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A NEW LOWER BOUND FOR THE NUMBER OF ROOTS OF MAPS BETWEEN GRAPHS

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Abstract. We shall present a new lower bound for the number of roots of maps between graphs in any given homotopy class. We also give an example showing that our new lower bound can be arbitrary larger than the number of essential root classes.

1. Introduction

Let $f: Y \to X$ be a map, and x_* a given point in X. The points in the set $f^{-1}(x_*)$ are said to be roots of f at x_*. A natural question is how to describe the set $f^{-1}(x_*)$. This topic is discussed in root theory, which is a branch of Nielsen fixed point theory (see [3] and [6]). It is well-known that a non-zero degree map between closed manifolds must be surjective. In other word, such maps always has a root at any point in the target manifold. Root theory provides much more information about the root set. There exists an estimation for the number of roots. The detail relation between root theory and degree can be found in [4].

Here, we give a brief account of root theory (see [8] for the details). Choose y_* in Y as a base point of Y and a base path w from x_* to $f(y_*)$ in X. The map f induces a homomorphism $\widetilde{f}_{\pi,w}: \pi_1(Y, y_*) \to \pi_1(X, x_*)$, which is given by

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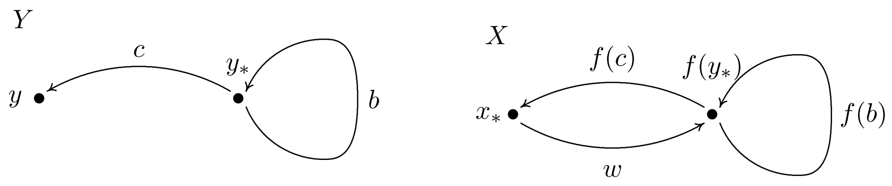
$\widetilde{f}_{\pi,w}(\langle b\rangle) = \langle wf(b)w^{-1}\rangle$, where b is a loop at y_*.



For any root y of f at x_*, pick a path c from y_* to y, we can define an element $\langle wf(c)\rangle \in \pi_1(X, x_*)$. The corresponding element in the right coset of $\mathrm{Im}\widetilde{f}_{\pi,w}(\pi_1(Y, y_*))$ in $\pi_1(X, x_*)$ is independent of the choice of the path c, because another path d from y_* to y will lead to the element

$$\langle wf(d)\rangle = \langle wf(d)f(c^{-1})w^{-1}\rangle\langle wf(c)\rangle = \widetilde{f}_{\pi,w}(\langle dc^{-1}\rangle)\langle wf(c)\rangle.$$

Such a correspondence is written as

$$\phi_w\colon f^{-1}(x_*) \to \pi_1(X, x_*)/\mathrm{Im}\widetilde{f}_{\pi,w}.$$

Two roots y_1 and y_2 are said to be in the same *root class* if $\phi_w(y_1) = \phi_w(y_2)$. This correspondence does depend on the choice of w, but the induced classification of the roots of $f^{-1}(x_*)$ is independent of the choice of w. It can be proved that the root set $f^{-1}(x_*)$ is divided into a disjoint union of finitely many root classes.

Consider a homotopy $H\colon Y \times I \to X$ from f to f', i.e. $f(y) = H(y, 0)$ and $f'(y) = H(y, 1)$ for all $y \in Y$. It gives a natural correspondence from the set of base paths of map f to the set of base paths of map f', which is given by $w \mapsto w' = w\{H(y_*, t)\}_{t=0}^1$. Notice that

$$f'(c) \simeq (\{H(y_*, t)\}_{t=0}^1)^{-1} f(c)\{H(y_*, t)\}_{t=0}^1 \ \mathrm{rel}\,\{0, 1\}$$

for any loop c at y_* in Y. We have that

$$\widetilde{f}'_{\pi,w'}(\langle c\rangle) = \langle w'f'(c)w'^{-1}\rangle = \langle wf(c)w^{-1}\rangle = \widetilde{f}_{\pi,w}(\langle c\rangle).$$

Thus, any homotopy H from f to f' induces a natural correspondence from root classes of f to root classes of f'. Two root classes under this correspondence are said to be *H-related*.

Each root class R of $f\colon Y \to X$ has an index homomorphism composed by

$$H_*(Y) \xrightarrow{j_*} H_*(Y, Y - R) \xleftarrow{e_*} H_*(N, N - R) \xrightarrow{f_*} H_*(X, X - x_*),$$

where N is a neighbourhood of R with $\mathrm{cl}(N) \cap f^{-1}(x_*) = R$, j_* comes from the homology long exact sequence of $(Y, Y - R)$, and e_* is the excision isomorphism. This homomorphism is a homotopy invariant in the sense that homotopy-related root classes have the same index homomorphism. Since any empty root set has a zero index homomorphism, a root class with non-zero index homomorphism can not be removed by any homotopy. Thus, the number of roots which have non-zero index homomorphism is a lower bound for the number of roots of maps in the homotopy class of f.

