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POSITIVE SOLUTIONS FOR A CLASS OF NONLOCAL IMPULSIVE BVPS VIA FIXED POINT INDEX

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Abstract. We study the existence of positive solutions for perturbed impulsive integral equations. Our setting is quite general and covers a wide class of impulsive boundary value problems. We also study other cases that can be treated in a similar manner. The main ingredient in our theory is the classical fixed point index theory for compact maps.

1. Introduction

The interest of researchers in the theory of impulsive differential equations has grown in the last decades. The study of problems of this type is driven not only by a theoretical interest, but also by the fact that several phenomena in engineering, physics and life sciences can be modelled with impulsive equations. For example, in the field of population control, this has been done by J. J. Nieto and co-authors [39], [41].

For an introduction to the theory of impulsive differential equations we refer to the books [4], [8], [24], [31], that also include a variety of examples and applications.

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In this paper we discuss the existence of positive solutions of the, fairly general, second order differential equations of the form

$$u''(t) + p(t)u'(t) + q(t)u(t) + g(t)f(t, u(t)) = 0, \quad t \in (0, 1),\ t \neq \tau, \tag{1.1}$$

with impulsive terms of the type

$$\begin{aligned} \Delta u|_{t=\tau} &= I(u(\tau)), \quad \tau \in (0, 1), \\ \Delta u'|_{t=\tau} &= N(u(\tau)), \end{aligned}$$

and nonlocal boundary conditions (BCs) of "Sturm–Liouville" kind

$$a_1 u(0) - b_1 u'(0) = \alpha[u], \quad a_2 u(1) + b_2 u'(1) = \beta[u]; \tag{1.2}$$

here $\Delta v|_{t=\tau}$ denotes the "jump" of the function v in $t = \tau$, that is

$$\Delta v|_{t=\tau} = v(\tau^+) - v(\tau^-),$$

where $v(\tau^-)$, $v(\tau^+)$ are the left and right limits of v in $t = \tau$, and $\alpha[\,\cdot\,]$, $\beta[\,\cdot\,]$ are linear functionals given by Stieltjes integrals, namely

$$\alpha[u] = \int_0^1 u(s)\, dA(s), \quad \beta[u] = \int_0^1 u(s)\, dB(s). \tag{1.3}$$

The formulation (1.3) is rather general and includes, as special cases, multi-point BCs and integral BCs that have been studied recently, in the case of impulsive equations, by many authors, see for example [5]–[7], [10]–[12], [22], [23], [27], [38] and references therein.

Second order differential equations of the form (1.1) with no impulsive term under the m-point BCs of the type

$$u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i), \tag{1.4}$$

have been studied by R. Ma and L. Ren [29], who utilized a careful equivalent integral formulation of the problem, combined with results of K. Q. Lan and J. R. L. Webb [26] and K. Q. Lan [25]. An equation similar to (1.1), subject to two multi-point BCs has been studied later by R. Ma [28] and, with two integral BCs, by Z. Yang [40]. A key ingredient in [28], [29] is that the coefficients involved in the BCs are non-negative and in [40] only positive Stieltjes measures are considered.

J. R. L. Webb [32], also with no impulsive term, studied (1.1) under the BCs (1.2), where the involved measures can be signed; this, in particular, allows some negative coefficients in (1.4). The method in [32] is to rewrite the boundary value problem (BVP) in an integral form of the type

$$u(t) = \gamma(t)\alpha[u] + \delta(t)\beta[u] + \int_0^1 k(t, s)g(s)f(s, u(s))\, ds, \tag{1.5}$$

and to make use of a general theory developed by J. R. L. Webb and G. Infante [33].

Our idea is to use the results of [32], [35], valid for the non-impulsive problem, as a starting point and to rewrite the impulsive boundary value problem (IBVP) (1.1)–(1.2) as a perturbation of (1.5), namely

$$u(t) = \gamma(t)\alpha[u] + \delta(t)\beta[u] + \int_0^1 k(t,s)g(s)f(s,u(s))\,ds + Gu(t),$$

where the term $Gu(t)$ takes care of the impulsive effect and is constructed in a very natural manner.

We use the classical theory of fixed point index, combined with some results from the paper [33], in order to prove the existence of one or more positive solutions. It is worth pointing out that in [32] the theory allows signed measures, here we focus, as in [17], on *positive* measures only, because we want our functionals to preserve some inequalities.

We extend the results of [33] to the context of impulsive equations and, doing so, we cover a *wide* class of IBVPs, including local ones. We stress that our methodology involves the construction of *new* Stieltjes measures that take into account, at the same time, the boundary conditions and the impulsive effect. This avoids long calculations in order to build the corresponding impulsive integral operator.

