Topological Methods in Nonlinear Analysis
Journal of the Juliusz Schauder Center
Volume 36, 2010, 311–326

EIGENVALUE CRITERIA FOR EXISTENCE OF POSITIVE SOLUTIONS OF SECOND-ORDER, MULTI-POINT, p-LAPLACIAN BOUNDARY VALUE PROBLEMS

Bryan P. Rynne

Abstract. In this paper we consider the existence and uniqueness of positive solutions of the multi-point boundary value problem

$$-(\phi_p(u')' + (a + g(x,u,u'))\phi_p(u) = 0, \quad \text{a.e. on } (-1,1), \tag{1}$$

$$u(\pm 1) = \sum_{i=1}^{m^{\pm}} \alpha_i^{\pm} u(\eta_i^{\pm}), \tag{2}$$

where $p > 1$, $\phi_p(s) := |s|^{p-2}s$, $s \in \mathbb{R}$, $m^{\pm} \geq 1$ are integers, and

$$\eta_i^{\pm} \in (-1,1), \quad \alpha_i^{\pm} > 0, \quad i = 1,\dots,m^{\pm}, \quad \sum_{i=1}^{m^{\pm}} \alpha_i^{\pm} < 1.$$

Also, $a \in L^1(-1,1)$, and $g\colon [-1,1] \times \mathbb{R}^2 \to \mathbb{R}$ is Carathéodory, with

$$g(x,0,0) = 0, \quad x \in [-1,1]. \tag{3}$$

Our criteria for existence of positive solutions of (1), (2) will be expressed in terms of the asymptotic behaviour of $g(x,s,t)$, as $s \to \infty$, and the principal eigenvalues of the multi-point boundary value problem consisting of the equation

$$-\phi_p(u')' + a\phi_p(u) = \lambda\phi_p(u), \quad \text{on } (-1,1), \tag{4}$$

2010 *Mathematics Subject Classification*. Primary 34B10; Secondary 34B18.
Key words and phrases. Positive solutions of nonlinear boundary value problems, principal eigenvalues.

where $\lambda \in \mathbb{R}$, together with the boundary conditions (2). The spectral properties of this eigenvalue problem are not fully known for general functions $a \in L^1(-1,1)$. We will show here that, for any $a \in L^1(-1,1)$, there exists a unique principal eigenvalue $\lambda_0(a)$ of (2), (4), and we obtain some properties of this eigenvalue.

We then consider a bifurcation-type problem and show that there exists a global continuum of positive solutions bifurcating from the principal eigenvalue. Finally, we use this result to give criteria for the existence, and uniqueness, of positive solutions of (1), (2).

1. Introduction

In this paper we consider the existence and uniqueness of positive solutions of the multi-point boundary value problem

$$-(\phi_p(u')' + (a + g(x,u,u'))\phi_p(u) = 0, \quad \text{a.e. on } (-1,1), \tag{1.1}$$

$$u(\pm 1) = \sum_{i=1}^{m^\pm} \alpha_i^\pm u(\eta_i^\pm), \tag{1.2}$$

where $p > 1$, $\phi_p(s) := |s|^{p-2}s$, $s \in \mathbb{R}$, $m^\pm \geq 1$ are integers, and

$$\eta_i^\pm \in (-1,1), \quad \alpha_i^\pm > 0, \quad i = 1,\dots,m^\pm, \quad \sum_{i=1}^{m^\pm} \alpha_i^\pm < 1. \tag{1.3}$$

We suppose throughout the paper that $a \in L^1(-1,1)$ and $g\colon [-1,1] \times \mathbb{R}^2 \to \mathbb{R}$ is a Carathéodory function (see Section 2.1 for the precise conditions), satisfying

$$g(x,0,0) = 0, \quad x \in [-1,1] \tag{1.4}$$

(a further condition on the asymptotic behaviour of $g(x,s,t)$ as $s \to \infty$ will be imposed in Section 5, in our main existence result).

Our criteria for existence of positive solutions of (1.1)–(1.2) will be expressed in terms of the asymptotic behaviour of $g(x,s,t)$, as $s \to \infty$, and the principal eigenvalues of the multi-point boundary value problem consisting of the equation

$$-\phi_p(u')' + a\phi_p(u) = \lambda\phi_p(u), \quad \text{on } (-1,1), \tag{1.5}$$

where $\lambda \in \mathbb{R}$, together with the boundary conditions (1.2). A number $\lambda \in \mathbb{R}$ is said to be an *eigenvalue* of (1.2), (1.5), if there exists a non-trivial solution u of this problem, which is then an *eigenfunction* corresponding to λ. We will say that an eigenvalue is a *principal* eigenvalue if it has a positive eigenfunction. The spectral properties of this eigenvalue problem are not fully known for general functions $a \in L^1(-1,1)$ (see [13] for the constant coefficient case). In Section 3 we will show that, for any $a \in L^1(-1,1)$, there exists a unique principal eigenvalue $\lambda_0(a)$ of (1.2), (1.5), and we obtain some properties of this eigenvalue. In Section 4 we consider a bifurcation-type problem and show that there exists a global continuum of positive solutions bifurcating from the principal eigenvalue. Next, in Section 5, we use the global bifurcation result to give eigenvalue criteria for the existence of positive solutions of (1.1)–(1.2). Finally, under a further monotonicity condition on g we also obtain a uniqueness result.

The existence of positive solutions to multi-point problems similar to (1.1)–(1.2) (with $p = 2$ and $p \neq 2$) have been considered in many recent papers, see for example [2], [8], [14], [15] and the references therein. The methods used have been a mixture of fixed point theorems and degree theory. Our results are closest in spirit to those of [14], although we use different methods. In particular, [14] considers the case $p = 2$ and obtains a principal eigenvalue of (1.2), (1.5) using the Krein–Rutman theorem for positive operators, which requires that $a \geq 0$ (in fact, [14] considers a general integral equation formulation of the problem). We consider the general case $p \neq 2$ and we obtain a principal eigenvalue using a bifurcation theory approach, which requires no conditions on the sign of a. Criteria for the existence of positive solutions of (1.1)–(1.2) are then obtained in [14] using degree theory, while bifurcation theory is again used here. However, the existence criteria in [14] and here are both expressed in terms of the relationship between the behaviour of the function g, as $u \to 0$ and $u \to \infty$, and the corresponding principal eigenvalues. Thus there is some similarity between these existence criteria (indeed, eigenvalue criteria of this general form are well-known in many other contexts).

