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ROOT PROBLEM FOR CONVENIENT MAPS

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Abstract. In this paper we study when the minimal number of roots of the so-called convenient maps from two-dimensional CW complexes into closed surfaces is zero. We present several necessary and sufficient conditions for such a map to be root free. Among these conditions we have the existence of specific liftings for the homomorphism induced by the map on the fundamental groups, existence of the so-called mutation of a specific homomorphism also induced by the map, and existence of particular solutions of specific systems of equations on free groups over specific subgroups.

1. Introduction

Let $f\colon K \to Y$ be a continuous map from a finite 2-dimensional CW complex into a closed surface. The root problem for such a map is concerned, roughly, with the study of its minimal number of roots, denoted by $\mu(f)$, which is defined to be the minimal cardinality of $g^{-1}(a)$ among all maps g homotopic to f, where $a \in Y$ is an arbitrary point. By [6], the number $\mu(f)$ is independent of the point $a \in Y$ and it is finite. When $\mu(f) = 0$, that is, f is homotopic to a map which is not onto, we say that f is *root free*.

The Nielsen root theory provides a number, called the Nielsen root number, denoted by $N(f)$, which is a lower bound for $\mu(f)$ (see [2] for details). D. L. Gonçalves and P. Wong [7] proved that, under the conditions assumed

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here, $N(f) = 0$ implies f is root free, what does not occur in general (when Y is not a surface, for example). In many cases, it is not easy to compute the number $N(f)$.

In this paper, we prove necessary and sufficient conditions for the mentioned map $f\colon K \to Y$ to be root free. The sufficient part of these conditions restrict the problem to a specific type of maps, the so-called *convenient maps* (Definition 2.2).

In Section 2, we study the first of such conditions which is about the triviality of the homomorphism $f_{\#_2}\colon \pi_2(K) \to \pi_2(Y)$, induced by f on the second homotopy groups, and the existence of a lifting for $f_{\#}\colon \pi_1(K) \to \pi_1(Y)$ through the homomorphism $l_{\#}\colon \pi_1(Y^1) \to \pi_1(Y)$ induced by the natural inclusion $l\colon Y^1 \to Y$ of the 1-skeleton Y^1 of the surface Y into Y. (Here we are considering surfaces with their minimal cellular decomposition). In Section 3 we present some consequences of the main result of Section 2, the Theorem 2.6.

In Section 4, we define a new concept *mutation of a homomorphism*. This is used to provide conditions for the existence of liftings of a homomorphism through an epimorphism from a free group into an arbitrary group. Such conditions will be used later, in Section 5, to provide conditions for maps to be root free.

A similar concept *symbolic mutation* is presented in Section 6. In fact, we prove that symbolic mutation is a kind of generalization of the concept of mutation. In Section 7, we use this new concept to show alternative ways to use the main results (theorems) of previous sections. Namely, we present results linking the annihilation of the roots of a map f with the existence of particular solutions of a system of equations on a free group.

We finish the paper presenting in Section 8 several examples to illustrate the applicability of the main results.

Throughout the text, we use the capital letter K to denote a finite and connected two-dimensional CW complex. We simplify two-dimensional CW complex by 2-complex. The capital letter Y is used to denote closed surfaces. We also simplify f is a continuous map by f is a map. The homotopy homomorphisms induced by f are denoted as $f_{\#}$ and homology homomorphisms as f_*.

2. Convenient maps

Let K and L be finite and connected 2-complexes and let $\Pi = \pi_1(K)$ and $\Xi = \pi_1(L)$. Let $f\colon K \to L$ be a map from K into L and let $\alpha = f_{\#}\colon \Pi \to \Xi$ be its induced homomorphism on fundamental groups. Since Ξ acts on the group $\pi_2(L)$ making it into a Ξ-module, a *change of ring procedure* defines an action of Π on the group $\pi_2(L)$ making it a Π-module. The procedure is the following: For each $\pi \in \Pi$ and each $\gamma \in \pi_2(M)$, we define the action $\pi \cdot \gamma = \alpha(\pi) \cdot \gamma$. To avoid confusion, when $\pi_2(L)$ is viewed as a Π-module through this procedure, we denote it by ${}_\alpha\pi_2(L)$.

Note that if α is the trivial homomorphism, then the action of Π on $\pi_2(L)$ is also trivial, that is, $\pi \cdot \gamma = \gamma$. This occur, in particular, if either K or L (or both) is simply connected.

Since ${}_\alpha\pi_2(L)$ is a Π-module, we have the second cohomology module of Π with coefficients in ${}_\alpha\pi_2(L)$, denoted by $H^2(\Pi; {}_\alpha\pi_2(L))$.

Let $[K, L]_\alpha$ be the set of the based homotopy classes of based maps from K into L inducing the homomorphism $\alpha\colon \Pi \to \Xi$ on fundamental groups.

The following result is Corollary 4.13 of [1, p. 95].

Theorem 2.1. *Homotopy classes $[f] \in [K, L]_\alpha$ are uniquely determined by their induced module homomorphisms $f_{\#_2}\colon \pi_2(K) \to {}_\alpha\pi_2(L)$ if and only if the cohomology module $H^2(\Pi; {}_\alpha\pi_2(L))$ is trivial.*

The condition $H^2(\Pi; {}_\alpha\pi_2(L)) = 0$ is an essential assumption in most of the results proved in this paper. Because of this, we introduce the following definition:

Definition 2.2. A map $f\colon K \to L$ inducing $\alpha\colon \Pi \to \Xi$ on fundamental groups is called a *convenient map* if $H^2(\Pi; {}_\alpha\pi_2(L)) = 0$.

Remark 2.3. In a sense, the class of convenient maps is really convenient, since, by Theorem 2.1, two such maps are homotopic if and only if the homomorphisms induced on π_1 and π_2 are equal.

Remark 2.4. We have the following results:

(a) If $\pi_2(L) = 0$, then every map $f\colon K \to L$ is convenient.

(b) If Y is a closed surface, $S^2 \neq Y \neq \mathbb{R}\mathrm{P}^2$, then $\pi_2(Y) = 0$ and every map $f\colon K \to Y$ is convenient.

(c) If $\pi_2(K) = 0$, then K is aspherical (see [1]) and so it is a $K(\Pi, 1)$-complex. Hence $H^2(\Pi; {}_\alpha\pi_2(L)) \approx H^2(K; {}_\alpha\pi_2(L))$. Thus, a map $f\colon K \to L$ is convenient if and only if $H^2(K; {}_\alpha\pi_2(L)) = 0$.

(d) If $\Pi = \pi_1(K)$ is a free group, say of rank p, then the bouquet $\vee^p S^1$ is a $K(\Pi, 1)$-complex and we have

$$H^2(\Pi; {}_\alpha\pi_2(L)) \approx H^2(\vee^p S^1; {}_\alpha\pi_2(L)) = 0.$$

Hence, every map $f\colon K \to L$ is convenient.

(e) If $K = K^1$ is a 1-complex, then $\Pi = \pi_1(K)$ is a free group and every map $f\colon K \to L$ is convenient.

(f) A map $f\colon K \to S^2$ is convenient if and only if $H^2(\Pi; \mathbb{Z}) = 0$.

(g) For a map $f\colon K \to \mathbb{R}\mathrm{P}^2$ to be convenient it is sufficient that $H^2(\Pi; \widetilde{\mathbb{Z}}) = 0$ for every local coefficient system defined by $\Pi \to \mathrm{Aut}(\mathbb{Z})$, which we denote $\widetilde{\mathbb{Z}}$.

(h) A constant map $\kappa\colon K \to Y$ is always convenient if $S^2 \neq Y \neq \mathbb{R}\mathrm{P}^2$ and, if $Y = S^2$ or $Y = \mathbb{R}\mathrm{P}^2$, then it is convenient if and only if $H^2(\Pi;\mathbb{Z}) = 0$.

The first five items of the above remark are easy. In order to justify the sixth and seventh items, we present a simple argument: If Y is either the 2-sphere or the 2-dimensional projective space, then $\pi_2(Y) \approx \mathbb{Z}$. Given a map $f\colon K \to Y$ inducing $\alpha = f_{\#}\colon \Pi \to \pi_1(Y)$ on fundamental groups, we have:

- If $Y = S^2$, then α is the trivial homomorphism and, in this case, the action of Π on $\pi_2(Y) \approx \mathbb{Z}$ is trivial.
- If $Y = \mathbb{R}\mathrm{P}^2$, then $\pi_1(Y) \approx \mathbb{Z}_2 = \{-1, 1\}$ and, in this case, for each $\pi \in \Pi$ and $\gamma \in \pi_2(Y)$, we have exactly two possibilities: either $\pi \cdot \gamma = \gamma$ or $\pi \cdot \gamma = -\gamma$.

This shows that the action of Π on $\pi_2(Y) \approx \mathbb{Z}$ defines a local coefficient system $\widetilde{\alpha}\colon \Pi \to \mathrm{Aut}(\mathbb{Z}) \approx \mathrm{Aut}(\pi_2(Y))$ for K, as well as for any $K(\Pi,1)$-complex. According to the two items above, we have:

- If $Y = S^2$, the system $\widetilde{\alpha}\colon \Pi \to \mathrm{Aut}(\mathbb{Z})$ is trivial, i.e. $\widetilde{\alpha}(\pi) = 1$, $\pi \in \Pi$.
- If $Y = \mathbb{R}\mathrm{P}^2$, the system $\widetilde{\alpha}\colon \Pi \to \mathrm{Aut}(\mathbb{Z})$ may be surjective or not.

This is enough to justify items (f) and (g) of Remark 2.4. The first part of item (h) is an immediate consequence of item (b) and the second part of it is a consequence of items (f) and (g), since a constant map induces the trivial homomorphism on fundamental groups.

Before we present the main theorem of this section, we present an important lemma which will be used in its proof.

Lemma 2.5. *Every homomorphism $\alpha\colon \Pi \to \Xi$ is obtained as an induced homomorphism on fundamental groups by a cellular map $f\colon K \to L$.*

Proof. Let $\varphi\colon K \to K_{\mathcal{P}}$ and $\psi\colon L_{\mathcal{Q}} \to L$ homotopy equivalences (which exists by Theorem 1.9 of [1, p. 61]), where $K_{\mathcal{P}}$ and $L_{\mathcal{Q}}$ are the model 2-complexes of group presentations $\mathcal{P} = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_m \rangle$ and $\mathcal{Q} = \langle y_1, \ldots, y_u \mid s_1, \ldots, s_v \rangle$, respectively. Then, the 1-skeletons of $K_{\mathcal{P}}$ an $L_{\mathcal{Q}}$ are, respectively, the bouquets

$$K^1_{\mathcal{P}} = \vee^n S^1 = e^0_K \cup e^1_{x_1} \cup \ldots \cup e^1_{x_n} \quad \text{and} \quad L^1_{\mathcal{Q}} = \vee^u S^1 = e^0_L \cup e^1_{y_1} \cup \ldots \cup e^1_{y_u}.$$

Denote $\mathbf{x} = \{x_1, \ldots, x_n\}$ and $\mathbf{y} = \{y_1, \ldots, y_u\}$ and let $F(\mathbf{x})$ and $F(\mathbf{y})$ be the free groups of rank n and u, generated by $\mathbf{x}$ and $\mathbf{y}$, respectively. Let $N(\mathbf{r})$ be the normal subgroup of $F(\mathbf{x})$ generated by the words $r_1, \ldots, r_m$, and let $N(\mathbf{s})$ be the normal subgroup of $F(\mathbf{y})$ generated by the words $s_1, \ldots, s_v$. Let $\Omega_\Pi\colon F(\mathbf{x}) \to \Pi = F(\mathbf{x})/N(\mathbf{r})$ and $\Omega_\Xi\colon F(\mathbf{y}) \to \Xi = F(\mathbf{y})/N(\mathbf{s})$ be the quotient homomorphisms.

