Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder University Centre
Volume 40, 2012, 315–336

DIMENSION OF ATTRACTORS AND INVARIANT SETS IN REACTION DIFFUSION EQUATIONS

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Abstract. Under fairly general assumptions, we prove that every compact invariant set $\mathcal{I}$ of the semiflow generated by the semilinear reaction diffusion equation

$$\begin{aligned} u_t + \beta(x)u - \Delta u &= f(x,u), && (t,x) \in [0,+\infty[\, \times \Omega, \\ u &= 0, && (t,x) \in [0,+\infty \times \partial\Omega \end{aligned}$$

in $H^1_0(\Omega)$ has finite Hausdorff dimension. Here Ω is an arbitrary, possibly unbounded, domain in $\mathbb{R}^3$ and $f(x,u)$ is a nonlinearity of subcritical growth. The nonlinearity $f(x,u)$ needs not to satisfy any dissipativeness assumption and the invariant subset $\mathcal{I}$ needs not to be an attractor. If Ω is regular, $f(x,u)$ is dissipative and $\mathcal{I}$ is the global attractor, we give an explicit bound on the Hausdorff dimension of $\mathcal{I}$ in terms of the structure parameter of the equation.

1. Introduction

In this paper we consider the reaction diffusion equation

$$\text{(1.1)} \qquad \begin{aligned} u_t + \beta(x)u - \Delta u &= f(x,u), && (t,x) \in [0,+\infty[\, \times \Omega, \\ u &= 0, && (t,x) \in [0,+\infty[\, \times \partial\Omega. \end{aligned}$$

2010 *Mathematics Subject Classification.* 35L70, 35B40, 35B65.
Key words and phrases. Reaction diffusion equation, invariant set, attractor, dimension.

Here Ω is an arbitrary (possibly unbounded) open set in $\mathbb{R}^3$, $\beta(x)$ is a potential such that the operator $-\Delta + \beta(x)$ is positive, and $f(x,u)$ is a nonlinearity of subcritical growth (i.e. of polynomial growth strictly less than five).

The assumptions on $\beta(x)$ and $f(x,u)$ will be made more precise in Section 2 below. Under such assumptions, equation (1.1) generates a local semiflow π in the space $H_0^1(\Omega)$. Suppose that the semiflow π admits a compact invariant set $\mathcal{I}$ (i.e. $\pi(t,\mathcal{I}) = \mathcal{I}$ for all $t \geq 0$). We do not make any structure assumption on the nonlinearity $f(x,u)$ and therefore we do not assume that $\mathcal{I}$ is the global attractor of equation (1.1): for example, $\mathcal{I}$ can be an unstable invariant set detected by Conley index arguments (see e.g. [16]).

Our aim is to prove that $\mathcal{I}$ has finite Hausdorff dimension and to give an explicit estimate of its dimension. The first results concerning the dimension of invariant sets of dynamical systems are due to Mallet–Paret [14] and Mañé [15]. For a comprehensive study of the subject, see e.g. [6], [12], [20], [23].

When Ω is a bounded domain and $f(x,u)$ satisfies suitable dissipativeness conditions, the existence of a finite dimensional compact global attractor for (1.1) is a classical achievement (see e.g. [6], [12], [23]). When Ω is unbounded, new difficulties arise due to the lack of compactness of the Sobolev embeddings. These difficulties can be overcome in several ways: by introducing weighted spaces (see e.g. [5], [9]), by developing suitable tail-estimates (see e.g. [24], [17]), by exploiting comparison arguments (see e.g. [3]).

Concerning the finite dimensionality of the attractor, in [5], [9], [24] and other similar works the potential $\beta(x)$ is always assumed to be just a positive constant. In [4] Arrieta et al. considered for the first time the case of a sign-changing potential. In their results the invariant set $\mathcal{I}$ does not need to be an attractor; however they need to make some structure assumptions on $f(x,u)$ which essentially resemble the conditions ensuring the existence of the global attractor. Moreover, in [4] the invariant set is a-priori assumed to be bounded in the L^∞-norm. In concrete situations, such a-priori estimate can be obtained through elliptic regularity combined with some comparison argument. This in turn requires to make some regularity assumption on the boundary of Ω.

In this paper we do not make any structure assumption on the nonlinearity $f(x,u)$, neither do we assume $\partial\Omega$ to be regular. Our only assumption is that the mapping $h \mapsto (\partial_u f(x,0))_+ h$ has to be a relatively form compact perturbation of $-\Delta + \beta(x)$. This can be achieved, e.g. by assuming that $\partial_u f(x,0)$ can be estimated from above by some positive L^r function, $r > 3/2$. Under this assumption, we shall prove that $\mathcal{I}$ has finite Hausdorff dimension. Also, we give an explicit estimate of the dimension of $\mathcal{I}$, involving the number $\mathcal{N}$ of negative eigenvalues of the operator $-\Delta + \beta(x) - \partial_u f(x,0)$. When Ω has a regular boundary, we can explicitly estimate $\mathcal{N}$ by mean of Cwickel–Lieb–Rozenblum inequality (see [21]);

as a consequence, if we also assume that $f(x,u)$ is dissipative, we recover the result of Arrieta et al. [4].

The paper is organized as follows. In Section 2 we introduce notations, we state the main assumptions and we collect some preliminaries about the semiflow generated by equation (1.1). In Section 3 we prove that the semiflow generated by equation (1.1) is uniformly L^2-differentiable on any compact invariant set $\mathcal{I}$. In Section 4 we recall the definition of Hausdorff dimension and we prove that any compact invariant set $\mathcal{I}$ has finite Hausdorff dimension in $L^2(\Omega)$ as well as in $H_0^1(\Omega)$. In Section 5 we compute the number of negative eigenvalues of the operator $-\Delta+\beta(x)-\partial_u f(x,0)$ by mean of Cwickel–Lieb–Rozenblum inequality. In Section 6 we specialize our result to the case of a dissipative equation and we recover the result of Arrieta et al. [4].

The results contained in this paper continue to hold if one replaces $-\Delta$ with the general second order elliptic operator in divergence form

$$-\sum_{i,j=1}^{3}\partial_{x_i}(a_{ij}(x)\partial_{x_j}).$$

2. Notation, preliminaries and remarks

Let $\sigma\geq 1$. We denote by $L^\sigma_{\mathrm{u}}(\mathbb{R}^N)$ the set of measurable functions $\omega\colon\mathbb{R}^N\to\mathbb{R}$ such that

$$|\omega|_{L^\sigma_{\mathrm{u}}}:=\sup_{y\in\mathbb{R}^N}\bigg(\int_{B(y)}|\omega(x)|^\sigma\,dx\bigg)^{1/\sigma}<\infty,$$

where, for $y\in\mathbb{R}^N$, $B(y)$ is the open unit cube in $\mathbb{R}^N$ centered at y.

In this paper we assume throughout that $N=3$, and we fix an open (possibly unbounded) set $\Omega\subset\mathbb{R}^3$. We denote by M_B the constant of the Sobolev embedding $H^1(B)\subset L^6(B)$, where B is any open unit cube in $\mathbb{R}^3$. Moreover, for $2\leq q\leq 6$, we denote by M_q the constant of the Sobolev embedding $H^1(\mathbb{R}^3)\subset L^q(\mathbb{R}^3)$.

Proposition 2.1. *Let $\sigma>3/2$ and let $\omega\in L^\sigma_{\mathrm{u}}(\mathbb{R}^3)$. Set $\rho:=3/2\sigma$. Then, for every $\varepsilon>0$ and for every $u\in H_0^1(\Omega)$,*

$$\int_\Omega|\omega(x)||u(x)|^2\,dx\leq|\omega|_{L^\sigma_{\mathrm{u}}}(\rho\varepsilon M_B^2|u|^2_{H^1}+(1-\rho)\varepsilon^{-\rho/(1-\rho)}|u|^2_{L^2}).$$

Moreover, for every $u\in H_0^1(\Omega)$,

$$\int_\Omega|\omega(x)||u(x)|^2\,dx\leq M_B^{2\rho}|\omega|_{L^\sigma_{\mathrm{u}}}|u|^{2\rho}_{H^1}|u|^{2(1-\rho)}_{L^2}.$$

Proof. See the proof of Lemma 3.3 in [18]. □

Let $\beta \in L^\sigma_{\rm u}(\mathbb{R}^3)$, with $\sigma > 3/2$. Let us consider the following bilinear form defined on the space $H^1_0(\Omega)$:

$$a(u,v) := \int_\Omega \nabla u(x)\cdot\nabla v(x)\,dx + \int_\Omega \beta(x)u(x)v(x)\,dx, \quad u,v\in H^1_0(\Omega).$$

Our first assumption is the following:

Hypothesis 2.2. *There exists $\lambda_1 > 0$ such that*

$$\int_\Omega |\nabla u(x)|^2\,dx + \int_\Omega \beta(x)|u(x)|^2\,dx \geq \lambda_1 |u|^2_{L^2}, \quad u\in H^1_0(\Omega). \tag{2.1}$$

Remark 2.3. Conditions on $\beta(x)$ under which Hypothesis 2.2 is satisfied are expounded e.g. in [1], [2].