It is natural to ask whether the number of roots which have non-zero index homomorphism is the best possible lower bound for the number of roots of maps in the homotopy class of give map $f: Y \to X$. The answer to this question is known to be "Yes" if X and Y are closed manifolds with $\dim X = \dim Y \neq 2$ (see [9] or [10]). It is easy to shown that both numbers are zero if the target manifold X has boundary, and hence we have a positive answer. As in Nielsen fixed point theory, such a question is more delicate for maps between closed surfaces ([2] and [5]). The sets of root of some special maps between the interval are also interesting, see [1].

In this paper, we study the roots of maps between graphs (1-dimensional polyhedra). The key point is the following phenomenon: some root classes will always have more than one roots. The number of roots in a given root class can be derived from the information about 1-dimensional index homomorphism. Hence, we give a new lower bound for the number of root of maps in any given homotopy class. We shall show, by using some examples, that our new lower bound is much better than existed ones.

We make following conventions:

(1) By a root, we mean the root of $f: Y \to X$ at the given point x_*.

(2) Either Y or X admits a triangulation, and x_* is a vertex of X.

(3) All homology groups are simplicial ones in integer coefficients.

(4) By using the notation in graph theory, 0- and 1-dimensional simplexes are said here to be vertices and edges respectively.

(5) If $v_1, \ldots, v_n$ are the vertices such that there is a unique edge from v_{j-1} to v_j for all possible j, we write $[v_1, \ldots, v_n]$ for the union of edges $[v_1, v_2], \ldots,$ $[v_{n-1}, v_n]$, and also for the element $[v_1, v_2] + \ldots + [v_{n-1}, v_n]$ in 1-dimensional chain.

(6) For a vertex v, we write St_v for the sub-complex consisting of all edges containing v, the number of edges in St_v is said to be the degree $\deg(v)$ of v in the underlying graph.

The paper is organized as follows. In section 2, we shall define special kinds of bases for $H_1(Y)$ and $H_1(X, X - x_*)$. A norm derived from a matrix expressing index homomorphism will be obtained in Section 3. Our main results lie in Section 4, we present a new lower bound for the number of roots for maps between graphs. Some examples will be given in the last section, and we show by an example our new lower bound is much better than existing ones.

2. Simple bases and standard bases

For a map $f\colon Y \to X$ between graphs, as in [11], we focus on the 1-dimensional part of the index homomorphism, i.e. the composition:

$$H_1(Y) \xrightarrow{j_*} H_1(Y, Y-R) \xleftarrow{e_*} H_1(N, N-R) \xrightarrow{f_*} H_1(X, X-x_*),$$

which keeps all the information of the original index homomorphism because $H_*(X, X-x_*)$ is almost trivial except for dimension one. In this section, we shall define special kinds of bases for $H_1(Y)$ and $H_1(X, X-x_*)$.

Since the 2-dimensional chain group of any graph is trivial, the 1-dimensional homology group is the same as its 1-cycle, and hence any element in 1-dimensional homology group has unique expression of the formal sum of some edges. So, we can define:

DEFINITION 2.1. Let Z be a graph. An element α in $H_1(Z)$ is said to be *simple* if it is written as a formal sum of some edges in Z such that any vertex in Z belongs to the boundaries of at most two edges in α.

A basis $\mathfrak{B} = \{\beta_1, \dots, \beta_n\}$ of $H_1(Z)$ is said to be a *simple basis* if each element β_j in $\mathfrak{B}$ is simple.

Note that any element of the 1-dimensional homology group is represented by some loops consisting of edges. These loops may have intersections. By definition, we have

PROPOSITION 2.2. *A basis $\mathfrak{B} = \{\beta_1, \dots, \beta_n\}$ of $H_1(Z)$ is a simple basis if and only if each β_j is represented by a disjoint union of simple closed curves.*

Consider a one cycle α in graph. Since the boundary of α is zero, any vertex appears in α an even number of times, so the number of edges in α containing a fixed vertex is even. If β is an simple element in the 1-dimensional homology group of a graph, any vertex belongs to the boundaries of 2 or 0 edges of β.

LEMMA 2.3. *The* 1*-dimensional homology group of any graph has a simple basis.*

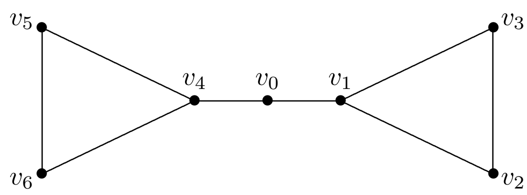
PROOF. Let $\{\beta_1, \dots, \beta_n\}$ be a basis of 1-dimensional homology of a graph. If it is not simple, then there will be a non-simple element β_j. Hence, there is a vertex v so that v belongs to the boundaries of more than two edges in β_j.

Let $[v, v_{j_1}]$ be an edge in β_j with v as its boundary. Since β_j is cycle, there must be another edge, say $[v_{j_1}, v_{j_2}]$, with v_{j_1} as its boundary. Repeat this step, we shall have a vertex $v_{j_k} = v$ such that v_{j_1}, v_{j_2}, and $v_{j_{k-1}}$ are different from v. Denote $\beta'_j = [v, v_{j_1}] + \dots + [v_{j_{k-1}}, v]$. Because $\beta_j - \beta'_j$ contains edges bounded by v, we get that $\beta_j \neq \beta'_j$.

