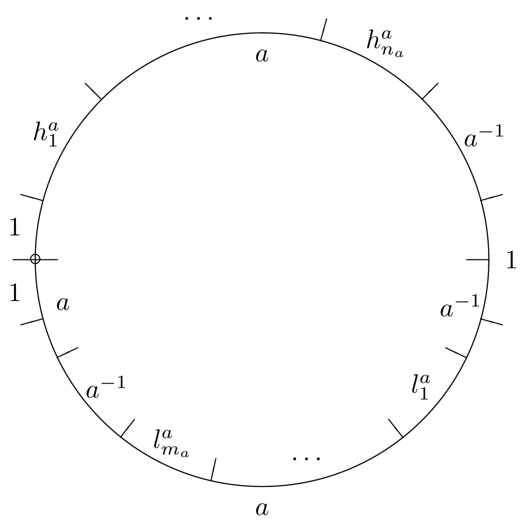

Figure 2. Diagram of the action of f and g on $|a|$. Outer labels give the action of f, while inner labels give the action of g.

Given this construction of f and g, we see that $\mathrm{Coin}(f, g)$ consists of the points x_0 and the various $[1/2]_a$, along with finitely many isolated coincidence points occuring away from the interval endpoints.

In order to compute the Reidemeister trace, we partition X into several intervals: let $I_1^a, \dots, I_{n_a}^a$ be the intervals of f in $(0, 1/2)_a$ labeled by $h_1^a, \dots, h_{n_a}^a$, and let $J_1^a, \dots, J_{m_a}^a$ be the intervals of g in $(1/2, 1)_a$ labeled by $l_1^a, \dots, l_{m_a}^a$. Let I_a be the interval of f containing $[1/2]_a$, and let I_0 be the union of all intervals of f having x_0 as an endpoint.

We define four more intervals to cover the remaining portions of X: let K_1^a be the interval of f in $(0, 1/2)_a$ which follows $I_{n_a}^a$ and is labeled by a^{-1}. Let V^a be the first interval of g in $(1/2, 1)_a$ (this interval is labeled by a^{-1}) and we set K_2^a to be the interior of $V^a - I_a$. Let K_3^a be the interval of g following $J_{m_a}^a$, this interval is labeled a^{-1} by g. Finally, let U^a be the last interval of g, and let K_4^a be the interior of $U^a - I_0$. Since our regular maps are constructed to have no coincidences at interval endpoints except for x_0 and the $[1/2]_a$, we have

$$\mathrm{Coin}(f', g') \subset I_0 \cup \bigcup_{a \in G_0} \left(I_a \cup \bigcup_{i=1}^{n_a} I_i^a \cup \bigcup_{j=1}^{m_a} J_j^a \cup \bigcup_{k=1}^{4} K_k^a \right),$$

and so we can compute $RT(f', \widetilde{f}', g', \widetilde{g}')$ as a sum of the Reidmeister traces over the various intervals in the above union.

The coincidence points at the various $[1/2]_a$ are removable, as g can be deformed away from x_0 in a neighborhood of $[1/2]_a$. Thus we have $RT(f, g, \widetilde{f}, \widetilde{g}, I_a)$ $= 0$ for each a. We can also observe that in a neighbourhood of x_0, the map g is homotopic to the identity, and thus that $\mathrm{ind}(f, g, I_0)$ is simply the fixed point index of a constant map, which is 1. Since $\widetilde{x}_0$ is a coincidence point of $\widetilde{f}$ and $\widetilde{g}$, we have $[x_0{}_{\widetilde{f},\widetilde{g}}] = \rho(1)$, and so $RT(f, \widetilde{f}, g, \widetilde{g}, I_0) = \rho(1)$.

The union above thus gives the Reidemeister trace as a sum:

$$RT(f, \widetilde{f}, g, \widetilde{g}) = \rho(1) + \sum_{a \in G_0} \left(\sum_{i=1}^{n_a} RT(I_i^a) + \sum_{j=1}^{m_a} RT(J_j^a) + \sum_{k=1}^{4} RT(K_k^a) \right), \tag{4.1}$$

where for brevity we write $RT(\,\cdot\,)$ for $RT(f, \widetilde{f}, g, \widetilde{g}, \,\cdot\,)$.