As a consequence of (2.1) and Proposition 2.1, we have:

Proposition 2.4. *There exist two positive constants λ_0 and Λ_0 such that*

$$\lambda_0|u|^2_{H^1} \leq \int_\Omega |\nabla u(x)|^2\,dx + \int_\Omega \beta(x)|u(x)|^2\,dx \leq \Lambda_0|u|^2_{H^1}, \quad u\in H^1_0(\Omega).$$

The constants λ_0 and Λ_0 can be computed explicitly in terms of λ_1, M_B and $|\beta|_{L^\sigma_{\rm u}}$.

Proof. Cf. Lemma 4.2 in [17]. □

It follows from Proposition 2.4 that the bilinear form $a(\,\cdot\,,\,\cdot\,)$ defines a scalar product in $H^1_0(\Omega)$, equivalent to the standard one. According to the results of Section 4 in [17], $a(\,\cdot\,,\,\cdot\,)$ induces a positive selfadjoint operator A in the space $L^2(\Omega)$. A is uniquely determined by the relation

$$\langle Au, v\rangle_{L^2} = a(u,v), \quad u\in D(A),\ v\in H^1_0(\Omega).$$

Notice that $Au = -\Delta u + \beta u$ in the sense of distributions, and $u \in D(A)$ if and only if $-\Delta u + \beta u \in L^2(\Omega)$. Set $X := L^2(\Omega)$, and let $(X^\alpha)_{\alpha\in\mathbb{R}}$ be the scale of fractional power spaces associated with A (see Section 2 in [17] for a short, self-contained, description of this scale of spaces). Here we just recall that $X^0 = L^2(\Omega)$, $X^1 = D(A)$, $X^{1/2} = H^1_0(\Omega)$ and $X^{-\alpha}$ is the dual of X^α for $\alpha \in\,]0,+\infty[$. For $\alpha\in\,]0,+\infty[$, the space X^α is a Hilbert space with respect to the scalar product

$$\langle u,v\rangle_{X^\alpha} := \langle A^\alpha u, A^\alpha v\rangle_{L^2}, \quad u,v\in X^\alpha.$$

Also, the space $X^{-\alpha}$ is a Hilbert space with respect to the scalar product $\langle\,\cdot\,,\,\cdot\,\rangle_{X^{-\alpha}}$ dual to the scalar product $\langle\,\cdot\,,\,\cdot\,\rangle_{X^\alpha}$, i.e.

$$\langle u',v'\rangle_{X^{-\alpha}} = \langle R_\alpha^{-1}u', R_\alpha^{-1}v'\rangle_{X^\alpha}, \quad u,v\in X^{-\alpha},$$

where $R_\alpha\colon X^\alpha\to X^{-\alpha}$ is the Riesz isomorphism $u\mapsto\langle\,\cdot\,,u\rangle_{X^\alpha}$. Finally, for every $\alpha\in\mathbb{R}$, A induces a selfadjoint operator $A_{(\alpha)}\colon X^{\alpha+1}\to X^\alpha$, such that $A_{(\alpha')}$ is an extension of $A_{(\alpha)}$ whenever $\alpha' \leq \alpha$, and $D(A^{\beta}_{(\alpha)}) = X^{\alpha+\beta}$ for $\beta \in [0,1]$. If $\alpha \in [0,1/2]$, $u \in X^{1-\alpha}$ and $v \in X^{1/2} \subset X^{\alpha}$, then

$$\langle v, A_{(-\alpha)}u\rangle_{(X^{\alpha},X^{-\alpha})} = \langle u, v\rangle_{X^{1/2}} = a(u,v).$$

Lemma 2.5. *Let $(X^{\alpha})_{\alpha\in\mathbb{R}}$ be as above.*

(a) *If $p \in [2,6[$, then $X^{\alpha} \subset L^p(\Omega)$ for $\alpha \in]3(p-2)/4p, 1/2]$. Accordingly, if $q \in]6/5, 2]$, then $L^q(\Omega) \subset X^{-\alpha}$ for $\alpha \in]3(2-q)/4q, 1/2]$.*
(b) *If $\sigma > 3/2$ and $\omega \in L^{\sigma}_{\mathrm{u}}(\Omega)$, then the assignment $u \mapsto \omega u$ defines a bounded linear map from $X^{1/2}$ to $X^{-\alpha}$ for $\alpha \in]3/4\sigma, 1/2]$.*

Proof. See Lemmas 5.1 and 5.2 and the proof of Proposition 5.3 in [17]. □

Our second assumption is the following:

Hypothesis 2.6.

(a) *$f:\Omega \times \mathbb{R} \to \mathbb{R}$ is such that, for every $u \in \mathbb{R}$, $f(\,\cdot\,,u)$ is measurable and, for almost every $x \in \Omega$, $f(x,\,\cdot\,)$ is of class C^2;*
(b) *$f(\,\cdot\,,0) \in L^q(\Omega)$, with $6/5 < q \leq 2$ and $\partial_u f(\,\cdot\,,0) \in L^{\sigma}_{\mathrm{u}}(\mathbb{R}^3)$, with $\sigma > 3/2$;*
(c) *there exist constants C and γ, with $C > 0$ and $2 \leq \gamma < 3$ such that $|\partial_{uu} f(x,u)| \leq C(1+|u|^{\gamma})$. Notice that, in view of Young's inequality, the requirement $\gamma \geq 2$ is not restrictive.*

We introduce the Nemitski operator $\hat{f}$ which associates with every function $u:\Omega \to \mathbb{R}$ the function $\widehat{f}(u)(x) := f(x,u(x))$.

Proposition 2.7. *Assume f satisfies Hypothesis* (2.6)*. Let α be such that*

$$\frac{1}{2} > \alpha > \max\left\{\frac{\gamma-1}{4}, \frac{3}{4}\frac{2-q}{q}, \frac{3}{4\sigma}\right\}.$$

Then the assignment $u \mapsto \mathbf{f}(u)$, where

$$\langle v, \mathbf{f}(u)\rangle_{(X^{\alpha},X^{-\alpha})} := \int_{\Omega} \hat{f}(u)(x)v(x)\,dx,$$

defines a map $\mathbf{f}: X^{1/2} \to X^{-\alpha}$ which is Lipschitzian on bounded sets.

Proof. See the proof of Proposition 5.3 in [17]. □

Setting $\mathbf{X} := X^{-\alpha}$ and $\mathbf{A} := A_{(-\alpha)}$, we have that $\mathbf{X}^{\alpha+1/2} = X^{1/2}$. We can rewrite equation (1.1) as an abstract parabolic problem in the space $\mathbf{X}$, namely

$$\dot{u} + \mathbf{A}u = \mathbf{f}(u). \tag{2.2}$$

By results in [11], equation (2.2) has a unique *mild solution* for every initial datum $u_0 \in \mathbf{X}^{\alpha+1/2} = H_0^1(\Omega)$, satisfying the *variation of constants* formula

$$u(t) = e^{-\mathbf{A}t}u_0 + \int_0^t e^{-\mathbf{A}(t-s)}\mathbf{f}(u(s))\,ds, \quad t \geq 0.$$

It follows that (2.2) generates a local semiflow π in the space $H_0^1(\Omega)$. Moreover, if $u(\,\cdot\,)\colon [0,T[\to \mathbf{X}^{\alpha+1/2}$ is a mild solution of (2.2), then $u(t)$ is differentiable into $\mathbf{X}^{\alpha+1/2} = H_0^1(\Omega)$ for $t \in \,]0,T[$, and it satisfies equation (2.2) in $\mathbf{X} = X^{-\alpha} \subset H^{-1}(\Omega)$. In particular, $u(\,\cdot\,)$ is a *weak solution* of (1.1).

Assume now that $\mathcal{I} \subset H_0^1(\Omega)$ is a compact invariant set for the semiflow π generated by (2.2). If $\mathcal{B}$ is a Banach space such that $H_0^1(\Omega) \subset \mathcal{B}$, we define

$$|\mathcal{I}|_{\mathcal{B}} := \max\{|u|_{\mathcal{B}} \mid u \in \mathcal{I}\}.$$

We end this section with a technical lemma that will be used later.

Lemma 2.8. *For every $T > 0$ there exists a constant $L(T)$ such that, whenever u_0 and $v_0 \in \mathcal{I}$, setting $u(t) := \pi(t,u_0)$ and $v(t) := \pi(t,v_0)$, $t \geq 0$, the following estimate holds:*

$$|u(t) - v(t)|_{H^1} \leq L(T)t^{-(\alpha+1/2)}|u_0 - v_0|_{L^2}, \quad t \in \,]0,T].$$

The constant $L(T)$ depends only on $|\mathcal{I}|_{H^1}$, and on the constants of Hypotheses 2.2 *and* 2.6.

Proof. We have

$$u(t) - v(t) = e^{-\mathbf{A}t}(u_0 - v_0) + \int_0^t e^{-\mathbf{A}(t-s)}(\mathbf{f}(u(s)) - \mathbf{f}(v(s)))\,ds;$$

it follows that

$$\begin{aligned} |u(t) &- v(t)|_{\mathbf{X}^{\alpha+1/2}} \\ &\leq t^{-(\alpha+1/2)}|u_0 - v_0|_{\mathbf{X}} + \int_0^t (t-s)^{-(\alpha+1/2)}|\mathbf{f}(u(s)) - \mathbf{f}(v(s))|_{\mathbf{X}}\,ds \\ &\leq t^{-(\alpha+1/2)}|u_0 - v_0|_{\mathbf{X}} + \int_0^t (t-s)^{-(\alpha+1/2)}C(|\mathcal{I}|_{H^1})|u(s) - v(s)|_{\mathbf{X}^{\alpha+1/2}}\,ds. \end{aligned}$$

By Henry's inequality [11, Theorem 7.1.1], this implies that

$$|u(t) - v(t)|_{\mathbf{X}^{\alpha+1/2}} \leq L(T)t^{-(\alpha+1/2)}|u_0 - v_0|_{\mathbf{X}}, \quad t \in]0,T],$$

and the thesis follows. □

3. Uniform differentiability

In this section we prove some technical results which will allow us to apply the methods of [23] for proving finite dimensionality of compact invariant sets. We assume throughout that $\mathcal{I} \subset H_0^1(\Omega)$ is a compact invariant set of the semiflow π generated by equation (2.2).