A simple argument shows that either $\{\beta'_j\} \cup (\{\beta_1, \dots, \beta_n\} - \{\beta_j\})$ or $\{\beta_j - \beta'_j\} \cup (\{\beta_1, \dots, \beta_n\} - \{\beta_j\})$ is linearly independent. Thus, we can find a new basis of homology so that the number of edges containing v as one of their boundary points is decreased.

Repeat this procedure until it can not work further. This implies that for any vertex v and any element β_i in the basis, v belongs to the boundaries of at most two edges in β_i. At this time, we already get a simple basis. $\square$

EXAMPLE 2.4. Let Y be the graph below.



Clearly, $H_1(Y) = \mathbb{Z} \oplus \mathbb{Z}$. We denote $\alpha = [v_1, v_2] - [v_1, v_3] + [v_2, v_3]$ and $\beta = [v_4, v_5] - [v_4, v_6] + [v_5, v_6]\}$. It is obvious that the set of all bases of $H_1(X)$ is:

$$\{\{p\alpha + q\beta,\ r\alpha + s\beta\} : ps + rq = \pm 1,\ p, q, r, s \text{ are integers}\}.$$

Notice that the number of edges in $m\alpha + n\beta$ containing v_i is 0 if $i = 0$, is $2|m|$ if $1 \le i \le 3$, and is $2|n|$ if $4 \le i \le 6$. It follows that $\{p\alpha + q\beta,\ r\alpha + s\beta\}$ is a simple basis if and only if each of p, q, r and s (with $ps + rq = \pm 1$) is either ± 1 or 0. Hence, there are 20 simple bases, which are $\{\pm\alpha, \pm\beta\}$, $\{\pm\alpha \pm \beta, \pm\beta\}$ and $\{\pm\alpha \pm \beta, \pm\alpha\}$.

Let us consider the bases of $H_1(X, X - x_*)$. Note that x_* is chosen as a vertex of the simplicial complex X. By excision theorem of relative homology groups, $H_1(X, X - x_*)$ is isomorphic to $H_1(X, X - St_{x_*})$, which is a free abelian group of rank $\deg(x_*) - 1$. Thanks to this isomorphism, we can define

DEFINITION 2.5. A basis of $H_1(X, X - x_*)$ is said to be a *standard basis* if it has the form $\{[v_{j_{i_0}}, x_*, v_{j_i}] : 1 \le i \le k = \deg(x_*), i \ne i_0\}$, where $\{v_{j_1}, \ldots, v_{j_k}\}$ is the set of vertices of St_{x_*} other than x_*.

It is obvious that $H_1(X, X - x_*)$ has $\deg(x_*)$ distinct standard bases.

3. Index matrix and its norm of an isolated root set

In this section, we shall present some invariants derived from the 1-dimensional index homomorphism.

DEFINITION 3.1. Let $\overrightarrow{m} = \{m_1, \ldots, m_q\}$ be a vector. A norm of $\overrightarrow{m}$ is defined by:

$$\|\overrightarrow{m}\|^* = \max\left\{\sum_{i=1}^{q} \frac{|m_i| + m_i}{2},\ \sum_{i=1}^{q} \frac{|m_i| - m_i}{2}\right\}.$$

By definition, $\|\overrightarrow{m}\|^*$ is the maximum of the sum of positive entries in $\overrightarrow{m}$ and the absolute value of the sum of negative entries in it. It is easy to check that $\|\cdot\|^*$ is a norm on Euclidian spaces.

DEFINITION 3.2. Let $M = (m_{ij})$ be a $(p \times q)$-matrix. Its norm is defined to be:

$$\|M\|^* = \max_{1 \le i \le p} \{\|\{m_{i1}, \dots, m_{iq}\}\|^*\}.$$

From the definition, we get immediately that

PROPOSITION 3.3. *The norm $\|\cdot\|^*$ satisfies the following the properties:*

(a) $\|M\|^* \ge 0$ *for any matrix M, and $\|M\|^* = 0$ if and only if M is a zero matrix;*
(b) $\|M + M'\|^* \le \|M\|^* + \|M'\|^*$ *for any two matrices M and M' of the same size;*
(c) $\|tM\|^* = |t|\|M\|^*$ *for any real number t.*

Let R be an isolated root set of $f\colon Y \to X$. Then there is a 1-dimensional index homomorphism $(f_* e_*^{-1} j_*)_R\colon H_1(Y) \to H_1(X, X - x_*)$. Choose a basis $\mathfrak{B}_Y = \{\beta_{Y,1}, \dots, \beta_{Y,m}\}$ of $H_1(Y)$ and a basis $\mathfrak{B}_X = \{\beta_{X,1}, \dots, \beta_{X,n}\}$ of $H_1(X, X - x_*)$, this index homomorphism is represented by a matrix, which is said to be the *index matrix* with respect to $\mathfrak{B}_Y$ and $\mathfrak{B}_X$, denoted $M(R; \mathfrak{B}_Y, \mathfrak{B}_X)$, i.e.

$$(f_* e_*^{-1} j_*)_R \begin{pmatrix} \beta_{Y,1} \\ \vdots \\ \beta_{Y,m} \end{pmatrix} = M(R; \mathfrak{B}_Y, \mathfrak{B}_X) \begin{pmatrix} \beta_{X,1} \\ \vdots \\ \beta_{X,n} \end{pmatrix}.$$

DEFINITION 3.4. The norm of an isolated root set R of $f\colon Y \to X$ at x_* is defined to be:

$$\|R\|^* = \max_{\mathfrak{B}_Y, \mathfrak{B}_X} \|M(R; \mathfrak{B}_Y, \mathfrak{B}_X)\|^*,$$

where $\mathfrak{B}_Y$ ranges over all simple bases of $H_1(Y)$ and $\mathfrak{B}_X$ ranges over all standard bases of $H_1(X, X - x_*)$.

