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ON THE SPECTRAL FLOW FOR PATHS OF ESSENTIALLY HYPERBOLIC BOUNDED OPERATORS ON BANACH SPACES

Daniele Garrisi

To my parents and my sister

Abstract. We give a definition of the spectral flow for paths of bounded essentially hyperbolic operators on a Banach space. The spectral flow induces a group homomorphism on the fundamental group of every connected component of the space of essentially hyperbolic operators. We prove that this homomorphism completes the exact homotopy sequence of a Serre fibration. This allows us to characterise its kernel and image and to produce examples of spaces where it is not injective or not surjective, unlike what happens for Hilbert spaces. For a large class of paths, namely the essentially splitting, the spectral flow of A coincides with $-\mathrm{ind}(F_A)$, the Fredholm index of the differential operator $F_A(u) = u' - Au$.

1. Introduction

The spectral flow first appeared in [7] for a family of elliptic and self-adjoint operators A_t, ascribed to the joint work of M. Atiyah and G. Lusztig. We outline their effective description as "net number of eigenvalues that change sign (from $-$ to $+$) while the parameter family is completing a period" in the definition given by J. Robbin and D. Salamon in [24, Theorem 4.21]: In a neighbourhood $[t_-, t_+]$ of the real line of a point $t \in \mathbb{R}$ (called a crossing) such that $0 \in \sigma(A_t)$

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(the spectrum of A_t), $\sigma(A_s)$ can be described as a finite family of continuously differentiable curves $\lambda_i\colon [t_-, t_+] \to \mathbb{R}$ such that $\lambda_i'(t) \neq 0$. The contribution to the spectral flow of a crossing is given by

$$\sum_{\lambda_i} \operatorname{sign}(\lambda_i(t_+)) - \operatorname{sign}(\lambda_i(t_-))$$

and the spectral flow is the sum of these contributions over all the crossings. The spectral flow was used to define a Morse index for 1-periodic hamiltonian orbits, as in [27, pp. 23], and then to define the Floer homology. The definition by J. Robbin and D. Salamon in [24] requires some differentiability hypotheses and transversality conditions. P. Rabier extended their work to unbounded families of operators in Banach spaces in [23]. In [21], J. Phillips simplified their definition as follows: $A_t \in \mathcal{F}^{sa}(H)$ is assumed to be a continuous path of Fredholm, bounded and self-adjoint operators, on $[0, 1]$. If U is a neighbourhood of the origin and $J = [t_-, t_+]$ is a closed interval such that

(1) $\sigma(A_s) \cap \partial U = \emptyset$ for every $s \in J$;
(2) $\sigma(A_s) \cap U$ is a finite set of eigenvalues,

then the contribution to the spectral flow from the interval J is defined as

$$\dim(P(A(t_+); U)) - \dim(P(A(t_-); U))$$

where $P(A; U)$ is the spectral projector of A relative to U. The spectral flow is the sum of all the contributions obtained over a partition of the unit interval, J_i, such that a neighbourhood U_i as in (1), (2) corresponds to each J_i. It is invariant for fixed-endpoint homotopies and defines a groups homomorphism on the fundamental group of each connected component of $\mathcal{F}^{sa}(H)$. If H is infinite-dimensional and separable, then there are exactly three connected components, corresponding to I and $-I$ (these are contractible to a point) and to $2P - I$, where P is a projector with infinite-dimensional kernel and image. On the fundamental group of the third one, denoted by $\mathcal{F}^{sa}_*$, the spectral flow

$$\operatorname{sf}\colon \pi_1(\mathcal{F}^{sa}_*) \to \mathbb{Z}$$

is a group isomorphism. In [8], J. Phillips later extended this definition to continuous families of unbounded, Fredholm and self-adjoint operators. In contrast with [24], no assumptions of differentiability or transversality are made. C. Zhu and Y. Long in [29] extended the definition in [21] to bounded, admissible operators (which are compact perturbations of hyperbolic operators) on Banach spaces: On every interval $J_i = [t^i_-, t^i_+]$ of a suitable partition of $[0, 1]$, they provide a path of projectors Q_i such that:

- $Q_i\colon J_i \to \mathcal{P}(\mathcal{L}(E))$ is continuous,
- $Q_i(t) - P^+(A(t))$ is compact for $t \in J_i$,

- $Q_i(t)$ is a spectral projector of $A(t)$.

The contribuition of J_i to the spectral flow is

$$[Q_i(t^i_-) - P^+(A(t^i_-))] - [Q_i(t^i_+) - P^+(A(t^i_+))],$$

where $[Q - P]$ denotes the Fredholm index of $P\colon \mathrm{Range}(Q) \to \mathrm{Range}(P)$. The spectral flow is defined as the sum of the quantity above as J_i varies over the partition of $[0, 1]$. We denote the spectral projector relative to the positive complex half-plane by $P^+(A)$, and $\mathcal{P}(\mathcal{L}(E))$ is the set of projectors of E. This work has three essential purposes:

(I) Further simplifying the definition of spectral flow. Given a path $A_t \in e\mathcal{H}(E)$ on $[0, 1]$ of essentially hyperbolic operators (which are compact perturbations of hyperbolic operators and correspond to the admissible ones used in [29]), there exists a continuous path P of projectors on $[0, 1]$ such that

$$P(t) - P^+(A(t)) \text{ is compact}$$

for every $t \in [0, 1]$. Thus, we define

$$\mathrm{sf}(A) = [P(0) - P^+(A(0))] - [P(1) - P^+(A(1))].$$

Therefore, we do not need to partition the unit interval (as in the definition in [29]), as long as we do not require $P(t)$ to be a spectral projector of $A(t)$.

The existence of such a path P follows from the homotopy lifting property of the Serre fibration

$$p\colon \mathcal{P}(\mathcal{L}(E)) \to \mathcal{P}(\mathcal{C}(E)).$$

We use $\mathcal{C}(E)$ to denote the quotient of the operator algebra $\mathcal{L}(E)$ by the ideal of the compact operators. $\mathcal{P}(\mathcal{C}(E))$ is the space of projectors of the Calkin algebra and p is the quotient projection. The definition we give of spectral flow coincides with the one in [29]. In Section 4, we show that the spectral flow is invariant for fixed-endpoint homotopies and that, given two continuous paths A and B such that $A(1) = B(0)$, there holds $\mathrm{sf}(A * B) = \mathrm{sf}(A) + \mathrm{sf}(B)$. These and other basic facts are outlined in Proposition 4.3.

(II) Studying the homomorphism properties of the spectral flow. We prove that $e\mathcal{H}(E)$ is homotopically equivalent to $\mathcal{P}(\mathcal{C}(E))$ and that the spectral flow completes the exact homotopy sequence of the fibration above. This allows us to characterise the kernel and the image of the spectral flow sf_P defined on the fundamental group of the connected component of $2P - I \in e\mathcal{H}(E)$. Precisely: an integer m belongs to the image of sf_P if and only if

(h1) there exists a projector Q connected to P by a path in the space of projectors $\mathcal{P}(\mathcal{L}(E))$, such that $Q - P$ is compact and $[P - Q] = m$;

$\ker(\mathrm{sf}_P) \cong \mathrm{im}(p_*)$. Hence, sf_P is injective if and only if

(h2) $\mathrm{im}(p_*) = \{0\}$.

The existence of a projector satisfying (h1) and (h2) depends heavily on the structure of the Banach space E. When E is Hilbert, J. Phillips proved in [21] that for every projector with infinite-dimensional range and kernel, (h1), with $m = 1$, and (h2) hold. We show that ℓ^p and ℓ^∞ have this property as well. In general, at least one projector (and then infinitely many) exists in the spaces satisfying the hypotheses of Propositions 5.6 and 5.7.

The question whether property (h1) holds for some projector is strongly related to the existence of complemented subspaces isomorphic to closed subspaces of co-dimension m. This relation is highlighted in Proposition 5.6. In fact, given a space E, isomorphic to its hyperplanes, the projector over each of the summands of $E \oplus E$ fulfills property (h1). Thus, sf_P is surjective. If E is isomorphic to its subspaces of co-dimension two, but not to its hyperplanes, the image of the spectral flow is $2\mathbb{Z}$, and so on. Examples of such spaces have been constructed by W. T. Gowers and B. Maurey in [15]. If E is not isomorphic to any of its proper subspaces, then sf_P is zero. Such a space was constructed by W. T. Gowers and B. Maurey in the celebrated paper [14].

We prove that in a Douady space (cf. [12]), there are projectors P such that sf_P is not injective and projectors with infinite-dimensional range and kernel such that the spectral flow is zero.

(III) Comparing the spectral flow with the Fredholm index of the differential operator

$$F_A\colon W^{1,p}(\mathbb{R}) \to L^p(\mathbb{R}), \quad u \mapsto \left(\frac{d}{dt} - A(t)\right)u.$$

In Section 6, we extend the definition of spectral flow for a path $A_t \in e\mathcal{H}(E)$ on $\mathbb{R}$ with hyperbolic limits at $\pm\infty$. We prove in Theorem 6.6 that for a large class of paths, which are essentially hyperbolic, with hyperbolic limits, and *essentially splitting* (cf. [2]), the equality

$$\mathrm{ind}(F_A) = -\mathrm{sf}(A)$$

holds. The equality above applies, for instance, in the special case where A is a continuous, compact perturbation of a path of hyperbolic operators with some boundary conditions (check [2, Theorem E]). Our theorem confirms the guess of A. Abbondandolo and P. Majer in §7 of [2] that for these paths, the equality above holds.

We remark that our work deals with *bounded* operators. The differential operator F_A arises naturally from the linearisation of a vector field $\xi \in C^1(E, E)$ on a solution $v'(t) = \xi(v(t))$, such that the endpoints are zeroes of ξ and $A(t) = D\xi(v(t))$. If F_A is Fredholm and surjective (and the zeroes are hyperbolic), then the set

$$W_\xi(p, q) = \{v\colon \mathbb{R} \to E : v'(t) = \xi(v(t)), v(-\infty) = p, v(\infty) = q\}$$

is a sub-manifold of dimension $\mathrm{ind}(F_A)$. This constitutes a landmark for the study of the Morse theory on Banach manifolds, as in [3]. A proof of this can be found in [2, §8]. If A fulfils the hypotheses of Theorem 6.6, then $\mathrm{sf}(A)$ determines the dimension of the manifold.