For each $1 \leq j \leq n$, choose $w_j \in F(\mathbf{y})$ such that $\Omega_\Xi(w_j) = (\alpha \circ \Omega_\Pi)(x_j)$. Let $\alpha^1: F(\mathbf{x}) \to F(\mathbf{y})$ be the unique homomorphism such that $\alpha^1(x_j) = w_j$. It is easy to see that $\alpha \circ \Omega_\Pi = \Omega_\Xi \circ \alpha^1$, that is, the square in the following diagram is commutative.

$$\begin{tikzcd}
0 \arrow[r] & N(\mathbf{r}) \arrow[r] & F(\mathbf{x}) \arrow[r, "\Omega_\Pi"] \arrow[d, "\alpha^1"'] & \Pi \arrow[r] \arrow[d, "\alpha"] & 0 \\
0 \arrow[r] & N(\mathbf{s}) \arrow[r] & F(\mathbf{y}) \arrow[r, "\Omega_\Xi"'] & \Xi \arrow[r] & 0
\end{tikzcd}$$

Let $f^1: K^1_{\mathcal{P}} \to L^1_{\mathcal{Q}}$ be the map which is defined so that its image on each $e^1_{x_j}$ is the loop which travels $L^1_{\mathcal{Q}}$ exactly as the homomorphism α^1 spells $\alpha^1(x_j)$ as a word in $F(\mathbf{y})$. It is obvious that there is a natural identification

$$\alpha^1 = f^1_\#: F(\mathbf{x}) \equiv \pi_1(K^1_{\mathcal{P}}) \longrightarrow \pi_1(L^1_{\mathcal{Q}}) \equiv F(\mathbf{y}).$$

Now, each relator r_i is a word in $F(\mathbf{x})$ (may be a word with a unique letter or even the empty word) such that $(\alpha \circ \Omega_\Pi)(r_i) = 0$ in Ξ, since $\Omega_\Pi(r_i) = 0$. Moreover, the model 2-complex $K_{\mathcal{P}}$ has m cells of dimension two, say $e^2_1, \dots, e^2_m$, indexed so that the 2-cell e^2_i is attached in $K^1_{\mathcal{P}}$ according to the relation word r_i. Let $l: L^1_{\mathcal{Q}} \hookrightarrow L_{\mathcal{Q}}$ be the natural inclusion. Then $(l \circ f^1)_\#(r_i) = (\Omega_\Xi \circ \alpha^1)(r_i) = (\alpha \circ \Omega_\Pi)(r_i) = 0$ for each $1 \leq i \leq m$. Hence, the composed map $l \circ f^1$ extends to each 2-cell e^2_i, defining a cellular map $f': K_{\mathcal{P}} \to L_{\mathcal{Q}}$ which satisfies, for each $1 \leq j \leq n$, $(f'_\# \circ \Omega_\Xi)(x_j) = (l \circ f^1)_\#(x_j) = (\Omega_L \circ \alpha^1)(x_j) = (\alpha \circ \Omega_\Pi)(x_j)$. This proves that $f'_\# = \alpha$.

To finalize, define $f = \psi \circ f' \circ \varphi: K \to L$. Since φ and ψ are homotopy equivalences, it follows that $f_\# = f'_\# = \alpha$. $\square$

Now, we present the main theorem of this section. For this, Y is a closed surface with minimal celular decomposition and Y^1 is its 1-skeleton. In addition, we write $l: Y^1 \to Y$ to be the natural inclusion.

Theorem 2.6. *A convenient map $f: K \to Y$ is root free if and only if the homomorphism $f_{\#_2}: \pi_2(K) \to \pi_2(Y)$ is trivial and there is a homomorphism ϕ making commutative the diagram below:*

$$\begin{tikzcd}
 & \pi_1(Y^1) \arrow[d, "l_\#"] \\
\pi_1(K) \arrow[ur, "\phi"] \arrow[r, "f_\#"'] & \pi_1(Y)
\end{tikzcd}$$

The "only if" part is true even if f is not a convenient map.

Proof. Suppose that f is root free. Then, let φ be a map homotopic to f and $a \in Y$ be a point such that $\varphi^{-1}(a) = \emptyset$. Up to composition of φ with a self-homeomorphism of Y homotopic to the identity map, we can consider that $a \in Y \setminus Y^1$. Thus, Y^1 is a strong deformation retract of $Y \setminus \{a\}$. Let $r\colon Y \setminus \{a\} \to Y^1$ be a retraction. Define $\overline{\varphi}\colon K \to Y^1$ to be the composition $\overline{\varphi} = r \circ \varphi$. Then, $l \circ \overline{\varphi}\colon K \to Y$ is a map homotopic to f. Now it is enough to define $\phi = \overline{\varphi}_{\#}$ to obtain $f_{\#} = l_{\#} \circ \phi$. Moreover, since $\pi_2(Y^1) = 0$, it is obvious that $f_{\#_2}$ is the trivial homomorphism.

In order to prove the "if" parte, suppose that $f_{\#_2}$ is trivial and $\phi\colon \pi_1(K) \to \pi_1(Y^1)$ is a homomorphism verifying $f_{\#} = l_{\#} \circ \phi$. By Lemma 2.5, there is a cellular map $\overline{\varphi}\colon K \to Y^1$ such that $\phi = \overline{\varphi}_{\#}\colon \pi_1(K) \to \pi_1(Y^1)$. Let $\varphi\colon K \to Y$ be the composition $\varphi = l \circ \overline{\varphi}$. Then $\varphi_{\#} = l_{\#} \circ \phi = f_{\#}$ and $f_{\#_2} = 0 = \varphi_{\#_2}$. Let $f^{\rm cel}\colon K \to Y$ be a cellular approximation of f and consider as the base point in Y its (unique) 0-cell and as a base point in K any of the its 0-cells. Since φ and $f^{\rm cel}$ are both cellular maps, they are based maps. Moreover, $\varphi_{\#} = f^{\rm cel}_{\#}$ and $\varphi_{\#_2} = f^{\rm cel}_{\#_2} = 0$. Now, from assumption, $H^2(\Pi; {}_\alpha\pi_2(M)) = 0$. From Theorem 2.1 it follows that $f^{\rm cel}$ is (based) homotopic to φ. Consequently, f is homotopic to φ (through a not necessarily based homotopy). Since φ is not surjective, f is root free. $\square$

When the homomorphism ϕ in Theorem 2.6 exists, we say that it is a *lifting of $f_{\#}$ through $l_{\#}$*. Optionally, we say that $f_{\#}$ *has a lifting through $l_{\#}$*.

It is obvious that if f is a convenient map and the homomorphism $f_{\#}$ is trivial, then the lifting ϕ exists, indeed, it is enough to define ϕ to be also the trivial homomorphism. Thus, in this case, the map f is root free if and only if the homomorphism $f_{\#_2}$ is also trivial.

The "if" part of Theorem 2.6 is not true, in general, if the map f is not a convenient map. We present now an example to illustrate this fact: Let $\mathbb{T}$ be the torus $S^1 \times S^1$. Since $\mathbb{T}$ is a $K(\pi_1(\mathbb{T}), 1)$-space, we have $H^2(\pi_1(\mathbb{T}); {}_\alpha\pi_2(S^2)) \approx H^2(\mathbb{T}; \mathbb{Z}) \approx \mathbb{Z}$ for every map $\mathbb{T} \to S^2$ inducing α (the trivial homomorphism) on fundamental groups. Therefore, *there are not convenient maps from $\mathbb{T}$ into the 2-sphere S^2*. However, it is clear that there is a map $f\colon \mathbb{T} \to S^2$ of degree 1, and such map is not root free. Now, it is obvious that $f_{\#}$ and $f_{\#_2}$ are trivial homomorphisms. In particular, there is a homomorphism $\phi\colon \pi_1(K) \to \pi_1(\mathbb{T}^1)$ satisfying $l_{\#} \circ \phi = f_{\#}$. In order to obtain another example, let $\mathfrak{p}_2\colon S^2 \to \mathbb{R}\mathrm{P}^2$ be the universal covering and let $\overline{f}\colon \mathbb{T} \to \mathbb{R}\mathrm{P}^2$ be the composition $\overline{f} = \mathfrak{p}_2 \circ f$. This map is not convenient and is not root free, in fact $\mu(\overline{f}\,) = 2$. However, $\overline{f}_{\#}$ and $\overline{f}_{\#_2}$ are trivial.

3. Some consequences of Theorem 2.6

Proposition 3.1. *Let G be a finite group such that $\mathrm{Hom}(G; \mathbb{Z}_2)$ is nontrivial. For each nontrivial homomorphism $\alpha\colon G \to \mathbb{Z}_2$ there is a finite 2-complex*

K, with $\pi_1(K) \approx G$, and there is a map $f_\alpha: K \to \mathbb{R}\mathrm{P}^2$, which is not root free, inducing α on fundamental groups.

PROOF. From assumption, G has a group presentation $\mathcal{P} = \langle \mathbf{x} \mid \mathbf{r} \rangle$, where $\mathbf{x}$ and $\mathbf{r}$ are finite. Let $K = K_{\mathcal{P}}$ be the model 2-complex of the group presentation $\mathcal{P}$ (see [1] for details on model 2-complex). Then $G \approx \pi_1(K)$ and, up to isomorphism, each homomorphism $\alpha \in \mathrm{Hom}(G; \mathbb{Z}_2)$ can be considered as a homomorphism from $\pi_1(K)$ into $\pi_1(\mathbb{R}\mathrm{P}^2)$. By Lemma 2.5, each such homomorphism α is realized as the induced homomorphism on fundamental groups by a cellular map $f_\alpha: K \to \mathbb{R}\mathrm{P}^2$. Then, since $\pi_1(\mathbb{R}\mathrm{P}^1) \approx \mathbb{Z}$ and $\pi_1(G)$ is a finite group, it is easy to see that there is a lifting $\phi_\alpha: \pi_1(K) \to \pi_1(\mathbb{R}\mathrm{P}^1)$ of $(f_\alpha)_\#$ through $l_\#$ if and only if $(f_\alpha)_\#$ is trivial. Now, since $\mathrm{Hom}(G; \mathbb{Z}_2) \neq 0$, by assumption, for each nontrivial homomorphism $\alpha \in \mathrm{Hom}(G; \mathbb{Z}_2)$, each map $f_\alpha: K \to \mathbb{R}\mathrm{P}^2$ is not root free, by Theorem 2.6. $\square$

To illustrate the applicability of this proposition, consider the *pseudo-projective plane* $\mathbb{P}^2_{2d}$ of degree $2d$ which is obtained by attaching a 2-cell in the 1-sphere by a map $S^1 \to S^1$ of degree $2d$. (Note that $\mathbb{R}\mathrm{P}^2 = \mathbb{P}^2_2$). It is well known that $\pi_1(\mathbb{P}^2_{2d}) \approx \mathbb{Z}_{2d}$ and so $\mathrm{Hom}(\pi_1(\mathbb{P}^2_{2d}); \mathbb{Z}_2) \approx \mathbb{Z}_2$. Let $\alpha: \mathbb{Z}_{2d} \approx \pi_1(\mathbb{P}^2_{2d}) \to \pi_1(\mathbb{R}\mathrm{P}^2) \approx \mathbb{Z}_2$ be the unique nontrivial homomorphism belonging to $\mathrm{Hom}(\pi_1(\mathbb{P}^2_{2d}); \mathbb{Z}_2)$. By the previous proposition, there is a map $f: \mathbb{P}^2_{2d} \to \mathbb{R}\mathrm{P}^2$ such that $f_\# = \alpha$ and f is not root free.

PROPOSITION 3.2. *Let K be a 2-complex with free fundamental group. A map $f: K \to Y$ is root free if and only if the homomorphism $f_{\#_2}$ is trivial.*

PROOF. Let $f: K \to Y$ be a map. By Remark 2.4, f is convenient. For each generator x_j of the free group $\pi_1(K)$, choose a word w_j in the free group $\pi_1(Y^1)$ such that $l_\#(w_j) = f_\#(x_j)$. Then, there is a (unique) homomorphism $\phi: \pi_1(K) \to \pi_1(Y^1)$ such that $\phi(x_j) = w_j$. It is clear that ϕ is a lifting of $f_\#$ through $l_\#$. By Theorem 2.6, f is root free if and only if $f_{\#_2}$ is trivial. $\square$

Another proof for this proposition can be constructed using a Theorem of Wall (see [1, p. 120]), which states that *every finite and connected 2-complex with free fundamental group is homotopy equivalent to a finite bouquet of* 1- *and* 2-*dimensional spheres.*

PROPOSITION 3.4. *Let K be a 2-complex with finite fundamental group and let $f: K \to Y$ be a convenient map. Then f is root free if and only if $f_\#$ and $f_{\#_2}$ are trivial. Additionally, if a constant map $\kappa: K \to Y$ is convenient (see Remark 2.4(h)), then f is root free if and only it is homotopic to a constant map.*

PROOF. Since $\pi_1(Y^1)$ is a free group and $\pi_1(K)$ is a finite group, the unique homomorphism from $\pi_1(K)$ into $\pi_1(Y^1)$ is the trivial homomorphism. Thus,

$f_\#\colon \pi_1(K) \to \pi_1(Y)$ has a lifting through $l_\#\colon \pi_1(Y^1) \to \pi_1(Y)$ if and only if $f_\#$ is trivial. It follows from Theorem 2.6 that f is root free if and only if $f_\#$ and $f_{\#_2}$ are both trivial. It proves the first part of the proposition. The second part is a consequence of the first part and Remark 2.3. $\square$

Note that the "only if" parts of Proposition 3.4 is true even if f is not convenient. An as a particular case of the second part of this proposition, we have the following corollary.

Corollary 3.5. *Let K be a 2-complex with finite fundamental group and let Y be a closed surface, $S^2 \neq Y \neq \mathbb{R}\mathrm{P}^2$. Then every map $f\colon K \to Y$ is homotopic to a constant map.*

Proof. Note that $\pi_2(Y) = 0$, so every map $f\colon K \to Y$ is convenient (see Remark 2.4) and Y is a finite $K(\pi_1(Y), 1)$-complex. It is well known that if G is a group which contains a torsion subgroup, then every $K(G, 1)$-complex is infinite (see Proposition II.3 of [8]). Therefore, the fundamental group $\pi_1(Y)$ is torsion free. So $\mathrm{Hom}(\pi_1(K); \pi_1(Y)) = 0$ and the result follows from the previous proposition, Remark 2.3 and item (h) of Remark 2.4. $\square$

Now, we will consider cases in which the fundamental group of K is abelian.