The homotopy invariance of the index homomorphism implies

PROPOSITION 3.5. *Let R and R' be root classes, respectively, of maps f and f' between graphs Y and X. If R and R' are homotopy-related, then*

$$\|R\|^* = \|R'\|^*.$$

By additivity of index homomorphism and Proposition 3.3, we have

PROPOSITION 3.6. *Let $R = R' \sqcup R''$ be a disjoint union of two isolated root sets R' and R''. Then $\|R\|^* \le \|R'\|^* + \|R''\|^*$.*

4. A new lower bound

In this section, we shall give a new lower bound for the number of roots for maps between graphs.

LENMMA 4.1. *Let $\mathfrak{B}_Y$ be a simple basis of $H_1(Y)$ and $\mathfrak{B}_X$ a standard basis of $H_1(X, X - x_*)$. Then $\|M(\{y\}; \mathfrak{B}_Y, \mathfrak{B}_X)\|^* = 0$ or 1 for any isolated root y of $f: Y \to X$ at x_*.*

PROOF. Pick a small contractible neighbourhood N_y of y in Y such that either N_y is contained in the edge containing y if y is not a vertex, or N_y is contained in St_y if y is a vertex of Y. Then $N_y - \{y\}$ is a disjoint union of line segments $N_y^1, \dots, N_y^k$ whose closures have endpoint y. Since N_y can be chosen arbitrary small, we may assume that $f(N_y) \subset St_{x_*}$. Since the index homomorphism is a homotopy invariant, we may assume that f is a piece-wise linear map. Thus, for each N_y^j, $f(N_y^j)$ lies in a unique edge of X starting at x_*.

Consider an arbitrary element γ in the simple basis $\mathfrak{B}_Y$ of $H_1(Y)$, we know that γ can be represented by a formal sum of some distinct edges. Then, under $j_*: H_1(Y) \to H_1(Y, Y - y)$ and the excision $e_*^{-1}: H_1(Y, Y - y) \to H_1(N_y, N_y - y)$, we have that $e_*^{-1} \circ j_*(\gamma) = \gamma \cap N_y$. Here, $\gamma \cap N_y$ is the sub path of γ in N_y.

Let $\{m_1, \dots, m_q\}$ be the row in $M(\{y\}; \mathfrak{B}_Y, \mathfrak{B}_X)$ corresponding to γ, i.e. $(f_* e_*^{-1} j_*)_{\{y\}}(\gamma) = \sum_{i=1}^q m_i \beta_i$, where $\mathfrak{B}_X = \{\beta_1, \dots, \beta_q\}$. It is sufficient to show that the norm $\|\{m_1, \dots, m_p\}\|^* \leq 1$.

(1) $\gamma \cap N_y = \emptyset$. We have that $\|\{m_1, \dots, m_q\}\|^* = 0$ because $m_1 = m_2 = \cdots = m_q = 0$.

(2) $\gamma \cap N_y \neq \emptyset$. Since each edge of Y appears at most once in γ, $\gamma \cap N_y$ contains exactly two components of $N_y - y$, say $N_y^{i^+}$ and $N_y^{i^-}$. We have that $\gamma \cap N_y = N_y^{i^+} \sqcup \{y\} \sqcup N_y^{i^-}$. There are two edges $[x_*, v_{i'}]$ and $[x_*, v_{i''}]$ such that $f(N_y^{i^+}) \subset [x_*, v_{i'}]$ and $f(N_y^{i^-}) \subset [x_*, v_{i''}]$. Thus, $f_* \circ e_*^{-1} \circ j_*(\gamma) = \pm[v_{i'}, x_*, v_{i''}]$.

By the definition of the standard basis, we can assume that

$$\mathfrak{B}_X = \{[v_{i_0}, x_*, v_{i_1}], \dots, [v_{i_0}, x_*, v_{i_p}]\}.$$

We have

$$f_* \circ e_*^{-1} \circ j_*(\gamma) = \begin{cases} \pm[v_{i_0}, x_*, v_{i''}] \mp [v_{i_0}, x_*, v_{i'}] & \text{if } i' \neq i_0, i'' \neq i_0, \\ \pm[v_{i_0}, x_*, v_{i''}] & \text{if } i' = i_0, i'' \neq i_0, \\ \mp[v_{i_0}, x_*, v_{i'}] & \text{if } i' \neq i_0, i'' = i_0, \\ 0 & \text{if } i' = i'' = i_0. \end{cases}$$

Thus, the vector $\{m_1, \dots, m_p\}$ has at most one entry which is 1 and has at most one entry which is -1. The other entries are all zero. So, we have that $\|\{m_1, \dots, m_p\}\|^* = 1$ if $i' \neq i''$; $\|\{m_1, \dots, m_p\}\|^* = 0$ if $i' = i''$. $\square$