The spectral flow also provides an index for 1-periodic solutions of $u'(t) = X(u(t))$. If $DX(u_t)$ is essentially hyperbolic, then $\mathrm{index}_X(u) = \mathrm{sf}(DX(u_t))$. Thus, in order to have a good Morse theory for periodic solutions, the question whether there are loops with non-trivial spectral flow becomes relevant.

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2. Preliminaries

Here we review some basic definitions and results on spectral theory and Fredholm operators. Our main references are [25, Chapter X] and [16, IV.4,5].

2.1. Spectral theory. A *Banach algebra* is an algebra $\mathcal{A}$ with unit, 1, over the real or complex field and a norm $\|\cdot\|$ such that:

- $(\mathcal{A}, \|\cdot\|)$ is a Banach space,
- $\|xy\| \leq \|x\| \cdot \|y\|$ for every $x, y \in \mathcal{A}$.

We denote by $G(\mathcal{A})$ the set of invertible elements of the algebra; it is an open subset of $\mathcal{A}$. Given $x \in \mathcal{A}$, the subset of the field

$$\sigma(x) = \{\lambda \in \mathbb{F} : x - \lambda \cdot 1 \notin G(\mathcal{A})\}$$

is called the *spectrum* of x. Let us review some properties of the spectrum.

Proposition 2.1. *For every $x \in \mathcal{A}$, $t \in \mathbb{F}$ and $\Omega \subset \mathbb{F}$ an open subset,*

(a) *if $\sigma(x)$ is non-empty, then it is closed and bounded;*
(b) *$\sigma(x + t) = \sigma(x) + t$, $\sigma(tx) = t\sigma(x)$;*
(c) *then there exists $\delta > 0$ such that $\sigma(y) \subset \Omega$ for every $y \in B(x, \delta)$;*
(d) *if $\mathcal{A}$ is complex, then $\sigma(x)$ is non-empty;*
(e) *if $f: \mathcal{A} \to \mathcal{B}$ is an algebras homomorphism such that $f(1) = 1$, then $\sigma(f(x)) \subseteq \sigma(x)$.*

Given a real algebra, we can consider the complex algebra associated with it, $\mathcal{A}_{\mathbb{C}} = \mathcal{A} \otimes_{\mathbb{R}} \mathbb{C}$. We have an inclusion of algebras

$$\mathcal{A} \hookrightarrow \mathcal{A}_{\mathbb{C}}, \quad x \mapsto x \otimes 1$$

and $\sigma(x) \subseteq \sigma(x \otimes 1)$. Hereafter, we will take $\sigma(x \otimes 1)$ as the spectrum of x.

Definition 2.2. A finite family of closed curves in the complex plane, $\Gamma = \{c_i : 1 \le i \le n\}$, is said to be *simple* if, for every $z \notin \bigcup_{i=1}^n \mathrm{im}(c_i)$,

$$\mathrm{ind}_\Gamma(z) := \frac{1}{2\pi i}\int_\Gamma \frac{d\zeta}{z-\zeta} = \frac{1}{2\pi i}\sum_{i=1}^n \int_{c_i} \frac{d\zeta}{z-\zeta} \in \{0,1\}.$$

We denote by $\Omega_0(\Gamma)$ and $\Omega_1(\Gamma)$ the subsets of the complex space such that $\mathrm{ind}_\Gamma(z)$ is 0 and 1, respectively.

Definition 2.3. An element $p \in \mathcal{A}$ is said to be a *projector* if $p^2 = p$. We can associate with it the sub-algebra $\mathcal{A}(p) = \{pxp : x \in \mathcal{A}\}$, with the unit p. We denote by $\sigma_p(y)$ the spectrum of an element $y \in \mathcal{A}(p)$.

If $xp = px$, we have the equality

$$\sigma(x) = \sigma_p(xp) \cup \sigma_{1-p}(x(1-p)). \tag{2.1}$$

Theorem 2.4. *Let $x \in \mathcal{A}$, $\Sigma \subset \sigma(x)$ open and closed in $\sigma(x)$. Then, there exists a projector, called a spectral projector relative to Σ, which we denote by $P(x;\Sigma)$, such that*

(a) $\sigma_P(PxP) = \Sigma$;
(b) $Px = xP$;
(c) $I - P = P(x;\Sigma^c)$;
(d) *given an algebra homomorphism,* $f\colon \mathcal{A} \to \mathcal{B}$, $f(P(x;\Sigma)) = P(f(x);\Sigma)$.

Given Γ simple such that $\Sigma = \Omega_1(\Gamma) \cap \sigma(x)$,

$$P(x;\Sigma) = P_\Gamma(x) := \frac{1}{2\pi i}\int_\Gamma (x-\zeta)^{-1}\, d\zeta.$$

On the open subset $\mathcal{A}(\Gamma) = \{x \in \mathcal{A} : \sigma(x) \cap \Gamma = \emptyset\}$, $P_\Gamma(x)$ is a continuous map.

For the proof and more details check [25, Theorem 10.27] or [16, Theorem 6.17].

2.2. Spaces of projectors. Given Banach spaces E and F, we denote by $\mathcal{L}(E,F)$ the space of bounded operators from E to F. When $E = F$ we use the notation $\mathcal{L}(E)$. We denote the subsets of compact operators by $\mathcal{L}_c(E,F)$ and $\mathcal{L}_c(E)$. The composition of operators endows the space $\mathcal{L}(E)$ with the structure of a Banach algebra (the identity operator being the unit), and the subspace of compact operators is a closed ideal. We denote the quotient algebra by $\mathcal{C}(E)$. It inherits the structure of a Banach algebra and is called the *Calkin algebra*. The quotient projection

$$p\colon \mathcal{L}(E) \to \mathcal{C}(E), \quad A \mapsto A + \mathcal{L}_c(E) \tag{2.2}$$

is an algebra homomorphism.

Definition 2.5. Given a Banach algebra $\mathcal{A}$, we define the following subsets

(a) $\mathcal{P}(\mathcal{A}) = \{p \in \mathcal{A} : p^2 = p\}$, *projectors*;
(b) $\mathcal{Q}(\mathcal{A}) = \{q \in \mathcal{A} : q^2 = 1\}$, *square roots of the unit*;
(c) $\mathcal{H}(\mathcal{A}) = \{x \in \mathcal{A} : \sigma(x) \cap i\mathbb{R} = \emptyset\}$, *hyperbolic* elements.

Properties and remarks. $\mathcal{P}$ and $\mathcal{Q}$ are closed subsets, locally path connected and analytic sub-manifolds. A proof can be found in [4, Lemma 1.5]. $\mathcal{P}$ and $\mathcal{Q}$ are diffeomorphic to each other through the diffeomorphism $p \mapsto 2p - 1$. By (c) of Proposition 2.1, applied with $\Omega = \{z : \mathrm{Re}(z) \neq 0\}$, the subset $\mathcal{H}(\mathcal{A}) \subset \mathcal{A}$ is open. We denote by $G_1(\mathcal{A})$ the connected component of $G(\mathcal{A})$ of the unit.

Theorem 2.6. *Given two projectors p, q such that either $\|p - q\| < 1$ or both are in the same connected component of $\mathcal{P}(\mathcal{A})$, there exists $u \in G_1(\mathcal{A})$ such that $up = qu$.*

For the proof and details, we refer to [22, Proposition 4.2] and [13, Proposition 2.2]. The theorem above has two consequences:

(c1) $\mathcal{P}(\mathcal{A})$ is locally path-connected. So is $\mathcal{Q}(\mathcal{A})$;
(c2) when $\mathcal{A} = \mathcal{L}(E)$, two projectors in the same connected component have isomorphic ranges and kernels.

The quotient projection p in (2.2) restricts to the subset of projectors and roots of the unit

$$\mathcal{P}(p): \mathcal{P}(\mathcal{L}(E)) \to \mathcal{P}(\mathcal{C}(E)), \qquad P \mapsto P + \mathcal{L}_c(E)$$
$$\mathcal{Q}(p): \mathcal{Q}(\mathcal{L}(E)) \to \mathcal{Q}(\mathcal{C}(E)), \qquad Q \mapsto Q + \mathcal{L}_c(E).$$

Definition 2.7. A continuous map $p: E \to B$ has the *homotopy lifting property* w.r.t. a topological space X if, given continuous maps

$$h: X \times [0,1] \to B, \qquad f: X \times \{0\} \to E,$$

there exists $H: X \times [0,1] \to E$ such that $H(x,0) = f(x,0)$ and $p \circ H = h$. If the homotopy lifting property holds w.r.t. $[0,1]^n$ for every $n \geq 0$, then p is called a *Serre fibration.*

Proposition 2.8. *The maps $\mathcal{P}(p)$ and $\mathcal{Q}(p)$ are surjective Serre fibrations.*

In general, every surjective algebra homomorphism induces a Serre fibration. For a proof, see [10, Theorem 2.4]. The surjectivity of $\mathcal{P}(p)$ and $\mathcal{Q}(p)$ follows from [4, Proposition 4.1]. In fact, $\mathcal{P}(p)$ and $\mathcal{Q}(p)$ are locally trivial fiber bundles, as follows from [4, Proposition 1.3] or [13, Theorem 4.2].

2.3. Fredholm operators and relative dimension. Let $T \in \mathcal{L}(E,F)$ be a bounded operator. If the image of T is a closed subspace, we have two Banach spaces associated with it, namely $\ker(T)$ and $E/\mathrm{Range}(T) = \mathrm{coker}(T)$.

Definition 2.9. An operator as above is said to be *semi-Fredholm* if either $\ker(T)$ or $\operatorname{coker}(T)$ is a finite-dimensional space. If both have finite dimension, T is called *Fredholm* and the integer

$$\operatorname{ind}(T) = \dim\ker(T) - \dim\operatorname{coker}(T)$$

is the *Fredholm index*. Otherwise, the index is defined to be $+\infty$ or $-\infty$ as long as $\ker(T)$ or $\operatorname{coker}(T)$ has infinite dimension.