We say that a subgroup H of a group G is *cyclic* (in G) if H is either trivial or can be generated by a single element.

Let A be an abelian group with torsion subgroup $\mathcal{T}$. Then $A \approx \mathcal{F} \oplus \mathcal{T}$, where $\mathcal{F}$ is the free abelian group $A/\mathcal{T}$. A group homomorphism $h\colon A \to B$ induces two group homomorphisms

$$h^{\mathcal{F}}\colon \mathcal{F} \to B \quad \text{and} \quad h^{\mathcal{T}}\colon \mathcal{T} \to B$$

in a natural way: For each $x \in \mathcal{F}$ we define $h^{\mathcal{F}}(x) = (h \circ \Lambda)(x, 0)$ and, for each $y \in \mathcal{T}$ we define $h^{\mathcal{T}}(y) = (\alpha \circ \Lambda)(0, y)$, where $\Lambda\colon \mathcal{F} \oplus \mathcal{T} \approx A$.

Lemma 3.6. *Let A be an abelian group and suppose that $A = \mathcal{F} \oplus \mathcal{T}$, where $\mathcal{F}$ is a free abelian group and $\mathcal{T}$ is an abelian torsion group. Let $h\colon A \to B$ be a group homomorphism and let $\xi\colon F \to B$ be an epimorphism from a (nonabelian) free group F onto B. There is a lifting $\phi\colon A \to F$ of h through ξ if and only if $h^{\mathcal{F}}(\mathcal{F})$ in cyclic and $h^{\mathcal{T}}(\mathcal{T})$ is trivial.*

Proof. Since F is free, it is obvious that $h^{\mathcal{T}}$ has a lifting $\phi^{\mathcal{T}}\colon \mathcal{T} \to F$ through ξ if and only if $h^{\mathcal{T}}$ is trivial and, in this case, also $\phi^{\mathcal{T}}$ is trivial.

Now, if there exists a lifting $\phi^{\mathcal{F}}\colon \mathcal{F} \to F$ of $h^{\mathcal{F}}$ through ξ, then the image $\phi^{\mathcal{F}}(\mathcal{F})$ is an abelian free subgroup of F, by Nielsen–Schreler Theorem (see [9]). Then, it is easy to check that $\phi^{\mathcal{F}}(\mathcal{F})$ is cyclic (see Chapter III of [3]). Therefore, since $h^{\mathcal{F}} = \xi \circ \phi^{\mathcal{F}}$, the subgrupo $h^{\mathcal{F}}(\mathcal{F})$ of B is also cyclic.

Conversely, suppose that $h^{\mathcal{F}}(\mathcal{F})$ is cyclic and let $\vartheta \in h^{\mathcal{F}}(\mathcal{F})$ be its generator. Let p be the rank of $\mathcal{F}$ and let $\mathcal{F}'$ be the free abelian group generated by $u_1, \dots, u_p$. There is an isomorphism $\eta: \mathcal{F}' \to \mathcal{F}$, such that $(\eta \circ h)(u_1) = \vartheta$ and $(\eta \circ h)(u_i) = 0$ for each $1 < i \leq p$. Since ξ is an epimorphism, we can select a word $w \in F$ such that $\xi(w) = \vartheta$. Let $\phi': \mathcal{F}' \to F$ be the (unique) homomorphism from $\mathcal{F}'$ into F satisfying $\phi'(u_1) = w$ and $\phi'(u_i) = \mathbb{1}$ for each $1 < i \leq p$. Now, define $\phi^{\mathcal{F}}: \mathcal{F} \to F$ to be the composition $\phi^{\mathcal{F}} = \phi' \circ \eta^{-1}$, where $\eta^{-1}: \mathcal{F} \to \mathcal{F}'$ denotes the inverse isomorphism of η. It is obvious that $\phi^{\mathcal{F}}$ is a lifting of $h^{\mathcal{F}}$ through ξ. $\square$

Proposition 3.7. *Let $f: K \to Y$ be a convenient map and suppose that $\pi_1(K) = \mathcal{F} \oplus \mathcal{T}$ is an abelian group, where $\mathcal{T}$ is its torsion subgroup. We have:*

(a) *If $S^2 \neq Y \neq \mathbb{R}\mathrm{P}^2$, then f is root free if and only if $f_{\#}^{\mathcal{F}}(\mathcal{F})$ is cyclic.*
(b) *If $Y = S^2$, then f is root free if and only if $f_{\#_2}$ is trivial.*
(c) *If $Y = \mathbb{R}\mathrm{P}^2$, then f is root free if and only if $f_{\#_2}$ and $f_{\#}^{\mathcal{T}}$ are trivial.*

Proof. We will prove each assertion separately.

(a) We have $\pi_2(Y) = 0$ and $\pi_1(Y)$ torsion free. Hence, the homomorphisms $f_{\#_2}: \pi_2(K) \to \pi_2(Y)$ and $f_{\#}^{\mathcal{T}}: \mathcal{T} \to \pi_1(Y)$ are both trivial. By Theorem 2.6, f is root free if and only if $f_{\#}^{\mathcal{F}}: \mathcal{F} \to \pi_1(Y)$ has a lifting through $l_{\#}: \pi_1(Y^1) \to \pi_1(Y)$. But by Lemma 3.6, this occurs if and only if $f_{\#}^{\mathcal{F}}(\mathcal{F})$ is cyclic.

(b) Since $f_{\#}: \pi_1(K) \to \pi_1(S^2)$ is trivial and so has a lifting through $l_{\#}$, it follows by Theorem 2.6 that f is root free if and only if $f_{\#_2}$ is trivial.

(c) Certainly $f_{\#}^{\mathcal{F}}(\mathcal{F})$ is cyclic and so $f_{\#}^{\mathcal{F}}$ has a lifting through $l_{\#}$, by Lemma 3.6. Again by this lemma, there is a lifting ϕ of $f_{\#}$ trough $l_{\#}$ if and only if $f_{\#}^{\mathcal{T}}$ is trivial. The result follows from Theorem 2.6. $\square$

From this proposition we can extract a particular result for the case in which the domain of the map is the torus $\mathbb{T}$.

Corollary 3.8. *A convenient map $f: \mathbb{T} \to Y$ from the torus into a closed surface is root free if and only if $f_{\#}$ is not injective.*

Proof. If $f_{\#}$ is injective, then it is clear that $f_{\#}\pi_1(\mathbb{T})$ is not cyclic and, by Proposition 3.7, f is not root free. Now, note that $f_{\#_2}: \pi_2(\mathbb{T}) \to \pi_2(Y)$ is trivial and suppose that $f_{\#}$ is not injective. Then, since $\pi_1(\mathbb{T}) \approx \mathbb{Z} \oplus \mathbb{Z}$, it is obvious that $f_{\#}\pi_1(\mathbb{T}) = f_{\#}^{\mathcal{F}}(\mathcal{F})$ is cyclic. By the previous proposition, f is root free. $\square$

Proposition 3.9. *Let K be an aspherical 2-complex and let $\mathcal{T}_1$ be the torsion subgroup of $H_1(K)$. If $\mathrm{Hom}(\mathcal{T}_1; \mathbb{Z}_2) = 0$ then every convenient map from K into $\mathbb{R}\mathrm{P}^2$ is root free.*

Proof. First, we remember that a (finite and connected) 2-complex K is aspherical if and only if $\pi_2(K) = 0$ (see [1] for details). Thus, for any map

$f\colon K \to \mathbb{R}\mathrm{P}^2$, the homomorphism $f_{\#_2}$ is trivial. Now, let $\rho\colon \pi_1(K) \to H_1(K)$ be the Hurewicz homomorphism (the abelianization homomorphism). Since $\pi_1(\mathbb{R}\mathrm{P}^2)$ is an abelian group, there is a unique homomorphism $f_{\circledast}\colon H_1(K) \to \pi_1(\mathbb{R}\mathrm{P}^2)$ such that $f_{\circledast} \circ \rho = f_{\#}$. By assumption, $f_{\circledast}^{\mathcal{T}_1}$ is trivial. Moreover, it is clear that the imagem of $f_{\circledast}^{\mathcal{F}_1}$ is cyclic, where $\mathcal{F}_1$ is the abelian free subgroup of $H_1(K)$ such that $H_1(K) \approx \mathcal{F}_1 \oplus \mathcal{T}_1$. By Lemma 3.6, there is a homomorphism $\phi'\colon H_1(K) \to \pi_1(\mathbb{R}\mathrm{P}^1)$ such that $l_{\#} \circ \phi' = f_{\circledast}$. Define $\phi\colon \pi_1(K) \to \pi_1(\mathbb{R}\mathrm{P}^1)$ to be the composition $\phi = \phi' \circ \rho$. Then ϕ is a lifting of $f_{\#}$ through $l_{\#}$. By Theorem 2.6, f is root free. $\square$

4. Mutation of homomorphisms and existence of liftings

In this section, we present a new concept which we call *mutation of homomorphisms*. We present its definition and some easy technical lemmas. We will show the relationship between the existence of $(\mathcal{P}, \theta)$-mutations of a given group homomorfismo $\tau\colon F(\mathbf{x}) \to G$ and the existence of liftings of a homomorphism $\alpha\colon \Pi \to \Xi$, through a group homomorphism $\theta\colon G \to \Xi$ verifying the commutativity $\alpha \circ \Omega = \theta \circ \tau$, where $\Omega\colon F(\mathbf{x}) \to \Pi$ is the quotient homomorphism given by the presentation group $\mathcal{P}$. In the next section we will use this relationship to study the root problem.

Let $\mathcal{P} = \langle \mathbf{x} \mid \mathbf{r} \rangle$ be a group presentation with alphabet $\mathbf{x} = \{x_1, \dots, x_n\}$ and set of relators $\mathbf{r} = \{r_1, \dots, r_m\}$. Let Π be the group presented by $\mathcal{P}$, that is, $\Pi = F(\mathbf{x})/N(\mathbf{r})$, and let $\Omega\colon F(\mathbf{x}) \to \Pi$ be the quotient homomorphism. Let $\theta\colon G \to \Xi$ be a group homomorphism with kernel $\ker(\theta)$.

Definition 4.1. Given a group homomorphism $\tau\colon F(\mathbf{x}) \to G$, by a $(\mathcal{P}, \theta)$-*mutation* of τ we mean a homomorphism $\mathcal{M}\tau\colon F(\mathbf{x}) \to G$ satisfying:

(a) For each $w \in F(\mathbf{x})$ there is $\mathcal{M}(w) \in \ker(\theta)$ such that $\mathcal{M}\tau(w) = \mathcal{M}(w)\tau(w)$;
(b) $\mathcal{M}\tau(r_i) = \mathfrak{e}_G$, the identity element of G, for every relator $r_i \in \mathbf{r}$.

$$\begin{tikzcd} F(\mathbf{x}) \arrow[rr, bend left, "\tau"] \arrow[rr, bend right, "\mathcal{M}\tau"'] \arrow[d, "\Omega"'] & & G \arrow[d, "\theta"] \\ \Pi & & \Xi \end{tikzcd}$$

If $\mathcal{M}\tau\colon F(\mathbf{x}) \to G$ is a $(\mathcal{P}, \theta)$-mutation of a homomorphism $\tau\colon F(\mathbf{x}) \to G$, then the function $\mathcal{M}\colon F(\mathbf{x}) \to G$ carrying $w \in F(\mathbf{x})$ into $\mathcal{M}(w) \in \ker(\theta)$ is such that $\mathcal{M}(\mathbb{1}) = \mathfrak{e}_G$ (the identity element of G) and $\mathcal{M}(r_i) = \tau(r_i)^{-1}$ for each $1 \le i \le m$.

On the other hand, suppose that $\mathcal{M}\colon F(\mathbf{x}) \to G$ is a function carrying $w \in F(\mathbf{x})$ into $\mathcal{M}(w) \in \ker(\theta)$ and satisfying the following three conditions: (a) $\mathcal{M}(\mathbb{1}) = \mathfrak{e}_G$, (b) $\mathcal{M}(r_i) = \tau(r_i)^{-1}$ for each $1 \le i \le m$ and (c) the function

$\mathcal{M}\tau\colon F(\mathbf{x}) \to G$ given by $\mathcal{M}\tau(w) = \mathcal{M}(w)\tau(w)$ is a group homomorphism. Then, it is easy to prove that $\mathcal{M}\tau\colon F(\mathbf{x}) \to G$ is a $(\mathcal{P}, \theta)$-mutation of $\tau\colon F(\mathbf{x}) \to G$.

This show that in order to construct a $(\mathcal{P}, \theta)$-mutation of a given group homomorphism $\tau\colon F(\mathbf{x}) \to G$, it is necessary and sufficient to define a function $\mathcal{M}\colon F(\mathbf{x}) \to G$, with $\mathcal{M}(w) \in \ker(\theta)$ for every $w \in F(\mathbf{x})$, satisfying the three conditions above. Such a function, when exists, will be called the $(\mathcal{P}, \theta)$-*mutator function* of τ to $\mathcal{M}\tau$.