LEMMA 3.1. *There exists a constant K such that, whenever u_0 and $v_0 \in \mathcal{I}$, setting $u(t) := \pi(t, u_0)$ and $v(t) := \pi(t, v_0)$, $t \geq 0$, the following estimate holds:*

$$|u(t) - v(t)|_{L^2}^2 + \lambda_0 \int_0^t |u(s) - v(s)|_{H^1}^2 \, ds \leq e^{Kt} |u_0 - v_0|_{L^2}^2.$$

The constant K depends only on $|\mathcal{I}|_{H^1}$, on λ_0 and Λ_0 (see Proposition 2.4), on $|\partial_u f(\,\cdot\,, 0)|_{L^\sigma_u}$, and on the constants C and γ (see Hypothesis 2.6).

PROOF. Set $z(t) = u(t) - v(t)$. Then

$$\frac{1}{2}\frac{d}{dt}|z(t)|_{L^2}^2 + \int_\Omega |\nabla z(t)(x)|^2 \, dx + \int_\Omega \beta(x)|z(t)(x)|^2 \, dx = \int_\Omega (f(x, u(t)(x)) - f(x, v(t)(x))z(t)(x) \, dx.$$

It follows from Proposition 2.4 and Hypothesis 2.6 that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|z(t)|_{L^2}^2 + \lambda_0 |z(t)|_{H^1}^2 \leq & \int_\Omega |\partial_u f(x,0)||z(t)(x)|^2 \, dx \\ & + C' \int_\Omega (1 + |u(t)(x)|^{\gamma+1} + |v(t)(x)|^{\gamma+1})|z(t)(x)|^2 \, dx \\ \leq & \int_\Omega |\partial_u f(x,0)||z(t)(x)|^2 \, dx + C'|z(t)|_{L^2}^2 \\ & + C'(|u(t)|_{L^6}^{\gamma+1} + |v(t)|_{L^6}^{\gamma+1}|z(t)|_{L^{12/(5-\gamma)}}^2, \end{aligned}$$

where C' is a constant depending only on C and γ. Notice that $2 < 12/(5-\gamma) < 6$. Therefore, by interpolation, we get that for every $\varepsilon > 0$ there exists a constant $c_\varepsilon > 0$ such that

$$|z(t)|_{L^{12/(5-\gamma)}}^2 \leq \varepsilon |z(t)|_{H^1}^2 + c_\varepsilon |z(t)|_{L^2}^2. \tag{3.1}$$

Now (3.1) and Proposition 2.1 imply that, for every $\varepsilon > 0$, there exists a constant C'_ε, depending on C', $|\mathcal{I}|_{H^1}$ and ε, such that

$$\frac{1}{2}\frac{d}{dt}|z(t)|_{L^2}^2 + \lambda_0 |z(t)|_{H^1}^2 \leq \varepsilon |z(t)|_{H^1}^2 + C'_\varepsilon |z(t)|_{L^2}^2. \tag{3.2}$$

Now choosing $\varepsilon = \lambda_0/2$ and multiplying (3.2) by $e^{-2C'_\varepsilon t}$ we get

$$\frac{d}{dt}(e^{-2C'_\varepsilon t}|z(t)|_{L^2}^2) + \lambda_0 e^{-2C'_\varepsilon t}|z(t)|_{H^1}^2 \leq 0. \tag{3.3}$$

Integrating (3.3) we obtain the thesis. □

Let $\overline{u}(\,\cdot\,)\colon \mathbb{R} \to H^1_0(\Omega)$ be a full bounded solution of (2.2) such that $\overline{u}(t) \in \mathcal{I}$ for $t \in \mathbb{R}$. Let us consider the non autonomous linear equation

$$\begin{aligned} u_t + \beta(x)u - \Delta u &= \partial_u f(x, \overline{u}(t))u, \quad && (t,x) \in [0,+\infty[\, \times \Omega, \\ u &= 0, && (t,x) \in [0,+\infty[\, \times \partial\Omega. \end{aligned} \tag{3.4}$$

We introduce the following bilinear form defined on on the space $H^1_0(\Omega)$:

$$\begin{aligned} a(t;u,v) := \int_\Omega \nabla u(x) \cdot \nabla v(x)\,dx + \int_\Omega \beta(x)u(x)v(x)\,dx \\ - \int_\Omega \partial_u f(x, \overline{u}(t)(x))u(x)v(x)\,dx, \quad u,v \in H^1_0(\Omega). \end{aligned} \tag{3.5}$$

Proposition 3.2. *There exist constants $\kappa_i > 0$, $i = 1,\ldots,4$, such that:*

(a) $|a(t;u,v)| \le \kappa_1 |u|_{H^1}|v|_{H^1}$, $u,v \in H^1_0(\Omega)$, $t \in \mathbb{R}$;
(b) $|a(t;u,u)| \ge \kappa_2 |u|^2_{H^1} - \kappa_3 |u|^2_{L^2}$, $u \in H^1_0(\Omega)$, $t \in \mathbb{R}$;
(c) $|a(t;u,v) - a(s;u,v)| \le \kappa_4 |t-s||u|_{H^1}|v|_{H^1}$, $u,v \in H^1_0(\Omega)$, $t,s \in \mathbb{R}$.

Proof. Properties (a) and (b) follow from Hypothesis 2.6 and Proposition 2.1. In order to prove point (c), we first observe that, by Theorem 3.5.2 in [11] (and its proof), the function $\overline{u}(\,\cdot\,)$ is differentiable into $H^1_0(\Omega)$, with $|\dot{\overline{u}}(\,\cdot\,)|_{H^1} \le L$, where L is a constant depending on $|\mathcal{I}|_{H^1}$ and on the constants in Hypotheses 2.2 and 2.6. Therefore we have:

$$\begin{aligned} |a(t;u,v) - a(s;u,v)| &\le \int_\Omega |\partial_u f(x,\overline{u}(t) - \partial_u f(x,\overline{u}(s))||u(x)||v(x)|\,dx \\ &\le \int_\Omega C(1 + |\overline{u}(t)(x)|^\gamma + |\overline{u}(s)(x)|^\gamma)|\overline{u}(t)(x) - \overline{u}(s)(x)||u(x)||v(x)|\,dx \\ &\le C'(1 + |\overline{u}(t)|^\gamma_{H^1} + |\overline{u}(s)|^\gamma_{H^1})|\overline{u}(t) - \overline{u}(s)|_{H^1}|u|_{H^1}|v|_{H^1} \\ &\le C'(1 + 2|\mathcal{I}|^\gamma_{H^1})L|t-s||u|_{H^1}|v|_{H^1}, \end{aligned}$$

and the proof is complete. □

Now let $A(t)$ be the self-adjoint operator determined by the relation

$$\langle A(t)u, v\rangle_{L^2} = a(t;u,v), \quad u \in D(A(t)),\ v \in H^1_0(\Omega). \tag{3.6}$$

We can apply Theorem 3.1 in [10] and get:

Proposition 3.3. *There exists a two parameter family of bounded linear operators $U(t,s)\colon L^2(\Omega) \to L^2(\Omega)$, $t \ge s$, such that:*

(a) *$U(s,s) = I$ for all $s \in \mathbb{R}$, and $U(t,s)U(s,r) = U(t,r)$ for all $t \ge s \ge r$;*
(b) *$U(t,s)h_0 \in D(A(t))$ for all $h_0 \in L^2(\Omega)$ and $t > s$;*
(c) *for every $h_0 \in L^2(\Omega)$ and $s \in \mathbb{R}$, the map $t \mapsto U(t,s)h_0$ is differentiable into $L^2(\Omega)$ for $t > s$, and*

$$\frac{\partial}{\partial t}U(t,s)h_0 = -A(t)U(t,s)h_0.$$

In particular, $U(t,s)h_0$ is a weak solution of (3.4).

Given $\overline{u}_0 \in \mathcal{I}$, we take a full bounded solution $\overline{u}(\,\cdot\,)$ of (2.2), whose trajectory is contained in $\mathcal{I}$, and such that $\overline{u}(0) = \overline{u}_0$. Then we define

$$\mathcal{U}(\overline{u}_0; t) := U(t,0), \quad t \geq 0, \tag{3.7}$$

where $U(t,s)$ is the family of operators given by Proposition 3.3. Notice that $\mathcal{U}(\overline{u}_0; t)$ does not depend on the choice of $\overline{u}(\,\cdot\,)$, due to forward uniqueness for equation (2.2).