We denote by $\operatorname{Fred}(E, F)$ and $\operatorname{Fred}(E)$ the subsets of Fredholm operators in $\mathcal{L}(E, F)$ and $\mathcal{L}(E)$, respectively; $\operatorname{Fred}_k(E, F)$ is the set of Fredholm operators of index k.

Proposition 2.10. *Let* $T \in \operatorname{Fred}(E, F)$, $U \in \operatorname{Fred}(F, G)$ *and* $K \in \mathcal{L}_c(E, F)$. *We have*

(a) $\operatorname{Fred}_k(E, F) \subseteq \mathcal{L}(E, F)$ *is an open subset;*
(b) $T + K \in \operatorname{Fred}(E, F)$ *and* $\operatorname{ind}(T + K) = \operatorname{ind}(T)$*;*
(c) $U \circ T \in \operatorname{Fred}(E, G)$ *and* $\operatorname{ind}(U \circ T) = \operatorname{ind}(U) + \operatorname{ind}(T)$*;*
(d) *given* $B \in \mathcal{L}(E, F)$, *there exists* $\varepsilon > 0$ *such that the maps* $\dim\ker(T + \lambda B)$ *and* $\dim\operatorname{coker}(T + \lambda B)$ *are constant on* $B(0, \varepsilon) \setminus \{0\}$*;*
(e) $T \in \operatorname{Fred}(E, F)$ *if and only if there exists* $S \in \mathcal{L}(F, E)$ *such that* T *and* S *are the essential inverse of each other, that is,* $T \circ S - I \in \mathcal{L}_c(F)$ *and* $S \circ T - I \in \mathcal{L}_c(E)$.

Statements (a), (e) are easy to check. Statements (a), (b), (d) are all stated and proved in [16, Chapter IV.5] in the more general setting of semi-Fredholm and unbounded operators.

Definition 2.11. A pair of closed subspaces (X, Y) is *semi-Fredholm* if and only if their sum is closed and either $X \cap Y$ or $E/(X + Y)$ has a finite dimension. If both have finite dimension, then the *Fredholm index* of the pair (X, Y) is defined as

$$\operatorname{ind}(X, Y) = \dim X \cap Y - \operatorname{codim} X + Y.$$

Otherwise, the index is $+\infty$ or $-\infty$, when either $X \cap Y$ or $E/(X + Y)$ has infinite dimension.

Two projectors P, Q are compact perturbations of each other if $P - Q \in \mathcal{L}_c(E)$. In this case, the restriction of Q to $\operatorname{Range}(P)$ is in $\operatorname{Fred}(\operatorname{Range}(P), \operatorname{Range}(Q))$. The *relative dimension* between P and Q is defined as

$$[P - Q] := \operatorname{ind}(Q\colon \operatorname{Range}(P) \to \operatorname{Range}(Q)).$$

This definition is meant to generalise the dimension gap between two finite-dimensional spaces to Banach spaces. The notation above is used by C. Zhu and Y. Long in [29]. Corresponding definitions are known in Hilbert spaces, considered by A. Abbondandolo and P. Majer in [1, Definition 1.1] (see also [9, Remark 4.9]). A definition of relative dimension for pairs of closed subspaces (X, Y), not necessarily complemented, can be found in [13, Definition 5.8].

Theorem 2.12. *Given pairs of projectors (P, Q) and (Q, R) with compact difference, we have*

(a) *if* $\mathrm{Range}(P)$ *and* $\mathrm{Range}(Q)$ *have finite dimension, then* $[P - Q] = \dim \mathrm{Range}(P) - \dim \mathrm{Range}(Q)$*;*
(b) $[P - R] = [P - Q] + [Q - R]$*;*
(c) *on the subset* $\{(P, Q) \in \mathcal{P}(\mathcal{L}(E)) \times \mathcal{P}(\mathcal{L}(E)) : P - Q \in \mathcal{L}_c(E)\}$*, the map* $[P - Q]$ *is continuous;*
(d) $[P - Q] = [(I - Q) - (I - P)]$*;*
(e) $(\mathrm{Range}(P), \ker(Q))$ *is a Fredholm pair and* $\mathrm{ind}(\mathrm{Range}(P), \ker(Q)) = [Q - P]$.

Property (c) follows from stability results for the index of semi-Fredholm pairs; see [16, Remark IV.4.31] and [13, Theorem 3.3]. For a proof of (d) and (e), see [29, Lemma 2.3] and [13, Proposition 5.13], respectively; (a) follows from the remarks after Definition 5.8 in [13].

3. Essentially hyperbolic operators

We recall that a bounded operator $A \in \mathcal{L}(E)$ – or more generally an element of a Banach algebra $\mathcal{A}$ – is called *hyperbolic* if its spectrum does not meet the imaginary axis. We denote by $\mathrm{GL}(E)$ the group of invertible operators on E and by $\mathrm{GL}_I(E)$ the connected component of the identity operator.

Given $A \in \mathcal{L}(E)$, the spectrum of $A + \mathcal{L}_c(E)$ is called the *essential* spectrum. It is usually denoted by $\sigma_e(A)$. By (e) of Proposition 2.10,

$$\sigma_e(A) = \{\lambda : A - \lambda \notin \mathrm{Fred}(E)\}. \tag{3.1}$$

Definition 3.1. An operator A is called *essentially hyperbolic* if $A + \mathcal{L}_c(E)$ is a hyperbolic element in $\mathcal{C}(E)$.

By the equality above, an operator $A \in \mathcal{L}(E)$ is essentially hyperbolic if and only if its essential spectrum does not meet the imaginary axis. A consequence of (3.1) is:

Lemma 3.2. *Let $D(A)$ be the set of all isolated points of $\sigma(A)$, and let $\partial\sigma(A)$ be the set of the boundary points of $\sigma(A)$. Then $\partial\sigma(A) \setminus D(A)$ is a subset of $\sigma_e(A)$.*

Proof. Let $\lambda \in \partial\sigma(A) \setminus D(A)$, and suppose that $\lambda \notin \sigma_e(A)$, thus $A - \lambda \in \mathrm{Fred}(E)$. Let $\varepsilon > 0$ as in (d) of Proposition 2.10, with $B = -I$. Therefore, for some $c, k \in \mathbb{Z}$

$$\dim \ker(A - z) = c, \ \dim \operatorname{coker}(A - z) = k, \quad \text{for every } z \in B(\lambda, \varepsilon) \setminus \{\lambda\}.$$

Because $\lambda \in \partial\sigma(A)$, there exists $w \in B(\lambda, \varepsilon) \setminus \{\lambda\}$ such that $A - w$ is invertible. Thus, $c = k = 0$ and $A - z$ is invertible for every $z \in B(\lambda, \varepsilon) \setminus \{\lambda\}$. Hence λ is isolated in $\sigma(A)$. □

We need a well-known fact about the topology of the real line:

PROPOSITION 3.3. *A closed proper subset of the real line with an empty boundary is discrete.*

COROLLARY 3.4. *If A is an essentially hyperbolic operator, the set $\sigma(A) \cap i\mathbb{R}$ is finite.*

PROOF. We show that the boundary of $\sigma(A) \cap i\mathbb{R}$ is empty. Suppose it is not and let $\lambda \in \partial(\sigma(A) \cap i\mathbb{R})$ be an arbitrary point. Hence, $\lambda \in \partial\sigma(A)$. Because $\sigma(A) \cap i\mathbb{R}$ is closed, $\lambda \in i\mathbb{R}$. Hence $\lambda \notin \sigma_e(A)$, because A is essentially hyperbolic. Thus, $\lambda \in \partial\sigma(A) \setminus \sigma_e(A)$, whence, by Lemma 3.2, $\lambda \in D(A)$. Hence, λ is isolated in $\sigma(A) \cap i\mathbb{R}$ in contradiction with the hypothesis that λ is a boundary point. By the proposition above, $\sigma(A) \cap i\mathbb{R}$ is discrete. Because it is also compact, it is a finite set. □

PROPOSITION 3.5. *If A is an essentially hyperbolic operator, each of the points of $\sigma(A) \cap \mathrm{i}\mathbb{R}$ is an eigenvalue of finite algebraic multiplicity.*

PROOF. Let $\lambda \in \sigma(A) \cap \mathrm{i}\mathbb{R}$. We infer that $A - \lambda \in \mathrm{Fred}_0(E)$. By (3.1), $A - \lambda \in \mathrm{Fred}_k(E)$ for some $k \in \mathbb{Z}$. Now, by (a) of Proposition 2.10, there exists a neighbourhood V of λ such that

$$A - z \in \mathrm{Fred}_k(E), \quad z \in V.$$

Because λ is isolated, there exists $z' \in V \setminus \{\lambda\}$ such that $A - z'$ is invertible, hence $k = 0$. Thus, because $A - \lambda$ is not invertible, $\ker(A - \lambda) \neq \{0\}$. Hence λ is an eigenvalue and, by hypothesis, isolated. These two conditions, by Theorems 5.10 and 5.28 of [16], imply that the spectral projector $P(A; \{\lambda\})$ has range of finite dimension, which is the algebraic multiplicity. □

Theorem 2.4 provides us with projectors $P_i = P(A; \{\lambda_i\})$ for every $\lambda_i \in \sigma(A) \cap \mathrm{i}\mathbb{R}$. Let $P = P(A; \sigma(A) \cap \{\mathrm{Re}(z) \neq 0\})$. We can write

$$A = \left(AP + \sum_{i=1}^{n} P_i\right) + (A - I)\sum_{i=1}^{n} P_i. \tag{3.2}$$

According to (a) and (b) of Theorem 2.4, the term in the brackets is hyperbolic. The last term has finite rank. Thus, we have proved that an essentially hyperbolic operator is a compact perturbation of a hyperbolic one. Conversely, a compact perturbation of a hyperbolic operator is essentially hyperbolic. In fact, let H, K be a hyperbolic and a compact operator, respectively: By (b) of Proposition 2.10 and (3.1), $\sigma_e(H + K) = \sigma_e(H) \subseteq \sigma(H)$. Because H is hyperbolic, $\sigma(H)$ does not meet the imaginary axis, so neither does $\sigma_e(H)$. Therefore, $H+K$ is essentially hyperbolic. Thus, by (3.2) and the remarks after it, we have proved the following

Theorem 3.6. *An operator is essentially hyperbolic if and only if it is a compact perturbation of a hyperbolic operator.*

We denote by $e\mathcal{H}(E)$ the set of essentially hyperbolic operators endowed with the topology induced by the operator norm.