LEMMA 4.2. *Let $\mathfrak{B}_Y$ be a simple basis of $H_1(Y)$ and $\mathfrak{B}_X$ a standard basis of $H_1(X, X - x_*)$. If R is an isolated root set of $f: Y \to X$ at x_*, then R contains at least $\|M(R; \mathfrak{B}_Y, \mathfrak{B}_X)\|^*$ roots.*

PROOF. If R contains infinitely many points, the conclusion is true automatically. If R contains s roots, say $y_1, \dots, y_s$, by Proposition 3.6 and the lemma above, we that

$$\|M(R; \mathfrak{B}_Y, \mathfrak{B}_X)\|^* \leq \sum_{j=1}^{s} \|\{y_j\}\|^* \leq s. \qquad \square$$

By the homotopy invariance of the norm of root classes, we get

PROPOSITION 4.3 (Main theorem). *Let $f: Y \to X$ be a map between graphs. Then any map homotopic to f has at least $\sum_R \|R\|^*$ roots at x_*, where R ranges over all root classes of f at x_*.*

With the same reason, we can prove

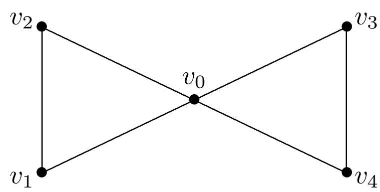
PROPOSITION 4.4. *Let $\mathfrak{B}_Y$ be a simple basis of $H_1(Y)$ and $\mathfrak{B}_X$ a standard basis of $H_1(X, X - x_*)$. Then any map homotopic to f has at least $\sum_R \|M(R; \mathfrak{B}_Y, \mathfrak{B}_X)\|^*$ roots at x_*, where R ranges over all root classes of f at x_*.*

5. Examples

In this section, we shall present some examples to show how to compute the norm $\|\cdot\|^*$ and our lower bound for a map between graphs.

The following example comes from [7, §4].

EXAMPLE 5.1. Let X be a graph shown as below.



A self map $f: X \to X$ is defined by

$$\begin{aligned} f([v_0, v_1]) &= [v_0, v_2, v_1], & f([v_0, v_2, v_1]) &= [v_0, v_1], \\ f([v_0, v_3]) &= [v_0, v_2, v_1, v_0], & f([v_3, v_4]) &= [v_0, v_3, v_4, v_0], \\ f([v_0, v_4]) &= [v_0, v_4, v_3, v_0]. \end{aligned}$$

The base point x_* is chosen as v_0.

Note that $\pi_1(X, x_*)$ is a free group of rank 2 with generators

$$\alpha = \langle [v_0, v_1, v_2, v_0] \rangle \quad \text{and} \quad \beta = \langle [v_0, v_3, v_4, v_0] \rangle.$$

By a suitable homotopy, we can assume that $f^{-1}(x_*) = \{v_0, v_3, v_4\}$. Pick the constant path w at x_* as base path from x_* to $f(x_*)$. We have that $\widetilde{f}_{\pi,w} = f_\pi\colon \pi_1(X, x_*) \to \pi_1(X, x_*)$. Then the elements in the right coset of $\operatorname{Im} \widetilde{f}_{\pi,w}(\pi_1(Y, y_*))$ in $\pi_1(X, x_*)$ corresponding to the three roots are:

$$\begin{aligned}
\phi_w(v_0) &= \langle f(w) \rangle = 1, \\
\phi_w(v_3) &= \langle f([v_0, v_3]) \rangle = [v_0, v_2, v_1, v_0] = \alpha^{-1}, \\
\phi_w(v_4) &= \langle f([v_0, v_4]) \rangle = [v_0, v_4, v_3, v_0] = \beta^{-1}.
\end{aligned}$$

Notice that

$$\begin{aligned}
f_\pi(\alpha) &= \langle f([v_0, v_1, v_2, v_0]) \rangle = \langle [v_0, v_2, v_1, v_0] \rangle = \alpha^{-1}, \\
f_\pi(\beta) &= \langle f([v_0, v_3, v_4, v_0]) \rangle = \langle [v_0, v_2, v_1, v_0, v_3, v_4, v_0, v_3, v_4, v_0] \rangle = \alpha^{-1}\beta^2,
\end{aligned}$$

we have that $\alpha^{-1} = f_\pi(\alpha) \in f_\pi(\pi_1(X, x_*))$, and that $\beta^{-1} \notin f_\pi(\pi_1(X, x_*))$. Thus, α^{-1} belongs to the right coset containing 1, but β^{-1} does not belong to it. This implies that v_0 and v_3 are in the same root class and v_4 lies in another root class.

Project down α and β into $H_1(X)$. We still write them as α and β. They are generators of the free abelian group $H_1(X)$. Any basis of $H_1(X)$ has the form $\{p\alpha + q\beta,\ r\alpha + s\beta\}$ where p, q, r, s are integers with $ps + rq = \pm 1$. Note that the number of edges in $p\alpha + q\beta$ and $r\alpha + s\beta$ containing the vertex v_0 are $|p| + |q|$ and $|r| + |s|$ respectively. We conclude that there are only 4 simple bases of $H_1(X)$, which are: $\{\alpha, \beta\}$, $\{\alpha, -\beta\}$, $\{-\alpha, \beta\}$ and $\{-\alpha, -\beta\}$.

