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GUIDING FUNCTIONS AND GLOBAL BIFURCATION OF PERIODIC SOLUTIONS OF FUNCTIONAL DIFFERENTIAL INCLUSIONS WITH INFINITE DELAY

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Abstract. In this paper, by using the topological degree theory for multivalued maps, we develop the method of guiding functions to deal with the problem of global structure of periodic solutions for functional differential inclusions with infinite delay. As example we consider the global structure of periodic solutions of feedback control systems with infinite delay.

1. Introduction

The bifurcation problem for inclusions with convex-valued multimaps was studied by J.C. Alexander and P.M. Fitzpatrick [2]. The authors of this work presented the sufficient conditions under which the set of all non-trivial solutions near the point $(0,0)$ admits a bifurcation to infinity, either bifurcation to the border of the considered domain, or bifurcation to some trivial solution of the inclusion. Some results on the bifurcation theory for inclusions and differential inclusions of various types are presented, e.g. in [7], [8], [9], [11], [18], [20], [21].

In the present paper, applying the topological degree theory for compact multivalued operators and the method of guiding functions we consider the global bifurcation problem of periodic solutions for functional differential inclusions

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with infinite delay. Let us mention that the method of guiding functions, developed by M.A. Krasnosel'skiĭ, A.I. Perov and others, is one of the most efficient tools for solving problems on periodic oscillations (see, e.g. [15]–[17]). Various modifications of this method were developed in [1], [5], [11], [19], [22].

The paper is organized in the following way. In the next section we recall some basic facts from the theory of Fredholm operators and from the theory of multivalued maps. In Section 3 the method of guiding functions is used to obtain the global structure of periodic solutions of functional differential inclusions with infinite delay. In the last section it is show how the abstract result can be apply to control systems with infinite delay.

2. Preliminaries

2.1. Multimaps. Let X and Y be Banach spaces. Denote by $P(Y)\setminus[Cv(Y), Kv(Y)]$ the collection of all nonempty (respectively: nonempty convex closed, nonempty convex compact) subsets of Y. By $B_X(0,r)$ ($\partial B_X(0,r)$) we denote a ball (a sphere) of radius r in X.

Definition 2.1 (see, e.g. [5], [11], [14]). A multimap $\mathcal{F}: X \to P(Y)$ is said to be:

(a) *upper semicontinuous* (u.s.c.), if for every open subset $V \subset Y$ the set

$$\mathcal{F}_+^{-1}(V) = \{x \in X : \mathcal{F}(x) \subset V\}$$

is open in X;

(b) *closed* if its graph $\{(x,y) : x \in X,\ y \in \mathcal{F}(x)\}$ is a closed subset of $X\times Y$;

(c) *compact*, if the set

$$\mathcal{F}(X') = \bigcup_{x\in X'} \mathcal{F}(x)$$

is relatively compact for every bounded subset $X' \subset X$.

Let $U \subset X$ be an open bounded subset and $\mathcal{F}: \overline{U} \to Kv(X)$ be an u.s.c. compact multimap. Denote by i the inclusion map and by ∂U the boundary of $\overline{U}$. If $\mathcal{F}$ has no fixed points ($x \notin \mathcal{F}(x)$) on ∂U, then the topological degree $\deg(i - \mathcal{F}, \overline{U})$ is well defined and has all usual properties (see e.g. [5], [11], [14]).

2.2. Fredholm operators.

Definition 2.2 (see e.g. [10]). A linear operator $L: \operatorname{dom} L \subseteq X \to Y$ is called Fredholm of index zero if

(a) $\operatorname{Im} L$ is closed in Y;

(b) $\operatorname{Ker} L$ and $\operatorname{Coker} L$ have the finite dimension and

$$\dim \operatorname{Ker} L = \dim \operatorname{Coker} L.$$

Let $L\colon \operatorname{dom} L \subseteq X \to Y$ be a Fredholm operator of index zero, then there exist projectors $P\colon X \to X$ and $Q\colon Y \to Y$ such that $\operatorname{Im} P = \operatorname{Ker} L$ and $\operatorname{Ker} Q = \operatorname{Im} L$. If the operator $L_P\colon \operatorname{dom} L \cap \operatorname{Ker} P \to \operatorname{Im} L$ is defined as the restriction of L on $\operatorname{dom} L \cap \operatorname{Ker} P$ then it's clear that L_P is an algeraic isomorphism and we may define $K_P\colon \operatorname{Im} L \to \operatorname{dom} L$ as $K_P = L_P^{-1}$.