Theorem 4.2. *Let* $\mathcal{P} = \langle \mathbf{x} \mid \mathbf{r} \rangle$ *be a group presentation for the group* Π *and let* $\theta\colon G \to \Xi$ *be a group homomorphism. Suppose that* $\tau\colon F(\mathbf{x}) \to G$ *and* $\alpha\colon \Pi \to \Xi$ *are group homomorphisms making commutative the diagram*

$$\begin{tikzcd}
F(\mathbf{x}) \arrow[r, "\tau"] \arrow[d, "\Omega"'] & G \arrow[d, "\theta"] \\
\Pi \arrow[r, "\alpha"'] & \Xi
\end{tikzcd}$$

where the left vertical arrow is the natural quotient homomorphism. Then α *has a lifting* $\phi\colon \Pi \to G$ *through* θ *if and only if* τ *has a* $(\mathcal{P}, \theta)$*-mutation. Moreover, the liftings of* α *through* θ *are in one-to-one correspondence with the* $(\mathcal{P}, \theta)$*-mutations of* τ.

Proof. Suppose that exists a lifting $\phi\colon \Pi \to G$ of τ through θ. Then, in the diagram below, the square and the lower triangle are commutative.

$$\begin{tikzcd}
F(\mathbf{x}) \arrow[r, "\tau"] \arrow[d, "\Omega"'] & G \arrow[d, "\theta"] \\
\Pi \arrow[r, "\alpha"'] \arrow[ur, "\phi"] & \Xi \arrow[u, phantom, "\curvearrowright"]
\end{tikzcd}$$

For each $w \in F(\mathbf{x})$, we have $(\theta \circ \phi \circ \Omega)(w) = (\alpha \circ \Omega)(w) = (\theta \circ \tau)(w)$. Thus, $\theta(\phi \circ \Omega(w)) = \theta(\tau(w))$, for every $w \in F(\mathbf{x})$. It shows that τ and $(\phi \circ \Omega)$ differ only by elements in $\ker(\theta)$, that is, for each $w \in F(\mathbf{x})$, the element $(\phi \circ \Omega)(w)\tau(w)^{-1}$ belongs to $\ker(\theta)$. Define the function $\mathcal{M}\colon F(\mathbf{x}) \to G$ to be $\mathcal{M}(w) = (\phi \circ \Omega)(w)\tau(w)^{-1}$. Then $\mathcal{M}(w) \in \ker(\theta)$ for each $w \in F(\mathbf{x})$ and we have:

- $\mathcal{M}(\mathbb{1}) = \mathbb{1}$;
- $\mathcal{M}(r_i) = (\phi \circ \Omega)(r_i)\tau(r_i)^{-1} = \tau(r_i)^{-1}$, for each r_i, since $\Omega(r_i) = \mathfrak{e}_\Pi$.
- $\mathcal{M}\tau\colon F(\mathbf{x}) \to G$ given by $\mathcal{M}\tau(w) = \mathcal{M}(w)\tau(w)$ is a group homomorphism, since by the definition we have $\mathcal{M}\tau(w) = (\phi \circ \Omega)(w)$.

It follows that $\mathcal{M}\tau$ is a $(\mathcal{P}, \theta)$-mutation of τ.

In order to prove the reciprocal, let $\mathcal{M}\tau\colon F(\mathbf{x}) \to G$ be a $(\mathcal{P}, \theta)$-mutation of τ. Then $\mathcal{M}\tau(r_i) = \mathfrak{e}_G$ for each relator $r_i \in \mathbf{r}$. Hence, $\mathcal{M}\tau(w) = \mathfrak{e}_G$ for every $w \in N(\mathbf{r})$, where $N(\mathbf{r})$ is the normal subgroup of $F(\mathbf{x})$ generated by the set of relators $\mathbf{r}$. Now, since $\ker(\Omega) = N(\mathbf{r})$, it follows that $\ker(\Omega) \subset \ker(\mathcal{M}\tau)$. Thus, there is a (unique) homomorphism $\phi\colon \Pi \to G$ satisfying $\phi \circ \Omega = \mathcal{M}\tau$. Such homomorphism is defined as follows: For each $\overline{w} \in \Pi$, choose $w \in F(\mathbf{x})$ such that $\overline{w} = \Omega(w)$. We define $\phi(\overline{w}) = \mathcal{M}\tau(w)$. It follows that, for each $\overline{w} = \Omega(w) \in \Pi$, we have $(\theta \circ \phi)(\overline{w}) = (\theta \circ \mathcal{M}\tau)(w) = \theta(\mathcal{M}(w)\tau(w)) = \theta(\mathcal{M}(w))\theta(\tau(w)) = (\theta\circ\tau)(w) = (\alpha\circ\Omega)(w) = \alpha(\overline{w})$. Therefore, $\phi\colon \Pi \to G$ is a lifting of α through θ.$\square$

5. Mutation for annihilation of roots

Let $f\colon K_{\mathcal{P}} \to Y$ be a convenient cellular map, where $K_{\mathcal{P}}$ is the model 2-complex of the group presentation $\mathcal{P} = \langle \mathbf{x} \mid \mathbf{r}\rangle$ and Y is a closed surface (See [1] for model 2-complex). Let Y^1 be the 1-skeleton of Y. Since f is cellular, its restriction on $K^1_{\mathcal{P}}$ provides a cellular map $f^1\colon K^1_{\mathcal{P}} \to Y^1$ making commutative the left diagram below, where the vertical arrows are the natural inclusions:

$$\begin{tikzcd} K^1_{\mathcal{P}} \arrow[r, "f^1"] \arrow[d] & Y^1 \arrow[d, "l"] \\ K_{\mathcal{P}} \arrow[r, "f"'] & Y \end{tikzcd} \qquad \begin{tikzcd} F(\mathbf{x}) \arrow[r, "f^1_{\#}"] \arrow[d, "\Omega"'] & \pi_1(Y^1) \arrow[d, "l_{\#}"] \\ \pi_1(K_{\mathcal{P}}) \arrow[r, "f_{\#}"'] & \pi_1(Y) \end{tikzcd}$$

The fundamental group $\pi_1(K_{\mathcal{P}})$ is that presented by $\mathcal{P}$. The right diagram above is that induced on fundamental groups by the left diagram. Considering the identification of $F(\mathbf{x})$ with $\pi_1(K^1_{\mathcal{P}})$, we have:

Theorem 5.1. *A convenient cellular map $f\colon K_{\mathcal{P}} \to Y$ is root free if and only if $f_{\#_2}\colon \pi_2(K_{\mathcal{P}}) \to \pi_2(Y)$ is trivial and $f^1_{\#}\colon \pi_1(K^1_{\mathcal{P}}) \to \pi_1(Y^1)$ has a $(\mathcal{P}, l_{\#})$-mutation. More precisely,*

(a) *If $\varphi\colon K_{\mathcal{P}} \to Y$ is a non-surjective cellular map homotopic to f, then $\varphi^1_{\#}\colon \pi_1(K^1_{\mathcal{P}}) \to \pi_1(Y^1)$ is a $(\mathcal{P}, l_{\#})$-mutation of $f^1_{\#}$.*

(b) *If $\mathcal{M}f^1_{\#}\colon \pi_1(K^1_{\mathcal{P}}) \to \pi_1(Y^1)$ is a $(\mathcal{P}, l_{\#})$-mutation of $f^1_{\#}$ and $f_{\#_2}$ is trivial, then there is a non-surjective cellular map $\varphi\colon K_{\mathcal{P}} \to Y$ homotopic to f such that $\varphi^1_{\#} = \mathcal{M}f^1_{\#}$.*

Proof. The first part is an immediate consequence of Theorems 2.6 and 4.2. In order to prove (a) and (b), we include some details.

(a) Suppose that $\varphi\colon K_{\mathcal{P}} \to Y$ is a non-surjective cellular map homotopic to f. We consider Y with its minimal cellular decomposition. Let $a \in Y$ be a point not belonging to the image of φ and belonging to the interior of the unique 2-cell of Y, (such a point exists, since $|K_{\mathcal{P}}|$ is compact and Y is Hausdorff and so the image of φ is a proper closed subset of Y). There is a strong deformation retraction $r\colon Y \setminus \{a\} \to Y^1$. Since $a \notin \mathrm{im}(\varphi)$ and $Y \setminus \{a\}$ is open in Y, the map $\overline{\varphi}\colon K_{\mathcal{P}} \to Y \setminus \{a\}$, obtained from φ by restriction of its range, is again a continuous map. Let $\psi\colon K_{\mathcal{P}} \to Y^1$ be the cellular map $\psi = r \circ \overline{\varphi}$. In the left diagram below, where j and the vertical arrows are natural inclusions, the square and the lower triangle are commutative. Moreover, r is a homotopy equivalence inducing the identity homomorphism on fundamental groups.

$$\begin{tikzcd}
K^1_{\mathcal{P}} \arrow[rr, "\varphi^1"] \arrow[dd, "i"'] & & Y^1 \arrow[dd, "l"] \\
& Y \setminus \{a\} \arrow[ur, "r"] \arrow[dr, "j"] & \\
K_{\mathcal{P}} \arrow[ur, "\overline{\varphi}"] \arrow[rr, "\varphi"'] & & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
\pi_1(K^1_{\mathcal{P}}) \arrow[r, "\varphi^1_{\#}"] \arrow[d, "\Omega"'] & \pi_1(Y^1) \arrow[d, "l_{\#}"] \\
\pi_1(K_{\mathcal{P}}) \arrow[ur, "\psi_{\#}"] \arrow[r, "\varphi_{\#}=f_{\#}"'] & \pi_1(Y)
\end{tikzcd}$$

It follows that $\varphi_{\#} = j_{\#} \circ \overline{\varphi}_{\#} = l_{\#} \circ r_{\#} \circ \overline{\varphi}_{\#} = l_{\#} \circ \psi_{\#}$. Hence, $\psi_{\#}$ is a lifting of $\varphi_{\#}$ through $l_{\#}$. On the other hand, we have $r \circ \overline{\varphi} \circ i = \varphi^1$. Thus, $\varphi^1_{\#} = \psi_{\#} \circ \Omega$. Therefore, the right diagram above is commutative. By the proof of Theorem 4.2, $\varphi^1_{\#}$ is a $(\mathcal{P}, l_{\#})$-mutation of $f^1_{\#}$. Note that the $(\mathcal{P}, l_{\#})$-mutator function $\mathcal{M}\colon \pi_1(K^1_{\mathcal{P}}) \to \pi_1(Y^1)$ which makes $\mathcal{M}f^1_{\#} = \varphi^1_{\#}$ is capriciously given by

$$\mathcal{M}(w) = (\psi_{\#} \circ \Omega)(w) f^1_{\#}(w)^{-1} = \varphi^1_{\#}(w) f^1_{\#}(w)^{-1}.$$

(b) Suppose that $f_{\#_2}$ is trivial and $\mathcal{M}f^1_{\#}\colon \pi_1(K^1_{\mathcal{P}}) \to \pi_1(Y^1)$ is a $(\mathcal{P}, l_{\#})$-mutation of $f^1_{\#}$. By Theorem 4.2, there is a lifting $\phi\colon \pi_1(K^1_{\mathcal{P}}) \to \pi_1(Y^1)$ of $f_{\#}$ through $l_{\#}$. Moreover, we have $\mathcal{M}f^1_{\#} = \phi \circ \Omega$. (See the diagram below). Let $\varphi_1\colon K^1_{\mathcal{P}} \to Y^1$ be a cellular map such that $(\varphi_1)_{\#} = \mathcal{M}f^1_{\#}$. (Such a map exists by Lemma 2.5).

$$\begin{tikzcd}
\pi_1(K^1_{\mathcal{P}}) \arrow[r, "f^1_{\#}"] \arrow[r, bend left=40, "\mathcal{M}f^1_{\#}"] \arrow[d, "\Omega"'] & \pi_1(Y^1) \arrow[d, "l_{\#}"] \\
\pi_1(K_{\mathcal{P}}) \arrow[ur, "\phi"] \arrow[r, "f_{\#}"'] \arrow[r, bend right=40, "\varphi_{\#}"'] & \pi_1(Y)
\end{tikzcd}$$

Since $\mathcal{M}f^1_{\#}(r_i) = \mathbb{1}$ for each relation word $r_i \in \mathbf{r}$, the cellular map φ_1 extends to a cellular map $\varphi_2\colon K_{\mathcal{P}} \to Y^1$. Let $\varphi\colon K_{\mathcal{P}} \to Y$ be the cellular map given by the composition $\varphi = l \circ \varphi_2$. It is clear that φ is non-surjective (in fact, $\operatorname{im}(\varphi) \subset Y^1$) and $\varphi^1 = \varphi_1$. Therefore, $\varphi^1_{\#} = \mathcal{M}f^1_{\#}$.