Proposition 3.4. *For every $t \geq 0$,*

$$\sup_{\overline{u}_0 \in \mathcal{I}} \|\mathcal{U}(\overline{u}_0; t)\|_{\mathcal{L}(L^2,L^2)} < +\infty.$$

Proof. Let $\overline{u}_0 \in \mathcal{I}$ and $h_0 \in L^2(\Omega)$. Set $h(t) := \mathcal{U}(\overline{u}_0; t)h_0$. Then, by property (c) of Proposition 3.3, for $t > 0$ we have

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}|h(t)|^2_{L^2} + \int_\Omega |\nabla h(t)(x)|^2\,dx + \int_\Omega \beta(x)|h(t)(x)|^2\,dx \\ = \int_\Omega \partial_u f(x, \overline{u}(t)(x))|h(t)(x)|^2\,dx,\end{aligned}$$

where $\overline{u}(\,\cdot\,)$ is a full bounded solution of (2.2), whose trajectory is contained in $\mathcal{I}$, and such that $\overline{u}(0) = \overline{u}_0$. It follows from Hypothesis 2.6 and Propositions 2.1 and 2.4 that for all $\varepsilon > 0$

$$\begin{aligned}&\frac{d}{dt}\frac{1}{2}|h(t)|^2_{L^2} + \lambda_0|h(t)|^2_{H^1} \\ &\leq \int_\Omega \partial_u f(x,0)|h(t)(x)|^2\,dx + \int_\Omega (\partial_u f(x,\overline{u}(t)(x)) - \partial_u f(x,0))|h(t)(x)|^2\,dx \\ &\leq \varepsilon|h(t)|^2_{H^1} + c_\varepsilon|h(t)|^2_{L^2} + \int_\Omega C(1 + |\overline{u}(t)(x)|^\gamma)|\overline{u}(t)(x)||h(t)(x)|^2\,dx \\ &\leq \varepsilon|h(t)|^2_{H^1} + c_\varepsilon|h(t)|^2_{L^2} + \int_\Omega C'(1 + |\overline{u}(t)(x)|^{\gamma+1})|h(t)(x)|^2\,dx \\ &\leq \varepsilon|h(t)|^2_{H^1} + (c_\varepsilon + C')|h(t)|^2_{L^2} + C'|\overline{u}(t)|^{\gamma+1}_{L^6}|h(t)|^2_{L^{12/(5-\gamma)}}.\end{aligned}$$

Since $2 < 12/(5-\gamma) < 6$, by interpolation we get that for every $\varepsilon > 0$ there exists a constant $c'_\varepsilon > 0$ such that

$$|h(t)|^2_{L^{12/(5-\gamma)}} \leq \varepsilon|h(t)|^2_{H^1} + c'_\varepsilon|h(t)|^2_{L^2}.$$

Therefore we have

$$\frac{d}{dt}\frac{1}{2}|h(t)|^2_{L^2} + \lambda_0|h(t)|^2_{H^1} \leq \varepsilon|h(t)|^2_{H^1} + C''(\varepsilon, |\mathcal{I}|_{H^1})|h(t)|^2_{L^2}. \tag{3.8}$$

Choosing $\varepsilon = \lambda_0$ and integrating (3.8) we obtain

$$|h(t)|^2_{L^2} \leq e^{2C''(\lambda_0,|\mathcal{I}|_{H^1})t}|h_0|^2_{L^2} \qquad \square$$

Proposition 3.5. *For every $t \geq 0$,*

$$\lim_{\varepsilon \to 0} \sup_{\substack{\overline{u}_0, \overline{v}_0 \in \mathcal{I} \\ 0 < |\overline{u}_0 - \overline{v}_0|_{L^2} < \varepsilon}} \frac{|\pi(t, \overline{v}_0) - \pi(t, \overline{u}_0) - \mathcal{U}(\overline{u}_0; t)(\overline{v}_0 - \overline{u}_0)|_{L^2}}{|\overline{v}_0 - \overline{u}_0|_{L^2}} = 0.$$

Proof. Let $\overline{u}_0, \overline{v}_0 \in \mathcal{I}$. Set $\overline{u}(t) := \pi(t, \overline{u}_0)$, $\overline{v}(t) := \pi(t, \overline{v}_0)$ and $\theta(t) := \overline{v}(t) - \overline{u}(t) - \mathcal{U}(\overline{u}_0; t)(\overline{v}_0 - \overline{u}_0)$, $t \geq 0$. A computation using property (c) of Proposition 3.3 shows that, for $t > 0$,

$$\begin{aligned}
&\frac{d}{dt}\frac{1}{2}|\theta(t)|_{L^2}^2 + \int_\Omega |\nabla\theta(t)(x)|^2\,dx + \int_\Omega \beta(x)|\theta(t)(x)|^2\,dx \\
&= \int_\Omega \partial_u f(x, \overline{u}(t)(x))|\theta(t)(x)|^2\,dx \\
&+ \int_\Omega (f(x, \overline{v}(t)(x)) - f(x, \overline{u}(t)(x)) - \partial_u f(x, \overline{u}(t)(x))(\overline{v}(t)(x) - \overline{u}(t)(x)))\theta(t)(x)\,dx.
\end{aligned}$$

Therefore, by Proposition 2.4

$$\frac{d}{dt}\frac{1}{2}|\theta(t)|_{L^2}^2 + \lambda_0|\theta(t)|_{H^1} \leq I_1(t) + I_2(t) + I_3(t),$$

where

$$\begin{aligned}
I_1(t) &:= \int_\Omega \partial_u f(x, 0)|\theta(t)(x)|^2\,dx, \\
I_2(t) &:= \int_\Omega (\partial_u f(x, \overline{u}(t)(x)) - \partial_u f(x, 0))|\theta(t)(x)|^2\,dx, \\
I_3(t) &= \int_\Omega (f(x, \overline{v}(t)) - f(x, \overline{u}(t)) - \partial_u f(x, \overline{u}(t))(\overline{v}(t) - \overline{u}(t)))\theta(t)\,dx.
\end{aligned}$$

Repeating the same computations of the proof of Proposition 3.4, for $\varepsilon > 0$ we get

$$I_1(t) + I_2(t) \leq \varepsilon|\theta(t)|_{H^1}^2 + C_1(\varepsilon, |\mathcal{I}|_{H^1})|\theta(t)|_{L^2}^2.$$

Concerning $I_3(t)$, for $\varepsilon > 0$ we have

$$\begin{aligned}
I_2(t) \leq &\int_\Omega C(1 + |\overline{u}(t)(x)|^\gamma + |\overline{v}(t)(x)|^\gamma)|\overline{v}(t)(x) - \overline{u}(t)(x)|^2\theta(t)(x)\,dx \\
\leq &\,C|\theta(t)|_{L^6}|\overline{v}(t) - \overline{u}(t)|_{L^{12/5}}^2 + C|\theta(t)|_{L^6}(|\overline{u}(t)|_{L^6}^\gamma \\
&+ |\overline{v}(t)|_{L^6}^\gamma)|\overline{v}(t) - \overline{u}(t)|_{L^{12/(5-\gamma)}}^2 \\
\leq &\,\varepsilon|\theta(t)|_{H^1}^2 + C_2(\varepsilon, |\mathcal{I}|_{H^1})(|\overline{v}(t) - \overline{u}(t)|_{L^{12/5}}^4 + |\overline{v}(t) - \overline{u}(t)|_{L^{12/(5-\gamma)}}^4) \\
\leq &\,\varepsilon|\theta(t)|_{H^1}^2 + C_3(\varepsilon, |\mathcal{I}|_{H^1})(|\overline{v}(t) - \overline{u}(t)|_{H^1}|\overline{v}(t) - \overline{u}(t)|_{L^2}^3 \\
&+ |\overline{v}(t) - \overline{u}(t)|_{H^1}^{1+\gamma}|\overline{v}(t) - \overline{u}(t)|_{L^2}^{3-\gamma})
\end{aligned}$$

Choosing $\varepsilon = \lambda_0/2$, we get

$$\begin{aligned}
&\frac{d}{dt}\frac{1}{2}|\theta(t)|_{L^2}^2 - C_1(\varepsilon, |\mathcal{I}|_{H^1})|\theta(t)|_{L^2}^2 \\
&\le C_3(\varepsilon, |\mathcal{I}|_{H^1})(|\overline{v}(t) - \overline{u}(t)|_{H^1}|\overline{v}(t) - \overline{u}(t)|_{L^2}^3 + |\overline{v}(t) - \overline{u}(t)|_{H^1}^{1+\gamma}|\overline{v}(t) - \overline{u}(t)|_{L^2}^{3-\gamma}) \\
&\qquad \le C_4(\varepsilon, |\mathcal{I}|_{H^1})(|\overline{v}(t) - \overline{u}(t)|_{L^2}^3 + |\overline{v}(t) - \overline{u}(t)|_{H^1}^2|\overline{v}(t) - \overline{u}(t)|_{L^2}^{3-\gamma}).
\end{aligned}$$

By Lemma 3.1, we get

$$\begin{aligned}
&\frac{d}{dt}\frac{1}{2}|\theta(t)|_{L^2}^2 - C_1(\varepsilon, |\mathcal{I}|_{H^1})|\theta(t)|_{L^2}^2 \\
&\qquad \le C_4(\varepsilon, |\mathcal{I}|_{H^1})(e^{3Kt}|\overline{v}_0 - \overline{u}_0|_{L^2}^3 + e^{(3-\gamma)Kt}|\overline{v}(t) - \overline{u}(t)|_{H^1}^2|\overline{v}_0 - \overline{u}_0|_{L^2}^{3-\gamma}).
\end{aligned}$$