2. Preliminary notation and results

2.1. Carathéodory functions and Nemitskii operators. For any integer $n \geq 0$, let $C^n[-1,1]$ denote the usual Banach space of n-times continuously differentiable functions on $[-1,1]$, with the usual sup-type norm, denoted by $|\cdot|_n$. Let $\|\cdot\|_1$ and $\|\cdot\|_{1,1}$ denote the standard norms on $L^1(-1,1)$ and the Sobolev space $W^{1,1}(-1,1)$, respectively.

We suppose that the function $g\colon [-1,1] \times \mathbb{R}^2 \to \mathbb{R}$ satisfies the following standard *Carathéodory* conditions:

(a) $g(x,s,t)$ is measurable in x for every fixed $(s,t) \in \mathbb{R}^2$, and continuous in (s,t) for almost every $x \in (-1,1)$;

(b) for any bounded set $B \subset \mathbb{R}^2$, there exists $h_B \in L^1(-1,1)$ such that

$$|g(x,s,t)| \leq h_B(x), \quad (x,s,t) \in [0,1] \times B. \tag{2.1}$$

Under these hypotheses g induces a bounded (in the sense that bounded sets are mapped to bounded sets), continuous Nemitskii operator $g\colon C^1[-1,1] \to L^1(-1,1)$ defined by $g(u)(x) := g(x,u(x),u'(x))$, $x \in [-1,1]$, for $u \in C^1[-1,1]$, see [1, Section 1.2] (using the same notation for the function and the operator should not cause any confusion). In particular, for any $a \in L^1(-1,1)$

the function $(x, s, t) \to a(x)\phi_p(s)$ satisfies the above conditions, so the Nemytskiĭ operator $u \to a\phi_p(u)\colon C^1[-1,1] \to L^1(-1,1)$ is bounded and continuous. More specifically, the function $\phi_p\colon \mathbb{R} \to \mathbb{R}$ also induces a continuous operator $\phi_p\colon C^0[-1,1] \to C^0[-1,1]$, with inverse $\phi_p^{-1} = \phi_{p^*}$, where $p^* := p/(p-1) > 1$.

Since we are searching for positive solutions of (4.1), in principle the function g need not be defined for $s < 0$. However, in some of the following proofs it is convenient to have the Nemytskiĭ operator g defined on the whole of $C^1[-1,1]$, so we assume that the function g is defined on $[0,1] \times \mathbb{R}^2$ and satisfies the above conditions. Hence, by (1.4), the Nemytskiĭ operator g satisfies

$$g(0) = 0. \tag{2.2}$$

2.2. The multi-point, p-Laplacian operator. A suitable space in which to search for solutions of (1.1) or (1.5), on which the differential operator in these equations makes sense and which incorporates the boundary conditions (1.2), is the space

$$D(\Delta_p) := \{u \in C^1[-1,1] : \phi_p(u') \in W^{1,1}[-1,1] \text{ and } u \text{ satisfies (1.2)}\},$$
$$\|u\|_{D(\Delta_p)} := |u|_1 + \|\phi_p(u')\|_1, \quad u \in D(\Delta_p).$$

We now define $\Delta_p\colon D(\Delta_p) \to L^1(-1,1)$ by

$$\Delta_p(u) := \phi_p(u')', \quad u \in D(\Delta_p).$$

By the definition of the spaces $D(\Delta_p)$, $L^1(-1,1)$, the operator Δ_p is well-defined and continuous. Combining [13, Theorem 3.1] and [13, Remark 3.6], together with our Carathéodory conditions on g, yields the following result. Here, an operator is *completely continuous* if it is continuous and maps bounded sets into relatively compact sets, while a sequence (h_n) in $L^1(-1,1)$ is *equi-integrable* if there exists h in $L^1(-1,1)$ such that $|h_n(x)| \le h(x)$ for all $n \ge 1$ and almost every $x \in [-1,1]$ (the condition (2.1) will yield equi-integrability of suitable sequences below).

THEOREM 2.1. *The operator $\Delta_p\colon D(\Delta_p) \to L^1(-1,1)$ is bijective, and the inverse operator $\Delta_p^{-1}\colon L^1(-1,1) \to D(\Delta_p)$ is continuous. In addition:*

(a) *if (h_n) is an equi-integrable sequence in $L^1(-1,1)$ which converges weakly to h_∞ then $\Delta_p^{-1}(h_n) \to \Delta_p^{-1}(h_\infty)$ in $C^1[-1,1]$;*
(b) *the operator $u \to \Delta_p^{-1}\big(g(u)\phi_p(u)\big)\colon C^1[-1,1] \to C^1[-1,1]$ is completely continuous (under the above Carathéodory conditions on g).*

REMARK 2.2. As a special case of Theorem 2.1(b), which will be used several times below, the operator $u \to \Delta_p^{-1}(a\phi_p(u))\colon C^1[-1,1] \to C^1[-1,1]$ is completely continuous.