Proposition 3.7. *$e\mathcal{H}(E)$ is an open subset of $\mathcal{L}(E)$, and homeomorphic to the product $\mathcal{H}(\mathcal{C}(E)) \times \mathcal{L}_c(E)$.*

Proof. By Definition 3.1,

$$e\mathcal{H}(E) = p^{-1}(\mathcal{H}(\mathcal{C})), \tag{3.3}$$

where p is the quotient projection defined in (2.2). Because the right term is an open subset of $\mathcal{L}(E)$, so is $e\mathcal{H}(E)$. Because p is linear, continuous and surjective, there exists $s\colon \mathcal{C}(E) \to \mathcal{L}(E)$ continuous such that $p \circ s = \mathrm{id}$. This follows from [4, Proposition A.1]. We define the continuous maps

$$\begin{aligned} &f\colon e\mathcal{H}(E) \to \mathcal{H}(\mathcal{C}(E)) \times \mathcal{L}_c(E), && A \mapsto (p(A), A - s(p(A))),\\ &g\colon \mathcal{H}(\mathcal{C}(E)) \times \mathcal{L}_c(E) \to e\mathcal{H}(E), && (x, K) \mapsto s(x) + K. \end{aligned}$$

Both are well defined. In fact, by (3.3), $p(A) \in \mathcal{H}(\mathcal{C}(E))$. By the property of s,

$$p(A - s(p(A))) = 0,$$

thus the $\mathcal{L}(E)$ component of f is compact, because $\mathcal{L}_c(E) = \ker(p)$. As for g, because $p(s(x)) = x \in \mathcal{H}(\mathcal{C}(E))$, by (3.3), $s(x) \in e\mathcal{H}(E)$, hence

$$\sigma_e(s(x)) \cap \mathrm{i}\mathbb{R} = \emptyset.$$

Because $\sigma_e(s(x)+K) = \sigma_e(s(x))$, $s(x)+K \in e\mathcal{H}(E)$. We conclude the proof by checking that f and g are the inverses of each other:

$$\begin{aligned} f \circ g(x, K) &= f(s(x) + K)\\ &= (p(s(x)+K), s(x) + K - s(p(s(x)+K))) = (x, K),\\ g \circ f(A) &= s(p(A)) + A - s(p(A)) = A. \end{aligned}$$

□

Definition 3.8. Let $x \in \mathcal{A}$ be such that $\sigma^+(x) = \sigma(x) \cap \{\mathrm{Re}(z) > 0\}$ is open and closed in $\sigma(x)$. We denote by $p^+(x)$ the projector $P(x; \{\mathrm{Re}(z) > 0\})$. Similarly, we define $p^-(x)$.

PROPOSITION 3.9. *The map $p^+ : \mathcal{H}(\mathcal{A}) \to \mathcal{P}(\mathcal{A})$ defines a homotopy equivalence, a homotopy inverse being the map $j : \mathcal{P}(\mathcal{A}) \to \mathcal{H}(\mathcal{A})$, $j(p) = 2p - 1$.*

PROOF. For $x \in \mathcal{H}(A)$, $\sigma^+(x)$ is open and closed in $\sigma(x)$. Thus, there exists a rectangle $Q = (0, a) \times (-b, b)$ such that $\sigma(x) \subset Q$. There is a continuous, closed and simple (in the sense of Definiton 2.2) curve c, such that $\mathrm{im}(c) = \partial Q^+$. Thus, $p^+(x) = P_c(x)$. By (c) of Proposition 2.1, there exists $\delta > 0$ such that, if $d(y, x) < \delta$, then $y \in \mathcal{H}(\mathcal{A})$ and $\sigma(y) \subset Q$. Thus $Q^+ \cap \sigma(y) = \sigma^+(y)$ and $p^+(y) = P_c(y)$. By Theorem 2.4, P_c is continuous on $B(x, \delta)$. Thus, p^+ is continuous on a neighbourhood of x, namely $B(x, \delta)$. Repeating the same argument for every x, we obtain that p^+ is continuous on $\mathcal{H}(\mathcal{A})$.

Given a projector p, $j(p)$ is a square root of the unit. In fact,

$$j(p)^2 = (2p - 1)^2 = 4p^2 - 4p + 1 = 1.$$

Thus $\sigma(j(p)) \subseteq \{-1, 1\}$, hence $j(p)$ is hyperbolic. Given $\zeta \in \mathbb{C} \setminus \{-1, 1\}$, we have

$$(j(p)-\zeta)^{-1} = \frac{\zeta}{1-\zeta^2} + \frac{j(p)}{1-\zeta^2} = \frac{1}{2}\left(-\frac{1}{\zeta+1} + \frac{1}{1-\zeta}\right) - \frac{1}{2}\left(-\frac{1}{\zeta+1} - \frac{1}{1-\zeta}\right) j(p).$$

Let c be a simple curve as $c(t) = 1 + e^{-2\pi i t}/2$. Thus, following the notations of Definition 2.2, we have

$$\sigma^+(j(p)) = \sigma(j(p)) \cap \Omega_1(c).$$

Therefore, we can compute the spectral projector relative to $1 \in \sigma(j(p))$ as in Theorem 2.4. By integrating both sides of the above equality,

$$\begin{aligned} p^+(j(p)) &= \frac{1}{2\pi i} \int_c (j(p) - \zeta)^{-1} d\zeta \\ &= \frac{1}{2}\big(-\operatorname{ind}_c(-1) + \operatorname{ind}_c(1)\big) - \frac{1}{2}\big(-\operatorname{ind}_c(-1) - \operatorname{ind}_c(1)\big) j(p) \\ &= \frac{1}{2}\big(0 + 1 + j(p)\big) = \frac{1}{2}(1 + 2p - 1) = p. \end{aligned}$$

The computation above shows that $p^+ \circ j$ is the identity map on $\mathcal{P}(\mathcal{A})$. To prove that $j \circ p^+$ is homotopically equivalent to the identity on $\mathcal{H}(\mathcal{A})$, we define

$$H(t, x) = ((1 - t)x + t)p^+(x) + ((1 - t)x - t)p^-(x). \tag{3.4}$$

Because x is hyperbolic, $\sigma^+(x) \cup \sigma^-(x) = \sigma(x)$, thus, by (c) of Theorem 2.4,

$$p^+(x) + p^-(x) = 1. \tag{3.5}$$

By (2.1) and by (b) of Theorem 2.4, we have

$$\begin{aligned}\sigma(H(t,x)) &= \sigma_{p^+(x)}((1-t)xp^+(x)+tp^+(x)) \\ &\quad \cup \sigma_{p^-(x)}((1-t)xp^-(x)-tp^-(x)) \\ &= \{(1-t)\sigma^+(x)+t\} \cup \{(1-t)\sigma^-(x)-t\}.\end{aligned} \tag{3.6}$$

The second equality follows from (a) of Theorem 2.4 applied to $p^+(x)$ (resp. $p^-(x)$) and $\sigma^+(x)$ (resp. $\sigma^-(x)$). Because the subsets of the complex plane $\{\mathrm{Re}(z) > 0\}$ and $\{\mathrm{Re}(z) < 0\}$ are convex, the sets in the second line of (3.6) do not meet the imaginary axis, and thus $H(t,x)$ is hyperbolic. Moreover,

$$\begin{aligned} H(0,x) &= p^+(x)+p^-(x) = 1, \\ H(1,x) &= p^+(x)-p^-(x) = 2p^+(x)-1 = j(p^+(x)) \end{aligned}$$

by (3.5). Hence H is a homotopy of $j \circ p^+$ with the identity map. □

Because $\mathcal{L}_c(E)$ is a vector space, thus it is contractible to a point, the projection onto the first factor in $\mathcal{H}(\mathcal{C}(E)) \times \mathcal{L}_c(E)$ is a homotopy equivalence. Together with the last two propositions, we have proved the following

COROLLARY 3.10. *The map* $\Psi: e\mathcal{H}(E) \to \mathcal{P}(\mathcal{C}(E))$, $A \mapsto p^+(A+\mathcal{L}_c(E))$ *is a homotopy equivalence.*

Given an essentially hyperbolic operator A, we denote by $P^+(A)$ and $P^-(A)$ the spectral projectors relative to $\{\mathrm{Re}(z) > 0\}$ and $\{\mathrm{Re}(z) < 0\}$, respectively.

PROPOSITION 3.11. *Given a connected component* $\mathcal{X} \subset e\mathcal{H}(E)$, *there exists* $P \in \mathcal{P}(\mathcal{L}(E))$ *such that* $2P - I \in \mathcal{X}$. *Moreover, two essentially hyperbolic operators* A, B *belong to the same connected component* $\mathcal{X}$, *if and only if there exists* $T \in \mathrm{GL}_I(E)$ *such that*

$$TP^+(A)T^{-1} - P^+(B) \in \mathcal{L}_c(E).$$

PROOF. By Proposition 3.7, $e\mathcal{H}(E)$ is an open subset of $\mathcal{L}(E)$. Thus, $\mathcal{X}$ is path-connected. Let $A \in \mathcal{X} \subset e\mathcal{H}(E)$ be an essentially hyperbolic operator. By Theorem 3.6, there exists a hyperbolic operator H such that

$$A - H \in \mathcal{L}_c(E).$$

By the converse of the same theorem, the continuous convex combination

$$\gamma: t \mapsto A + t(H-A)$$

lies in $e\mathcal{H}(E)$. Because $\gamma(0) = A \in \mathcal{X}$, $\gamma(1) = H \in \mathcal{X}$. By Proposition 3.9, $j \circ p^+(H) = 2P^+(H) - I$ is path-connected to H, a path being defined as in (3.4). Thus $2P^+(H) - I \in \mathcal{X}$, and this concludes the first part of the proposition.