Now, we will prove that φ is homotopic to f. Since $f_{\#_2}$ and $\varphi_{\#_2}$ are trivial homomorphisms (the first from assumption and the second by construction), due to Theorem 2.1 it is enough to prove that $f_{\#} = \varphi_{\#}$. But this is a consequence of the following identities, where we use the definition of ϕ as in Theorem 4.2:

For each $\overline{w} = \Omega(w) \in \pi_1(K_{\mathcal{P}})$, we have

$$\begin{aligned}\varphi_{\#}(\overline{w}) &= (l \circ \varphi_2)_{\#}(\overline{w}) = (l_{\#} \circ (\varphi_2)_{\#} \circ \Omega)(w) = (l_{\#} \circ (\varphi_1)_{\#})(w) \\ &= (l_{\#} \circ \mathcal{M}f^1_{\#})(w) = (l_{\#} \circ \phi \circ \Omega)(w) = (l_{\#} \circ \phi)(\overline{w}) = f_{\#}(\overline{w}).\end{aligned}$$

This concludes the proof. $\square$

Remark 5.2. The whole result of Theorem 5.1 is not true, in general, for non convenient maps. However, the "only if" part is always true (due to Theorem 2.6), that is, if $f\colon K_{\mathcal{P}} \to Y$ is an arbitrary cellular map and f is root free, then the homomorphism $f^1_{\#}$ has a $(\mathcal{P}, l_{\#})$-mutation. Also, item (a) is true even if f is not convenient.

Theorem 5.3. *Let $K_{\mathcal{P}}$ be the model 2-complex of the group presentation $\mathcal{P} = \langle \mathbf{x} \mid \mathbf{r} \rangle$ and let $f\colon K_{\mathcal{P}} \to Y$ be a map. If f is root free, then $f_{\#_2}\colon \pi_2(K_{\mathcal{P}}) \to \pi_2(Y)$ is trivial and there exists a map $\varphi\colon K_{\mathcal{P}} \to Y$ homotopic to f such that $\varphi^1_{\#}(r_i) = \mathbb{1}$ for each relator $r_i \in \mathbf{r}$. The reciprocal is true if f is a convenient map.*

Proof. Suppose that $f\colon K_{\mathcal{P}} \to Y$ is root free. Then, there is a cellular map $\varphi\colon K_{\mathcal{P}} \to Y$ homotopic to f such that $\mathrm{im}(\varphi) \subset Y^1$. Let $f_{\mathrm{cel}}\colon K_{\mathcal{P}} \to Y$ be a cellular approximation of f. By the item (a) of Theorem 5.1, the homomorphism $\varphi^1_{\#}$ is a $(\mathcal{P}, l_{\#})$-mutation of $(f^1_{\mathrm{cel}})_{\#}$. Therefore, $\varphi^1_{\#}(r_i) = \mathbb{1}$ for each $r_i \in \mathbf{r}$.

Now, suppose that f is a convenient map and suppose that $f_{\#_2}$ is trivial and $\varphi\colon K_{\mathcal{P}} \to Y$ is a cellular map homotopic to f such that $\varphi^1_{\#}(r_i) = \mathbb{1}$ for each $r_i \in \mathbf{r}$. Then, the map $\varphi^1\colon K^1_{\mathcal{P}} \to Y^1$ extends to a cellular map $\overline{\varphi}\colon K_{\mathcal{P}} \to Y^1$. Thus, the upper triangles of both diagrams below are commutative.

$$\begin{tikzcd} K^1_{\mathcal{P}} \arrow[r, "\varphi^1"] \arrow[d] & Y^1 \arrow[d, "l"] \\ K_{\mathcal{P}} \arrow[r, "\varphi"'] \arrow[ur, "\overline{\varphi}"] & Y \end{tikzcd} \qquad \begin{tikzcd} \pi_1(K^1_{\mathcal{P}}) \arrow[r, "\varphi^1_{\#}"] \arrow[d, "\Omega"'] & \pi_1(Y^1) \arrow[d, "l_{\#}"] \\ \pi_1(K_{\mathcal{P}}) \arrow[r, "\varphi_{\#}=f_{\#}"'] \arrow[ur, "\overline{\varphi}_{\#}"] & \pi_1(Y) \end{tikzcd}$$

Now, for each $\overline{w} = \Omega(w) \in \pi_1(K_{\mathcal{P}})$, we have $(l_{\#} \circ \overline{\varphi}_{\#})(\overline{w}) = (l_{\#} \circ \overline{\varphi}_{\#} \circ \Omega)(w) = (l_{\#} \circ \varphi^1_{\#})(w) = (\varphi_{\#} \circ \Omega)(w) = \varphi_{\#}(\overline{w}) = f_{\#}(\overline{w})$. Then, the lower triangle of the right diagram is also commutative. This means that $\overline{\varphi}_{\#}$ is a lifting of $f_{\#}$ through $l_{\#}$. By Theorem 2.6, f is root free. $\square$

6. Symbolic mutation

Let $\{\mathfrak{X}\}_n = \{\mathfrak{X}_1, \dots, \mathfrak{X}_n\}$ be a list of n abstract symbols and let G be a nontrivial group with identity element $\mathfrak{e}_G$. Let $W(G, \{\mathfrak{X}\}_n)$ be the set of all (reduced and of finite length) words of the form

$$g_0 \mathfrak{X}^{\delta_1}_{\lambda_1} g_1 \mathfrak{X}^{\delta_2}_{\lambda_2} g_2 \dots \mathfrak{X}^{\delta_k}_{\lambda_k} g_k,$$

where each g_s, for $0 \leq s \leq k$, is an element of the group G, each δ_s is an integer and each $\lambda_s \in \{1, \ldots, n\}$. We require that every word in $W(G, \{\mathfrak{X}\}_n)$ contains at least one symbol of the list $\{\mathfrak{X}\}_n$ and an element of the group G, which can be the identity element. However, the identity element of G can be omitted when we spell the words. Thus, each symbol $\mathfrak{X}_j$ is itself an element of $W(G, \{\mathfrak{X}\}_n)$. In addition, we also consider as an element of $W(G, \{\mathfrak{X}\}_n)$ the "empty" word, which we denote by $\dot{\mathbb{1}}$. Such element can be identified with $\mathfrak{e}_G$, since this last can be omitted. Hence, also e_G can be see as an element of $W(G, \{\mathfrak{X}\}_n)$. The others elements of G are not itself elements of $W(G, \{\mathfrak{X}\}_n)$.

On the set $W(G, \{\mathfrak{X}\}_n)$ we define the natural multiplication: the product of two words is formed simply by writing one after the other and by reducing the word obtained, that is, given $\Lambda = g_0\mathfrak{X}_{\lambda_1}^{\delta_1} g_1 \mathfrak{X}_{\lambda_2}^{\delta_2} g_2 \ldots \mathfrak{X}_{\lambda_k}^{\delta_k} g_k$ and $\Gamma = \widetilde{g}_0 \mathfrak{X}_{\gamma_1}^{\varepsilon_1} \widetilde{g}_1 \ldots \mathfrak{X}_{\gamma_t}^{\varepsilon_t} \widetilde{g}_t$ two arbitrary elements in $W(G, \{\mathfrak{X}\}_n)$, we define the product $\Lambda\Gamma$ to be the word obtained by reducing the word

$$g_0\mathfrak{X}_{\lambda_1}^{\delta_1} g_1 \mathfrak{X}_{\lambda_2}^{\delta_2} g_2 \ldots \mathfrak{X}_{\lambda_k}^{\delta_k} (g_k \widetilde{g}_0) \mathfrak{X}_{\gamma_1}^{\varepsilon_1} \widetilde{g}_1 \ldots \mathfrak{X}_{\gamma_t}^{\varepsilon_t} \widetilde{g}_t.$$

The *semi-group* $W(G, \{\mathfrak{X}\}_n)$, equipped with this product is a (*non-abelian*) *group* with identity element $\dot{\mathbb{1}}$ and natural inversion given by

$$[g_0\mathfrak{X}_{\lambda_1}^{\delta_1} g_1 \mathfrak{X}_{\lambda_2}^{\delta_2} g_2 \ldots \mathfrak{X}_{\lambda_k}^{\delta_k} g_k]^{-1} = g_k^{-1} \mathfrak{X}_{\lambda_k}^{-\delta_k} \ldots g_2^{-1} \mathfrak{X}_{\lambda_2}^{-\delta_2} g_1^{-1} \mathfrak{X}_{\lambda_1}^{-\delta_1} g_0^{-1}.$$

Definition 6.1. Let $\tau\colon F(\mathbf{x}) \to G$ be a group homomorphism, where $\mathbf{x} = \{x_1, \ldots, x_n\}$ is an alphabet with n letters. A *symbolic mutation* of τ with respect to the list $\{\mathfrak{X}\}_n = \{\mathfrak{X}_1, \ldots, \mathfrak{X}_n\}$ is a function (not necessarily a homomorphism)

$$\mathfrak{X}\colon F(\mathbf{x}) \to W(G, \{\mathfrak{X}\}_n),$$

verifying the following conditions:

(a) $\mathfrak{X}(\mathbb{1}) = \dot{\mathbb{1}}$;
(b) $\mathfrak{X}(x_j) = \mathfrak{X}_j$ for each index $1 \leq j \leq n$;
(c) $\mathfrak{X}(w_1 w_2) = \mathfrak{X}(w_1)\tau(w_1)\mathfrak{X}(w_2)\tau(w_1)^{-1}$ for every $w_1, w_2 \in F(\mathbf{x})$.

Note that these three conditions define completely the function $\mathfrak{X}$ exclusively in terms of the homomorphism τ, since each $w \in F(\mathbf{x})$ is a word of the form

$$w = \prod_{k=1}^{s} (x_1^{p_{1k}} x_2^{p_{2k}} \ldots x_n^{p_{nk}}).$$

The proof that the image of the function $\mathfrak{X}$ is (in fact) in the group $W(G, \{\mathfrak{X}\}_n)$ is easy and will be omitted.

Lemma 6.2. *Let $\mathcal{M}\tau\colon F(\mathbf{x}) \to G$ be a $(\mathcal{P}, \theta)$-mutation of the group homomorphism $\tau\colon F(\mathbf{x}) \to G$. Then, the $(\mathcal{P}, \theta)$-mutator function $\mathcal{M}\colon F(\mathbf{x}) \to G$ can* *be seen as a symbolic mutation* $F(\mathbf{x}) \to W(G, \{\mathcal{M}\}_n)$ *of* τ *with respect to the list of symbols* $\{\mathcal{M}\}_n = \{\mathcal{M}(x_1), \dots, \mathcal{M}(x_n)\}$.

Proof. Since $\mathcal{M}(\mathbb{1})$ is the identity element of G, which is identified with the empty word $\dot{\mathbb{1}}$, it is sufficient to prove that

$$\mathcal{M}(w_1w_2) = \mathcal{M}(w_1)\tau(w_1)\mathcal{M}(w_2)\tau(w_1)^{-1}.$$

Now, since τ and $\mathcal{M}\tau$ are homomorphisms, we have

$$\begin{aligned} \mathcal{M}(w_1w_2)\tau(w_1)\tau(w_2) &= \mathcal{M}(w_1w_2)\tau(w_1w_2) = \mathcal{M}\tau(w_1w_2) \\ &= \mathcal{M}\tau(w_1)\mathcal{M}\tau(w_2) = \mathcal{M}(w_1)\tau(w_1)\mathcal{M}(w_2)\tau(w_2). \end{aligned}$$

Multiplying both sides on the right by the element $\tau(w_2)^{-1}\tau(w_1)^{-1}$ of G, we obtain the desired formula. □

By using the definition of symbolic mutation and induction argument, we can prove the following result:

Proposition 6.3. *Let* $\mathfrak{X}\colon F(\mathbf{x}) \to W(G, \{\mathfrak{X}\}_n)$ *be a symbolic mutation of* τ. *For any integer* $p > 0$, *we have*

$$\mathfrak{X}(w^{-p}) = [\mathfrak{X}(w)\tau(w)]^{-p}\tau(w)^p \quad \text{for every } w \in F(\mathbf{x})$$

and for $w_1, \dots, w_s \in F(\mathbf{x})$ *and nonnegative integers* $p_1, \dots, p_s$, *we have*

$$\begin{aligned} &\mathfrak{X}(w_1^{p_1} \dots w_s^{p_s}) \\ &\quad = \bigg(\prod_{i=1}^{s-1}[\mathfrak{X}(w_i)\tau(w_i)]^{p_i}\bigg)[\mathfrak{X}(w_s)\tau(w_s)]^{p_s-1}\mathfrak{X}(w_s)\tau(w_1^{p_1} \dots w_{s-1}^{p_{s-1}}w_s^{p_s-1})^{-1}. \end{aligned}$$

In particular, $\mathfrak{X}(x_j^{-1}) = \tau(x_j)^{-1}\mathfrak{X}_j^{-1}\tau(x_j)$ *and so* $\mathfrak{X}_j^{-1} = \tau(x_j)\mathfrak{X}(x_j^{-1})\tau(x_j)^{-1}$.

These identities show that the structure of the group $W(G, \{\mathfrak{X}\}_n)$, which exists independently on the homomorphism τ, is "compatible" with this homomorphism in the following way:

$$\dot{\mathbb{1}} = \mathfrak{X}_j\mathfrak{X}_j^{-1} = \mathfrak{X}(x_j)\tau(x_j)\mathfrak{X}(x_j^{-1})\tau(x_j)^{-1} = \mathfrak{X}(x_jx_j^{-1}) = \mathfrak{X}(\mathbb{1}) = \dot{\mathbb{1}}.$$

In the following results, $K_{\mathcal{P}}$ is a model 2-complex of a group presentation $\mathcal{P} = \langle \mathbf{x} \mid \mathbf{r} \rangle$, where $\mathbf{x} = \{x_1, \dots, x_n\}$ is the alphabet and $\mathbf{r} = \{r_1, \dots, r_m\}$ is the set of relators. We consider the natural identification $\pi_1(K_{\mathcal{P}}^1) \equiv F(\mathbf{x})$. Furthermore, Y is a closed surface and $l_\#\colon \pi_1(Y^1) \to \pi_1(Y)$ denotes the homomorphism induced on fundamental groups by the natural inclusion.