Writing C_1 for $C_1(\varepsilon, |\mathcal{I}|_{H^1})$ and C_4 for $C_4(\varepsilon, |\mathcal{I}|_{H^1})$, we have

$$\begin{aligned}
\text{(3.9)}\quad &\frac{d}{dt}\frac{1}{2}(e^{-C_1 t}|\theta(t)|_{L^2}^2) \\
&\le C_4(e^{(3K-C_1)t}|\overline{v}_0 - \overline{u}_0|_{L^2}^3 + e^{((3-\gamma)K-C_1)t}|\overline{v}(t) - \overline{u}(t)|_{H^1}^2|\overline{v}_0 - \overline{u}_0|_{L^2}^{3-\gamma}).
\end{aligned}$$

Finally, integrating (3.9), recalling that $\theta(0) = 0$ and taking into account Lemma 3.1, we get the existence of two increasing functions $\Phi_1(t)$ and $\Phi_2(t)$ such that

$$|\theta(t)|_{L^2}^2 \le \Phi_1(t)|\overline{v}_0 - \overline{u}_0|_{L^2}^3 + \Phi_2(t)|\overline{v}_0 - \overline{u}_0|_{L^2}^{5-\gamma},$$

and the thesis follows. □

4. Dimension of invariant sets

Let $\mathcal{X}$ be a complete metric space and let $\mathcal{K} \subset \mathcal{X}$ be a compact set. For $d \in \mathbb{R}^+$ and $\varepsilon > 0$ one defines

$$\mu_H(\mathcal{K}, d, \varepsilon) := \inf\bigg\{\sum_{i\in I} r_i^d \;\bigg|\; \mathcal{K} \subset \bigcup_{i\in I} B(x_i, r_i),\ r_i \le \varepsilon\bigg\},$$

where the infimum is taken over all the finite coverings of $\mathcal{K}$ with balls of radius $r_i \le \varepsilon$. Observe that $\mu_H(\mathcal{K}, d, \varepsilon)$ is a non increasing function of ε and d. The d-dimensional Hausdorff measure of $\mathcal{K}$ is by definition

$$\mu_H(\mathcal{K}, d) := \lim_{\varepsilon\to 0}\mu_H(\mathcal{K}, d, \varepsilon) = \sup_{\varepsilon>0}\mu_H(\mathcal{K}, d, \varepsilon).$$

One has:

(1) $\mu_H(\mathcal{K}, d) \in [0, +\infty]$;
(2) if $\mu_H(\mathcal{K}, \overline{d}) < \infty$, then $\mu_H(\mathcal{K}, d) = 0$ for all $d > \overline{d}$;
(3) if $\mu_H(\mathcal{K}, \overline{d}) > 0$, then $\mu_H(\mathcal{K}, d) = +\infty$ for all $d < \overline{d}$.

The Hausdorff dimension of $\mathcal{K}$ is the smallest d for which $\mu_H(\mathcal{K}, d)$ is finite, i.e.

$$\dim_H(\mathcal{K}) := \inf\{d > 0 \mid \mu_H(\mathcal{K}, d) = 0\}.$$

As pointed up in [22], the Hausdorff dimension is in fact an intrinsic metric property of the set $\mathcal{K}$. Moreover, if $\mathcal{Y}$ is another complete metric space and $\ell\colon \mathcal{K} \to \mathcal{Y}$ is a Lipschitzian map, then $\dim_H(\ell(\mathcal{K})) \le \dim_H(\mathcal{K})$.

There is a well developed technique to estimate the Hausdorff dimension of an invariant set of a map or a semigroup. We refer the reader e.g. to [23] and [12]. The geometric idea consists in tracking the evolution of a d-dimensional volume under the action of the linearization of the semigroup along solutions lying in the invariant set. One looks then for the smallest d for which any d-dimensional volume contracts asymptotically as $t \to \infty$.

Let $\overline{u}_0 \in \mathcal{I}$ and let $\overline{u}(\,\cdot\,)\colon \mathbb{R} \to H^1_0(\Omega)$ be a full bounded solution of (2.2) such that $\overline{u}(0) = \overline{u}_0$ and $\overline{u}(t) \in \mathcal{I}$ for $t \in \mathbb{R}$. For $t \ge 0$, we denote by $a_{\overline{u}_0}(t; u, v)$ the bilinear form defined by (3.5), and by $A_{\overline{u}_0}(t)$ the self-adjoint operator determined by the relation (3.6). Given a d-dimensional subspace E_d of $L^2(\Omega)$, with $E_d \subset H^1_0(\Omega)$, we define the operator $A_{\overline{u}_0}(t \mid E_d)\colon E_d \to E_d$ by

$$\langle A_{\overline{u}_0}(t \mid E_d)\phi, \psi\rangle_{L^2} := a_{\overline{u}_0}(t; \phi, \psi), \quad \phi, \psi \in E_d.$$

Notice that, if $E_d \subset D(A_{\overline{u}_0}(t))$, then one has $A_{\overline{u}_0}(t \mid E_d) = P_{E_d} A_{\overline{u}_0}(t) P_{E_d}|_{E_d}$, where $P_{E_d}\colon L^2(\Omega) \to E_d$ is the L^2-orthogonal projection onto E_d. We define

$$\mathrm{Tr}_d(A_{\overline{u}_0}(t)) := \inf_{\substack{E_d \subset H^1_0(\Omega) \\ \dim E_d = d}} \mathrm{Tr}(A_{\overline{u}_0}(t \mid E_d)).$$

Let $\overline{u}_0 \in \mathcal{I}$, let $d \in \mathbb{N}$ and let $v_{0,i} \in L^2(\Omega)$, $i = 1, \ldots, d$. Set $v_i(t) := \mathcal{U}(\overline{u}_0; t)v_{0,i}$, $t \ge 0$, where $\mathcal{U}(\overline{u}_0; t)$ is defined by (3.7). We denote by $G(t)$ the d-dimensional volume delimited by $v_1(t), \ldots, v_d(t)$ in $L^2(\Omega)$, that is

$$G(t) := |v_1(t) \wedge v_2(t) \wedge \cdots \wedge v_d(t)|_{\wedge^d L^2} = (\det(\langle v_i(t), v_j(t)\rangle_{L^2})_{ij})^{1/2}.$$

An easy computation using Leibnitz rule and Proposition 3.3 shows that, for $t > 0$, $G(t)$ satisfies the ordinary differential equation

$$G'(t) = -\mathrm{Tr}(A_{\overline{u}_0}(t \mid E_d(t))G(t),$$

where $E_d(t) := \mathrm{span}(v_1(t), \ldots, v_d(t))$. It follows from Propositions 3.4 and 3.5 and from the results in [23, Chapter V] that the Hausdorff dimension $\dim_H(\mathcal{I})$ of $\mathcal{I}$ in $L^2(\Omega)$ is finite and less than or equal to d, provided

$$\limsup_{t \to \infty} \sup_{\overline{u}_0 \in \mathcal{I}} \frac{1}{t} \int_0^t -\mathrm{Tr}_d(A_{\overline{u}_0}(s))\, ds < 0.$$

In order to prove that $\dim_H(\mathcal{I}) \le d$, we are lead to estimate $-\mathrm{Tr}_d(A_{\overline{u}_0}(t))$. To this end, we notice that, whenever E_d is a d-dimensional subspace of $L^2(\Omega)$, and $B\colon E_d \to E_d$ is a selfadjoint operator, then

$$\mathrm{Tr}(B) = \sum_{i=1}^d \langle B\phi_i, \phi_i\rangle_{L^2},$$

where $\phi_1, \ldots, \phi_d$ is any L^2-orthonormal basis of E_d. So let $E_d \subset H_0^1(\Omega)$ be a d-dimensional space and let $\phi_1, \ldots, \phi_d$ be an L^2-orthonormal basis of E_d. Fix $0 < \delta < 1$. It follows that

$$\begin{aligned}
&\operatorname{Tr}(A_{\overline{u}_0}(t \mid E_d)) \\
&\quad = \sum_{i=1}^{d} \biggl((1-\delta)\biggl(\int_\Omega |\nabla\phi_i|^2\,dx + \int_\Omega \beta(x)|\phi_i|^2\,dx\biggr) - \int_\Omega \partial_u f(x,0)|\phi_i|^2\,dx\biggr) \\
&\qquad + \delta\sum_{i=1}^{d}\biggl(\int_\Omega |\nabla\phi_i|^2\,dx + \int_\Omega \beta(x)|\phi_i|^2\,dx\biggr) \\
&\qquad + \sum_{i=1}^{d}\int_\Omega (\partial_u f(x,\overline{u}(t)) - \partial_u f(x,0))|\phi_i|^2\,dx.
\end{aligned}$$

We introduce the following bilinear form defined on the space $H_0^1(\Omega)$:

$$\begin{aligned}
a_\delta(u,v) := (1-\delta)\biggl(\int_\Omega \nabla u(x)\cdot\nabla v(x)\,dx + \int_\Omega \beta(x)u(x)v(x)\,dx\biggr)& \\
- \int_\Omega \partial_u f(x,0)u(x)v(x)\,dx, \quad u,v \in H_0^1(\Omega).&
\end{aligned}$$

Let A_δ be the self-adjoint operator determined by the relation

$$\langle A_\delta u, v\rangle_{L^2} = a_\delta(u,v), \quad u \in D(A_\delta), v \in H_0^1(\Omega).$$