Given $A, B \in \mathcal{X}$, there exists a path A_t such that $A(0) = A$ and $A(1) = B$. Thus, the path

$$\alpha := \Psi \circ A \in \mathcal{P}(\mathcal{C}(E))$$

connects $\alpha(0) = \Psi(A)$ to $\alpha(1) = \Psi(B)$. By Proposition 2.8, the fibration $(\mathcal{P}(\mathcal{L}(E), \mathcal{P}(\mathcal{C}(E), p)$ satisfies the homotopy lifting property w.r.t. to the unit interval $[0,1]$. Thus, there exists a path of projectors P such that

$$P(0) = P^+(A), \qquad p(P(t)) = \alpha(t). \tag{3.7}$$

By Theorem 2.6, there exists $T \in \mathrm{GL}_I(E)$ such that

$$TP^+(A)T^{-1} = P(1). \tag{3.8}$$

From (3.7) with $t = 1$, we obtain

$$p(P(1)) = \Psi(B) = p^+(B + \mathcal{L}_c(E)) = P^+(B) + \mathcal{L}_c(E) = p(P^+(B)).$$

The second equality follows from the definition Ψ and the third one from (d) of Proposition 2.4. Thus, comparing the first and the last terms in the chain of equalities above, we obtain

$$P(1) - P^+(B) \in \mathcal{L}_c(E).$$

Hence, by (3.8),

$$TP^+(A)T^{-1} - P^+(B) \in \mathcal{L}_c(E). \qquad \square$$

4. The spectral flow

Let $A\colon [0,1] \to e\mathcal{H}(E)$ be a continuous path. By Proposition 3.10, $\Psi(A(t))$ is a continuous path in $\mathcal{P}(\mathcal{C}(E))$. This path can be lifted to a path of projectors P, such that

$$p(P(t)) = \Psi(A(t)) = p(P^+(A(t)))$$

by Proposition 2.8. We define the integer

$$\mathrm{sf}(A; P) := [P(0) - P^+(A(0))] - [P(1) - P^+(A(1))]. \tag{4.1}$$

Proposition 4.1. *The integer* $\mathrm{sf}(A;P)$ *does not depend on the choice of the path of projectors* P.

Proof. Let Q be a path of projectors such that $p(Q(t)) = p(P(t))$. Thus, $Q(0) - P(0)$ and $Q(1) - P(1)$ are compact operators. By (b) of Theorem 2.12, we have

$$\begin{aligned}\mathrm{sf}(A;Q) &= [Q(0) - P^+(A(0))] - [Q(1) - P^+(A(1))] \\ &= [Q(0) - P(0)] + [P(0) - P^+(A(0))] \\ &\quad - [Q(1) - P(1)] - [P(1) - P^+(A(1))] = \mathrm{sf}(A;Q)\end{aligned}$$

By (c) of Theorem 2.12, $[Q(t) - P(t)]$ is constant. Thus, the third equality follows. □

DEFINITION 4.2. Given $A\colon [0,1] \to e\mathcal{H}(E)$ continuous, we define the *spectral flow* as the integer $\mathrm{sf}(A;P)$ where P is any of the paths of projectors such that $p(P(t)) = p(P^+(A(t)))$. We denote it by $\mathrm{sf}(A)$.

Given $T \in \mathcal{L}(E)$ and $S \in \mathcal{L}(F)$, we refer to $T \oplus S$ as the linear operator on $E \oplus F$ such that $T \oplus S(x,y) = (Tx, Sy)$. Given two paths A and B such that $A(1) = B(0)$, we denote by $A * B$ the continuous path

$$A * B(t) = \begin{cases} A(2t) & \text{if } 0 \le t \le 1/2, \\ B(2t-1) & \text{if } 1/2 \le t \le 1. \end{cases}$$

PROPOSITION 4.3. *The spectral flow satisfies the following properties:*

(a) *Given two paths A and B such that $A(1) = B(0)$,* $\mathrm{sf}(A * B) = \mathrm{sf}(A) + \mathrm{sf}(B)$*;*

(b) *the spectral flow of a constant path or a path in $\mathcal{H}(\mathcal{L}(E))$ is zero;*

(c) *it is invariant for homotopies with endpoints in $\mathcal{H}(\mathcal{L}(E))$ and for fixed-endpoint homotopies in $e\mathcal{H}(E)$;*

(d) *if $A_i \in C([0,1], e\mathcal{H}(E_i))$ for $1 \le i \le n$, then*

$$\mathrm{sf}\biggl(\bigoplus_{i=1}^{n} A_i\biggr) = \sum_{i=1}^{n} \mathrm{sf}(A_i);$$

(e) *if E is an n-dimensional linear space, then for every integer $-n \le k \le n$, there is a path such that* $\mathrm{sf}(A) = k$*;*

(f) *if E has infinite dimension, then for every k there is A such that*

$$\mathrm{sf}(A) = k.$$

PROOF. (a) Let A, B be two paths such that $A(1) = B(0)$. We can choose paths of projectors P and Q such that $p(P(t)) = p(P^+(A(t))$ and $p(Q(t)) = p(P^+(B(t)))$, with $Q(0) = P(1)$. Denote by C and R the paths $A * B$ and $P * Q$, respectively. Then,

$$\begin{aligned} \mathrm{sf}(A * B) &= [R(0) - P^+(C(0))] - [R(1) - P^+(C(1))] \\ &= [P(0) - P^+(A(0))] - [Q(1) - P^+(B(1))] = [P(0) - P^+(A(0))] \\ &\quad - [P(1) - P^+(A(1))] + [Q(0) - P^+(B(0))] - [Q(1) - P^+(B(1))] \\ &= \mathrm{sf}(A) + \mathrm{sf}(B). \end{aligned}$$

(b) If A is constant, $P^+(A(t))$ is constant; if A is hyperbolic, $P^+(A(t))$ is continuous. In both cases, $P^+(A(t))$ is a continuous path and can be chosen as a lifting path of $p(P^+(A(t)))$. Therefore,

$$\mathrm{sf}(A) = [P^+(A(0)) - P^+(A(0))] - [P^+(A(1)) - P^+(A(1))] = 0.$$

(c) Let $H\colon I \times I \to e\mathcal{H}(E)$ be a continuous map. By the homotopy lifting property of the fibre bundle $p\colon \mathcal{P}(\mathcal{L}(E)) \to \mathcal{P}(\mathcal{C}(E))$ w.r.t. I^2, there exists $P\colon I \times I \to \mathcal{P}(\mathcal{L}(E))$ such that

$$P(t,s) - P^+(H(t,s)) \in \mathcal{L}_c(E), \quad \text{for every } t, s.$$

Let $H(\,\cdot\,,0) = A$ and $H(\,\cdot\,,1) = B$. We have

$$\mathrm{sf}(A) = [P(0,0) - P^+(H(0,0))] - [P(1,0) - P^+(H(1,0))].$$

For $i = 0,1$ and every s, the operator $P(i,s) - P^+(H(i,s))$ is compact. For a fixed i, the right summand is constant or continuous, whether the homotopy has fixed endpoints in $e\mathcal{H}(E)$ or lies in $\mathcal{H}(\mathcal{L}(E))$. In both cases, is continuous. By (c) of Theorem 2.12, there are integers k_1, k_2 such that

$$[P(i,s) - P^+(H(i,s))] = k_i$$

for every $0 \le s \le 1$ and $i = 0,1$. Thus, $\mathrm{sf}(A) = k_0 - k_1 = \mathrm{sf}(B)$.

(d) Let P_i be continuous paths of projectors such that $P_i(t) - P^+(A_i(t)) \in \mathcal{L}_c(E_i)$.

$$\begin{aligned}\mathrm{sf}\bigg(\bigoplus_{i=1}^n A_i\bigg) &= \bigg[\bigoplus_{i=1}^n P_i(0) - \bigoplus_{i=1}^n P^+(A_i(0))\bigg] - \bigg[\bigoplus_{i=1}^n P_i(1) - \bigoplus_{i=1}^n P^+(A_i(1))\bigg] \\ &= \sum_{i=1}^n [P_i(0) - P^+(A_i(0))] - [P_i(1) - P^+(A_i(1))] = \sum_{i=1}^n \mathrm{sf}(A_i).\end{aligned}$$

(e) We denote the identity on $\mathbb{R}^k$ by I_k. Given $0 \le k \le n$, the spectral flow of

$$A(t) = (2t-1)I_k \oplus I_{n-k}$$

can be computed using $P(t) \equiv I_n$. Because $P^+(A(1)) = I_n$ and $P^+(A(0)) = 0 \oplus I_{n-k}$, we have

$$\mathrm{sf}(A; I_n) = [I_n - 0 \oplus I_{n-k}] - [I_n - I_n] = k$$

by (a) of Theorem 2.12. We define $\overline{A}(t) := A(1-t)$. By property (a), proved above, $\mathrm{sf}(\overline{A}; I_n) = -k$.

(f) Given $k \in \mathbb{Z}$, let $E = X^k \oplus R_k$ where X^k is a closed subspace and $\dim(R_k) = k$. Thus, the spectral flow of $A(t) = (2t-1)I_{R_k} \oplus I_{X^k}$ can be computed with $P(t) \equiv I$. We obtain $\mathrm{sf}(A; I) = k$ and $\mathrm{sf}(\overline{A}; I) = -k$. □

Spectral sections. The definition of spectral flow we used corresponds to the one given by C. Zhu and Y. Long in [29] for paths of *admissible* operators (see [29, Definition 2.3]), which are essentially hyperbolic. We recall the definition of s-section:

Definition 4.4. An *s-section* for a path of projectors Q on $J \subset [0,1]$ is a continuous path P such that $P(t) - Q(t) \in \mathcal{L}_c(E)$.