Theorem 6.4. *Let* $f\colon K_{\mathcal{P}} \to Y$ *be a cellular map and*

$$\mathfrak{X}\colon F(\mathbf{x}) \to W(\pi_1(Y^1), \{\mathfrak{X}\}_n)$$

be a symbolic mutation of the obvious homomorphism $f^1_\#\colon F(\mathbf{x}) \to \pi_1(Y^1)$. *If* f *is root free, then the homomorphism* $f_{\#_2}$ *is trivial and there is a function* $\mathfrak{A}\colon \{\mathfrak{X}\}_n \to \pi_1(Y^1)$ *satisfying the following conditions:*

(a) $\mathfrak{A}(\mathfrak{X}_j) \in \ker(l_\#)$ *for every index* $1 \le j \le n$;

(b) $\mathfrak{A}(\mathfrak{X}(r_i))f^1_\#(r_i) = \mathbb{1}$ $(=$ *identity element of* $\pi_1(Y^1))$ *for every index* $1 \le i \le m$;

where $\mathfrak{A}(\mathfrak{X}(w))$ *is the word obtained from* $\mathfrak{X}(w)$ *by replacing each symbol* $\mathfrak{X}_j$ *by* $\mathfrak{A}(\mathfrak{X}_j)$. *The reciprocal is true if* f *is a convenient map.*

Proof. Suppose that f is root free. Then, by Theorem 5.1 and Remark 5.2, $f_{\#_2}$ is trivial and $f^1_\#$ has a $(\mathcal{P}, l_\#)$-mutation $\mathcal{M}f^1_\#\colon F(\mathbf{x}) \to \pi_1(Y^1)$, being $\mathcal{M}f^1_\#(w) = \mathcal{M}(w)f^1_\#(w)$. Define the function $\mathfrak{A}\colon \{\mathfrak{X}\}_n \to \pi_1(Y^1)$ by $\mathfrak{A}(\mathfrak{X}_j) = \mathcal{M}(x_j)$, for each $1 \le j \le n$. Then $\mathfrak{A}(\mathfrak{X}(w))$ is the word obtained from $\mathfrak{X}(w) \in W(\pi_1(Y^1), \{\mathfrak{X}\}_n)$ by replacing each symbol $\mathfrak{X}_j$ by the word $\mathcal{M}(x_j) \in \pi_1(Y^1)$. By Lemma 6.2,

$$\mathfrak{A}(\mathfrak{X}(w)) = \mathcal{M}(w) \quad \text{for every } w \in F(\mathbf{x}).$$

In particular, $\mathfrak{A}(\mathfrak{X}(w)) \in \ker(l_\#)$ for every $w \in F(\mathbf{x})$ and, moreover,

$$\mathfrak{A}(\mathfrak{X}(r_i))f^1_\#(r_i) = \mathcal{M}(r_i)f^1_\#(r_i) = \mathcal{M}f^1_\#(r_i) = \mathbb{1}$$

(the identity element of $\pi_1(Y^1)$) for every $1 \le i \le m$.

Now, suppose that f is a convenient map and suppose that $f_{\#_2}$ is trivial and there exists a function $\mathfrak{A}\colon \{\mathfrak{X}\}_n \to \pi_1(Y^1)$ satisfying the conditions (a) and (b).

First, we will prove that $\mathfrak{A}(\mathfrak{X}(w)) \in \ker(l_\#)$ for every $w \in F(\mathbf{x})$. For this, we note that if $\mathfrak{A}(\mathfrak{X}(w_1))$ and $\mathfrak{A}(\mathfrak{X}(w_2))$ belong to $\ker(l_\#)$, then $\mathfrak{A}(\mathfrak{X}(w_1w_2))$ belongs to $\ker(l_\#)$. In fact, if $\mathfrak{A}(\mathfrak{X}(w_1))$ and $\mathfrak{A}(\mathfrak{X}(w_2))$ belong to $\ker(l_\#)$, then

$$l_\#(\mathfrak{A}(\mathfrak{X}(w_1w_2))) = l_\#(\mathfrak{A}(\mathfrak{X}(w_1))\tau(w_1)\mathfrak{A}(\mathfrak{X}(w_2))\tau(w_1)^{-1}) = 1_{\pi_1(Y)},$$

where $1_{\pi_1(Y)}$ is the identity element of $\pi_1(Y)$. This proves that $\mathfrak{A}(\mathfrak{X}(w_1w_2)) \in \ker(l_\#)$.

Now, since for each $1 \le j \le n$, $\mathfrak{A}(\mathfrak{X}(x_j)) = \mathfrak{A}(\mathfrak{X}_j) \in \ker(l_\#)$, and each $w \in F(\mathbf{x})$ is a word of the form

$$w = \prod_{k=1}^{s} (x_1^{p_{1k}} x_2^{p_{2k}} \dots x_n^{p_{nk}}),$$

by repeating the previous argument, we have that $\mathfrak{A}(\mathfrak{X}(w)) \in \ker(l_\#)$ for every $w \in F(\mathbf{x})$. Define the function $\mathcal{M}\colon F(\mathbf{x}) \to \pi_1(Y^1)$ by $\mathcal{M}(w) = \mathfrak{A}(\mathfrak{X}(w)) \in \pi_1(Y^1)$. Then $\mathcal{M}(w) \in \ker(l_\#)$ for every $w \in F(\mathbf{x})$ and, moreover,

- $\mathcal{M}(\mathbb{1}) = \mathfrak{A}(\mathfrak{X}(\mathbb{1})) = \mathbb{1}$;

- $\mathcal{M}(r_i) = \mathfrak{A}(\mathfrak{X}(r_i)) = f^1_\#(r_i)^{-1}$;
- Define $\mathcal{M}f^1_\#: F(\mathbf{x}) \to \pi_1(Y^1)$ by $\mathcal{M}f^1_\# = \mathcal{M}(w)f^1_\#(w)$. Then, certainly, $\mathcal{M}f^1_\#(\mathbb{1}) = \mathbb{1}$ and, given $w_1, w_2 \in F(\mathbf{x})$, we have

$$\begin{aligned}\mathcal{M}f^1_\#(w_1w_2) &= \mathfrak{A}(\mathfrak{X}(w_1w_2))f^1_\#(w_1w_2)\\ &= \mathfrak{A}(\mathfrak{X}(w_1)f^1_\#(w_1)\mathfrak{X}(w_2)f^1_\#(w_2)^{-1})f^1_\#(w_1w_2)\\ &= \mathfrak{A}(\mathfrak{X}(w_1))f^1_\#(w_1)\mathfrak{A}(\mathfrak{X}(w_2))f^1_\#(w_1)^{-1}f^1_\#(w_1)f^1_\#(w_2)\\ &= \mathfrak{A}(\mathfrak{X}(w_1))f^1_\#(w_1)\mathfrak{A}(\mathfrak{X}(w_2))f^1_\#(w_2)\\ &= \mathcal{M}(w_1)f^1_\#(w_1)\mathcal{M}(w_2)f^1_\#(w_2) = \mathcal{M}f^1_\#(w_1)\mathcal{M}f^1_\#(w_2),\end{aligned}$$

showing that $\mathcal{M}f^1_\#$ is a group homomorphism.

This is enough to prove that $\mathcal{M}f^1_\#$ is a $(\mathcal{P}, l_\#)$-mutation of the homomorphism $f^1_\#$. Therefore, by Theorem 5.1, f is root free. $\square$

Theorem 6.5. *Let $f: K_\mathcal{P} \to Y$ be a cellular map and let $f^1_\#: \pi_1(K^1_\mathcal{P}) \to \pi_1(Y^1)$ be the obvious homomorphism. If f is root free, then $f_{\#_2}$ is trivial and the following $m \times n$ system of equations on the free group $\pi_1(Y^1)$, with unknowns $\mathfrak{X}_1, \ldots, \mathfrak{X}_n$, has a solution over* $\ker(l_\#)$*:*

$$\{\mathcal{S}\mathfrak{X}f\} : \begin{cases} \mathfrak{X}(r_1)f^{(1)}_\#(r_1) = \mathbb{1}, \\ \qquad\vdots \\ \mathfrak{X}(r_m)f^{(1)}_\#(r_m) = \mathbb{1}. \end{cases}$$

The reciprocal is true if f is a convenient map.

Proof. Suppose that f is root free. Then $f_{\#_2}$ is trivial and, by Theorem 6.4, there is a function $\mathfrak{A}: \{\mathfrak{X}\}_n \to \pi_1(Y^1)$ such that $\mathfrak{A}(\mathfrak{X}_j) \in \ker(l_\#)$ for every $1 \le j \le n$ and, furthermore, $\mathfrak{A}(\mathfrak{X}(r_i))f^1_\#(r_i) = \mathbb{1}$ for every $1 \le i \le m$. Then, the n-vector $(\mathfrak{A}(\mathfrak{X}_1), \ldots, \mathfrak{A}(\mathfrak{X}_n))$ is a solution of the system $\{\mathcal{S}\mathfrak{X}f\}$, with each coordinate belonging to $\ker(l_\#)$.

On the other hand, suppose that $(\mathfrak{s}_1, \ldots, \mathfrak{s}_n)$ is a solution of the system $\{\mathcal{S}\mathfrak{X}f\}$, with each coordinate $\mathfrak{s}_j \in \ker(l_\#)$. Define the function $\mathfrak{A}: \{\mathfrak{X}\}_n \to \pi_1(Y^1)$ by $\mathfrak{A}(\mathfrak{X}_j) = \mathfrak{s}_j$ for each $1 \le j \le n$. Then, it is clear that $\mathfrak{A}(\mathfrak{X}_j) \in \ker(l_\#)$ for each $1 \le j \le n$ and, furthermore, $\mathfrak{A}(\mathfrak{X}(r_i))f^1_\#(r_i) = \mathbb{1}$ for each $1 \le i \le m$. Now, we apply Theorem 6.4 and the result follows. $\square$

7. Making the results applicable

In this section, we develop results which are more suitable for applications.

Lemma 7.1. *Let $\mathfrak{X}: F(\mathbf{x}) \to W(G, \{\mathfrak{X}\}_n)$ be a symbolic mutation of the group homomorphism $\tau: F(\mathbf{x}) \to G$. Then, for each word of the type $x_1^{p_1} \ldots x_n^{p_n}$*

in $F(\mathbf{x})$, *we have*

$$\mathfrak{X}(x_1^{p_1}\dots x_n^{p_n})\tau(x_1^{p_1}\dots x_n^{p_n}) = \prod_{j=1}^{n}[\mathfrak{X}_j\tau(x_j)]^{p_j}.$$

Proof. Suppose that $p_1,\dots,p_n$ are all nonnegative. By Proposition 6.3,

$$\begin{aligned}\mathfrak{X}(x_1^{p_1}\dots x_n^{p_n}) &= \left(\prod_{j=1}^{n-1}[\mathfrak{X}(x_j)\tau(x_j)]^{p_j}\right)\\ &\quad\cdot[\mathfrak{X}(x_n)\tau(x_n)]^{p_n-1}\mathfrak{X}(x_n)\tau(x_1^{p_1}\dots x_{n-1}^{p_{n-1}}x_n^{p_n-1})^{-1}\\ &= \left(\prod_{j=1}^{n-1}[\mathfrak{X}_j\tau(x_j)]^{p_j}\right)[\mathfrak{X}_n\tau(x_n)]^{p_n-1}\mathfrak{X}_n\tau(x_1^{p_1}\dots x_{n-1}^{p_{n-1}}x_n^{p_n-1})^{-1}.\end{aligned}$$

Multiplying both sides on the right by $\tau(x_1^{p_1}\dots x_n^{p_n})$ we obtain

$$\mathfrak{X}(x_1^{p_1}\dots x_n^{p_n})\tau(x_1^{p_1}\dots x_n^{p_n}) = \prod_{i=1}^{n}[\mathfrak{X}_i\tau(x_i)]^{p_i}.$$

Now, by the formulas of Proposition 6.3, for any integer $p > 0$ and $w \in F(\mathbf{x})$, we have, $\mathfrak{X}(w^{-p})\tau(w^{-p}) = [\mathfrak{X}(w)\tau(w)]^{-p}\tau(w)^p\tau(w^{-p}) = [\mathfrak{X}(w)\tau(w)]^{-p}$.

The general case also follows similarly using formulas of Proposition 6.3. □

Next, we consider $\mathfrak{X}_1,\dots,\mathfrak{X}_n$ as unknowns of equations on the free group $\pi_1(Y^1)$.