Now if we let $\operatorname{Coker} L = Y/\operatorname{Im} L$ and $\Pi\colon Y \to \operatorname{Coker} L$ be the canonical surjection $\Pi(z) = z + \operatorname{Im} L$ and $\Lambda\colon \operatorname{Coker} L \to \operatorname{Ker} L$ be a one-to-one linear mapping, then equation $Lx = y$ for $y \in Y$ is equivalent to equation

$$(i - P)x = (\Lambda\Pi + K_{P,Q})y,$$

where $K_{P,Q}\colon Y \to X$ be defined as $K_{P,Q} = K_P(i - Q)$.

2.3. Phase space. We will use an axiomatical definition of the *phase space* $\mathcal{B}$, introduced by J.K. Hale and J. Kato (see [12], [13]) for treating of functional differential equations and inclusions with infinite delay. The space $\mathcal{B}$ will be considered as a linear topological space of functions mapping $(-\infty, 0]$ into $\mathbb{R}^n$ endowed with a seminorm $\|\cdot\|_{\mathcal{B}}$.

For any function $y\colon (-\infty; T] \to \mathbb{R}^n$ and for every $t \in [0, T]$, y_t represents the function from $(-\infty, 0]$ into $\mathbb{R}^n$ defined by

$$y_t(\theta) = y(t + \theta), \quad \theta \in (-\infty; 0].$$

We will assume that $\mathcal{B}$ satisfies the following axioms.

($\mathcal{B}1$) If $y\colon (-\infty; T] \to \mathbb{R}^n$ is such that $y_{|[0,T]} \in C([0, T]; \mathbb{R}^n)$ and $y_0 \in \mathcal{B}$, then we have:
- (a) $y_t \in \mathcal{B}$ for $t \in [0, T]$;
- (b) function $t \in [0, T] \mapsto y_t \in \mathcal{B}$ is continuous;
- (c) $\|y_t\|_{\mathcal{B}} \le K(t) \sup_{0 \le \tau \le t} \|y(\tau)\| + N(t)\|y_0\|_{\mathcal{B}}$ for $t \in [0, T]$, where $K(\cdot), N(\cdot)\colon [0; \infty) \to [0; \infty)$ are independent of y, $K(\cdot)$ is strictly positive and continuous, and $N(\cdot)$ is bounded.

($\mathcal{B}2$) There exists $l > 0$ such that $\|\psi(0)\|_{\mathbb{R}^n} \le l\,\|\psi\|_{\mathcal{B}}$ for all $\psi \in \mathcal{B}$.

Let us mention that under above hypotheses the space C_{00} of all continuous functions from $(-\infty, 0]$ into $\mathbb{R}^n$ with compact support is a subset of each phase space $\mathcal{B}$ ([13, Proposition 1.2.1]). We will assume, additionally, that the following hypothesis holds true.

($\mathcal{B}3$) If a uniformly bounded sequence $\{\psi_n\}_{n=1}^{+\infty} \subset C_{00}$ converges to a function ψ compactly (i.e. uniformly on each compact subset of $(-\infty, 0]$), then $\psi \in \mathcal{B}$ and $\lim\limits_{n\to+\infty} \|\psi_n - \psi\|_{\mathcal{B}} = 0$.

The hypothesis ($\mathcal{B}3$) yields that the Banach space $BC((-\infty, 0]; \mathbb{R}^n)$ of bounded continuous functions is continuously embedded into $\mathcal{B}$.

We may consider the following examples of phase spaces satisfying all above properties.

Example 2.3. For $\nu > 0$, let $\mathcal{B} = C_\nu$ be the space of functions $\psi\colon (-\infty; 0] \to \mathbb{R}^n$ such that:

(a) $\psi_{|[-r,0]} \in C([-r,0];\mathbb{R}^n)$ for each $r > 0$;
(b) the limit $\lim\limits_{\theta\to-\infty} e^{\nu\theta}\|\psi(\theta)\|$ is finite.

Then we set

$$\|\psi\|_{\mathcal{B}} = \sup_{-\infty<\theta\leq 0} e^{\nu\theta}\|\psi(\theta)\|.$$

Example 2.4 (Spaces of "fading memory"). Let $\mathcal{B} = C_\rho$ be the space of functions $\psi\colon (-\infty; 0] \to \mathbb{R}^n$ such that

(a) $\psi \in C([-r;0];\mathbb{R}^n)$ for some $r > 0$;
(b) ψ is Lebesgue measurable on $(-\infty;-r)$ and there exists a positive Lebesgue integrable function $\rho\colon (-\infty;-r) \to \mathbb{R}^+$ such that $\rho\psi$ is Lebesgue integrable on $(-\infty;-r)$; moreover, there exists a locally bounded function $P\colon (-\infty;0] \to \mathbb{R}^+$ such that, for all $\xi \leq 0$, $\rho(\xi+\theta) \leq P(\xi)\rho(\theta)$ almost every $\theta \in (-\infty;-r)$. Then,

$$\|\psi\|_{\mathcal{B}} = \sup_{-r\leq\theta\leq 0}\|\psi(\theta)\| + \int_{-\infty}^{-r} \rho(\theta)\|\psi(\theta)\| d\theta.$$