3. Principal eigenvalues of $-\Delta_p + a\phi_p$

We consider the eigenvalue problem

$$-\Delta_p(u) + a\phi_p(u) = \lambda\phi_p(u), \quad u \in D(\Delta_p), \tag{3.1}$$

with $a \in L^1(-1,1)$. We say that λ is an *eigenvalue* of $-\Delta_p + a\phi_p$ if (3.1) has a non-trivial solution u, which will be termed an *eigenfunction* of $-\Delta_p + a\phi_p$. If λ is an eigenvalue of $-\Delta_p + a\phi_p$, with corresponding eigenfunction u, then tu is also an eigenfunction for all non-zero $t \in \mathbb{R}$, and we say that λ is *simple* if every eigenfunction corresponding to λ is of this form (for linear problems, "simple" eigenvalues usually have some further properties, but here we will use the term in the above sense for all $p > 1$, even in the linear case $p = 2$). A function u is *positive* if $u \not\equiv 0$ and $u \geq 0$ on $[-1,1]$, and u is *strictly positive* if $u > 0$ on $[-1,1]$ (for brevity we will also apply this terminology to a solution (λ, u) of (3.1) without specifically referring to the function u). An eigenvalue λ is a *principal eigenvalue* of $-\Delta_p + a\phi_p$ if it has a positive eigenfunction.

We first show that a positive eigenfunction is in fact strictly positive.

Lemma 3.1. *If* (λ, u) *is a positive solution of* (3.1) *then* u *is strictly positive.*

Proof. Suppose that u is not strictly positive. If $u(x_0) = 0$ at some $x_0 \in (-1,1)$ then, by positivity, $u'(x_0) = 0$, which implies that $u = 0$ (by the uniqueness of the solution of the initial value problem for the differential equation (3.1), for arbitrary initial conditions, see [3, Lemma 3.1]). However, this contradicts the non-triviality of u. On the other hand, if $u(e) = 0$ at an end point $e \in \{\pm 1\}$ then it follows from the boundary conditions (1.2) that $u(x_0) = 0$ at some $x_0 \in (-1,1)$ (since u is positive and $\alpha_i^{\pm} > 0$, $i = 1, \ldots, m^{\pm}$, by (1.3)). Hence, the preceding discussion shows that this case also cannot hold, which completes the proof. □

We now show the existence of a principal eigenvalue, and derive some of its properties. We note that, by Theorem 2.1, the eigenvalue problem (3.1) can be rewritten as the equivalent equation

$$u + \Delta_p^{-1}((\lambda - a)\phi_p(u)) = 0, \quad u \in C^1[-1,1]. \tag{3.2}$$

This formulation will be useful in the following proofs.

Theorem 3.2. *For any* $a \in L^1(-1,1)$, $-\Delta_p + a\phi_p$ *has a unique principal eigenvalue* $\lambda_0(a)$, *with a strictly positive eigenfunction* $u_0(a)$. *This eigenvalue has the following properties:*

(a) $\lambda_0(a)$ *is simple;*
(b) *there exists* $d(a) > 0$ *such that if* $\lambda \neq \lambda_0(a)$ *is an eigenvalue of* $-\Delta_p + a\phi_p$ *then* $\lambda \geq \lambda_0(a) + d(a)$*;*

(c) *the function* $\lambda_0(\,\cdot\,)\colon L^1(-1,1) \to \mathbb{R}$ *is continuous;*
(d) *if* $b \in L^1(-1,1)$ *satisfies* $b \geq a$ *almost everywhere in* $(-1,1)$, *with strict inequality on a set of positive measure, then* $\lambda_0(b) > \lambda_0(a)$.

REMARK 3.3. Of course, the eigenvalue $\lambda_0(a)$ and the number $d(a)$ also depend on p, but for simplicity we will omit this in the notation.

PROOF. To prove existence of a principal eigenvalue of $-\Delta_p + a\phi_p$ we will consider the auxiliary problem

$$-\Delta_p(u) + \chi(|u|_0)a\phi_p(u)\big) = \mu\phi_p(u), \quad u \in D(\Delta_p), \tag{3.3}$$

where $\chi\colon \mathbb{R} \to \mathbb{R}$ is an increasing, C^∞ function with $\chi(s) = 0$, $s \leq 1$ and $\chi(s) = 1$, $s \geq 2$. The term $\chi(|u|_0)a\phi_p(u)$ in (2.1) is a continuous function of $u \in D(\Delta_p)$ and is zero when $|u|_0 \leq 1$, so (2.1) can be regarded as a bifurcation (from $u = 0$) problem. When $|u|_0 \leq 1$, equation (3.3) reduces to the (constant coefficient) eigenvalue problem (3.1) with $a = 0$, which was considered in [13]. In particular, [13, Theorem 5.1] shows that this problem has a unique principal eigenvalue, which we denote by $\lambda_0(0)$, and this eigenvalue is simple. On the other hand, when $|u|_0 \geq 2$, equation (3.3) reduces to the eigenvalue problem (3.1). Hence, if we can find a positive solution (μ, u) of (3.3) with $|u|_0 \geq 2$ then μ will be a principal eigenvalue of $-\Delta_p + a\phi_p$. We will do this by a global bifurcation argument. We then prove that the principal eigenvalue has the results stated in the theorem by a sequence of further lemmas.

LEMMA 3.4. *There exists a closed, connected set* $\mathcal{C}_0^+ \subset \mathbb{R} \times C^1[-1,1]$ *of solutions of* (3.3) *such that:*

(a) $(\lambda_0(0), 0) \in \mathcal{C}_0^+$*;*
(b) *if* $(\lambda, u) \in \mathcal{C}_0^+ \setminus \{(\lambda_0(0), 0)\}$ *then* u *is strictly positive;*
(c) $\mathcal{C}_0^+$ *is unbounded in* $\mathbb{R} \times C^1[-1,1]$.

PROOF. We first rewrite (3.3) in the equivalent form

$$u + \Delta_p^{-1}((\mu - \chi(|u|_0)a)\phi_p(u)) = 0, \quad u \in C^1[-1,1]. \tag{3.4}$$

The proof is now a modification of the proof of [13, Theorem 6.2], which considered a bifurcation problem of the form $u + \Delta_p^{-1}(\lambda f(u)) = 0$, with $f\colon \mathbb{R} \to \mathbb{R}$ independent of x (the details of the proof of this theorem are given in [4, Section 4], which considered a similar problem with a standard Dirichlet condition at one end point). The proof of [13, Theorem 6.2] is based on the proof of Rabinowitz' well-known global bifurcation theorem in [10]. We will simply sketch the necessary modifications here.