Theorem 7.2. *Let* $K_{\mathcal{P}}$ *be the model* 2*-complex of a group presentation* $\mathcal{P} = \langle \mathbf{x} \mid \mathbf{r}\rangle$, *with* $\mathbf{x} = \{x_1,\dots,x_n\}$ *and* $\mathbf{r} = \{r_1,\dots,r_m\}$, *where the relators are in the generic form*

$$\begin{aligned} r_1 &= (x_1^{\delta_{11}^{(1)}}\dots x_n^{\delta_{1n}^{(1)}})\dots(x_1^{\delta_{11}^{(k_1)}}\dots x_n^{\delta_{1n}^{(k_1)}});\\ r_m &= (x_1^{\delta_{m1}^{(1)}}\dots x_n^{\delta_{mn}^{(1)}})\dots(x_1^{\delta_{m1}^{(k_m)}}\dots x_n^{\delta_{mn}^{(k_m)}}).\end{aligned}$$

Let $f\colon K_{\mathcal{P}} \to Y$ *be a map into a closed surface. If* f *is root free, then the homomorphism* $f_{\#_2}$ *is trivial and the following* $m \times n$ *system, on the free group* $\pi_1(Y^1)$, *with unknowns* $\mathfrak{X}_1,\dots,\mathfrak{X}_n$, *has a solution over* $\ker(l_{\#})$*:*

$$\{\mathcal{S}\mathfrak{X}f\}:\left\{\begin{array}{l}\displaystyle\prod_{\lambda=1}^{k_1}\prod_{j=1}^{n}[\mathfrak{X}_jf_{\#}^1(x_j)]^{\delta_{1j}^{(\lambda)}} = \mathbb{1},\\ \qquad\qquad\vdots\\ \displaystyle\prod_{\lambda=1}^{k_m}\prod_{j=1}^{n}[\mathfrak{X}_jf_{\#}^1(x_j)]^{\delta_{mj}^{(\lambda)}} = \mathbb{1}.\end{array}\right.$$

The reciprocal is true if f *is a convenient map.*

Proof. The prove is a consequence of Theorem 6.5 and Lemma 7.1. □

By Theorem 7.2 we have that if $\mathcal{P} = \langle x_1, \ldots, x_n \mid x_1^{p_1} \ldots x_n^{p_n} \rangle$ is a group presentation with n generators and only one relator of the form above, then a convenient cellular map $f\colon K_{\mathcal{P}} \to Y$ is root free if and only if $f_{\#_2}$ is trivial and the equation $\prod_{j=1}^{n} [\mathfrak{X}_j f_{\#}^1(x_j)]^{p_j} = \mathbb{1}$, with unknowns $\mathfrak{X}_1, \ldots, \mathfrak{X}_n$, has a solution over $\ker(l_{\#})$.

8. Examples and applications

In this section, we present some examples and applications of the main results, in special Theorem 7.2. The examples are presented in some subsections, separated accordingly to the closed surface Y, the range of the map being studied.

8.1. Maps into the torus.

Example 8.1. Let $K_{\mathcal{P}}$ be the model 2-complex of $\mathcal{P} = \langle x, y, z \mid x^2y^3z^5 \rangle$. Let $\tau\colon F(x, y, z) \to F(a, b)$ be the group homomorphism between free groups given by

$$\tau(x) = ab, \quad \tau(y) = ab \quad \text{and} \quad \tau(z) = (ba)^{-1}.$$

Consider the 1-skeleton $K_{\mathcal{P}}^1 = S_x^1 \vee S_y^1 \vee S_z^1 = e^0 \cup e_x^1 \cup e_y^1 \cup e_z^1$. Let $\mathbb{T}$ be the torus and consider its 1-skeleton $\mathbb{T}^1 = S_a^1 \vee S_b^1 = c^0 \cup c_a^1 \cup c_b^1$. Let $f^1\colon K_{\mathcal{P}}^1 \to \mathbb{T}^1$ be the cellular map which carries e_s^1 into $\mathbb{T}^1$ exactly as τ carries s into $F(a, b)$, for $s = x, y, z$. Then $f_{\#}^1 = \tau$, up to the identifications $F(x, y, z) \equiv \pi_1(K_{\mathcal{P}}^1)$ and $F(a, b) \equiv \pi_1(\mathbb{T}^1)$. Let $l\colon \mathbb{T}^1 \to \mathbb{T}$ be the natural inclusion and let $l_{\#}\colon F(a, b) \to \pi_1(\mathbb{T}) \approx \mathbb{Z} \oplus \mathbb{Z}$ be the homomorphism induced by l on fundamental groups. (Note that $l_{\#}$ is the abelianization homomorphism). We have:

$$(l_{\#} \circ \tau)(x^2y^3z^5) = l_{\#}((ab)^2(ab)^3(ba)^{-5}) = 0.$$

Hence, $l \circ f^1\colon K_{\mathcal{P}}^1 \to \mathbb{T}$ extends to a cellular map $f\colon K_{\mathcal{P}} \to \mathbb{T}$. Since $\pi_2(\mathbb{T}) = 0$, such map is convenient (see Remark 2.4). By Theorem 7.2, f is root free if and only if the equation

$$[\mathfrak{X}_1 ab]^2[\mathfrak{X}_2 ab]^3[\mathfrak{X}_3 (ba)^{-1}]^5 = \mathbb{1}$$

has a solution over $\ker(l_{\#})$. Now, it is easy to check that $([b, a], [b, a], \mathbb{1})$ is such a solution, where, as usual, $[b, a] = bab^{-1}a^{-1}$. Therefore, f is root free.

Let $\mathcal{M}\tau\colon F(x, y, z) \to F(a, b) \equiv \pi_1(\mathbb{T}^1)$ be the homomorphism defined by

$$\begin{aligned}
\mathcal{M}\tau(x) &= [b, a]\tau(x) = [b, a]ab = ba, \\
\mathcal{M}\tau(y) &= [b, a]\tau(y) = [b, a]ab = ba, \\
\mathcal{M}\tau(z) &= \mathbb{1}\tau(z) = (ab)^{-1}.
\end{aligned}$$

By previous results, we have the following conclusions:

- $\mathcal{M}\tau$ is a $(\mathcal{P}, l_{\#})$-mutation of the homomorphism $\tau = f_{\#}^1$.

- There is only one homomorphism $\phi\colon \pi_1(K_{\mathcal{P}}) \to \pi_1(\mathbb{T}^1)$ such that $\phi \circ \Omega = \mathcal{M}\tau$.
- The homomorphism ϕ is a lifting of $f_{\#}$ through $l_{\#}$.

EXAMPLE 8.2. Let $\mathcal{P} = \langle x_1, x_2, x_3, x_4 \mid [x_1, x_2], [x_1, x_3], [x_3, x_4]\rangle$ be a group presentation and let $K_{\mathcal{P}}$ be the model 2-complex of $\mathcal{P}$. This complex is the 2-complex K of Example 2.5 of [4], obtained by attaching two torus $\mathbb{T}_1$ and $\mathbb{T}_2$ through of the longitudinal closed 1-cell and, next, by attaching the longitudinal closed 1-cell of a torus $\mathbb{T}_3$ into the meridional closed 1-cell of the torus $\mathbb{T}_2$. Let $\widetilde{f}\colon K_{\mathcal{P}} \to \mathbb{T}$ be the map of Example 2.5 of [4], which restricted to each torus $\mathbb{T}_i \subset K_{\mathcal{P}}$, for $i = 1, 2, 3$, is a cellular homomorphism. Let $p_n\colon \mathbb{T} \to \mathbb{T}$ be the longitudinal n-fold covering. Let $f\colon K_{\mathcal{P}} \to \mathbb{T}$ be the composition $f = p_n \circ \widetilde{f}$. Then the restricted map

$$f^1\colon K^1_{\mathcal{P}} = S^1_{x_1} \vee S^1_{x_2} \vee S^1_{x_3} \vee S^1_{x_4} \longrightarrow S^1_a \vee S^1_b = \mathbb{T}^1$$

is such that $f^1_{\#}(x_1) = f^1_{\#}(x_4) = a^n$ and $f^1_{\#}(x_2) = f^1_{\#}(x_3) = b$. Now, by Theorem 2.7 of [4], we have $\mu(f) = 2n - 1$, which means that f is not root free. Since f is a convenient map (see Remark 2.4), Theorem 7.2 implies that the following system on $F(a, b)$ has no solution over the kernel of the abelianization $l_{\#}\colon F(a, b) \to \mathbb{Z} \oplus \mathbb{Z}$:

$$\begin{cases} \mathfrak{X}_1 a^n \mathfrak{X}_2 b a^{-n} \mathfrak{X}_1^{-1} b^{-1} \mathfrak{X}_2^{-1} = \mathbb{1}, \\ \mathfrak{X}_1 a^n \mathfrak{X}_3 b a^{-n} \mathfrak{X}_1^{-1} b^{-1} \mathfrak{X}_3^{-1} = \mathbb{1}, \\ \mathfrak{X}_3 b \mathfrak{X}_4 a^n b^{-1} \mathfrak{X}_3^{-1} a^{-n} \mathfrak{X}_4^{-1} = \mathbb{1}. \end{cases}$$

Indeed, since $\widetilde{f}|_{\mathbb{T}_i}$ is a homeomorphism for each $i = 1, 2, 3$, each map $f|_{\mathbb{T}_i} = p_n \circ \widetilde{f}|_{\mathbb{T}_i}$ is not root free (in fact, $\mu(f|_{\mathbb{T}_1}) = n$). Then, none of the equations of the system above has a solution over $\ker(l_{\#})$.

8.2. Maps into the Klein bottle. The Klein bottle is usually meant as the square with identification of reciprocal sides one of them twisted, being so given by relation $aca^{-1}c$. However, by performing a cut on the diagonal of the square, which we indexed with the letter b, and pasting properly two of the sides of the square (exactly the sides corresponding to the letter c), we see that the Klein bottle can be given by relation a^2b^2.

EXAMPLE 8.3. Let $K_{\mathcal{P}}$ be the model 2-complex of $\mathcal{P} = \langle x, y, z \mid x^3y^2z^7\rangle$. Let $\tau\colon F(x, y, z) \to F(a, b)$ be the group homomorphism between free groups given by

$$\tau(x) = a^{10}, \quad \tau(y) = b \quad \text{and} \quad \tau(z) = a^{-4}.$$

Consider the 1-skeleton $K^1_{\mathcal{P}} = S^1_x \vee S^1_y \vee S^1_z = e^0 \cup e^1_x \cup e^1_y \cup e^1_z$. Let $\mathbb{K}$ be the Klein bottle and consider its 1-skeleton $\mathbb{K}^1 = S^1_a \vee S^1_b = c^0 \cup c^1_a \cup c^1_b$. Let $f^1\colon K^1_{\mathcal{P}} \to \mathbb{K}^1$ be the map which carries e^1_s into $\mathbb{K}^1$ exactly as τ carries s into

$F(a,b)$, for $s = x, y, z$. Then, up to the identifications $F(x,y,z) \equiv \pi_1(K^1_{\mathcal{P}})$ and $F(a,b) \equiv \pi_1(\mathbb{K}^1)$, we have $f^1_\# = \tau$. Let $l\colon \mathbb{K}^1 \to \mathbb{K}$ be the natural inclusion and let $l_\#\colon F(a,b) \to \pi_1(\mathbb{K})$ be the homomorphism induced by l on fundamental groups. Note that $\pi_1(\mathbb{K})$ has a presentation $\langle a, b \mid a^2b^2\rangle$ and $l_\#\colon F(a,b) \to \pi_1(\mathbb{K})$ is the quotient homomorphism accordingly to this group presentation. We have:

$$(l_\# \circ \tau)(x^3y^2z^7) = l_\#(a^{30}b^2a^{-28}) = l_\#(a^{28}(a^2b^2)a^{-28}) = 0.$$

Hence, $l \circ f^1\colon K^1_{\mathcal{P}} \to \mathbb{K}$ extends to a cellular map $f\colon K_{\mathcal{P}} \to \mathbb{K}$. Since $\pi_2(\mathbb{K}) = 0$, the map f is convenient (see Remark 2.4). Then, by Theorem 7.2, f is root free if and only if the equation

$$[\mathfrak{X}_1a^{10}]^3[\mathfrak{X}_2b]^2[\mathfrak{X}_3a^{-4}]^7 = \mathbb{1}$$

has a solution over $\ker(l_\#)$. Now, it is easy to check that $(b^{-10}a^{-10}, \mathbb{1}, b^4a^4)$ is such a solution. Therefore, $f\colon K_{\mathcal{P}} \to \mathbb{K}$ is root free.

8.3. Maps into the projective plane. The fact that the 1-skeleton $\mathbb{R}\mathrm{P}^1$ of the projective plane $\mathbb{R}\mathrm{P}^2$ is homeomorphic to the sphere S^1, which has fundamental group isomorphic to the infinite cyclic group $\mathbb{Z}$, can be used with great advantage to study the solubility of the system $\{\mathcal{S}\mathfrak{X}f\}$ of Theorem 7.2.

We start by studying the case in which the (model) 2-complex has a single cell of dimension two.