A simple example of such a space can be obtained by taking the function

$$\rho(\theta) = e^{\mu\theta}, \quad \text{for } \mu \in \mathbb{R}.$$

2.4. A global bifurcation theorem. Consider the following one-parameter family of inclusions

$$x \in \mathcal{F}(x,\mu), \tag{2.1}$$

where $\mathcal{F}\colon X\times\mathbb{R} \to Kv(Y)$ is a multimap.

Assume that:

($\mathcal{F}1$) $\mathcal{F}$ is an u.s.c. and compact multimap and $0 \in \mathcal{F}(0,\mu)$ for all $\mu \in \mathbb{R}$;
($\mathcal{F}2$) for each μ, $0 < |\mu-\mu_0| < r_0$, there is $\delta_\mu > 0$ such that $x \notin \mathcal{F}(x,\mu)$ when $0 < \|x\| \leq \delta_\mu$, where μ_0, r_0 are given numbers;

A point $(0,\mu_*)$ is said to be a bifurcation point of inclusion (2.1) if for every open subset $U \subset X\times\mathbb{R}$ with $(0,\mu_*) \in U$ there exists a point $(x,\mu) \in U$ such that $x \neq 0$ and $x \in \mathcal{F}(x,\mu)$.

From ($\mathcal{F}1$)–($\mathcal{F}2$) it follows that for each μ, $0 < |\mu-\mu_0| < r_0$ the topological degree

$$\deg(i - \mathcal{F}(\,\cdot\,,\mu), B_X(0,\delta_\mu))$$

is well defined. Then the bifurcation index of the multimap $\mathcal{F}$ at $(0,\mu_0)$ may be defined as

$$\mathrm{Bi}(\mathcal{F}(0,\mu_0)) = \lim_{\mu\to\mu_0^+} \deg(i-\mathcal{F}(\,\cdot\,,\mu), B_X(0,\delta_\mu)) - \lim_{\mu\to\mu_0^-} \deg(i-\mathcal{F}(\,\cdot\,,\mu), B_X(0,\delta_\mu)).$$

Let us denote by $\mathcal{S}$ the set of all non-trivial solutions to inclusion (2.1), i.e.

$$\mathcal{S} = \{(x,\mu)\in X\times\mathbb{R} : x\neq 0 \text{ and } x\in\mathcal{F}(x,\mu)\}.$$

The following assertion can be easily followed from the global bifurcation theorems presented in [9], [18].

Theorem 2.5. *Under conditions* $(\mathcal{F}1)$–$(\mathcal{F}2)$, *assume that* $\mathrm{Bi}(\mathcal{F}(0,\mu_0))\neq 0$. *Then there exists a connected subset* $\mathcal{R}\subset\mathcal{S}$ *such that* $(0,\mu_0)\in\overline{\mathcal{R}}$ *and one of the following occurs:*

(a) $\mathcal{R}$ *is unbounded;*
(b) $(0,\mu_*)\in\overline{\mathcal{R}}$ *for some* $\mu_*\neq\mu_0$.

3. Main result

3.1. The statement of the problem. Let us denote by I the interval $[0,T]$; by $C(I,\mathbb{R}^n)$ $(L^2(I,\mathbb{R}^n))$ we denote the spaces of all continuous (respectively, square summable) functions $u\colon I\to\mathbb{R}^n$ with usual norms:

$$\|u\|_C = \max_{t\in I}\|u(t)\|_{\mathbb{R}^n} \quad\text{and}\quad \|u\|_2 = \bigg(\int_0^T \|u(t)\|^2_{\mathbb{R}^n}\,dt\bigg)^{1/2}.$$

Consider the space of all absolutely continuous functions $u\colon I\to\mathbb{R}^n$ whose derivatives belong to $L^2(I,\mathbb{R}^n)$. It is known (see e.g. [3]) that this space can be identified with the Sobolev space $W^{1,2}(I,\mathbb{R}^n)$ with the norm

$$\|u\|_W = \|u\|_2 + \|u'\|_2,$$

and the embedding $W^{1,2}(I,\mathbb{R}^n)\hookrightarrow C(I,\mathbb{R}^n)$ is compact for every $n\geq 1$. By $W_T^{1,2}(I,\mathbb{R}^n)$ $(C_T(I,\mathbb{R}^n))$ we denote the spaces of all functions $x\in W^{1,2}(I,\mathbb{R}^n)$ (respectively, $C(I,\mathbb{R}^n)$)satisfying the boundary condition $x(0)=x(T)$. The symbols $B_{C_T}(0,r)$ $(B_{\mathbb{R}^n}(0,r))$ denote the ball of radius r in the space $C_T(I,\mathbb{R}^n)$ (respectively, $\mathbb{R}^n$). The Banach space $BC((-\infty,0];\mathbb{R}^n)$ of bounded continuous functions will be denoted by $\mathcal{B}C(\mathbb{R}^n)$.