Given a d-dimensional subspace E_d of $L^2(\Omega)$, with $E_d \subset H_0^1(\Omega)$, we define the operator $A_\delta(E_d)\colon E_d \to E_d$ by

$$\langle A_\delta(E_d)\phi, \psi\rangle_{L^2} := a_\delta(\phi,\psi), \quad \phi,\psi \in E_d.$$

It follows that

$$\begin{aligned}
\operatorname{Tr}(A_{\overline{u}_0}(t \mid E_d)) = \operatorname{Tr}(A_\delta(E_d)) + \delta\sum_{i=1}^{d}\biggl(\int_\Omega |\nabla\phi_i|^2\,dx& \\
+ \int_\Omega \beta(x)|\phi_i|^2\,dx\biggr) + \sum_{i=1}^{d}\int_\Omega (\partial_u f(x,\overline{u}(t)) - \partial_u f(x,0))|\phi_i|^2\,dx.&
\end{aligned}$$

We introduce the *proper values* of the operator A_δ:

$$\mu_j(A_\delta) := \sup_{\psi_1,\ldots,\psi_{j-1}\in H_0^1(\Omega)} \inf_{\substack{\psi\in[\psi_1,\ldots,\psi_{j-1}]^\perp \\ |\psi|_{L^2}=1,\,\psi\in H_0^1(\Omega)}} a_\delta(\psi,\psi) \quad j = 1,2,\ldots$$

We recall (see e.g. Theorem XIII.1 in [19]) that:

Proposition 4.1. *For each fixed n, either*

(a) *there are at least n eigenvalues (counting multiplicity) below the bottom of the essential spectrum of A_δ and $\mu_n(A_\delta)$ is the nth eigenvalue (counting multiplicity);*

or

(b) *$\mu_n(A_\delta)$ is the bottom of the essential spectrum and in that case $\mu_{n+j}(A_\delta) = \mu_n(A_\delta)$, $j = 1, 2, \ldots$ and there are at most $n-1$ eigenvalues (counting multiplicity) below $\mu_n(A_\delta)$.* □

Let $\mu_j(A_\delta(E_d))$, $j = 1, \ldots, d$, be the *eigenvalues* of $A_\delta(E_d)$. By Theorem XIII.3 in [19], we have that

$$\mu_j(A_\delta(E_d)) \geq \mu_j(A_\delta), \quad j = 1, \ldots, d.$$

It follows that

$$\begin{aligned}\mathrm{Tr}(A_{\overline{u}_0}(t \mid E_d)) \geq \sum_{i=1}^{d} \mu_i(A_\delta) + \delta \sum_{i=1}^{d} \bigg(\int_\Omega |\nabla \phi_i|^2 \, dx + \int_\Omega \beta(x) |\phi_i|^2 \, dx \bigg) \\ + \sum_{i=1}^{d} \int_\Omega (\partial_u f(x, \overline{u}(t)) - \partial_u f(x, 0)) |\phi_i|^2 \, dx.\end{aligned}$$

To proceed further, we need to recall the Lieb–Thirring inequality (see [13]).

Proposition 4.2. *Let $N \in \mathbb{N}$ and let $p \in \mathbb{R}$, with $\max\{N/2, 1\} \leq p \leq 1 + N/2$. There exists a constant $K_{p,N} > 0$ such that, if $\phi_1, \ldots, \phi_d \in H^1(\mathbb{R}^N)$ are pairwise L^2-orthonormal, then*

$$\sum_{i=1}^{d} \int_{\mathbb{R}^N} |\nabla \phi_i(x)|^2 \, dx \geq \frac{1}{K_{p,N}} \bigg(\int_{\mathbb{R}^N} \rho(x)^{p/(p-1)} \, dx \bigg)^{2(p-1)/N}, \tag{4.1}$$

where $\rho(x) := \sum_{i=1}^{d} |\phi_i(x)|^2$.

Now we have:

Lemma 4.3. *Let $\overline{u} \in \mathcal{I}$ and let $\phi_1, \ldots, \phi_d \in H^1(\mathbb{R}^N)$ be pairwise L^2-orthonormal. Then*

$$\begin{aligned}\delta \sum_{i=1}^{d} \bigg(\int_\Omega |\nabla \phi_i|^2 \, dx + \int_\Omega \beta(x) |\phi_i|^2 \, dx \bigg) \\ + \sum_{i=1}^{d} \int_\Omega (\partial_u f(x, \overline{u}(x)) - \partial_u f(x, 0)) |\phi_i|^2 \, dx \geq -D(\gamma, \lambda_0, \delta, |\mathcal{I}|_{H^1}),\end{aligned}$$

where

$$\begin{aligned}D(\gamma, \lambda_0, \delta, |\mathcal{I}|_{H^1}) = \frac{5}{2} \bigg(\frac{3}{5} \frac{2}{\delta \lambda_0} \bigg)^{3/2} (C |\mathcal{I}|_{L^{5/2}} K_{5/2,3})^{5/2} \\ + \frac{3-\gamma}{4} \bigg(\frac{\gamma+1}{4} \frac{2}{\delta \lambda_0} \bigg)^{(\gamma+1)/(3-\gamma)} (C |\mathcal{I}|_{L^6}^{\gamma+1} K_{6/(\gamma+1),3})^{4/(3-\gamma)}.\end{aligned}$$

PROOF. We observe first that

$$\delta\sum_{i=1}^{d}\biggl(\int_\Omega|\nabla\phi_i|^2\,dx+\int_\Omega\beta(x)|\phi_i|^2\,dx\biggr)\geq\delta\lambda_0\sum_{i=1}^{d}\int_\Omega|\nabla\phi_i|^2\,dx.$$

On the other hand,

$$\begin{aligned}\biggl|\int_\Omega\biggl(\partial_uf(x,\overline{u}(x))-\partial_uf(x,0)\biggr)\rho(x)\,dx\biggr|&\\ \leq\int_\Omega C(1+|\overline{u}|^\gamma)|\overline{u}||\rho|\,dx\leq C|\overline{u}|_{L^{5/2}}|\rho|_{L^{5/3}}+C|\overline{u}|_{L^6}^{\gamma+1}|\rho|_{L^{6/(5-\gamma)}}.&\end{aligned}$$

By Lieb–Thirring inequality (4.1), we have

$$\begin{aligned}\biggl|\int_\Omega(\partial_uf(x,\overline{u}(x))-\partial_uf(x,0))\rho(x)\,dx\biggr|&\\ \leq C|\mathcal{I}|_{L^{5/2}}K_{5/2,3}\biggl(\sum_{i=1}^{d}\int_{\mathbb{R}^N}|\nabla\phi_i|^2\,dx\biggr)^{3/5}&\\ +C|\mathcal{I}|_{L^6}^{\gamma+1}K_{6/(\gamma+1),3}\biggl(\sum_{i=1}^{d}\int_{\mathbb{R}^N}|\nabla\phi_i|^2\,dx\biggr)^{(\gamma+1)/4}&.\end{aligned}$$

The conclusion follows by a simple application of Young's inequality. □

Thanks to Lemma 4.3, we finally get:

$$\mathrm{Tr}(A_{\overline{u}_0}(t\mid E_d))\geq\sum_{i=1}^{d}\mu_i(A_\delta)-D(\gamma,\lambda_0,\delta,|\mathcal{I}|_{H^1}).$$

Therefore, in order to conclude that $\dim_H(\mathcal{I})$ is finite, we are lead to make some assumption which guarantees that $\sum_{i=1}^{d}\mu_i(A_\delta)$ can be made positive and as large as we want, by choosing d sufficiently large. This is equivalent to the fact that the bottom of the essential spectrum of A_δ be strictly positive. We make the following assumption:

HYPOTHESIS 4.4. *For every $\varepsilon>0$ there exists $V_\varepsilon\in L^r(\Omega)$, $r>3/2$, $V_\varepsilon\geq0$, such that $\partial_uf(x,0)\leq V_\varepsilon(x)+\varepsilon$, for $x\in\Omega$.*

We need the following lemmas:

LEMMA 4.5. *Let $r>3/2$ and let $V\in L^r(\Omega)$. If $r>3$ let $p:=2$; if $r\leq3$ let $p:=6/5$. Then the assignment $u\mapsto Vu$ defines a compact map from $H_0^1(\Omega)$ to $L^p(\Omega)$, and hence to $H^{-1}(\Omega)$.*

PROOF. Let $B\subset H_0^1(\Omega)$ be bounded. If $\mathcal{B}$ is a Banach space such that $H_0^1(\Omega)\subset\mathcal{B}$, we define $|B|_{\mathcal{B}}:=\sup\{|u|_{\mathcal{B}}\mid u\in B\}$. If $u\in H_0^1(\Omega)$ we denote by $\widetilde{u}$ its trivial extension to the whole $\mathbb{R}^3$. Similarly, we denote by $\widetilde{V}$ the trivial extension of V to $\mathbb{R}^3$. For $k > 0$, let χ_k be the characteristic function of the set $\{x \in \mathbb{R}^3 \mid |x| \le k\}$. Now, for $u \in B$ and $k > 0$, we have:

$$\int_{\mathbb{R}^3} |(1-\chi_k)\widetilde{V}\widetilde{u}|^p\,dx \le \left(\int_{|x|\ge k} |\widetilde{V}|^r\,dx\right)^{p/r} \left(\int_{|x|\ge k} |\widetilde{u}|^{pr/(r-p)}\,dx\right)^{(r-p)/r}.$$