Let $K_{\mathcal{P}}$ be a model 2-complex having a single 2-cell. Then, the group presentation $\mathcal{P}$ is of the form $\mathcal{P} = \langle x_1, \dots, x_n \mid r_1\rangle$. Let $f\colon K_{\mathcal{P}} \to \mathbb{R}\mathrm{P}^2$ be a cellular map and let $l\colon \mathbb{R}\mathrm{P}^1 \to \mathbb{R}\mathrm{P}^2$ be the natural inclusion. ($\mathbb{R}\mathrm{P}^1 \cong S^1$ is the 1-skeleton of $\mathbb{R}\mathrm{P}^2$). We will study the solubility of the equation below, with unknowns $\mathfrak{X}_1, \dots, \mathfrak{X}_n$, over the subgroup $\ker(l_\#)$ of $\pi_1(\mathbb{R}\mathrm{P}^1)$:

$$\mathfrak{X}(r_1)f^1_\#(r_1) = \mathbb{1}. \tag{8.1}$$

Let $f^1\colon K^1_{\mathcal{P}} \to \mathbb{R}\mathrm{P}^1$ be the obvious map obtained by restriction of f and let $f^1_\#\colon \pi_1(K^1_{\mathcal{P}}) \approx F(x_1, \dots, x_n) \to F(a) \approx \pi_1(\mathbb{R}\mathrm{P}^1)$ be the homomorphism induced by f^1 on fundamental groups. Let $\Omega\colon F(x_1, \dots, x_n) \to \pi_1(K_{\mathcal{P}})$ be the natural quotient homomorphism, which identifies naturally with that induced by the inclusion $K^1_{\mathcal{P}} \hookrightarrow K_{\mathcal{P}}$. Then, we have the commutativity $l_\# \circ f^1_\# = f_\# \circ \Omega$. In particular, $l_\#(f^1_\#(r_1)) = 0$, that is, $f^1_\#(r_1) \in \ker(l_\#)$, where, of course, $\ker(l_\#)$ is the subgroup of $F(a)$ generated by a^2. It follows that $f^1_\#(r_1) = a^{2d}$ for some integer d.

On the other hand, for each $1 \le j \le n$, there is an integer p_j such that $f^1_\#(x_j) = a^{p_j}$. Suppose that the relator r_1 has the following generic form:

$$r_1 = (x_1^{\delta_1^{(1)}} \dots x_n^{\delta_n^{(1)}}) \dots (x_1^{\delta_1^{(k_1)}} \dots x_n^{\delta_n^{(k_1)}}).$$

Then, since $F(a)$ is an abelian group (the infinite cyclic group), we obtain

$$a^{2d} = f^1_{\#}(r_1) = (a^{p_1})^{\delta_1} \dots (a^{p_n})^{\delta_n},$$

where, for each $1 \le j \le n$, we define δ_j to be the integer $\delta_j = \sum_{\lambda=1}^{k_1} \delta_j^{(\lambda)}$.

Again since $F(a)$ is an abelian group, we have

$$\mathfrak{X}(r_1) f^1_{\#}(r_1) = \prod_{\lambda=1}^{k_1} \prod_{j=1}^{n} [\mathfrak{X}_j a^{p_j}]^{\delta_j^{(\lambda)}} = \prod_{j=1}^{n} [\mathfrak{X}_j a^{p_j}]^{\delta_j}.$$

Thus, equation (8.1) is equivalent to

$$[\mathfrak{X}_1 a^{p_1}]^{\delta^1} \dots [\mathfrak{X}_n a^{p_n}]^{\delta_n} = \mathbb{1}.$$

Now, to show that this equation has a solution over $\ker(l_{\#})$ is equivalent to show that there are integer $q_1, \dots, q_n$ such that

$$[a^{2q_1} a^{p_1}]^{\delta^1} \dots [a^{2q_n} a^{p_n}]^{\delta_n} = \mathbb{1}.$$

What, in turn, is equivalent to show that there are integers $q_1, \dots, q_n$ such that

$$\delta_1(2q_1 + p_1) + \dots + \delta_n(2q_n + p_n) = 0. \tag{8.2}$$

But we know that $(a^{p_1})^{\delta_1} \dots (a^{p_n})^{\delta_n} = a^{2d}$ and, therefore, $\delta_1 p_1 + \dots + \delta_n p_n = 2d$. It follows that to find a solution for equation (8.2) is equivalent to find a solution for

$$\delta_1 q_1 + \dots + \delta_n q_n = -d. \tag{8.3}$$

Up to this point, we can conclude at least the following:

(1) If $\delta_1, \dots, \delta_n$ are relatively prime, then (8.3) has infinite many solutions. Therefore, equation (8.1) has infinite many solutions over $\ker(l_{\#})$.

(2) If $K_{\mathcal{P}}$ is an orientable closed surface (with minimal cellular decomposition), then the alphabet is $\{x_1, y_1, \dots, x_g, y_g\}$ and the (unique) relator is of the form $r_1 = [x_1, y_2] \dots [x_g, y_g]$, where $[\,\cdot\,,\,\cdot\,]$ denotes the commutator and g is the genus of the surface. Thus, $\delta_1 = \dots = \delta_{2g} = 0$. Therefore, in accordance with (8.3), (8.1) has a solution over $\ker(l_{\#})$ if and only if $d = 0$, that is, $f^1_{\#}(r_1) = 1$. If this is the case, any $2g$-upple of elements in $\ker(l_{\#})$ is a solution over $\ker(l_{\#})$ of (8.1).

(3) If $K_{\mathcal{P}}$ is a nonorientable closed surface (with minimal cellular decomposition), then the alphabet is $\{x_1, , \dots, x_g\}$ and the (unique) relator is of the form $r_1 = x_1^2 \dots x_g^2$, where g is the genus of the surface. Thus, $\delta_1 = \dots = \delta_g = 2$. Therefore, in accordance with (8.3), (8.1) has a solution over $\ker(l_{\#})$ if and only if d is even.

Proposition 8.4. *Let $f\colon X \to \mathbb{R}P^2$ be a cellular map, where X is an orientable (respectively, nonorientable) closed surface with canonical presentation $\mathcal{P} = \langle \mathbf{x} \mid r_1\rangle$. If f is root free, then $f_{\#_2}$ is trivial and $f^1_{\#}(r_1) = 1$ (respectively, $f^1_{\#}(r_1) \equiv 0 \mod 4$). The reciprocal is true if f is a convenient map.*

Proof. It follows from Theorem 7.2 and items (2) and (3) above. Note that we identified $F(a) \approx \pi_1(\mathbb{R}P^1)$ with $\mathbb{Z}$ by identifying $a^k \in F(a)$ with $k \in \mathbb{Z}$. $\square$

Using item (3) above, we can construct examples of maps $f\colon N_g \to \mathbb{R}P^2$, from the nonorientable closed surface of genus $g > 1$ into the projective plane, which are not root free.

Exemple 8.5. The nonorientable closed surface of genus g, which we denote by N_g, is the model 2-complex of the (minimal) group presentation $\mathcal{P} = \langle x_1, \dots, x_g \mid x_1^2 \dots x_g^2\rangle$. Let $\tau\colon F(x,y) \to F(a)$ be the homomorphism between free groups given by

$$\tau(x_j) = a^2 \quad \text{for } 1 \le j \le g-1 \quad \text{and} \quad \tau(x_g) = a.$$

Let $l\colon \mathbb{R}P^1 \to \mathbb{R}P^2$ be the natural inclusion. Then, up to identifications, the homomorphism induced by l on fundamental groups is the obvious quotient homomorphism $l_{\#}\colon F(a) \to \mathbb{Z}_2$, where we consider the cyclic group $\mathbb{Z}_2$ being presented by $\langle a \mid a^2\rangle$. Then $\ker(l_{\#})$ is the normal subgroup of $F(a)$ generated by a^2. It follows that $l_{\#}(\tau(x_1^2 \dots x_g^2)) = l_{\#}(a^{2(2g-1)}) = 0$. Therefore, there is a cellular map $f\colon N_g \to \mathbb{R}P^2$ such that, up to identifications, $f^1_{\#} = \tau$. Now, we have $f^1_{\#}(x_1^2 \dots x_g^2) = a^{2(2g-1)}$ and, obviously, $2(2g-1) \not\equiv 0 \mod 4$. By the previous proposition, f is not root free.

Now we generalize the construction above for the case in which the (model) 2-complex has more than one cell of dimension two.

Let $K_{\mathcal{P}}$ be the model 2-complex of the group presentation $\mathcal{P} = \langle \mathbf{x} \mid \mathbf{r}\rangle$, with $\mathbf{x} = \{x_1, \dots, x_n\}$ and $\mathbf{r} = \{r_1, \dots, r_m\}$, where the relation words are in the generic form

$$\begin{aligned} r_1 &= (x_1^{\delta_{11}^{(1)}} \dots x_n^{\delta_{1n}^{(1)}}) \dots (x_1^{\delta_{11}^{(k_1)}} \dots x_n^{\delta_{1n}^{(k_1)}}); \\ &\ \ \vdots \\ r_m &= (x_1^{\delta_{m1}^{(1)}} \dots x_n^{\delta_{mn}^{(1)}}) \dots (x_1^{\delta_{m1}^{(k_m)}} \dots x_n^{\delta_{mn}^{(k_m)}}). \end{aligned}$$

Let $f\colon K_{\mathcal{P}} \to \mathbb{R}P^2$ be a cellular map and let $f^1\colon K_{\mathcal{P}} \to \mathbb{R}P^1$ be its obvious restriction. Then:

- For each $1 \le j \le n$, there is an integer p_j such that $f^1_{\#}(x_j) = a^{p_j}$;
- For each $1 \le i \le m$, there is an integer d_i such that $f^1_{\#}(r_i) = a^{2d_i}$.

- For each $1 \le i \le m$, we have $\delta_{i1}p_1 + \ldots + \delta_{in}p_n = 2d_i$, where each integer $\delta_{ij} = \sum_{\lambda=1}^{k_i} \delta_{ij}^{(\lambda)}$ is the sum of the powers of the letter x_j in the relator r_i.

With the same argument of the previous construction, we prove that in order f to be root free is necessary (but not sufficient, in general) that there exists integers $q_1, \ldots, q_n$ satisfying the following system of diophantine equations:

$$\begin{bmatrix} \delta_{11} & \ldots & \delta_{1n} \\ \vdots & \ddots & \vdots \\ \delta_{m1} & \ldots & \delta_{mn} \end{bmatrix} \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} = - \begin{bmatrix} d_1 \\ \vdots \\ d_m \end{bmatrix}.$$

If f is a convenient map, this condition is also sufficient if, in addition, we ask $f_{\#_2}$ to be the trivial homomorphism. If we denote $\Delta_{\mathcal{P}} = (\delta_{ij})_{m\times n}$ and $\vec{d} = (d_1, \ldots, d_m)^{\mathrm{T}}$, where the superscript T indicates transposition of matrices, we have:

Proposition 8.6. *Let $f: K_{\mathcal{P}} \to \mathbb{R}\mathrm{P}^2$ be a cellular map. If f is root free, then $f_{\#_2}$ is trivial and the diophantine linear system $\Delta_{\mathcal{P}} Y = \vec{d}$ has an integer solution. The reciprocal is true if f is a convenient map.*

In [5] we proved a similar result for the case of maps from 2-complexes into the 2-sphere. It is interesting to compare these two results.

The next example shows a not convenient map, which is root free, such that the associated system $\Delta_{\mathcal{P}} Y = \vec{d}$ has an integer solution.

Example 8.7. Let $\omega: \mathbb{T} \to S^2$ be the cellular map from the torus into the 2-sphere which collapses the whole 1-skeleton $\mathbb{T}^1$ of $\mathbb{T}$ onto the 0-cell of S^2. Let $\mathfrak{p}_2: S^2 \to \mathbb{R}\mathrm{P}^2$ be the universal covering map. Define the (cellular) map $f: \mathbb{T} \to \mathbb{R}\mathrm{P}^2$ to be the composition $f = \mathfrak{p}_2 \circ \omega$. Then f is not convenient and is not root free (in fact $\mu(f) = 2$). However, the diophantine linear system $\Delta_{\mathcal{P}} Y = \vec{d}$, in this case, is simply the equation $0Y = 0$, which has trivial solution.

Proposition 8.6 can be used to construct more sophisticated examples of maps from 2-complex into the projective plane which is not root free.

Example 8.8. Let $K_{\mathcal{P}}$ be the model 2-complex of the group presentation $\mathcal{P} = \langle x, y \mid x^2y^2, xy^3 \rangle$. Let $\tau: F(x,y) \to F(a)$ be the group homomorphism between free groups given by $\tau(x) = a$ and $\tau(y) = a$. Let $l: \mathbb{R}\mathrm{P}^1 \to \mathbb{R}\mathrm{P}^2$ be the natural inclusion and $l_\#: F(a) \equiv \pi_1(\mathbb{R}\mathrm{P}^1) \to \pi_1(\mathbb{R}\mathrm{P}^2) \equiv \mathbb{Z}_2$ as in Example 8.5. Then,

$$l_\#(\tau(x^2y^2)) = l_\#(a^4) = 0 \quad \text{and} \quad l_\#(\tau(xy^3)) = l_\#(a^4) = 0.$$

Hence, there is a cellular map $f: K_{\mathcal{P}} \to \mathbb{R}\mathrm{P}^2$ such that $f^1_\# = \tau$. By Proposition 8.6, if f is root free then the diophantine linear system

$$\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = - \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

should have an integer solution. But it is easy to check that this system has no integer solution. Therefore, f is not root free.

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