Consider a functional differential inclusion of the following form:

$$x'(t)\in F(t,x_t,\mu) \quad \text{for a.e. } t\in[0,T], \tag{3.1}$$

where the parameter $\mu\in\mathbb{R}$ and $F\colon\mathbb{R}\times\mathcal{B}C(\mathbb{R}^n)\times\mathbb{R}\to Kv(\mathbb{R}^n)$ is a multimap.

Assuming that the multimap F satisfies the next conditions:

(F_T) multimap F is T-periodic with respect to the first argument, i.e.

$$F(t,\varphi,\mu) = F(t+T,\varphi,\mu) \quad \text{for all } (\varphi,\mu) \in \mathcal{B}C(\mathbb{R}^n) \times \mathbb{R} \text{ and a.e. } t \in \mathbb{R};$$

(F1) for every $(\varphi,\mu) \in \mathcal{B}C(\mathbb{R}^n)\times\mathbb{R}$ multifunction $F(\,\cdot\,,\varphi,\mu)\colon[0,T]\to Kv(\mathbb{R}^n)$ admits a measurable selection;

(F2) for almost every $t \in [0,T]$ multimap $F(t,\,\cdot\,,\,\cdot\,)\colon \mathcal{B}C(\mathbb{R}^n)\times\mathbb{R}\to Kv(\mathbb{R}^n)$ is u.s.c.;

(F3) for every bounded subset $\Omega \subset C_T(I,\mathbb{R}^n)\times\mathbb{R}$ there exists a positive function $\nu_\Omega \in L^2[0,T]$ such that for each $(\varphi,\mu)\in\Omega$

$$\|F(t,\widetilde{\varphi}_t,\mu)\|_{\mathbb{R}^n} = \max_{y\in F(t,\widetilde{\varphi}_t,\mu)} \|y\|_{\mathbb{R}^n} \le \nu_\Omega(t), \quad \text{for a.e. } t\in[0,T],$$

where $\widetilde{x}$ denotes the T-periodic extension of x on $(-\infty,T]$;

(F4) $0 \in F(t,0,\mu)$ for all $\mu\in\mathbb{R}$ and almost every $t\in[0,T]$.

It is well know (see e.g. [5], [11], [14]) that under conditions (F1)–(F3) the superposition multioperator

$$\mathcal{P}_F\colon C_T(I,\mathbb{R}^n)\times\mathbb{R}\to Cv(L^2(I,\mathbb{R}^n)),$$
$$\mathcal{P}_F(x,\mu) = \{f\in L^2(I;\mathbb{R}^n) : f(s)\in F(s,\widetilde{x}_s,\mu) \text{ for a.e. } t\in I\},$$

is well-defined and closed.

Let

$$\ell\colon W_T^{1,2}(I,\mathbb{R}^n)\to L^2(I,\mathbb{R}^n), \qquad \ell(x) = x'.$$

Then we will treat the global bifurcation problem of T-periodic solutions of inclusion (3.1) as the global bifurcation problem of solutions of the following operator inclusion

$$\ell(x)\in\mathcal{P}_F(x,\mu). \tag{3.2}$$

From (F4) it follows that problem (3.2) has trivial solutions $(0,\mu)$ for all $\mu\in\mathbb{R}$. Let us denote by $\mathcal{S}$ the set of all nontrivial $T-$periodic solutions of (3.2).

A continuous differentiable function $V\colon\mathbb{R}^n\to\mathbb{R}$ is said to be a non-degenerate potential if there exists $r_0>0$ such that

$$\nabla V(x) = \left(\frac{\partial V(x)}{\partial x_1},\dots,\frac{\partial V(x)}{\partial x_n}\right) \neq 0$$

for all $x=(x_1,\dots,x_n)\in\mathbb{R}^n : 0<\|x\|_{\mathbb{R}^n}\le r_0$.

It is clear that the topological degree $\deg(\nabla V, B_{\mathbb{R}^n}(0,r'))$, where $0<r'\le r_0$, is defined and does not depend on r'. This degree is called index of the non-degenerate potential V and denoted by $\operatorname{ind} V$.