The standard bases of $H_1(X, X - x_*)$ are

$$\begin{aligned}
\mathfrak{B}_1 &= \{[v_1, v_0, v_2], [v_1, v_0, v_3], [v_1, v_0, v_4]\}, \\
\mathfrak{B}_2 &= \{[v_2, v_0, v_1], [v_2, v_0, v_3], [v_2, v_0, v_4]\}, \\
\mathfrak{B}_3 &= \{[v_3, v_0, v_1], [v_3, v_0, v_2], [v_3, v_0, v_4]\}, \\
\mathfrak{B}_4 &= \{[v_4, v_0, v_1], [v_4, v_0, v_2], [v_4, v_0, v_3]\}.
\end{aligned}$$

Let us consider 1-dimensional index homomorphisms of the two root classes $\{v_0, v_3\}$ and $\{v_4\}$.

For the root v_0, we choose a small regular neighbourhood U_0 such that $U_0 \cup f(U_0) \subset St_{x_*}$. We then have

$$\begin{aligned}
\alpha = [v_0, v_1, v_2, v_0] &\xmapsto{j_*} [v_0, v_1, v_2, v_0] \xmapsto{e_*^{-1}} [v_0, v_1, v_2, v_0] \cap U_0 = [v_2, v_0, v_1] \cap U_0 \\
&\xmapsto{f_*} [v_1, v_0, v_2] \in H_1(X, X - x_*), \\
\beta = [v_0, v_3, v_4, v_0] &\xmapsto{j_*} [v_0, v_3, v_4, v_0] \xmapsto{e_*^{-1}} [v_0, v_3, v_4, v_0] \cap U_0 = [v_4, v_0, v_3] \cap U_0 \\
&\xmapsto{f_*} [v_4, v_0, v_2] \in H_1(X, X - x_*).
\end{aligned}$$

For the root v_3, we choose a small regular neighbourhood U_3 such that $U_3 \subset [v_0, v_3, v_4]$ and $f(U_3) \subset St_{x_*}$. Then

$$\begin{aligned}
\alpha = [v_0, v_1, v_2, v_0] &\xmapsto{j_*} [v_0, v_1, v_2, v_0] \xmapsto{e_*^{-1}} [v_0, v_1, v_2, v_0] \cap U_3 = 0, \\
&\xmapsto{f_*} 0 \in H_1(X, X - x_*), \\
\beta = [v_0, v_3, v_4, v_0] &\xmapsto{j_*} [v_0, v_3, v_4, v_0] \xmapsto{e_*^{-1}} [v_0, v_3, v_4, v_0] \cap U_3 = [v_0, v_3, v_4] \cap U_0 \\
&\xmapsto{f_*} [v_1, v_0, v_3] \in H_1(X, X - x_*).
\end{aligned}$$

Combining the two index homomorphisms, we get that

$$\begin{aligned}
(f_* e_*^{-1} j_*)_{\{v_0, v_3\}}(\alpha) &= [v_1, v_0, v_2], \\
(f_* e_*^{-1} j_*)_{\{v_0, v_3\}}(\beta) &= [v_4, v_0, v_2] + [v_1, v_0, v_3].
\end{aligned}$$

Under simple bases $\{\varepsilon_\alpha \alpha, \varepsilon_\beta \beta\}$, where $\varepsilon_\alpha, \varepsilon_\beta = \pm 1$, of $H_1(X)$ and standard bases $\mathfrak{B}_i$, we can compute the index matrices

$$\begin{aligned}
M(\{v_0, v_3\}, \{\varepsilon_\alpha \alpha, \varepsilon_\beta \beta\}, \mathfrak{B}_1) &= \begin{pmatrix} \varepsilon_\alpha & 0 & 0 \\ \varepsilon_\beta & \varepsilon_\beta & -\varepsilon_\beta \end{pmatrix}, \\
M(\{v_0, v_3\}, \{\varepsilon_\alpha \alpha, \varepsilon_\beta \beta\}, \mathfrak{B}_2) &= \begin{pmatrix} -\varepsilon_\alpha & 0 & 0 \\ -\varepsilon_\beta & \varepsilon_\beta & -\varepsilon_\beta \end{pmatrix}, \\
M(\{v_0, v_3\}, \{\varepsilon_\alpha \alpha, \varepsilon_\beta \beta\}, \mathfrak{B}_3) &= \begin{pmatrix} -\varepsilon_\alpha & \varepsilon_\alpha & 0 \\ -\varepsilon_\beta & \varepsilon_\beta & -\varepsilon_\beta \end{pmatrix}, \\
M(\{v_0, v_3\}, \{\varepsilon_\alpha \alpha, \varepsilon_\beta \beta\}, \mathfrak{B}_4) &= \begin{pmatrix} -\varepsilon_\alpha & \varepsilon_\alpha & 0 \\ -\varepsilon_\beta & \varepsilon_\beta & \varepsilon_\beta \end{pmatrix}.
\end{aligned}$$

It follows that $\|\{v_0, v_3\}\|^* = 2$.