The $RT(I_i^a)$ terms can be computed as follows: given a coincidence point $x \in I_i^a$, we have $RT(I_i^a) = \mathrm{ind}(f, g, I_i^a)[x_{\widetilde{f},\widetilde{g}}]$. The index is computed as ± 1 depending on the sign of $dg_x - df_x$ (which in turn depends on whether h_i^a is a or a^{-1}). A standard covering space argument shows that $[x_{\widetilde{f},\widetilde{g}}]$ is either $h_1^a \dots h_{i-1}^a$ or $h_1^a \dots h_i^a$, again depending on whether h_i^a is a or a^{-1}. In particular, it is straightforward to show that

$$RT(I_i^a) = -\rho\left(h_1^a \dots h_{i-1}^a \frac{\partial}{\partial a} h_i^a \right).$$

Summing over i gives

$$\sum_{i=1}^{n_a} RT(I_i^a) = \sum_{i=1}^{n_a} -\rho\left(\frac{\partial}{\partial a} (h_1 \dots h_{n_a}^a) \right) = -\rho\left(\frac{\partial}{\partial a} \varphi(a) \right).$$

Completely analagous arguments will show that

$$RT(J_j^a) = \rho\left(\varphi(a) a^{-1} \left(a a^{-1} l_a^a \dots l_{j-1}^a \frac{\partial}{\partial a} l_j^a \right)^{-1} \right),$$

and thus that

$$\begin{aligned} \sum_{j=1}^{m_a} RT(J_j^a) = \sum_{j=1}^{m_a} \rho\left(\varphi(a) a^{-1} \left(a a^{-1} l_a^a \dots l_{j-1}^a \frac{\partial}{\partial a} l_j^a \right)^{-1} \right) \\ = -\rho\left(\varphi(a) a^{-1} i \left(\frac{\partial}{\partial a} \psi(a) \right) \right), \end{aligned}$$

where $i\colon \mathbb{Z}G \to \mathbb{Z}G$ is the involution defined by

$$i\Big(\sum_k c_k a_k\Big) = \sum_k c_k a_k^{-1}.$$

Similar computations of Reidemeister classes and indices will show that $RT(K_1^a)$ and $RT(K_2^a)$ will cancel in (4.1), and that

$$RT(K_3^a) = -\rho(\varphi(a)\psi(a)^{-1}).$$

To complete the computation of (4.1) we note that the coincidence point in K_4^a is removable by a homotopy, and thus we obtain

Theorem 4.1. *Let $f, g\colon X \to X$ be maps which induce the homomorphisms $\varphi, \psi\colon G \to G$. Then there are lifts $\widetilde{f}$ and $\widetilde{g}$ such that*

$$RT(f, \widetilde{f}, g, \widetilde{g}) = \rho\bigg(1 - \sum_{a\in G_0}\bigg(\frac{\partial}{\partial a}\varphi(a) + \varphi(a)\psi(a)^{-1} - \varphi(a)a^{-1}i\Big(\frac{\partial}{\partial a}\psi(a)\Big)\bigg)\bigg).$$

We end this section with an interpretation of the above formula with respect to a variation on the Fox calculus. For generators $\{x_i\}$ of G, define the operators $\frac{\Delta}{\Delta x_i}\colon G \to \mathbb{Z}G$ as follows:

$$\frac{\Delta}{\Delta x_i}1 = 0, \quad \frac{\Delta}{\Delta x_i}x_j = \delta_{ij}, \quad \frac{\Delta}{\Delta x_i}(uv) = \Big(\frac{\Delta}{\Delta x_i}u\Big)v + \frac{\Delta}{\Delta x_i}v.$$

Analogous to the properties given for the Fox calculus we can derive:

$$\frac{\Delta}{\Delta x_i}x_i^{-1} = -x_i, \qquad \frac{\Delta}{\Delta x_i}(h_n \dots h_1) = \sum_{k=1}^n \Big(\frac{\Delta}{\Delta x_i}h_k\Big)h_{k-1}\dots h_1$$