Definition 3.1. For a given $\mu \in \mathbb{R}$, a continuous differentiable function $V_\mu\colon \mathbb{R}^n \to \mathbb{R}$ is said to be an integral guiding function to problem (3.1) if there exists $\delta_\mu > 0$ such that for every $x \in W_T^{1,2}(I,\mathbb{R}^n)$, from $0 < \|x\|_2 < \delta_\mu$ it follows that

$$\int_0^T \langle \nabla V_\mu(x(s)), f(s)\rangle \, ds > 0$$

for all $f \in \mathcal{P}_F(x,\mu)$, where $\langle \,\cdot\,,\,\cdot\,\rangle$ denotes the inner product in $\mathbb{R}^n$.

Notice that guiding function V_μ is non-degenerate. In fact, for every $y \in \mathbb{R}^n$, $0 < \|y\|_{\mathbb{R}^n} < \delta_\mu/\sqrt{T}$, assuming that y is a constant function we obtain $0 < \|y\|_2 < \delta_\mu$. Hence

$$\int_0^T \langle \nabla V_\mu(y), f(s)\rangle \, ds = \left\langle \nabla V_\mu(y), \int_0^T f(s)\, ds \right\rangle = T\langle \nabla V_\mu(y), \Pi f\rangle > 0$$

for all $f \in \mathcal{P}_F(y,\mu)$. Therefore, $\nabla V_\mu(y) \neq 0$, and so there exists its index $\operatorname{ind} V_\mu$.

3.2. Global structure of $\mathcal{S}$.

Theorem 3.2. *Let conditions* (F1)–(F4) *and* (F_T) *hold. Assume that for each μ, $0 < |\mu - \mu_0| < \varepsilon_0$, where μ_0, ε_0 are given constants, there exists an integral guiding function V_μ to problem* (3.1) *such that*

$$\lim_{\mu \to \mu_0^+} \operatorname{ind} V_\mu - \lim_{\mu \to \mu_0^-} \operatorname{ind} V_\mu \neq 0.$$

Then there is a connected subset $\mathcal{R} \subset \mathcal{S}$ such that $(0,\mu_0) \in \overline{\mathcal{R}}$ and either $\mathcal{R}$ is unbounded or $\overline{\mathcal{R}} \ni (0,\mu_*)$ for some $\mu_* \neq \mu_0$.

Proof. At first, from the definition of operator ℓ it follows that ℓ is a linear Fredholm operator of index zero and

$$\operatorname{Ker} \ell \cong \mathbb{R}^n \cong \operatorname{Coker} \ell.$$

So the spaces $W_T^{1,2}(I,\mathbb{R}^n)$ and $L^2(I,\mathbb{R}^n)$ may be decomposed as

$$W_T^{1,2}(I,\mathbb{R}^n) = \mathcal{W}_{(0)} \oplus \mathcal{W}_{(1)} \quad \text{and} \quad L^2(I,\mathbb{R}^n) = \mathcal{L}_{(0)} \oplus \mathcal{L}_{(1)},$$

where $\mathcal{W}_{(0)} \cong \mathbb{R}^n \cong \mathcal{L}_{(0)}$, $\mathcal{W}_{(1)} = \mathcal{W}_{(0)}^{\perp}$, $\mathcal{L}_{(1)} = \mathcal{L}_{(0)}^{\perp}$. The corresponding decompositions of elements $u \in W_T^{1,2}(I,\mathbb{R}^n)$ and $f \in L^2(I,\mathbb{R}^n)$ will be denoted by $u = u_0 + u_1$ and $f = f_0 + f_1$.

Recall also that the projection $\Pi\colon L^2(I,\mathbb{R}^n) \to \mathbb{R}^n$ given as

$$\Pi f = \frac{1}{T}\int_0^T f(s)\, ds.$$

The inclusion (3.2) is equivalent to $x \in G(x,\mu)$, where

$$G\colon C_T(I,\mathbb{R}^n) \times \mathbb{R} \to Kv(C_T(I,\mathbb{R}^n)),$$
$$G(x,\mu) = Px + (\Lambda\Pi + K_{P,Q}) \circ \mathcal{P}_F(x,\mu).$$

We will show that the multimap G satisfies conditions in Theorem 2.5.

Step 1. From (F4) it follows that $0 \in G(0,\mu)$ for all $\mu \in \mathbb{R}$. We will show that G is u.s.c. and compact. Indeed, from the fact that the multioperator $\mathcal{P}_F$ is closed and the operator $\Lambda\Pi + K_{P,Q}$ is linear and continuous it follows that the multimap $(\Lambda\Pi + K_{P,Q}) \circ \mathcal{P}_F$ is closed (see e.g. Theorem 1.5.30 in [5]). Further, from (F3) it follows that for every bounded subset $U \subset C_T(I,\mathbb{R}^n) \times \mathbb{R}$ the set $(\Lambda\Pi + K_{P,Q}) \circ \mathcal{P}_F(U)$ is bounded in $W_T^{1,2}(I,\mathbb{R}^n)$, and by the Sobolev embedding theorem [6] it is relatively compact subset of $C_T(I,\mathbb{R}^n)$. Closed and compact multimap $(\Lambda\Pi + K_{P,Q}) \circ \mathcal{P}_F$ is u.s.c. (see e.g. [5], [14]) and now the assertion follows from the fact that P is continuous and has a finite-dimensional range. So condition $(\mathcal{F}1)$ holds true.