Other cases that do not fit directly in our theory, can be treated in a similar manner. Here we modify some techniques given by J. R. L. Webb and G. Infante [33] and J. R. L. Webb and M. Zima [37] in order to deal with both the impulses and two nonlocal conditions. We do this in three cases. The results are new and the IBVP (1.1)–(1.2) is, as far as we know, studied for the first time.

2. The non-impulsive case

We begin by recalling some results of J. R. L. Webb [32] and J. R. L. Webb and G. Infante [35], valid for the non-impulsive case. Consider the second order equation

$$u''(t) + p(t)u'(t) + \overline{q}(t)u(t) + \overline{g}(t)f(t,u(t)) = 0, \tag{2.1}$$

where $\overline{g}$, f are non-negative functions, p and $\overline{q}$ are continuous and $\overline{q}(t) \le 0$, so that the maximum principle holds. The nonlocal boundary conditions are of the general form

$$a_1u(0) - b_1u'(0) = \alpha[u], \quad a_2u(1) + b_2u'(1) = \beta[u], \tag{2.2}$$

where $a_1, b_1, a_2, b_2 \in [0,\infty)$, $a_1+b_1 \neq 0, a_2+b_2 \neq 0$ and $\lambda = 0$ is not an eigenvalue of the problem

$$u''(t) + p(t)u'(t) + \overline{q}(t)u(t) + \lambda u(t) = 0,$$
$$a_1u(0) - b_1u'(0) = 0, \quad a_2u(1) + b_2u'(1) = 0;$$

we assume that these conditions are satisfied throughout the paper, unless we state otherwise.

Let γ, δ be the unique solutions of

$$\begin{aligned} \gamma''(t) + p(t)\gamma'(t) + \overline{q}(t)\gamma(t) &= 0, & a_1\gamma(0) - b_1\gamma'(0) &= 1, & a_2\gamma(1) + b_2\gamma'(1) &= 0,\\ \delta''(t) + p(t)\delta'(t) + \overline{q}(t)\delta(t) &= 0, & a_1\delta(0) - b_1\delta'(0) &= 0, & a_2\delta(1) + b_2\delta'(1) &= 1; \end{aligned}$$

their properties are given in the following lemma, whose proof utilizes some results from [30].

Lemma 2.1 ([32], [35]). *The functions γ, δ are well-defined C^2 functions that are positive on $(0,1)$, γ is non-increasing and δ is non-decreasing on $[0,1]$.*

By multiplying by $e^{P(t)} := \exp(\int_0^t p(s)\,ds)$, equation (2.1) can be rewritten as

$$(u'(t)e^{P(t)})' + q(t)u(t) + g(t)f(t,u(t)) = 0, \tag{2.3}$$

where $q(t) = \overline{q}(t)\,e^{P(t)}$ and $g(t) = \overline{g}(t)\,e^{P(t)}$. When there are no nonlocal BCs, the well-known Green's function is then given by the following result, see for example [9, §5.7] or [16, XI, §1–2].

Lemma 2.2. *Solutions of (2.3) with BCs $a_1u(0) - b_1u'(0) = 0$, $a_2u(1) + b_2u'(1) = 0$ are given by solutions of the integral equation*

$$u(t) = \int_0^1 k(t,s)g(s)f(s,u(s))\,ds := Fu(t) \tag{2.4}$$

with Green's function given by

$$k(t,s) = \frac{1}{W(0)}\begin{cases} \gamma(t)\delta(s) & \text{if } s \le t,\\ \gamma(s)\delta(t) & \text{if } s \ge t, \end{cases}$$

where $W(t)$ is the Wronskian, $W(t) = \gamma(t)\delta'(t) - \delta(t)\gamma'(t)$, and $W(t) > 0$ by Lemma 2.1.

J. R. L. Webb [32] studied the existence of positive solutions of the nonlocal BVP (2.1)–(2.2) by seeking fixed points of the perturbed Hammerstein integral operator

$$\Lambda u(t) := \gamma(t)\alpha[u] + \delta(t)\beta[u] + \int_0^1 k(t,s)g(s)f(s,u(s))\,ds, \tag{2.5}$$

in a suitable cone of positive functions in the space $C[0,1]$ and utilizing the theory developed in [33]. Key ingredients in [32] are the following upper and lower bounds for the functions k, γ, δ. For an arbitrary $[a,b]\subset(0,1)$ one has

$$\begin{aligned}
&k(t,s)\le \Phi(s):=\frac{1}{W(0)}\gamma(s)\delta(s) &&\text{for } t\in[0,1] \text{ and } s\in[0,1],\\
&k(t,s)\ge c_1\Phi(s) &&\text{for } t\in[a,b] \text{ and } s\in[0,1],\\
&\gamma(t)\ge c_2\|\gamma\|, \quad \delta(t)\ge c_3\|\delta\| &&\text{for } t\in[a,b],
\end{aligned}$$

where

$$c_1:=\min_{t\in[a,b]}\left\{\frac{\gamma(t)}{\|\gamma\|},\frac{\delta(t)}{\|\delta\|}\right\},\qquad c_2:=\min_{t\in[a,b]}\frac{\gamma(t)}{\|\gamma\|},\quad c_3:=\min_{t\in[a,b]}\frac{\delta(t)}{\|\delta\|}.$$

We also make use of these estimates in the next sections.