It follows that

$$|(1-\chi_k)\widetilde{V}\widetilde{u}|_{L^p} \le |B|_{L^{pr/(r-p)}}|(1-\chi_k)\widetilde{V}|_{L^r}, \quad u \in B,\ k > 0.$$

Similarly, we have:

$$|\chi_k\widetilde{V}\widetilde{u}|_{L^p} \le |\widetilde{V}|_{L^r}|\chi_k\widetilde{u}|_{L^{pr/(r-p)}}, \quad u \in H^1_0(\Omega),\ k > 0. \tag{4.2}$$

Now, given $\varepsilon > 0$, we choose $k > 0$ so large that $|(1-\chi_k)\widetilde{V}|_{L^r} \le \varepsilon$. Then

$$\begin{aligned}\{\widetilde{V}\widetilde{u} \mid u \in B\} &= \{\chi_k\widetilde{V}\widetilde{u} + (1-\chi_k)\widetilde{V}\widetilde{u} \mid u \in B\}\\ &\subset \{\chi_k\widetilde{V}\widetilde{u} \mid u \in B\} + \{(1-\chi_k)\widetilde{V}\widetilde{u} \mid u \in B\}\\ &\subset \{v \in L^p(\mathbb{R}^3) \mid |v|_{L^p} \le \varepsilon\} + \{\chi_k\widetilde{V}\widetilde{u} \mid u \in B\}.\end{aligned}$$

We notice that $2 \le pr/(r-p) < 6$: therefore, By Rellich's Theorem, $H^1(B_k(0))$ is compactly embedded into $L^{pr/(r-p)}$. It follows that the set $\{\chi_k\widetilde{u} \mid u \in B\}$ is precompact in $L^{pr/(r-p)}$. By (4.2), we deduce that $\{\chi_k\widetilde{V}\widetilde{u} \mid u \in B\}$ is precompact in $L^p(\mathbb{R}^3)$. A simple *measure of non compactness argument* shows then that the set $\{\widetilde{V}\widetilde{u} \mid u \in B\}$ is precompact in $L^p(\mathbb{R}^3)$ and this in turn implies that the set $\{Vu \mid u \in B\}$ is precompact in $L^p(\Omega)$. □

Lemma 4.6. *Let V be as in Lemma 4.5. Let $A+V$ be the selfadjoint operator determined by the bilinear form $a(u,v) + \int_\Omega Vuv\,dx$, $u,v \in H^1_0(\Omega)$. Then, for sufficiently large $\lambda > 0$, $(A+\lambda)^{-1} - (A+V+\lambda)^{-1}$ is a compact operator in $L^2(\Omega)$.*

Proof. Take $\lambda > 0$ so large that $A+V+\lambda$ be strictly positive. Let $u \in L^2(\Omega)$. Set $v := (A+V+\lambda)^{-1}u$, $w := (A+\lambda)^{-1}u$ and $z := v-w$. This means that

$$a(v,\phi) + \lambda(v,\phi) + \int_\Omega Vv\phi\,dx = \int_\Omega u\phi\,dx, \quad \text{for all } \phi \in H^1_0(\Omega)$$

and

$$a(w,\phi) + \lambda(w,\phi) = \int_\Omega u\phi\,dx, \quad \text{for all } \phi \in H^1_0(\Omega).$$

It follows that

$$a(z,\phi) + \lambda(z,\phi) + \int_\Omega Vv\phi\,dx = 0, \quad \text{for all } \phi \in H^1_0(\Omega).$$

Choosing $\phi := z$, Proposition 2.4 and Lemma 4.5 imply

$$\lambda_0|z|^2_{H^1} \le |z|_{H^1}|Vv|_{H^{-1}} \le \frac{\lambda_0}{2}|z|^2_{H^1} + K_{\lambda_0}|Vv|^2_{H^{-1}}.$$

Therefore we obtain the estimate

$$|(A+\lambda)^{-1}u-(A+V+\lambda)^{-1}u|_{H^1}\le K_{\lambda_0}|V(A+V+\lambda)^{-1}u|_{H^{-1}},\quad u\in L^2(\Omega),$$

and the conclusion follows from Lemma 4.5. □

Now we can prove:

PROPOSITION 4.7. *Assume Hypothesis* 4.4 *is satisfied. Then the essential spectrum of A_δ is contained in* $[(1-\delta)\lambda_1,+\infty[$.

PROOF. Hypothesis 4.4 and Proposition 4.1 imply that, for every $\varepsilon>0$, the bottom of the essential spectrum of A_δ is larger than or equal to the bottom of the essential spectrum of $(1-\delta)A-\varepsilon-V_\varepsilon(x)$. We observe that the spectrum of $(1-\delta)A-\varepsilon$ is contained in $[(1-\delta)\lambda_1-\varepsilon,+\infty[$. By Lemma 4.6 and Weyl's Theorem (see [19, Theorem XIII.14]), the essential spectrum of $(1-\delta)A-\varepsilon-V_\varepsilon(x)$ coincides with that of $(1-\delta)A-\varepsilon$. It follows that the bottom of the essential spectrum of A_δ is larger than or equal to $(1-\delta)\lambda_1-\varepsilon$ for arbitrary small $\varepsilon>0$, and the conclusion follows. □

Whenever Hypothesis 4.4 is satisfied, for $0<\delta<1$ and $\lambda<(1-\delta)\lambda_0$ we introduce the following quantity:

$$\mathcal{N}(\delta,\lambda):=\#\,\text{eigenvalues of } A_\delta \text{ below } \lambda.$$

Then, for $d\ge\mathcal{N}(\delta,(\frac{1-\delta}{2}\lambda_1)$ we have:

$$\sum_{i=1}^{d}\mu_i(A_\delta)\ge\mathcal{N}\bigg(\delta,\frac{1-\delta}{2}\lambda_1\bigg)\mu_1(\delta)+\bigg(d-\mathcal{N}\bigg(\delta,\frac{1-\delta}{2}\lambda_1\bigg)\bigg)\frac{1-\delta}{2}\lambda_1.$$

We have thus proved our first main result:

THEOREM 4.8. *Assume Hypotheses* 2.2, 2.6 *and* 4.4 *are satisfied. Let $\mathcal{I}\subset H_0^1(\Omega)$ be a compact invariant set for the semiflow π generated by equation* (2.2) *in $H_0^1(\Omega)$. Then the Hausdorff dimension of $\mathcal{I}$ in $L^2(\Omega)$ is finite and less than or equal to d, provided d is an integer number larger than* $\max\{d_1,d_2\}$, *where*

$$d_1:=\mathcal{N}\bigg(\delta,\frac{1-\delta}{2}\lambda_1\bigg)$$

and

$$d_2:=\frac{2}{(1-\delta)\lambda_1}\bigg(\mathcal{N}\bigg(\delta,\frac{1-\delta}{2}\lambda_1\bigg)\bigg(\frac{1-\delta}{2}\lambda_1-\mu_1(A_\delta)\bigg)+D(\gamma,\lambda_0,\delta,|\mathcal{I}|_{H^1})\bigg).$$

REMARK 4.9. The first proper value $\mu_1(A_\delta)$ of A_δ can be estimated from below in terms of λ_0 and $|\partial_u f(\cdot,0)|_{L^\sigma_{\mathrm{u}}}$. The explicit computations are left to the reader.

REMARK 4.10. By Lemma 2.8, also the Hausdorff dimension of $\mathcal{I}$ in $H_0^1(\Omega)$ is finite and it is equal to the Hausdorff dimension of $\mathcal{I}$ in $L^2(\Omega)$.

5. Estimate of $\mathcal{N}\big(\delta, \frac{1-\delta}{2}\lambda_1\big)$

In this section we shall obtain an explicit estimate for the number $\mathcal{N}\big(\delta, \frac{1-\delta}{2}\lambda_1\big)$ in terms of the dominating potential V_ε of Hypothesis 4.4. Our main tool is the celebrated Cwickel–Lieb–Rozenblum inequality, in its abstract formulation due to Rozenblum and Solomyak (see [21]). In order to exploit the CLR inequality, we need to make some assumption on the regularity of the open domain Ω. Namely, we make the following assumption:

Hypothesis 5.1. *The open set Ω is a uniformly C^2 domain in the sense of Browder* [7, p. 36].

As a consequence, by elliptic regularity we have that

$$D(-\Delta) = H^2(\Omega) \cap H^1_0(\Omega) \subset L^\infty(\Omega).$$

In this situation, if $\omega \in L^\sigma_{\mathrm{u}}(\mathbb{R}^3)$ then the assignment $u \mapsto \omega u$ defines a relatively bounded perturbation of $-\Delta$ and therefore $D(-\Delta + \omega) = H^2(\Omega) \cap H^1_0(\Omega)$. It follows that $X^\alpha \subset L^\infty(\Omega)$ for $\alpha > 3/4$ (see [11, Theorem 1.6.1]).

Set $\overline{\varepsilon} := (1-\delta)\lambda_1/4$. Define the bilinear forms

$$\widetilde{a}_{\delta,\overline{\varepsilon}}(u,v) := (1-\delta)\left(\int_\Omega \nabla u \cdot \nabla v \, dx + \int_\Omega \beta u v \, dx\right) - 3\,\overline{\varepsilon} \int_\Omega uv\,dx,$$

for $u, v \in H^1_0(\Omega)$, and

$$b_{\delta,\overline{\varepsilon}}(u,v) := -\int_\Omega V_{\overline{\varepsilon}}\, uv\,dx.$$

Moreover, set

$$a_{\delta,\overline{\varepsilon}}(u,v) := \widetilde{a}_{\delta,\overline{\varepsilon}}(u,v) + b_{\delta,\overline{\varepsilon}}(u,v)$$

and denote by $\widetilde{A}_{\delta,\overline{\varepsilon}}$ and $A_{\delta,\overline{\varepsilon}}$ the selfadjoint operators induced by $\widetilde{a}_{\delta,\overline{\varepsilon}}$ and $a_{\delta,\overline{\varepsilon}}$, respectively.