Step 2. For each μ such that $0 < |\mu - \mu_0| < \varepsilon_0$, choose an arbitrary $0 < \pi_\mu < \min\{\delta_\mu, \delta_\mu/\sqrt{T}\}$. Let us show that $x \notin G(x,\mu)$ provided $0 < \|x\|_C \le \pi_\mu$.

In contrary assume that there is $x \in C_T(I,\mathbb{R}^n)$, $0 < \|x\|_C \le \pi_\mu$, such that (x,μ) is a nontrivial solution of inclusion (3.2). Then there exists $f \in \mathcal{P}_F(x,\mu)$ such that $x'(t) = f(t)$ for a.e. $t \in [0,T]$. From $0 < \|x\|_2 \le \|x\|_C\sqrt{T} < \delta_\mu$ it follows that

$$\begin{aligned}0 < \int_0^T \langle \nabla V_\mu(x(s)), f(s)\rangle\, ds &= \int_0^T \langle \nabla V_\mu(x(s)), x'(s)\rangle\, ds \\ &= V_\mu(x(T)) - V_\mu(x(0)) = 0,\end{aligned}$$

that is a contradiction. So condition $(\mathcal{F}2)$ holds true.

Step 3. In this step we will evaluate the bifurcation index $\mathrm{Bi}(G(0,\mu_0))$. For this purpose, we fix μ, $0 < |\mu - \mu_0| < \varepsilon_0$, and choose π_μ as in Step 2. Consider the following family of inclusions

$$x \in \Sigma_\mu(x,\lambda) \tag{3.3}$$

where $\Sigma_\mu\colon C_T(I,\mathbb{R}^n) \times [0,1] \to Kv(C_T(I,\mathbb{R}^n))$ is defined by

$$\Sigma_\mu(x,\lambda) = Px + (\Lambda\Pi + K_{P,Q}) \circ \alpha(\mathcal{P}_F(x,\mu),\lambda),$$

and $\alpha\colon L^2(I,\mathbb{R}^n) \times [0,1] \to L^2(I,\mathbb{R}^n)$ is defined by

$$\alpha(f,\lambda) = f_0 + \lambda f_1, \quad f_0 \in \mathcal{L}_{(0)}, \quad f_1 \in \mathcal{L}_{(1)}, \quad f = f_0 + f_1.$$

Following Step 1 we can easily prove that the multimap Σ_μ is u.s.c. and compact.

Assume $(x^*,\lambda^*) \in \partial B_{C_T}(0,\pi_\mu) \times [0,1]$ be a solution of (3.3), then there is a function $f^* \in \mathcal{P}_F(x^*,\mu)$ such that

$$x^* = Px^* + (\Lambda\Pi + K_{P,Q}) \circ \alpha(f^*,\lambda^*)$$

or, equivalently,

$$\begin{cases} \ell x^* = \lambda^* f_1^*, \\ 0 = f_0^*, \end{cases}$$

where $f_0^* + f_1^* = f^*$, $f_0^* \in \mathcal{L}_{(0)}$ and $f_1^* \in \mathcal{L}_{(1)}$.

From the choice of π_μ it follows that $0 < \|x^*\|_2 < \delta_\mu$, and hence

$$\int_0^T \langle \nabla V_\mu(x^*(s)), f(s)\rangle\, ds > 0 \quad \text{for all } f \in \mathcal{P}_F(x^*, \mu).$$

If $\lambda^* \neq 0$ then

$$\begin{aligned}\int_0^T \langle \nabla V_\mu(x^*(s)), f^*(s)\rangle\, ds &= \int_0^T \left\langle \nabla V_\mu(x^*(s)), \frac{1}{\lambda^*} x^{*\prime}(s)\right\rangle ds \\ &= \frac{1}{\lambda^*}\big(V_\mu\big(x^*(T)\big) - V_\mu(x^*(0))\big) = 0,\end{aligned}$$

giving a contradiction.