For the root v_4, we choose a small regular neighbourhood U_4 such that $U_4 \subset [v_3, v_4, v_0]$ and $f(U_4) \subset St_{x_*}$. Then, under the index homomorphism $(f_* e_*^{-1} j_*)_{\{v_4\}}$,

$$\begin{aligned}
\alpha = [v_0, v_1, v_2, v_0] &\xmapsto{j_*} [v_0, v_1, v_2, v_0] \xmapsto{e_*^{-1}} [v_0, v_1, v_2, v_0] \cap U_4 = 0 \\
&\xmapsto{f_*} 0 \in H_1(X, X - x_*), \\
\beta = [v_0, v_3, v_4, v_0] &\xmapsto{j_*} [v_0, v_3, v_4, v_0] \xmapsto{e_*^{-1}} [v_0, v_3, v_4, v_0] \cap U_4 = [v_3, v_4, v_0] \cap U_0 \\
&\xmapsto{f_*} [v_4, v_0, v_3] \in H_1(X, X - x_*).
\end{aligned}$$

So, with respect to the simple bases $\{\varepsilon_\alpha \alpha, \varepsilon_\beta \beta\}$, where $\varepsilon_\alpha, \varepsilon_\beta = \pm 1$, of $H_1(X)$ and standard bases $\mathfrak{B}_i$, we can compute the index matrices

$$\begin{aligned}
M(\{v_4\}, \{\varepsilon_\alpha \alpha, \varepsilon_\beta \beta\}, \mathfrak{B}_1) &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & \varepsilon_\beta & -\varepsilon_\beta \end{pmatrix}, \\
M(\{v_4\}, \{\varepsilon_\alpha \alpha, \varepsilon_\beta \beta\}, \mathfrak{B}_2) &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & \varepsilon_\beta & -\varepsilon_\beta \end{pmatrix},
\end{aligned}$$

$$M(\{v_4\},\{\varepsilon_\alpha\alpha,\varepsilon_\beta\beta\},\mathfrak{B}_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\varepsilon_\beta \end{pmatrix},$$
$$M(\{v_4\},\{\varepsilon_\alpha\alpha,\varepsilon_\beta\beta\},\mathfrak{B}_4) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \varepsilon_\beta \end{pmatrix}.$$

It follows that $\|\{v_4\}\|^* = 1$.

From our main theorem (Theorem 4.3), any map homotopic to f has at least $\|\{v_0,v_3\}\|^* + \|\{v_4\}\|^* = 3$ roots at x_*.

Compare with the graph Y in Example 2.4 with the graph X in this example, we know that they have the same homotopy type. But, the simple bases are different. So, the norm of roots is not a homotopy type invariant. It follows that our new lower bound is not a homotopy type invariant even in the category of maps between graphs.

Example 5.2. Let Y be the graph as in Example 2.4, and let X be a complete graph with 3 vertices, i.e. a triangle with vertices u_0, u_1 and u_2. A map $f\colon Y \to X$ is defined by

$$f([v_4,v_0,v_1]) = u_0, \qquad f([v_1,v_2]) = f([v_4,v_5]) = [u_0,u_1],$$
$$f([v_2,v_3]) = f([v_5,v_6]) = [u_1,u_2], \qquad f([v_1,v_3]) = f([v_4,v_6]) = [u_0,u_2].$$

The point x_* is chosen as u_1.

Using the notations in Example 2.4, $H_1(Y)$ is a free abelian group generated by $\alpha = [v_1,v_2,v_3,v_1]$ and $\beta = [v_4,v_5,v_6,v_4]$. Note that $f^{-1}(x_*) = \{v_2,v_5\}$. Pick a neighbourhood $St_{v_2} = [v_1,v_2,v_3]$ of v_2, the behavior of the 1-dimensional index homomorphism $(f_*e_*^{-1}j_*)_{\{v_2\}}\colon H_1(Y) \to H_1(X, X - x_*)$ is given by

$$\alpha \xmapsto{j_*} [v_1,v_2,v_3,v_1] \xmapsto{e_*^{-1}} [v_1,v_2,v_3,v_1] \cap St_{v_2} = [v_1,v_2,v_3] \xmapsto{f_*} [u_0,u_1,u_2],$$
$$\beta \xmapsto{j_*} [v_4,v_5,v_6,v_4] \xmapsto{e_*^{-1}} [v_4,v_5,v_6,v_4] \cap St_{v_2} = 0 \xmapsto{f_*} 0.$$

Similarly, pick a neighbourhood $St_{v_5} = [v_4,v_5,v_6]$ of v_6, the behavior of the 1-dimensional index homomorphism $(f_*e_*^{-1}j_*)_{\{v_5\}}\colon H_1(Y) \to H_1(X, X - x_*)$ is given by

$$\alpha \xmapsto{j_*} [v_1,v_2,v_3,v_1] \xmapsto{e_*^{-1}} [v_1,v_2,v_3,v_1] \cap St_{v_5} = 0 \xmapsto{f_*} 0,$$
$$\beta \xmapsto{j_*} [v_4,v_5,v_6,v_4] \xmapsto{e_*^{-1}} [v_4,v_5,v_6,v_4] \cap St_{v_5} = [v_4,v_5,v_6] \xmapsto{f_*} [u_0,u_1,u_2].$$