Given the properties of the eigenvalue $\lambda_0(0)$, the proof of [13, Theorem 6.2] can be applied to equation (3.4) to yield a set $\mathcal{C}_0^+$ such that:

- $\mathcal{C}_0^+$ has the properties in the lemma, except (b) and (c), and satisfying the usual Rabinowitz-type global alternatives;
- there is a neighbourhood V of $(\lambda_0(0), 0)$ in $\mathbb{R} \times C^1[-1,1]$ such that if $(\lambda, u) \in V \cap \mathcal{C}_0^+ \setminus \{(\lambda_0(0), 0)\}$ then u is strictly positive.

Now suppose that (μ_n, u_n), $n = 1, 2, \ldots$, is a sequence of positive solutions of (3.4) with $|u_n|_0 \geq 1$ and $(\mu_n, u_n) \to (\mu_\infty, u_\infty)$ in $\mathbb{R} \times C^1[-1,1]$. Then, by continuity, (μ_∞, u_∞) is a positive solution of (3.4), and hence of (3.3), so by Lemma 3.1, u_∞ is strictly positive. The usual Rabinowitz-type argument based on preservation of nodal properties (in this case, positivity) now proves properties (b) and (c) in the lemma. □

Some of the following proofs will use the p-Laplacian form of the Prüfer transformation. This is a standard technique, although there are slight variations in the precise definitions and functions used. For details, see for example, [3, Section 2], [11, Section 3] and [12, Section 3]. Our uses of this technique will be short, and standard, so for brevity we will not describe the details (or even the notation) here, but will simply refer to appropriate points in these papers for the results and arguments used.

Lemma 3.5. *There exists a constant $M_0(\|a\|_1)$ (depending only on $\|a\|_1$) such that if $(\mu, u) \in \mathcal{C}_0^+$ then $|\mu| \leq M_0(\|a\|_1)$.*

Proof. Suppose that (μ, u) is a solution of (3.3) with $\mu > 0$. Here, we will use the form of Prüfer angle for u defined in the proof of [3, Lemma 2.5] and, as in [3], we denote this angle by ϕ. Using the notation in [3], we set $s = 1$, $\widetilde{\mu} = \mu(p-1)^{-1}$ and $f = \widetilde{\mu}^{-1/p}$, and then, by [3, p. 381], the differential equation for ϕ is

$$\phi' = \widetilde{\mu}^{1/p}(1 - \widetilde{\mu}^{-1}\chi(|u|_0)a|S(\phi)|),$$

where S is the p-Laplacian sine function used in [3] (note that, for a fixed u, the term $\chi(|u|_0)$ in equation (3.3) is simply a number, so the Prüfer technique can be applied to this equation). Integrating this equation for ϕ from -1 to 1 shows that if $\widetilde{\mu} > \pi_p^p + 2\|a\|_1$ then

$$\phi(1) - \phi(-1) \geq \widetilde{\mu}^{1/p} > \pi_p,$$

so u must have a zero in $[-1, 1]$. Hence, if $(\mu, u) \in \mathcal{C}_0^+$ then

$$\mu \leq (p-1)(\pi_p^p + 2\|a\|_1).$$

We now obtain a lower bound for μ. Suppose that $(-\widetilde{\mu}, u)$ is a solution of (3.3) with $\widetilde{\mu} > 0$. It follows from (1.2) and (1.3) that u does not attain its maximum at ± 1 (for details, see the proof of [13, Lemma 5.2]) so, without loss of generality, we suppose that this maximum is attained at $x_0 \in (-1, 0]$ (a similar proof holds when $x_0 \in [0, 1)$). That is, $u(x_0) = |u|_0$. Integrating equation (3.3) now yields,

$$\phi_p(u')(x) \geq -\|a\|_1 |u|_0^{p-1} + \widetilde{\mu} \int_{x_0}^{x} \phi_p(u), \quad x \geq x_0, \tag{3.5}$$

and so, since $\widetilde{\mu}$ and u are positive,

$$u'(x) \geq -\|a\|_1^{1/(p-1)} |u|_0, \quad x \geq x_0. \tag{3.6}$$

We now suppose that $\|a\|_1^{-1/(p-1)} \leq 1$; if this is not true then in the rest of the proof we simply replace the quantity $\|a\|_1^{-1/(p-1)}$ by 1. Setting $x_1 := x_0 + (1/2)\|a\|_1^{-1/(p-1)} \leq 1/2$, it follows from (3.6) that

$$u(x) \geq \frac{1}{2}|u|_0, \quad x_0 \leq x \leq x_1. \tag{3.7}$$

Now suppose that $\widetilde{\mu}$ satisfies

$$\widetilde{\mu}\frac{1}{2}\|a\|_1^{-1/(p-1)} \left(\frac{1}{2}|u|_0\right)^{p-1} > (2 + \|a\|_1)|u|_0^{p-1}. \tag{3.8}$$

Then by (3.5) and (3.7),

$$\phi_p(u')(x_1) > 2|u|_0^{p-1},$$

that is, u is increasing at x_1. It now follows from (3.5) that the inequality (3.7) in fact holds for $x_0 \leq x \leq 1$, and also

$$\phi_p(u')(x) > 2|u|_0^{p-1}, \quad x \in [x_1, 1]. \tag{3.9}$$

Since $x_1 \leq 1/2$, it now follows from (3.7) and (3.9) that $u(1) > (3/2)|u|_0$. This contradiction shows that $\widetilde{\mu}$ cannot satisfy (3.8), and so yields the desired lower bound for μ. This completes the proof of Lemma 3.5. □

COROLLARY 3.6. *There exists a principal eigenvalue of $-\Delta_p + a\phi_p$.*

PROOF. It follows from Lemmas 3.4 and 3.5 that equation (3.3) has a strictly positive solution (μ, u) with $|u|_0 > 2$ (and $|\mu| \leq M_0$), and so (μ, u) satisfies (3.1). □

We now prove the other properties of the principal eigenvalue. The uniqueness part of the following lemma justifies the notation $\lambda_0(a)$ for the principal eigenvalue, and given this we can clearly select a suitable strictly positive eigenfunction, which we denote by $u_0(a)$.