If $\lambda^* = 0$ then $\ell x^* = 0$. Therefore $x^* \equiv a$ for some $a \in \mathbb{R}^n$, $\|a\|_{\mathbb{R}^n} = \pi_\mu$. For every $f \in \mathcal{P}_F(a, \mu)$ we have

$$\int_0^T \langle \nabla V_\mu(a), f(s)\rangle\, ds = \left\langle \nabla V_\mu(a), \int_0^T f(s)\, ds\right\rangle = T\langle \nabla V_\mu(a), \Pi f\rangle > 0. \tag{3.4}$$

Consequently, $\Pi f \neq 0$ for all $f \in \mathcal{P}_F(a, \mu)$, in particular, $\Pi f^* \neq 0$. But $\Pi f^* = \Pi f_0^* = 0$. That is a contradiction.

Thus multimap Σ_μ is a homotopy connecting the multimaps $\Sigma_\mu(\,\cdot\,, 1) = G(\,\cdot\,, \mu)$ and $\Sigma_\mu(\,\cdot\,, 0) = P + \Pi\mathcal{P}_F(\,\cdot\,, \mu)$. By virtue of the homotopy invariance of the topological degree we obtain that

$$\deg(i - G(\,\cdot\,, \mu), B_{C_T}(0, \pi_\mu)) = \deg(i - P - \Pi\mathcal{P}_F(\,\cdot\,, \mu), B_{C_T}(0, \pi_\mu)).$$

The multimap $P + \Pi\mathcal{P}_F(\,\cdot\,, \mu)$ has its range in $\mathbb{R}^n$ so

$$\begin{aligned}\deg(i - P - \Pi\mathcal{P}_F(\,\cdot\,, \mu), B_{C_T}(0, \pi_\mu)) &= \deg(i - P - \Pi\mathcal{P}_F(\cdot, \mu), \\ B_{\mathbb{R}^n}(0, \pi_\mu)) &= \deg(-\Pi\mathcal{P}_F(\cdot, \mu), B_{\mathbb{R}^n}(0, \pi_\mu)),\end{aligned}$$

where $B_{\mathbb{R}^n}(0, \pi_\mu) = B_{C_T}(0, \pi_\mu) \cap \mathbb{R}^n$.

From (3.4) it follows that the vector fields $\Pi\mathcal{P}_F(\,\cdot\,, \mu)$ and ∇V_μ are homotopic on $B_{\mathbb{R}^n}(0, \pi_\mu)$. So we obtain

$$\deg(-\Pi\mathcal{P}_F(\,\cdot\,, \mu), B_{\mathbb{R}^n}(0, \pi_\mu)) = \deg(-\nabla V_\mu, B_{\mathbb{R}^n}(0, \pi_\mu)) = (-1)^n \operatorname{ind} V_\mu.$$

Now we can evaluate the bifurcation index by following

$$\begin{aligned}&\operatorname{Bi}(G(0, \mu_0)) \\ &\quad = \lim_{\mu \to \mu_0^+} \deg(i - G(\,\cdot\,, \mu), B_{C_T}(0, \pi_\mu)) - \lim_{\mu \to \mu_0^-} \deg(i - G(\,\cdot\,, \mu), B_{C_T}(0, \pi_\mu)) \\ &\quad = \lim_{\mu \to \mu_0^+} \deg(-\Pi\mathcal{P}_F(\,\cdot\,, \mu), B_{\mathbb{R}^n}(0, \pi_\mu)) - \lim_{\mu \to \mu_0^-} \deg(-\Pi\mathcal{P}_F(\,\cdot\,, \mu), B_{\mathbb{R}^n}(0, \pi_\mu)) \\ &\quad = (-1)^n \left(\lim_{\mu \to \mu_0^+} \operatorname{ind} V_\mu - \lim_{\mu \to \mu_0^-} \operatorname{ind} V_\mu \right) \neq 0.\end{aligned}$$

To complete the proof we need only to apply Theorem 2.5. □

4. Application to a feedback control system

Consider the following control system with infinite delay

$$(4.1)\qquad \begin{cases} x'(t) = \mu a x(t) + f(x_t, u(t), \mu) & \text{for a.e. } t \in [0,T], \\ u(t) \in U(x(t)) & \text{for a.e. } t \in [0,T], \\ x(0) = x(T), \end{cases}$$

where $a > 0$, $\mu \in \mathbb{R}$, a multimap $U: \mathbb{R}^n \to Kv(\mathbb{R}^m)$ is u.s.c. and a map $f: \mathcal{B}C(\mathbb{R}^n) \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^n$ is continuous, $n, m \in \mathbb{N}$ and n is an odd number.