We will use one further property, relating our new operator to the ordinary Fox calculus operator, whose proof is left as an exercise:

$$\frac{\Delta}{\Delta x_j}w = x_j^{-1}i\Big(\frac{\partial}{\partial x_j}w\Big)w.$$

The identity $\rho(\varphi(w)z) = \rho(z\psi(w)$ can be used in Theorem 4.2 to obtain:

Corollary 4.2. *With notation as in Theorem 4.1, we have*

$$RT(f, \widetilde{f}, g, \widetilde{g}) = \rho\bigg(1 - \sum_{a\in G_0}\bigg(\frac{\partial}{\partial a}\varphi(a) - \frac{\Delta}{\Delta a}\psi(a) + \varphi(a)\psi(a)^{-1}\bigg)\bigg).$$

5. Some examples

First we will note that Theorem 1.1 is a simple consequence of our main result. Letting ψ be the identity map gives $\rho(\varphi(a)a^{-1}) = \rho(a^{-1}\psi(a)) = \rho(1)$, and Theorem 1.1 immediately follows from Theorem 4.1.

Our formula also gives the classical formula for the coincidence Nielsen number on the circle:

EXAMPLE 5.1. Let $G = \langle a \rangle$, and let f and g be maps which induce the homomorphisms

$$\varphi(a) = a^n, \qquad \psi(a) = a^m.$$

Without loss of generality, we will assume that $n \geq m$. Our formula then gives

$$\begin{aligned} RT(f, \widetilde{f}, g, \widetilde{g}) &= \rho\bigg(1 - \frac{\partial}{\partial a}a^n + \frac{\Delta}{\Delta a}a^m - a^n a^{-m}\bigg) \\ &= \rho(1 - (1 + \ldots + a^{n-1}) + (1 + \ldots + a^{m-1}) - a^{n-m}) \\ &= \rho(1 - a^m - \ldots - a^{n-1} - a^{n-m}). \end{aligned}$$

A simple calculation shows that $\rho(a^i) = \rho(a^j)$ if and only if $i \equiv j \mod n - m$. Thus $\rho(1) = \rho(a^{n-m})$, and all other terms in the above sum are in distinct Reidemeister classes. Thus the Nielsen number is $n - m$, as desired.

We conclude with one nontrivial computation of a coincidence Reidemeister trace of two selfmaps of the bouquet of three circles.

EXAMPLE 5.2. Let X be a space with fundamental group $G = \langle a, b, c \rangle$, and let f and g be maps which induce homomorphisms as follows:

$$\varphi : \begin{array}{ccc} a & \mapsto & acb^{-1} \\ b & \mapsto & ab \\ c & \mapsto & b \end{array} \qquad \psi : \begin{array}{ccc} a & \mapsto & a^{-1}cb^{-1} \\ b & \mapsto & c \\ c & \mapsto & b^{-1}a \end{array}$$

Our formula gives

$$\begin{aligned} RT(f, \widetilde{f}, g, \widetilde{g}) &= \rho(1 - (1 + a^{-1}cb^{-1} + a^2) - (a + abc^{-1}) - (ba^{-1}b)) \\ &= \rho(-a - a^2 - abc^{-1} - ba^{-1}b - a^{-1}cb^{-1}), \end{aligned}$$

and we must decide the twisted conjugacy of the above elements. We use the technique from [4] of abelian and nilpotent quotients.

Checking in the abelianization suffices to show that a is not twisted conjugate to any of the other terms. We also see that a^2 and $ba^{-1}b$ are twisted conjugate in the abelianization, and our computation reveals that $\rho(a^2) = \rho(ba^{-1}b)$ with conjugating element $\gamma = ac^{-1}$. Similarly, we find that $\rho(a^2) = \rho(abc^{-1})$ by the element $\gamma = ab^{-1}$.

It remains to decide whether or not a^2 and $a^{-1}cb^{-1}$ are twisted conjugate, and a check in the class 2 nilpotent quotient shows that they are not. Thus

$$RT(f, \widetilde{f}, g, \widetilde{g}) = -\rho(a) - 3\rho(a^2) - \rho(a^{-1}cb^{-1})$$

is a fully reduced expression for the Reidemeister trace, and so in particular the Neilsen number is 3.

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