Lemma 3.7. *For any $a \in L^1(-1,1)$, the principal eigenvalue $\lambda_0(a)$ of $-\Delta_p + a\phi_p$ is unique, and satisfies the properties* (a), (b), (d) *of Theorem* 3.2.

Proof. We first prove part (d). Let u, v denote (strictly) positive eigenfunctions corresponding to $\lambda_0(a)$, $\lambda_0(b)$, respectively. We first note that, by the homogeneity of (3.1), in the following arguments we may rescale u and v by positive constants without any loss of generality. We also note that if u and v are linearly dependent then, from (3.1), $\lambda_0(b) - \lambda_0(a) = b - a \geq 0$, so the result is trivial in this case — hereafter we suppose that u and v are linearly independent.

We now prove the result by dealing successively with various specific cases.

(A) $u < v$ on an interval (x, y), with $u(x) = v(x)$, $u(y) = v(y)$.

By [7, Lemma 3.3],

$$-\int_x^y \{(\lambda_0(a) - \lambda_0(b)) + (b - a)\}|v|^p \geq 0,$$

which implies the result in this case.

(B) $u > v$ or $u < v$ on $(-1,1)$, and $u(e) = v(e)$, at an end point $e \in \{\pm 1\}$.

Neither of these cases can occur, by (1.2) and (1.3) at e.

(C) $u \geq v$, with $u(x_0) = v(x_0)$ and $u(y) > v(y)$, for some $x_0, y \in (-1,1)$ (this can be achieved by suitably scaling u and v, and by case (B) we can choose x_0 to be an interior point).

In this case, if there are points y_1, y_2 such that $y_1 < x_0 < y_2$ and $u(y_i) > v(y_i)$, $i = 1, 2$, then scaling u downwards slightly yields case (A), and so the result holds. Hence, we may suppose that $u \equiv v$ on an interval $[x_{\mathrm{r}}, 1]$, with $x_{\mathrm{r}} < 1$ (or on an interval $[-1, x_{\mathrm{l}}]$, for which a similar argument holds). Furthermore, by considering (3.1) on $[x_{\mathrm{r}}, 1]$, we see that $\lambda_0(b) - \lambda_0(a) = b - a \geq 0$. Hence, from now on we may suppose that $\lambda_0(b) = \lambda_0(a)$.

(D) $u \leq v$, with $u(x_1) = v(x_1)$ at some $x_1 \in (-1, x_{\mathrm{r}})$, and $u(1) < v(1)$ (a further rescaling of u and v, and appeal to cases (B) and (C), now yields this).

To deal with this case we use the Prüfer transformation, as defined in [3, Sections 2, 3] and [12, Section 3], and we let θ_u, θ_v, denote the Prüfer angles associated with the solutions u, v. By the choice of x_1, we have $u'(x_1) = v'(x_1)$, so we may choose the Prüfer angles such that $\theta_u(x_1) = \theta_v(x_1)$, and then, by the linear dependence of u, v near 1, and the positivity of u and v on $[-1,1]$, we also have $\theta_u(1) = \theta_v(1)$.

Next, we note that it follows from the assumption that $b \geq a$, and the differential equations for θ_u, θ_v, that $\theta_v' \leq \theta_u'$ (see [11, Lemma 4] and the proof of [12, Theorem 3.2], and recalling that in this case $\lambda_0(b) = \lambda_0(a)$). Furthermore, since $u(1) < v(1)$, we must have $b > a$ on a set of positive measure in $[x_1, 1]$, so $\theta'_v < \theta'_u$ on a set of positive measure in $[x_1, 1]$, and hence

$$\theta_u(1) - \theta_u(x_1) > \theta_v(1) - \theta_v(x_1).$$

This contradiction shows that this case cannot occur.

Cases (A)–(D) have dealt with all possible scenarios — either showing that the result is true for the specific case, or that the case cannot in fact occur. Hence, we have completed the proof of part (d) of Theorem 3.2.

Next, we observe that a similar proof also proves part (a) of the theorem, and shows that part (b) holds with $d(a) = 0$, so it remains to show that we can choose $d(a) > 0$. Suppose the contrary. Then there exists a sequence of solutions (λ_n, u_n) of (3.2) such that $\lambda_n \to \lambda_0(a)$ and, for each $n \geq 1$, $|u_n|_1 = 1$ and u_n changes sign. By Remark 2.2 we may suppose that $u_n \to u_\infty$ in $C^1[-1, 1]$, where u_∞ is non-trivial and $(\lambda_0(a), u_\infty)$ satisfies (3.2), so that u_∞ is an eigenfunction corresponding to $\lambda_0(a)$. However, by its construction, u_∞ cannot be strictly positive, so it cannot be a multiple of $u_0(a)$, which contradicts the simplicity of the principal eigenvalue $\lambda_0(a)$. This contradiction completes the proof of Lemma 3.7. □

We now prove part (c) of Theorem 3.2.

LEMMA 3.8. *The function* $\lambda_0(\,\cdot\,)\colon L^1(-1, 1) \to \mathbb{R}$ *is continuous.*

PROOF. Suppose that there exists some $a_0 \in L^1(-1, 1)$ at which $\lambda_0(\,\cdot\,)$ is not continuous. Then there exists $\varepsilon > 0$ and a sequence (a_n) in $L^1(-1, 1)$ such that $\|a_n - a_0\|_1 \to 0$ and $|\lambda_0(a_n) - \lambda_0(a_0)| \geq \varepsilon$, $n = 1, 2, \ldots$ Combining the bounds obtained in the proof of Lemma 3.5 with the convergence of the sequence (a_n) yields a uniform bound for the sequence $(|\lambda_0(a_n)|)$, so we may suppose that $\lambda_0(a_n) \to \lambda_\infty$, for some $\lambda_\infty \in \mathbb{R}$ (by choosing a subsequence if necessary).