We shall assume the following conditions:

(f1) there exist $\beta > 1$, $\gamma \geq 1$ and $b > 0$ such that

$$\|f(\widetilde{\varphi}_t, y, \mu)\|_{\mathbb{R}^n} \leq b\,|\mu|^\beta\, \|\varphi\|_2^\gamma (1 + \|y\|_{\mathbb{R}^m})$$

for all $(\varphi, y, \mu) \in C_T(I, \mathbb{R}^n) \times \mathbb{R}^m \times \mathbb{R}$ and almost every $t \in [0,T]$;

(U1) for every $(\varphi, \mu) \in \mathcal{B}C(\mathbb{R}^n) \times \mathbb{R}$ the set $f(\varphi, U(\varphi(0)), \mu)$ is convex;

(U2) there exists $c > 0$ such that

$$\|U(y)\|_{\mathbb{R}^m} \leq c\,(1 + \|y\|_{\mathbb{R}^n}) \quad \text{for all } y \in \mathbb{R}^n.$$

Define a multimap $F: \mathcal{B}C(\mathbb{R}^n) \times \mathbb{R} \to Kv(\mathbb{R}^n)$ by

$$F(\varphi, \mu) = \mu a \varphi(0) + f(\varphi, U(\varphi(0)), \mu).$$

Then we will treat the problem of global bifurcation of T-periodic solutions of inclusion (4.1) by the problem of global bifurcation of T-periodic solutions of the following:

$$\begin{cases} x'(t) \in F(x_t, \mu) & \text{for a.e. } t \in I, \\ x(0) = x(T). \end{cases}$$

Let us denote by $\mathcal{S}$ the set of all nontrivial T-periodic solutions of (4.1).

Theorem 4.1. *Let conditions* (f1) *and* (U1)–(U2) *hold. Then there is an unbounded connected subset* $\mathcal{R} \subset \mathcal{S}$ *such that* $(0,0) \in \overline{\mathcal{R}}$.

Proof. It is easy to see that multimap F satisfies all conditions (F_T) and (F1)–(F4) in Theorem 3.2. For each $\mu \neq 0$ we will show that the function

$$V_\mu: \mathbb{R}^n \to \mathbb{R}, \qquad V_\mu(y) = \frac{1}{2}\,\mu\,\langle y, y\rangle$$

is an integral guiding function for problem (4.1).

In fact, letting $x \in W_T^{1,2}(I, \mathbb{R}^n)$ and choosing an arbitrary $g \in \mathcal{P}_F(x, \mu)$, then there exists $u \in L^2(I, \mathbb{R}^m)$ such that $u(s) \in U(x(s))$ for almost every $s \in I$ and

$$g(s) = \mu\, a x(s) + f(\widetilde{x}_s, u(s), \mu) \quad \text{for a.e. } s \in I.$$

We have that

$$\begin{aligned}
&\int_0^T \langle \nabla V_\mu(x(t)), g(t)\rangle\, dt \\
&\qquad = \int_0^T \langle \mu x(t), \mu a x(t) + f(\widetilde{x}_t, u(t), \mu)\rangle\, dt \\
&\qquad \geq a\,\mu^2\, \|x\|_2^2 - |\mu| \int_0^T \|x(t)\|_{\mathbb{R}^n} \|f(\widetilde{x}_t, u(t), \mu)\|_{\mathbb{R}^n}\, dt \\
&\qquad \geq a\,\mu^2 \|x\|_2^2 - b\,|\mu|^{1+\beta} \|x\|_2^\gamma \int_0^T \|x(t)\|_{\mathbb{R}^n} (1 + \|u(t)\|_{\mathbb{R}^m})\, dt \\
&\qquad \geq a\,\mu^2 \|x\|_2^2 - b\,|\mu|^{1+\beta} \|x\|_2^\gamma \int_0^T \|x(t)\|_{\mathbb{R}^n} (1 + c + c\|x(t)\|_{\mathbb{R}^n})\, dt \\
&\qquad \geq a\,\mu^2 \|x\|_2^2 - b\,(1+c)\sqrt{T} |\mu|^{1+\beta} \|x\|_2^{1+\gamma} - bc|\mu|^{1+\beta} \|x\|_2^{2+\gamma} \\
&\qquad = \mu^2 \|x\|_2^2 (a - b(1+c)\sqrt{T} |\mu|^{\beta-1} \|x\|_2^{\gamma-1} - bc|\mu|^{\beta-1} \|x\|_2^\gamma) > 0
\end{aligned}$$

for $\mu \neq 0$ and sufficiently small $\|x\|_2 \neq 0$. Thus, for every $\mu \neq 0$, V_μ is an integral guiding function of problem (4.1). Since

$$\lim_{\mu \to 0^+} \operatorname{ind} V_\mu - \lim_{\mu \to 0^-} \operatorname{ind} V_\mu = 1 - (-1)^n = 2$$

and the last inequality it follows that $(0,0)$ is the unique bifurcation point of problem (4.1). Now, to complete the proof we need only apply Theorem 3.2. □

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