Given a continuous path $A\colon [0,1] \to e\mathcal{H}(E)$, the authors show in [29, Lemma 2.5] and [29, Corollary 2.1] that there exists a partition of the unit interval $(J_k)_{k=1}^n$ and $P_k\colon J_k \to \mathcal{P}(\mathcal{L}(E))$ such that

$$P_k \text{ is an s-section for } P^+(A) \text{ on } J_k, \tag{4.2}$$

$$P_k(t) \text{ is a spectral projector of } A(t). \tag{4.3}$$

Then, they define

$$\mathrm{sf}(A) = \sum_{k=1}^{n} \mathrm{sf}(A_k; P_k)$$

where A_k is the restriction of A to J_k. In our definition we do not need to partition the unit interval because we dropped the requirement (4.3). This allows us to simplify the definition of spectral flow and to provide simpler proofs of well known properties – such as the homotopy invariance – than the original ones in [29, Proposition 2.2] or in [21].

We conclude by showing that there exists a path A such that $P^+(A(t))$ does not admit an s-section fulfilling (4.3).

Example 4.5. Consider the decomposition $E = R_1 \oplus X_- \oplus X_+$, where E is a Banach space, X_- and X_+ are closed, infinite-dimensional subspaces and $\dim(R_1) = 1$. Denote by P_1, P_-, P_+ the projectors onto R_1, X_- and X_+, respectively. Define

$$A\colon [0,1] \to e\mathcal{H}(E), \quad A(t) = P_+ - P_- + (2t-1)P_1.$$

Then $A(t) \in e\mathcal{H}(E)$, and no continuous s-section satisfying (4.3) exists.

Proof. For every $t \in [0,1]$ we can write $A(t) = P_+ - (P_- + P_1) + 2tP_1$; because $P_+ - P_- - P_1 \in \mathcal{H}(\mathcal{L}(E))$ (in fact, is a square root of the identity) and P_1 is compact, $A(t) \in e\mathcal{H}(E)$. By contradiction, suppose that such P exists. We have

$$A(0) = P_+ - P_- - P_1, \quad P^+(A(0)) = P_+, \quad \sigma(A(0)) = \{-1, 1\}.$$

Because $P(0)$ is spectral, there exists $\Sigma_0 \subset \sigma(A(0))$ such that $P(0) = P(A(0); \Sigma_0)$. The only choice is $\Sigma_0 = \{1\}$, thus $P(0) = P(A(0); \{1\}) = P_+$. On the other endpoint,

$$A(1) = P_+ - P_- + P_1, \quad P^+(A(1)) = P_+ + P_1, \quad \sigma(A(1)) = \{-1, 1\}.$$

As above, $P(1)$ is spectral and $P(1) = P(A(1); \{1\}) = P_+ + P_1$. Because P is an s-section and $P^+(A(t)) - P^+(A(s))$ is compact for every $0 \leq t, s \leq 1$, $P(t) - P(s)$ is also compact. By (c) of Theorem 2.12, $m(t) := [P(t) - P(0)]$ is constant. Because $m(0) = 0$, $m(1) = 0$. But,

$$0 = m(1) = [P(1) - P(0)] = [(P_+ + P_1) - P_+] = [P_1] = 1$$

where the last equality follows from (a) of Theorem 2.12. Thus, we obtained a contradiction. □

5. Spectral flow as group homomorphism

By (c) and (a) of Proposition 4.3, the spectral flow determines a $\mathbb{Z}$-valued group homomorphism on the fundamental group of each connected component of $e\mathcal{H}(E)$. Given a projector P, we denote by sf_P the spectral flow on the fundamental group of the connected component of $2P - I$.

The fiber of $p \colon \mathcal{P}(\mathcal{L}(E)) \to \mathcal{P}(\mathcal{C}(E))$ over a point of the base space, $P + \mathcal{L}_c(E)$ is the set

$$\mathcal{P}_c(E; P) = \{Q \in \mathcal{P}(\mathcal{L}(E)) : Q - P \in \mathcal{L}_c(E)\}$$
$$i \colon \mathcal{P}_c(E; P) \hookrightarrow \mathcal{P}(\mathcal{L}(E)).$$

Proposition 5.1. *The connected components of $\mathcal{P}_c(E; P)$ correspond to $\mathbb{Z}$ through the bijection $Q \mapsto [P - Q]$ for every projector P. Moreover, if the range and the kernel have infinite dimension, $\pi_1(\mathcal{P}_c(E; P), P) \cong \mathbb{Z}_2$.*

The two facts follow from [13, Theorem 6.3] and [13, Theorems 7.2,7.3].

Because p induces a Serre fibration, the sequence of homomorphisms

$$\pi_1(\mathcal{P}_c(E; P), P) \xrightarrow{i_*} \pi_1(\mathcal{P}(\mathcal{L}(E)), P) \xrightarrow{p_*} \pi_1(\mathcal{P}(\mathcal{C}(E)), P + \mathcal{L}_c(E)) \tag{5.1}$$

is exact. The homotopy equivalence Ψ defined in Corollary 3.10 determines a group isomorphism

$$\Psi_* \colon \pi_1(e\mathcal{H}(E), 2P - I) \to \pi_1(\mathcal{P}(\mathcal{C}(E)), P + \mathcal{L}_c(E)).$$

Theorem 5.2. *There exists a homomorphism δ_P such that the sequence*

$$\pi_1(\mathcal{P}(\mathcal{L}(E)), P) \xrightarrow{p_*} \pi_1(\mathcal{P}(\mathcal{C}(E)), P + \mathcal{L}_c(E)) \xrightarrow{\delta_P} \mathbb{Z}, \tag{5.2}$$

is exact and $\delta_P \circ \Psi_* = \mathrm{sf}_P$.

Proof. Let a be a loop at the base point $P + \mathcal{L}_c(E)$, and let Q be a path of projectors such that $Q(0) = P$ and $p(Q(t)) = a(t)$. Because $a(0) = a(1)$,

$Q(1) - P$ is compact. We define

$$\delta_P([a]) = [Q(1) - P].$$

Arguing as in Proposition 4.1, δ_P is well defined. Let A be a closed path in $e\mathcal{H}(E)$ and Q a path of projectors such that

$$p(Q(t)) = \Psi(A(t)), \quad Q(0) = P.$$

Hence, $Q(t) - P^+(A(t))$ is compact for every $t \in [0,1]$. By (4.1), $\mathrm{sf}(A) = [Q(1) - P]$. By the definition above, $\delta_P([\Psi \circ A]) = [Q(1) - P]$, thus $\delta_P \circ \Psi_* = \mathrm{sf}_P$. Because Ψ_* is invertible, δ_P is a homomorphism. We prove that δ_P is exact.

$\ker(\delta_P) \subseteq \mathrm{im}(p_*)$: Let a be a loop at the base point $P + \mathcal{L}_c(E)$ such that $[a] \in \ker(\delta_P)$. Then,

$$[Q(1) - P] = 0.$$

P and $Q(1)$ are in the same connected component in $\mathcal{P}_c(E; P)$ by Proposition 5.1. Thus, there exists a continuous path of projectors R such that

$$R(0) = Q(1), \quad R(1) = P, \quad R(t) - P \in \mathcal{L}_c(E) \quad \text{for every } t.$$

Set $S := Q * R$. It is a closed path of projectors, and $p \circ S = a * c_{p(P)}$, where $c_{p(P)}$ is the constant path $p(P)$. Thus, $[a] \in \mathrm{im}(p_*)$.

$\mathrm{im}(p_*) \subseteq \ker(\delta_P)$: Given a loop $P_t \in \mathcal{P}(\mathcal{L}(E))$ at the base point P, we have

$$\delta_P(p_*(P_t)) = [P(1) - P] = 0$$

because $P(1) = P$. □

PROPOSITION 5.3. *Given a projector P, $m \in \mathrm{im}(\mathrm{sf}_P)$ if and only if there exists a projector Q such that $Q - P$ is compact, $[Q - P] = m$ and is path-connected to P.*

By the previous theorem, $\mathrm{im}(\mathrm{sf}_P) = \mathrm{im}(\delta_P)$. Therefore, the proposition follows from the definition of δ_P.

We define the following properties:

(h1) P is path-connected to a projector Q such that $Q - P$ is compact and $[Q - P] = m$.

(h2) the image of $p_*\colon \pi_1(\mathcal{P}(\mathcal{L}(E)), P) \to \pi_1(\mathcal{P}(\mathcal{C}(E), p(P))$ is trivial.

COROLLARY 5.4. *Given $P \in \mathcal{P}(\mathcal{L}(E))$, we characterise the kernel and the image of the spectral flow* sf_P*:*

(a) $m \in \mathrm{im}(\mathrm{sf}_P)$ *if and only if P fulfills property* (h1).

(b) $\mathrm{im}(p_*) \cong \ker(\mathrm{sf}_P)$. sf_P *is injective if and only if P fulfills property* (h2).

The isomorphism classes of the kernel and the image of sf_P depend only on the conjugacy class of $P + \mathcal{L}_c(E)$ in $\mathcal{P}(\mathcal{C}(E))$. We show that in many cases we can find a projector P such that sf_P is an isomorphism.

Lemma 5.5. *Let E be a Banach space, and $X, Y \subset E$ closed subspaces such that $X \cong Y$ and $X \oplus Y = E$. Then, the projectors P_X, P_Y with ranges X and Y respectively, are connected by a continuous path in $\mathcal{P}(\mathcal{L}(E))$.*

A proof of this can be found in [22, §9] or in [19].