Take $y_* = v_0$. The base path w form x_* to $f(y_*)$ is chosen as $[u_1,u_0]$. Thus, $\widetilde{f}_{\pi,w}\colon \pi_1(Y,y_*) \to \pi_1(X,x_*)$ is surjective. It follows that f has unique root class, which is $\{v_2,v_5\}$. From the computation above and its additivity, the index homomorphism of this root class is given by $(f_*e_*^{-1}j_*)_{\{v_2,v_5\}}(\alpha) = [u_0,u_1,u_2]$ and $(f_*e_*^{-1}j_*)_{\{v_2,v_5\}}(\beta) = [u_0,u_1,u_2]$.

Let $\gamma = [w_0, w_1, w_2] \in H_1(X, X - x_*)$. Then its basis of is either $\{\gamma\}$ or $\{-\gamma\}$, both are standard. We can compute easily the index matrices with respect to the bases of $H_1(Y)$ and $H_1(X, X - x_*)$:

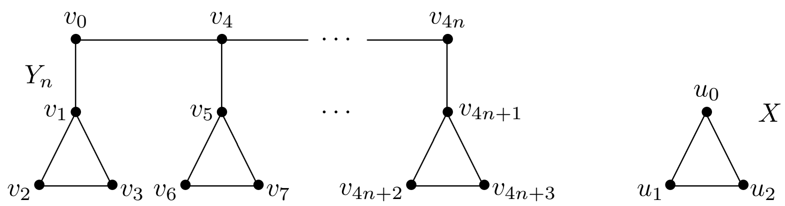
$$M(\{v_2, v_5\}, \{p\alpha + q\beta, r\alpha + s\beta\}, \{\pm\gamma\}) = \pm \begin{pmatrix} p+q \\ r+s \end{pmatrix}.$$

When the bases of $H_1(Y)$ are simple (cf. Example 2.4) and the bases of $H_1(X, X - x_*)$ are standard, any corresponding index matrix will have one of the following 40:

$$\pm \begin{pmatrix} \pm 1 \\ \pm 1 \end{pmatrix}, \ \pm \begin{pmatrix} \pm 1 \pm 1 \\ \pm 1 \end{pmatrix}, \ \pm \begin{pmatrix} \pm 1 \\ \pm 1 \pm 1 \end{pmatrix}.$$

It follows that $\|\{v_2, v_5\}\|^* = 2$. Thus, any map homotopic to f has at least two roots at $x_* = u_1$.

Example 5.3. Let Y_n be a graph with $n + 1$ loops and let X be a triangle with vertices u_0, u_1 and u_2.



Let $f\colon Y_n \to X$ be the piece-wise linear map such that for $0 \le k \le n$, the edges $[v_{4k+1}, v_{4k+2}]$, $[v_{4k+2}, v_{4k+3}]$ and $[v_{4k+1}, v_{4k+3}]$ are mapped into $[u_0, u_1]$, $[u_1, u_2]$ and $[u_0, u_2]$ respectively, and $f(Y_n - \cup_{i=0}^n [v_{4i+1}, v_{4i+2}, v_{4i+3}, v_{4i+1}]) = u_0$. The point x_* is chosen as u_1.

Clearly, $f^{-1}(x_*) = \{v_2, v_6, \dots, v_{4n+2}\}$. The homology $H_1(Y_n)$ is a free abelian group of rank $n + 1$ with generators $\alpha_i = [v_{4i+1}, v_{4i+2}, v_{4i+3}, v_{4i+1}]$, $i = 0, 1, \dots, n$. Using the same argument as in last example, we can prove that f has one root class, and that for any root v_{4k+2},

$$(f_* e_*^{-1} j_*)_{\{v_{4k+2}\}}(\alpha_i) = \begin{cases} [u_0, u_1, u_2] & \text{if } i = k, \\ 0 & \text{if } i \ne k. \end{cases}$$

Notice that $\mathfrak{B} = \{(\sum_{i=0}^n \alpha_i), \alpha_1, \dots, \alpha_n\}$ is a simple basis of $H_1(Y_n)$. The index matrix of this unique root class with respect to the simple basis $\mathfrak{B}$ and the standard basis $\{[u_0, u_1, u_2]\}$ of $H_1(X, X - x_*)$ is:

$$M(\{v_2, v_6, \dots, v_{4n+2}\}; \mathfrak{B}, \{[u_0, u_1, u_2]\}) = \begin{pmatrix} n+1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

As its norm is $n+1$, we have that $\|\{v_2, v_6, \dots, v_{4n+2}\}\|^* \geq n+1$. On the other hand, by Proposition 4.4, $\|\{v_2, v_6, \dots, v_{4n+2}\}\|^*$ is less or equal to the number of roots of f, that is $n+1$. It follows that $\|\{v_2, v_6, \dots, v_{4n+2}\}\|^* = n+1$.

Note that n can be an arbitrary positive integer, thus the difference between our new lower bound and the number of root classes can be arbitrary large. It implies that the difference between the minimal root number in a given homotopy class and the number of root classes can also be arbitrary large.

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