Now, for each $n \geq 1$, let u_n be the positive eigenfunction corresponding to $\lambda_0(a_n)$ with $|u_n|_0 = 1$. Since $(\lambda_0(a_n), u_n)$ satisfies (3.2) and the operator $\Delta_p^{-1}\colon C^1[-1, 1] \to C^1[-1, 1]$ is completely continuous (see Theorem 2.1), we may suppose that $u_n \to u_\infty$ in $C^1[-1, 1]$, for some $u_\infty \in C^1[-1, 1]$, and

$$u_\infty + \Delta_p^{-1}(\lambda_\infty \phi_p(u_\infty) - a_0 \phi_p(u_\infty)) = 0.$$

These results imply that $|\lambda_\infty - \lambda_0(a_0)| \geq \varepsilon$ and $\lambda_\infty = \lambda_0(a_0)$ (by the uniqueness of the principal eigenvalue). This contradiction completes the proof of the lemma. □

These results prove Theorem 3.2. □

REMARK 3.9. The eigenvalue λ_0 depends on the coefficients $\alpha^\pm := (\alpha_1, \ldots, \alpha_{m^\pm})$, $\eta^\pm := (\eta_1, \ldots, \eta_{m^\pm})$, and p, although we have suppressed this dependence here since we regard these coefficients as fixed. However, in other settings it may be useful to regard some, or all, of these coefficients as variable. The proof of

Lemma 3.8 can be extended to show that λ_0 depends continuously on $\alpha^{\pm}$, $\eta^{\pm}$, and p, so long as (1.3) holds and $p \in (1,\infty)$.

Now, defining the operator

$$K_{\lambda,a} := \Delta_p^{-1} \circ ((\lambda - a)\phi_p(\,\cdot\,)) \colon C^1[-1,1] \to C^1[-1,1],$$

we can rewrite the eigenvalue problem (3.2) in the form

$$u + K_{\lambda,a}(u) = 0, \quad u \in C^1[-1,1]. \tag{3.10}$$

In particular, (3.10) has a non-trivial solution u if and only if λ is an eigenvalue of $-\Delta_p + a\phi_p$. Furthermore, the operator $K_{\lambda,a}$ is completely continuous (by Theorem 2.1), and homogeneous (in the sense that $K_{\lambda,a}(tu) = tK_{\lambda,a}(u)$, for any $t \in \mathbb{R}$ and $u \in C^1[-1,1]$). Thus, if λ is not an eigenvalue of $-\Delta_p + a\phi_p$ then the Leray–Schauder degree $\deg(I + K_{\lambda,a}, B_r, 0)$ is well defined for any $r > 0$, where B_r denotes the open ball in $C^1[-1,1]$, centred at 0 with radius r.

Theorem 3.10. *For any $a \in L^1(-1,1)$ and any $r > 0$,*

$$\deg(I + K_{\lambda,a}, B_r, 0) = \begin{cases} 1 & \text{if } \lambda < \lambda_0(a), \\ -1 & \text{if } \lambda_0(a) < \lambda < \lambda_0(a) + d(a), \end{cases}$$

where $d(a)$ is as in Theorem 3.2(b).

Proof. We prove the result by a homotopy argument. Regarding a and p as fixed, it follows from the proof of Theorem 3.2(b) and (c) that we can choose $\delta \in (0, d(a))$ such that, for each $t \in [0,1]$, if $\lambda_0(ta) < \lambda \le \lambda_0(ta) + \delta$ then λ is not an eigenvalue of $-\Delta_p + ta\phi_p$. Now, the homotopy

$$H(t,u) := K_{\lambda_0(ta)+\delta,ta}(u) \colon [0,1] \times C^1[-1,1] \to C^1[-1,1]$$

is completely continuous and, for each $t \in [0,1]$, the equation $u + H(t,u) = 0$ has no solution $u \ne 0$. Hence, by the homotopy invariance of the degree and Theorem 3.8 of [4],

$$\deg(I + K_{\lambda_0(a)+\delta,a}, B_r, 0) = \deg(I + K_{\lambda_0(0)+\delta,0}, B_r, 0) = -1.$$

This proves the result when $\lambda = \lambda_0(a) + \delta$, and hence, by continuity of the degree, when $\lambda_0(a) < \lambda < \lambda_0(a) + d(a)$; the proof when $\lambda < \lambda_0(a)$ is similar. □

4. Global bifurcation results

In this section we consider the bifurcation problem

$$-\Delta_p(u) + (a + g(u))\phi_p(u) = \lambda\phi_p(u), \quad (\lambda, u) \in \mathbb{R} \times D(\Delta_p). \tag{4.1}$$

It follows from our standing assumptions on $g \colon [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ (viz. (1.4) and the Carathéodory conditions described in Section 2.1) that the Nemytskiĭ operator

$g: C^1[-1,1] \to L^1(-1,1)$ is continuous and $g(0) = 0$. Hence $u = 0$ is a solution of (4.1), for all $\lambda \in \mathbb{R}$. Also, if (λ, u) is a non-trivial, positive solution of (4.1) then, since $a + g(u) \in L^1(-1,1)$, Lemma 3.1 shows that u is strictly positive. We will show that there is an unbounded continuum of positive solutions of (4.1) bifurcating from the principal eigenvalue $\lambda_0(a)$.