Proposition 5.6. *Let $X, Y \subset E$ be as above. Suppose that X is isomorphic to its closed subspaces of co-dimension m. Let P be the projector onto X with kernel Y. Then P satisfies the property* (h1) *w.r.t. m.*

Proof. Let $X^m, R_m \subset X$ be closed subspaces such that $\dim(R_m) = m$ and $X^m \cong X$. We have the following decompositions and isomorphism:

$$E = R_m \oplus X^m \oplus Y, \quad X^m \cong Y, \quad R_m \oplus X^m = X.$$

By applying Lemma 5.5 to $X^m \oplus Y$ and subspaces X^m and Y, we obtain that P_{X^m} is connected to P_Y. By applying it a second time to E and subspaces X and Y, we obtain that P_X is connected to P_Y. Hence, P_X is connected to P_{X^m}.□

In Proposition 5.6, we required E to be isomorphic to a cartesian product of a space X with itself, but it suffices that E has a complemented subspace F fulfilling the requirements of Proposition 5.6. In fact, if $A_t \in e\mathcal{H}(F)$ is such that $\mathrm{sf}(A_t) = m$, then $\mathrm{sf}(I \oplus A_t) = m$, by (d) of Proposition 4.3.

Proposition 5.7. *Given $P \in \mathcal{P}(\mathcal{L}(E))$, the map $\pi\colon \mathrm{GL}(E) \to \mathcal{P}(\mathcal{L}(E))$, $T \mapsto TPT^{-1}$ defines a principal bundle with fiber $\mathrm{GL}(X) \times \mathrm{GL}(Y)$, where $X = \mathrm{Range}(P)$ and $Y = \ker(P)$.*

A proof of this can be found in [10, Theorem 2.1] or in [4, Proposition 1.2]. Both theorems are stated in the more general setting of Banach algebras.

Corollary 5.8. *If $\mathrm{GL}(E)$ is simply-connected and $\mathrm{GL}(X), \mathrm{GL}(Y)$ are connected, then sf_P is injective.*

Proof. Because a locally trivial bundle is a Serre fibration, we have a long exact sequence of homomorphisms that ends

$$\pi_1(\mathrm{GL}(E), I) \xrightarrow{\pi_*} \pi_1(\mathcal{P}(\mathcal{L}(E)), P) \xrightarrow{\Delta} \pi_0(\mathrm{GL}(X) \times \mathrm{GL}(Y), I).$$

Thus, if $\mathrm{GL}(E)$ is simply connected and $\mathrm{GL}(X)$ and $\mathrm{GL}(Y)$ are connected, the middle group is trivial, hence in (5.2) p_* is the trivial map, thus δ_P is injective and sf_P is injective. □

Then, we have sufficient conditions for a Banach space to have at least one projector P such that sf_P is an isomorphism.

Theorem 5.9. *Let $E = X \oplus X$ be such that X is isomorphic to its hyperplanes,* $\mathrm{GL}(E)$ *is simply-connected and* $\mathrm{GL}(X)$ *is connected. Then* sf_{P_X} *is an isomorphism.*

Proof. That sf_{P_X} is surjective follows from Proposition 5.6. From the corollary above, sf_{P_X} is also injective. $\square$

Let us consider the particular case, where E is isomorphic to $E \times E$ and to its hyperplanes, and $\mathrm{GL}(E)$ is *contractible* to a point. This, in fact, is the case of the most common infinite-dimensional spaces as separable Hilbert spaces, c_0, ℓ^p with $p \geq 1$, and $L^p(\Omega, \mu)$ with $p > 1$, $C(K, F)$ for large classes of compact spaces K and Banach spaces F, and many others. For a richer list, see Theorem 2 of [28] and [17], [6], [20], [19]. Sequence spaces ℓ^p, ℓ^∞ and c_0 are also *prime* (see [5, Theorem 2.2.4] and [18]), that is, they are isomorphic to their complemented, infinite-dimensional subspaces. Thus, for *every* projector P such that $\mathrm{Range}(P)$ and $\ker(P)$ have infinite dimension, sf_P is an isomorphism.

Trivial spectral flow. When P is a projector with a finite-dimensional range or kernel, $P + \mathcal{L}_c(E)$ is 0 or 1, then its connected component in $\mathcal{P}(\mathcal{C}(E))$ is $\{0\}$ or $\{1\}$. Hence, $\mathrm{sf}_P = 0$. This is the case of finite-dimensional spaces. A space is said to be *undecomposable* if the only projectors are as above. In [14], W. T. Gowers and B. Maurey showed the existence of an infinite-dimensional, undecomposable space.

Non-trivial and not surjective spectral flow. W. T. Gowers and B. Maurey proved in [15] the existence of a space isomorphic to their subspaces of codimension two, but not their hyperplanes. If we denote by X a space with such a property and by P the projector onto the first factor in $E = X \oplus X$, then $2 \in \mathrm{im}(\mathrm{sf}_P)$ by Proposition 5.6. However, if $1 \in \mathrm{im}(\mathrm{sf}_P)$, by Proposition 5.3 and (c2) in Section 2.2, X would be isomorphic to its hyperplanes.

The Douady space. We show the existence of a projector P such that δ_P, and thus sf_P, is not injective.

Proposition 5.10. *Let $E = X \oplus X$. Given $T \in \mathrm{GL}(X)$, there exists a loop x in the space of projectors such that*

$$\Delta(x) = \begin{pmatrix} T & 0 \\ 0 & T^{-1} \end{pmatrix},$$

where Δ is the connecting homomorphism in the sequence of Corollary 5.8.

Proof. Let M be the operator defined in the line above. Let $U_t \in \mathrm{GL}(E)$ be such that $U(1) = M$ and $U(0) = I$. The existence of U follows from the fact that $T \oplus T'$ is connected to $TT' \oplus I$ (see [19]). Because M commutes with P_X, the path

$$P(t) = U(t) P_X U(t)^{-1}$$

is a loop in $\mathcal{P}(\mathcal{L}(E))$ with base point P_X. We denote its homotopy class by x. The path U_t is a lifting path for P. Hence $\Delta(x) = U(1) = M$. $\square$

Let F and G be Banach spaces such that

(i) every bounded map $G \to F$ is compact;

(ii) both F and G are isomorphic to their hyperplanes.

The next lemma follows from a more general result of A. Douady, [12, Proposition 1]. We briefly sketch the proof by B. S. Mitjagin in [19].

Lemma 5.11. *Let $X = F \oplus G$, F and G as above. Then, there exists a continuous, surjective homomorphism $j\colon \mathrm{GL}(X) \to \mathbb{Z}$.*

Proof. Let $T \in \mathrm{GL}(X)$ be an invertible operator and S be its inverse. We have

$$\begin{pmatrix} I_F & 0 \\ 0 & I_G \end{pmatrix} = TS = \begin{pmatrix} T_{11}S_{11} + T_{12}S_{21} & T_{11}S_{12} + T_{12}S_{22} \\ T_{21}S_{11} + T_{22}S_{21} & T_{22}S_{22} + T_{21}S_{12} \end{pmatrix}.$$

A similar equality holds for ST. Taking the first element of the diagonals of TS and ST, respectively, we have the following relations

$$T_{11}S_{11} + T_{12}S_{21} = I_F, \qquad S_{11}T_{11} + S_{12}T_{21} = I_F.$$

Because S_{21} and T_{21} are compact operators, T_{11} and S_{11} are the essential inverse of each other. According to (e) of Proposition 2.10, T_{11} is a Fredholm operator. We define

$$j(T) = \mathrm{ind}(T_{11}).$$

By (a) of Proposition 2.10, there exists $\varepsilon > 0$ such that, if $\|T'_{11} - T_{11}\| < \varepsilon$, then T'_{11} is a Fredholm operator and $\mathrm{ind}(T'_{11}) = \mathrm{ind}(T_{11})$. This proves the continuity. Moreover, given two invertible operators T and S, we have

$$j(TS) = \mathrm{ind}(TS)_{11} = \mathrm{ind}(T_{11}S_{11} + T_{12}S_{21}) = \mathrm{ind}(T_{11}S_{11}) = \mathrm{ind}(T_{11}) + \mathrm{ind}(S_{11}),$$

where (b) and (c) of Proposition 2.10 have been used. Thus, j is a group homomorphism. Let F^1 and G^1 be hyperplanes of F and G, respectively. We define

$$\begin{aligned} &\sigma\colon F \to F^1, && F = \langle v \rangle \oplus F^1, \\ &\tau\colon G^1 \to G, && G = \langle w \rangle \oplus G^1, \\ &B\colon G \to F, && tw + y \mapsto tv. \end{aligned}$$

where σ, τ are isomorphisms, which exist by (i). We define

$$T(x,y) = (\sigma(x) + B(y), \tau(Py))$$

where $P\colon G \to G$ is a projector onto G^1. T is invertible and $\mathrm{ind}(T_{11}) = \mathrm{ind}(\sigma) = -1$. Because j is a homomorphism, it is surjective. $\square$

Proposition 5.12. *If $E = X \oplus X$, where X is a direct sum of two spaces F and G as in* (i) *and* (ii) *above, then sf_{P_X} is surjective, but not injective.*

Proof. From the lemma above, for every $k \in \mathbb{Z}$, there exists $T_k \in \mathrm{GL}(X)$ such that $j(T_k) = k$. By Proposition 5.10, there exists $x_k \in \pi_1(\mathcal{P}(\mathcal{L}(E)), P_X)$ such that $\Delta(x_k) = T_k \oplus T_k^{-1}$ and $x_k \neq x_h$ if $k \neq h$. Then $\pi_1(\mathcal{P}(\mathcal{L}(E)), P_X)$ has infinitely many elements, while $\pi_1(\mathcal{P}_c(E; P), P_X)$ is a finite group, by Proposition 5.1. Hence, in (5.1), i_* is not surjective, thus $x_k \notin \mathrm{im}(i_*)$ for infinitely many $k \in \mathbb{Z}$. Hence,

$$p_*(x_k) \neq 0, \quad \delta_{P_X}(p_*(x_k)) = 0.$$

because the sequence (5.1) is exact; therefore, the kernel of δ_{P_X} is not trivial. Because E and X fulfill the hypothesis of Proposition 5.6, sf_{P_X} is surjective. $\square$

By [26, Theorem 4.23], examples of pairs of Banach spaces as in (i) and (ii) are given by (ℓ^p, ℓ^2), with $p > 2$.

In the next proposition, we show the existence of a projector P whose range and kernel are isomorphic to their hyperplanes, but $\mathrm{sf}_P = 0$.

Proposition 5.13. *Let $X = F \oplus G$, where F and G fulfill properties* (i) *and* (ii). *Then,* $\mathrm{sf}_{P_F} = 0$.