A simple computation shows that

$$\mathcal{N}\left(\delta, \frac{1-\delta}{2}\lambda_1\right) \leq n_{\delta,\overline{\varepsilon}},$$

where $n_{\delta,\overline{\varepsilon}}$ is the number of negative eigenvalues of $A_{\delta,\overline{\varepsilon}}$.

By Theorem 1.3.2 in [8], the operator $\widetilde{A}_{\delta,\overline{\varepsilon}}$ is positive (with $\widetilde{A}_{\delta,\overline{\varepsilon}} \geq \overline{\varepsilon} I$) and order preserving. Moreover, since $D(A^\alpha_{\delta,\overline{\varepsilon}}) \subset L^\infty(\Omega)$ for $\alpha > 3/4$, then for every such α and $\gamma < \overline{\varepsilon}$ we have

$$|e^{-t\widetilde{A}_{\delta,\overline{\varepsilon}}} u|_{L^\infty} \leq M_{\alpha,\gamma} t^{-\alpha} e^{-\gamma t} |u|_{L^2}, \quad u \in L^2(\Omega),$$

where $M_{\alpha,\gamma}$ is a constant depending only on α, γ and on the embedding constant of $H^2(\Omega)$ into $L^\infty(\Omega)$. It follows that

$$M_{\widetilde{A}_{\delta,\overline{\varepsilon}}}(t) := \|e^{-(t/2)\widetilde{A}_{\delta,\overline{\varepsilon}}}\|^2_{\mathcal{L}(L^2,L^\infty)} \leq M^2_{\alpha,\gamma} 2^{2\alpha} t^{-2\alpha} e^{-\gamma t}.$$

We are now in a position to apply Theorem 2.1 in [21]. We have thus proved the following theorem:

Theorem 5.2. *Assume that Hypotheses* 2.2, 2.6, 4.4 *and* 5.1 *are satisfied. Let* $\overline{\varepsilon} := (1-\delta)\lambda_1/4$. *Then*

$$\mathcal{N}\left(\delta, \frac{1-\delta}{2}\lambda_1\right) \leq n_{\delta,\overline{\varepsilon}} \leq C_{q/2} M_{q/2,\gamma} \int_\Omega V_{\overline{\varepsilon}}(x)^q \, dx,$$

where C_α *is a constant depending only on* α, *for* $\alpha > 3/4$.

6. Dissipative equations: dimension of the attractor

In this section we specialize our results to the case of a dissipative equation. We make the following assumption:

Hypothesis 6.1. *There exists a non negative function* $D \in L^q(\Omega)$, $2 \geq q > 3/2$, *such that*

$$f(x,u)u \leq D(x)|u|, \quad (x,u) \in \Omega \times \mathbb{R}. \tag{6.1}$$

Remark 6.2. Hypotheses 6.1 and 2.2 together are equivalent to the structure assumption of Theorem 4.4 in [4].

An easy computation shows that $|f(x,0)| \leq D(x)$ for $x \in \Omega$, and that $F(x,u) := \int_0^u f(x,s)\,ds$ satisfies

$$F(x,u) \leq D(x)|u|, \quad (x,u) \in \Omega \times \mathbb{R}.$$

By slightly modifying some technical arguments in [17], one can prove that the semiflow π generated by equation (2.2) in $H_0^1(\Omega)$ possesses a compact global attractor $\mathcal{A}$. Moreover, π is gradient-like with respect to the Lyapunov functional

$$\mathcal{L}(u) := \int_\Omega |\nabla u|^2 \, dx + \int_\Omega \beta(x)|u|^2 \, dx - \int_\Omega F(x,u)\, dx, \quad u \in H^1(\Omega).$$

Assuming Hypothesis 6.1, we shall give an explicit estimate for $|\mathcal{A}|_{H^1}$ in terms of $|D|_{L^q}$. Moreover, we shall prove that Hypothesis 6.1 implies Hypothesis 4.4, and we explicitly compute the dominating potential V_ε in terms of D. Therefore, we are able to obtain an explicit estimate for the number $\mathcal{N}\left(\delta, \frac{1-\delta}{2}\lambda_1\right)$ in terms of $|D|_{L^q}$. As a consequence, the estimate of the dimension of $\mathcal{A}$ given by Theorem 4.8 can be made completely explicit in terms of the structure parameters of equation (1.1).

We have the following theorem:

Theorem 6.3. *Assume Hypotheses* 2.2*,* 2.6 *and* 6.1 *are satisfied.*

(a) *Let* $\phi \in H_0^1(\Omega)$ *be an equilibrium of* π*. Then*

$$|\phi|_{H^1} \leq \frac{M_{q'}}{\lambda_0} |D|_{L^q},$$

where $M_{q'}$ *is the embedding constant of* $H_0^1(\mathbb{R}^3)$ *into* $L^{q'}(\mathbb{R}^3)$*.*

(b) *There exists a constant* $S > 0$ *such that*

$$|u|_{H^1} \leq S \quad \text{for all } u \in \mathcal{A}.$$

The constant S *can be explicitly computed and depends only on* C*,* γ*,* σ*,* λ_0*,* Λ_0*,* $|D|_{L^q}$*,* $|\partial_u f(\,\cdot\,,0)|_{L^\sigma_{\rm u}}$ *and on the constants of Sobolev embeddings.*

Proof. Let $\phi \in H_0^1(\Omega)$ be an equilibrium of π. Then, for $\varepsilon > 0$, we have

$$\begin{aligned}\lambda_0|\phi|_{H^1}^2 &\leq \int_\Omega |\nabla\phi|^2\,dx + \int_\Omega \beta(x)|\phi|^2\,dx = \int_\Omega f(x,\phi)\phi\,dx \leq \int_\Omega D(x)|\phi|\,dx \\ &\leq |D|_{L^q}|\phi|_{L^{q'}} \leq \varepsilon|\phi|_{L^{q'}}^2 + \frac{1}{4\varepsilon}|D|_{L^q}^2 \leq \varepsilon M_{q'}^2|\phi|_{H^1}^2 + \frac{1}{4\varepsilon}|D|_{L^q}^2;\end{aligned}$$

choosing $\varepsilon := \lambda_0/(2M_{q'}^2)$ we get property (a). In order to prove (b), we notice that, since $\mathcal{L}$ is a Lyapunov functional for π and $\mathcal{A}$ is compact in $H_0^1(\Omega)$, there exists an equilibrium ϕ such that, for every $u \in \mathcal{A}$,

$$\begin{aligned}&\int_\Omega |\nabla u|^2\,dx + \int_\Omega \beta(x)|u|^2\,dx - \int_\Omega F(x,u)\,dx \\ &\qquad\leq \int_\Omega |\nabla\phi|^2\,dx + \int_\Omega \beta(x)|\phi|^2\,dx - \int_\Omega F(x,\phi)\,dx.\end{aligned}$$

Then, for $\varepsilon > 0$, we have:

$$\begin{aligned}\lambda_0|u|_{H^1}^2 &\leq \int_\Omega D(x)|u|\,dx + \Lambda_0|\phi|_{H^1}^2 + \int_\Omega F(x,\phi)\,dx \\ &\leq \varepsilon M_{q'}^2|u|_{H^1}^2 + \frac{1}{4\varepsilon}|D|_{L^q}^2 + \Lambda_0|\phi|_{H^1}^2 + \int_\Omega F(x,\phi)\,dx.\end{aligned}$$

We choose $\varepsilon := \lambda_0/(2M_{q'}^2)$ and the conclusion follows. □

Finally, we have:

Theorem 6.4. *Assume that Hypotheses* 2.6 *and* 6.1 *are satisfied. Then for every* $0 < \varepsilon \leq 1$*,*

$$\partial_u f(x,0) \leq \frac{2}{\varepsilon}D(x) + \frac{\varepsilon}{2}C(1+\varepsilon^\gamma).$$

Proof. For $\varepsilon > 0$ we have:

$$f(x,\varepsilon) = f(x,0) + \partial_u f(x,0)\varepsilon + \int_0^\varepsilon \left(\int_0^s \partial_{uu} f(x,r)\,dr\right) ds.$$

It follows that

$$f(x,0)\varepsilon + \partial_u f(x,0)\varepsilon^2 + \varepsilon \int_0^\varepsilon \left(\int_0^s \partial_{uu} f(x,r)\,dr \right) ds = f(x,\varepsilon)\varepsilon \le D(x)\varepsilon.$$

Therefore

$$\partial_u f(x,0) \le \frac{D(x) + |f(x,0)|}{\varepsilon} + \frac{1}{\varepsilon} \int_0^\varepsilon \left(\int_0^s C(1+|r|^\gamma)\,dr \right) ds,$$

and the conclusion follows. □

REMARK 6.5. Theorem 6.4 shows that Hypotheses 2.6 and 6.1 together imply Hypothesis 4.4, with $V_\varepsilon(x) = \frac{2C}{\varepsilon} D(x)$.

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Manuscript received June 21, 2011

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