Let $\mathcal{S} \subset \mathbb{R} \times D(\Delta_p)$ denote the set of non-trivial solutions of (4.1), and let $\overline{\mathcal{S}}$ denote its closure. Let $\mathcal{C}_0$ denote the component of $\overline{\mathcal{S}}$, in $\mathbb{R} \times D(\Delta_p)$, containing the point $(\lambda_0(a), 0)$, and let

$$\mathcal{C}_0^{\pm} := \{(\lambda_0(a), 0)\} \cup \{(\lambda, u) \in \mathcal{C}_0 : \pm u \text{ is positive}\}.$$

THEOREM 4.1. *The set $\mathcal{C}_0^+$ is closed, connected and unbounded in $\mathbb{R} \times D(\Delta_p)$.*

PROOF. Equation (4.1) is equivalent to the problem

$$u = G(\lambda, u) := -\Delta_p^{-1}((\lambda - a - g(u))\phi_p(u)), \quad (\lambda, u) \in \mathbb{R} \times C^1[-1,1].$$

In [10], P. H. Rabinowitz deals with a similar problem, where the operator $G(\lambda, u)$ has the form $\lambda L u + H(\lambda, u)$, with L linear and compact, and H is completely continuous with $\lim_{\|u\| \to 0} \|H(\lambda, u)\|/\|u\| = 0$, uniformly on compact λ intervals (for suitable norms). With our hypotheses on g it follows from Theorem 2.1 that $G: \mathbb{R} \times C^1[-1,1] \to C^1[-1,1]$ is completely continuous and $\lim_{|u|_1 \to 0} \|g(u)\|_1 = 0$, but we have homogeneity of the mapping $u \to \Delta_p^{-1}((\lambda - a)\phi_p(u))$, rather than linearity. However, by some slight amendments of the proofs in [10], these conditions are sufficient to prove the above result. We will sketch some of the details of the amended proof.

Firstly, we observe that an analogue of the basic Lemma 1.24 in [10] holds here, with a similar proof (essentially, this lemma states that if (λ_n, u_n), for $n = 1, 2, \ldots$, is a sequence of positive solutions of (4.1) with $(\lambda_n, u_n) \to (\lambda_\infty, 0)$, then $\lambda_\infty = \lambda_0(a)$, and u_n must approach zero in the "direction" of the corresponding positive eigenfunctions). Next, since any non-trivial, positive solution of (4.1) is strictly positive, the argument in the proof of [10, Theorem 2.3] regarding preservation of the nodal structure of solutions of (4.1) along continua can be used here (the nodal structure here is simply positivity). This then shows that the set $\mathcal{C}_0 \setminus \{(\lambda_0(a), 0)\}$ contains only positive solutions. All the results of the theorem now follow immediately from this and the definition of the set $\mathcal{C}_0^+$, except the unboundedness of this set. To prove this we require the following result.

PROPOSITION 4.2. *If $\lambda \neq \lambda_0(a)$ and $\lambda < \lambda_0(a) + d(a)$ then $u = 0$ is an isolated zero of the operator $I - G(\lambda, \cdot)$, and the index $\operatorname{ind}(I - G(\lambda, \cdot), 0)$ of this zero changes as λ crosses $\lambda_0(a)$.*

PROOF. Since $\|g(u)\|_1 \to 0$ as $|u|_1 \to 0$ (for $u \in C^1[-1,1]$), a slight extension of the proof of Theorem 3.2(b) shows that if $r > 0$ is sufficiently small and $|u|_1 \leq r$

and $t \in [0,1]$, then λ is not an eigenvalue of $-\Delta_p + a + tg(u)$. Hence, the only zero of $I - G(\lambda, \cdot\,)$ in $\overline{B}_r$ is $u = 0$, and a standard homotopy invariance argument (cf. the proof of Theorem 3.10) shows that

$$\operatorname{ind}(I - G(\lambda, \cdot\,), 0) = \deg(I - G(\lambda, \cdot\,), B_r, 0) = \deg(I + K_{\lambda,a}, B_r, 0).$$

The result now follows from Theorem 3.10. □

Using the index jump result of Proposition 4.2 we can now follow the proof of [10, Theorem 1.3] to show that $\mathcal{C}_0$ is unbounded in $\mathbb{R} \times D(\Delta_p)$. It follows immediately from this and a minor adaptation of the reflection argument in the proof of [10, Theorem 1.27] that $\mathcal{C}_0^+$ is unbounded. □

Remark 4.3. We could also allow g to depend on λ in a suitable manner (see [10, Theorem 2.3], for the case $p = 2$).

5. Eigenvalue criteria for existence of positive solutions

In this section we will consider the problem (1.1)–(1.2), which we can rewrite in the form

$$-\Delta_p(u) + (a + g(u))\phi_p(u) = 0, \quad u \in D(\Delta_p). \tag{5.1}$$

In addition to our standing assumptions on g, in this section we also suppose that for $(x, s, t) \in [-1,1] \times [0,\infty) \times \mathbb{R}$,

$$\psi(x) - E(x,s,t) \le g(x,s,t) \le \Psi(x) + E(x,s,t), \tag{5.2}$$

where $\psi, \Psi \in L^1(-1,1)$ and $E(x,s,t) = \zeta(x)e(|s| + |t|)$, with $\zeta \in L^1(-1,1)$, $\zeta \ge 0$ and the function $e\colon [0,\infty) \to [0,\infty)$ is bounded, with $\lim_{r\to\infty} e(r) = 0$. Clearly, (5.2) implies (2.1).

Remark 5.1. In essence, the condition (5.2) describes the asymptotic behaviour of the function $g(x,s,t)$ as $s \to \infty$, and hence the asymptotic behaviour of the function $f(x,s,t) := (a(x) + g(x,s,t))\phi_p(s)$ in (5.1). The conditions on E yield a precise "uniformity" condition for this asymptotic behaviour. In particular, if $\psi = \Psi$ then f behaves like $(a(x) + \psi(s))s^{p-1}$ as $s \to \infty$. In addition, by (2.2), f behaves like $a(x)s^{p-1}$ as $s \to 0$. Our criteria for the existence of positive solutions of (5.1) will be based on the relationship between these asymptotic behaviours.