Proof. Let $0 \geq -m \in \mathrm{im}(\mathrm{sf}_{P_F})$. Then, by (a) of Corollary 5.4, P_F is connected to a projector $Q \in \mathcal{P}(\mathcal{L}(X))$ such that $Q - P_F$ is compact and $[P_F - Q] = m$. Let $P_m \in \mathcal{P}(\mathcal{L}(X))$ be a projector onto a subspace $F^m \subset F$ of codimension m such that $P_m(I - P_F) = 0$. Thus, $P_F - P_m$ is compact and $[P_F - P_m] = m$. Therefore, $Q - P_m$ is compact and $[P_m - Q] = 0$. By Proposition 5.1, Q is connected to P_m. Hence, P_F is connected to P_m. By Theorem 2.6, there exists a continuous path $U_t \in \mathrm{GL}(X)$ such that

$$U(0) = I, \quad U(1)P_F = P_m U(1).$$

From these relations, it follows that $U(1)_{11}(F) = F^m$, hence $j(U(1)) = -m$, and $j(U(0)) = 0$. By the lemma above, $j(U(t))$ is constant, therefore $m = 0$. $\square$

6. The Fredholm index of F_A and the spectral flow

Definition 6.1. A path $A\colon \mathbb{R} \to \mathcal{L}(E)$ of bounded operators is said to be *asymptotically hyperbolic* if the limits $A(+\infty)$ and $A(-\infty)$ exist and are hyperbolic operators.

If A is also essentially hyperbolic, we can define the spectral flow as follows: Because the set of hyperbolic operators is an open subset of $\mathcal{L}(E)$, there exists $\delta > 0$ such that $A(t)$ is hyperbolic for every $t \in (-\infty, -\delta] \cup [\delta, +\infty)$. We set

$$\mathrm{sf}(A) = \mathrm{sf}(A, [-\delta, \delta]). \tag{6.1}$$

The definition does not depend on the choice of δ by (a) and (b) of Proposition 4.3. Let $P^+(A)$ be the spectral projector $P(A;\{\mathrm{Re}(z)>0\})$.

Definition 6.2. An asymptotically hyperbolic path is called *essentially splitting* if the following properties

(a) $P^+(A(+\infty)) - P^+(A(-\infty))$ is compact;

(b) $[A(t), P^+(A(+\infty))]$ is compact for every $t \in \mathbb{R}$,

hold, where $[A,P] = AP - PA$.

The definition above corresponds to the one given in [2, Theorem 6.3] when (a) holds. For short, we will refer to essentially splitting as a path satisfying (a) and (b).

Lemma 6.3. *Let A be an asymptotically hyperbolic and essentially hyperbolic path. It is also essentially splitting if and only if $P^+(A(t)) - P^+(A(s))$ is compact for every $t,s \in \mathbb{R}$.*

Proof. Suppose A is essentially splitting. We denote by E^+ and E^- the ranges of $P^+(A(+\infty))$ and $P^-(A(+\infty))$, respectively. With respect to the decomposition $E = E^+ \oplus E^-$, we can write $A(t)$ block-wise:

$$A(t) = \begin{pmatrix} A_+(t) & K_\pm(t) \\ K_\mp(t) & A_-(t) \end{pmatrix}$$

where $K_\pm(t)$ and $K_\mp(t)$ are compact operators by (a). Then $A(t)$ is a compact perturbation of $A_+(t) \oplus A_-(t)$. Therefore

$$P^+(A(t)) - P^+(A_+(t)) \oplus P^+(A_-(t)) \in \mathcal{L}_c(E) \quad \text{for every } t \in \mathbb{R}. \tag{6.2}$$

Because $A(+\infty)$ commutes with $P^+(A(+\infty))$ and $P^-(A(+\infty))$, $K_\pm(+\infty)$, $K_\mp(+\infty)$ are null operators, hence $A(+\infty) = A_+(+\infty) \oplus A_-(+\infty)$. By the definition of essential spectrum and (b) of Proposition 2.10, $\sigma_e(A(t)) = \sigma_e(A_+(t)) \cup \sigma_e(A_-(t))$. Therefore,

$$P^+(A_+(+\infty)) = I_{E^+}, \quad A_+(t) \in e\mathcal{H}(E^+), \tag{6.3}$$

$$P^+(A_-(+\infty)) = 0_{E^-}, \quad A_-(t) \in e\mathcal{H}(E^-). \tag{6.4}$$

From the first part of (6.3) and Proposition 3.11,

$$A_+(+\infty) \in \mathcal{X}_{I_{E^+}} \subset e\mathcal{H}(E^+).$$

where $\mathcal{X}_{I_{E^+}}$ is the connected component of I_{E^+} in $e\mathcal{H}(E^+)$. Because $e\mathcal{H}(E^+)$ is locally path-connected, $\mathcal{X}_{I_{E^+}}$ is open. Then, by the second part of (6.3)

$$A_+(t) \in \mathcal{X}_{I_{E^+}} \quad \text{for every } t \in \mathbb{R}.$$

By Proposition 3.11,

$$P^+(A_+(t)) - I_{E^+} \in \mathcal{L}_c(E) \quad \text{for every } t \in \mathbb{R}. \tag{6.5}$$

Similarly, from (6.4) and Proposition 3.11, we obtain

$$P^+(A_-(t)) \in \mathcal{L}_c(E) \quad \text{for every } t \in \mathbb{R}. \tag{6.6}$$

By the definition of $P^+(A(+\infty))$, $I_{E^+} \oplus 0_{E^-} = P^+(A(+\infty))$. Thus, by (6.2), (6.5) and (6.6),

$$P^+(A(t)) - P^+(A(+\infty)) \in \mathcal{L}_c(E) \quad \text{for every } t \in \mathbb{R}.$$

Conversely, suppose that each of the projectors of the set $\{P^+(A(t)) : t \in \mathbb{R}\}$ is a compact perturbation of the others. Let $a > 0$ be such that $A(s)$ is hyperbolic for every $|s| > a$. By the continuity of P^+, it follows that

$$P^+(A(+\infty)) - P^+(A(-\infty)) = \lim_{\substack{s\to+\infty \\ |s|>a}} (P^+(A(s)) - P^+(A(-s))).$$

because A is asymptotically hyperbolic. Hence, the left member is the limit of a sequence of compact operators. Because $\mathcal{L}_c(E)$ is a closed subset of $\mathcal{L}(E)$, we have proved (a). Property (b) follows from the equality

$$[A(t), P^+(A(s))] = [A(t), P^+(A(t))] + [A(t), P^+(A(s)) - P^+(A(t))],$$

where the first summand of the right member is zero and the second one is compact by hypothesis. In particular, the equality holds for $s > a$, so we finish our proof by taking the limit on the left member as $s \to +\infty$. □

The spectral flow and the Fredholm index of F_A. Given a continuous, bounded path $A_t \in \mathcal{L}(E)$, we denote by F_A the differential operator

$$F_A\colon W^{1,p}(\mathbb{R}, E) \to L^p(\mathbb{R}, E), \quad F_A(u) = \frac{du}{dt} - A(\,\cdot\,)u.$$

Theorem 6.4. *Let A be an asymptotically hyperbolic and essentially splitting path. Then F_A is a Fredholm operator and*

$$\operatorname{ind}(F_A) = [P^-(A(-\infty)) - P^-(A(+\infty))]$$

A. Abbondandolo and P. Majer proved it in [2, Theorem 6.3] where E is a Hilbert space and $p = 2$. However, the theorem, like much of the content of their work, can be generalised to Banach spaces as in [13, Theorem 3.3].

Theorem 6.5. *Let A be an asymptotically hyperbolic, essentially splitting and essentially hyperbolic path. Then,*

$$\operatorname{sf}(A) = -[P^-(A(-\infty)) - P^-(A(+\infty))]$$

Proof. Let $\delta > 0$ as in (6.1). By Lemma 6.3, the constant path $P \equiv P^+(A(\delta))$ is an s-section for $P^+(A_t)$ on $[-\delta, \delta]$ in the sense of Definition 4.4. Hence,

$$\operatorname{sf}(A) = [P^+(A(\delta)) - P^+(A(-\delta))].$$

Because A is hyperbolic on $(-\infty - \delta] \cup [\delta, +\infty)$, the path $P^+(A_t)$ is continuous on this subset. By (c) of Theorem 2.12,

$$[P^-(A(-\infty)) - P^-(A(+\infty))] = -[P^+(A(\delta)) - P^+(A(-\delta))] = -\mathrm{sf}(A). \qquad \square$$

From Theorems 6.5 and 6.4 we have the final result.

Theorem 6.6. *If A is essentially hyperbolic, essentially splitting and asymptotically hyperbolic, then*

$$\mathrm{ind}(F_A) = -\mathrm{sf}(A). \tag{6.7}$$

Let $A(t) = A_0(t) + K(t)$, where $A_0(t)$ is hyperbolic and A_0, A are asymptotically hyperbolic. $K(t)$ is compact and

$$\begin{aligned} A_0(t)E^- &\subseteq E^-, & A_0(t)E^+ &\subseteq E^+, \\ E^- &= E^-(A_0(\pm\infty)), & E^+ &= E^+(A_0(\pm\infty)). \end{aligned}$$

The second line tells us that $P^+(A_0(+\infty)) = P^+(A_0(-\infty))$ and thus $P^+(A(+\infty)) - P^+(A(-\infty))$ is compact. From the first line, it follows that $[A(t), P^+(A(+\infty))]$ is compact. Thus, by Theorem 6.6, we confirm the guess of A. Abbondandolo and P. Majer in [2, §7], that for paths satisfying the hypotheses of [2, Theorem E], corresponding to those listed above, the relation (6.7) holds.

When A is not essentially splitting, the authors provided in [2, Example 6.7] counterexamples to (6.7).

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Daniele Garrisi
Pohang Mathematics Institute
Department of Mathematics
Postech San 31
Hyojadong, Namgu, Pohang
Gyungbuk 790-784, REPUBLIC OF KOREA