Theorem 5.2. *Suppose that g satisfies* (5.2) *and one of the inequalities*

$$\lambda_0(a) < 0 < \lambda_0(a + \psi) \quad \text{or} \quad \lambda_0(a + \Psi) < 0 < \lambda_0(a), \tag{5.3}$$

holds. Then (5.1) *has at least one positive solution.*

Proof. Let $\mathcal{C}_0^+$ be as in Theorem 4.1. Choose a sequence $(\lambda_n, u_n) \in \mathcal{C}_0^+$, $n \ge 1$, such that $|\lambda_n| + |u_n|_1 \to \infty$. By (5.2) there exists $A \in L^1(-1,1)$ such that for all $n \geq 1$, $|g(u_n)(x)| \leq A(x)$, for almost every $x \in [0,1]$, that is, the set $\{g(u_n)\}$ is equi-integrable (see [12, Section 4]). Hence, by Theorem 3.2,

$$\lambda_0(-|a| - A) \leq \lambda_n \leq \lambda_0(|a| + A),$$

and so, by taking a subsequence, we may suppose that $\lambda_n \to \lambda_\infty$ and $|u_n|_1 \to \infty$. Also, defining $v_n := u_n/|u_n|_1$, for each $n \geq 1$, we have

$$v_n + \Delta_p^{-1}\big((\lambda_n - a - g(u_n))\phi_p(v_n)\big) = 0, \tag{5.4}$$

and so (after taking a subsequence if necessary) it follows that:

- the sequence $(g(u_n))$ is equi-integrable (by (5.2)) and converges weakly in $L^1(-1,1)$ (by [12, Lemma 2.1]);
- the sequence (v_n) converges strongly in $C^1[-1,1]$ to some non-trivial, positive v_∞ (by Theorem 2.1);
- there exists $m_\infty \in L^1(-1,1)$, with $\psi \leq m_\infty \leq \Psi$, such that

  $$g(u_n)\phi_p(v_n) \rightharpoonup m_\infty v_\infty$$

  (by (5.2) and [12, Lemma 5.2]).

Hence, letting $n \to \infty$ in (5.4) yields

$$v_\infty + \Delta_p^{-1}((\lambda_\infty - m_\infty)\phi_p(v_\infty)) = 0,$$

and so, by Theorem 4.5,

$$\lambda_0(\psi) \leq \lambda_\infty = \lambda_0(m_\infty) \leq \lambda_0(\Psi). \tag{5.5}$$

It follows from this, together with $(\lambda_0(a), 0) \in \mathcal{C}_0^+$, (5.3) and the connectedness of $\mathcal{C}_0^+$, that $\mathcal{C}_0^+$ intersects the set $\{0\} \times C^1[-1,1]$, which proves the result. □

REMARK 5.3. If (1.4) and (5.2) hold then (5.5) gives an estimate of the values of λ at which the continuum $\mathcal{C}_0^+$ "meets infinity". In particular, if $\psi = \Psi$ then $\mathcal{C}_0^+$ "meets infinity" precisely at $\lambda = \lambda_0(\psi)$.

6. Uniqueness of positive solutions

Finally, we prove a simple uniqueness result for positive solutions of equation (5.1). We will now suppose that g is independent of t and satisfies the monotonicity condition:

$$0 < s_1 < s_2 \;\Rightarrow\; g(x, s_1) < g(x, s_2), \quad \text{a.e. } x \in [-1,1]. \tag{6.1}$$

This condition is standard, see for example, [9] or [16] (in fact, in [16] F. Wong allows g to depend on t but for brevity we omit this here), but the usual proof has to be modified to deal with the multi-point boundary conditions. Note that we do not require (5.2) for the following uniqueness result (and in fact we do not require (1.4)).

THEOREM 6.1. *Suppose that g satisfies* (6.1). *Then equation* (5.1) *has at most one positive solution.*

PROOF. If (5.1) has two distinct, positive solutions u, v, then by definition,

$$\lambda_0(a + g(u)) = \lambda_0(a + g(v)) = 0$$

(the functions $g(u),\ g(v) \in L^1(-1,1)$). Also, by the monotonicity condition (6.1), if $u < v$ on some interval I then $g(u) < g(v)$ almost everywhere on I (similarly if $u > v$).

We now observe that none of the following cases can occur:

(A) $u \le v$ or $u \ge v$ on $[-1,1]$ (by Theorem 3.2(d));

(B) $u < v$ or $u > v$ on an interval (x,y), with $u(x) = v(x)$, $u(y) = v(y)$ (by adapting the argument in case (A) in the proof of Lemma 3.7);

(C) $u < v$ or $u > v$ on $(-1,1)$ and $u(e) = v(e)$ at some $e \in \{\pm 1\}$ (by (1.2)).

Combining these results shows that $u - v$ must change sign so, without loss of generality, we may suppose that there exists $x_0 \in (-1,1)$ such that $u(x_0) = v(x_0)$ and $u < v$ on $(x_0, 1]$. Hence, the Prüfer angles of u and v satisfy $\theta_u(x_0) \ge \theta_v(x_0)$, and $\theta_u' > \theta_v'$ almost everywhere on $(x_0, 1]$ (since $g(u) < g(v)$).

Next, choosing γ_{m} such that

$$\max_{x \in [x_0, 1]} \{\gamma_{\mathrm{m}} v - u\} = 0,$$

it follows from the choice of x_0 and from (1.2) that this maximum is attained at some point $x_{\mathrm{m}} \in (x_0, 1)$, and hence

$$\theta_v(x_{\mathrm{m}}) = \theta_{\gamma_{\mathrm{m}} v}(x_{\mathrm{m}}) = \theta_u(x_{\mathrm{m}}).$$

Combining these properties of the Prüfer angles θ_u, θ_v yields a contradiction, which completes the proof of the theorem. □

If we do impose the condition (5.2) (and (1.4)) we obtain the following existence and uniqueness result.

COROLLARY 6.2. *Suppose that g satisfies the conditions of Theorem* 5.2 *and* (6.1). *Then equation* (5.1) *has exactly one positive solution.*

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Manuscript received April 30, 2010

Bryan P. Rynne
Department of Mathematics
and the Maxwell Institute
for Mathematical Sciences
Heriot-Watt University
Edinburgh EH14 4AS, SCOTLAND

E-mail address: bryan@ma.hw.ac.uk