3. The impulsive case

We can now consider the second order impulsive differential problem

$$\begin{aligned}
(u'(t)e^{P(t)})'+q(t)u(t)+g(t)f(t,u(t))&=0, && t\in(0,1),\ t\ne\tau,\\
\Delta u|_{t=\tau}&=I(u(\tau)), && \tau\in(0,1),\\
\Delta u'|_{t=\tau}&=N(u(\tau)),
\end{aligned}\tag{3.1}$$

with the nonlocal BCs

$$a_1u(0)-b_1u'(0)=\alpha[u],\quad a_2u(1)+b_2u'(1)=\beta[u].\tag{3.2}$$

We work in the Banach space

$$\begin{aligned}
PC_\tau[0,1]:=\{u{:}[0,1]\to\mathbb{R}\mid u \text{ is continuous in } t\in[0,1]\setminus\{\tau\},\\
\text{there exist } u(\tau^-)=u(\tau) \text{ and } |u(\tau^+)|<\infty\},
\end{aligned}$$

endowed with the usual supremum norm $\|u\|=\sup\{|u(t)|:t\in[0,1]\}$.

Our idea is to seek a solution of the IBVP (3.1)–(3.2) as a fixed point of a perturbation of the operator (2.5), namely

$$Tu(t):=\gamma(t)\alpha[u]+\delta(t)\beta[u]+\int_0^1 k(t,s)g(s)f(s,u(s))\,ds+Gu(t),\tag{3.3}$$

where

$$Gu(t):=\gamma(t)\chi_{(\tau,1]}(d_1I+e_1N)(u(\tau))+\delta(t)\chi_{[0,\tau]}(d_2I+e_2N)(u(\tau)),$$

and the coefficients d_1, e_1, d_2, e_2 are given by the following lemma.

Lemma 3.1. *The solutions of the integral equation* $u(t) = Tu(t)$ *with*

$$d_1 = \frac{\delta'(\tau)}{W(\tau)}, \quad e_1 = \frac{-\delta(\tau)}{W(\tau)}, \quad d_2 = \frac{\gamma'(\tau)}{W(\tau)}, \quad e_2 = \frac{-\gamma(\tau)}{W(\tau)},$$

are solutions of the IBVP (3.1)–(3.2).

Proof. Let u be a fixed point of the operator T. From the choice of the coefficients d_1, e_1, d_2, e_2 it follows that

$$\Delta u|_{t=\tau} = I(u(\tau)) \quad \text{and} \quad \Delta u'|_{t=\tau} = N(u(\tau)).$$

The rest of the proof follows from the choice of γ and δ and from Lemma 2.2. $\square$

We now fix $[a,b] \subset (\tau,1)$ and make the following assumptions on the functions f, g, I, N and the functionals α, β that appear in (3.1)–(3.2).

(C_1) $f\colon [0,1] \times [0,\infty) \to [0,\infty)$ satisfies Carathéodory conditions, that is, for each u, $t \mapsto f(t,u)$ is measurable and for almost every t, $u \mapsto f(t,u)$ is continuous and for every $r > 0$ there exists a L^∞-function $\phi_r\colon [0,1] \to [0,\infty)$ such that

$$f(t,u) \le \phi_r(t) \quad \text{for almost all } t \in [0,1] \text{ and all } u \in [0,r].$$

(C_2) $g\,\Phi \in L^1[0,1]$, $g \ge 0$ almost everywhere, and $\int_a^b \Phi(s) g(s)\,ds > 0$.

(C_3) $I\colon [0,\infty) \to \mathbb{R}$ and $N\colon [0,\infty) \to \mathbb{R}$ are continuous functions and there exist $h_1, h_2 > 0$ and $h_3 \ge 0$ such that for $w \in [0,\infty)$

$$h_1 w \le (d_1 I + e_1 N)(w) \le h_2 w, \quad \text{and} \quad 0 \le (d_2 I + e_2 N)(w) \le h_3 w.$$

(C_4) $\alpha[\,\cdot\,]$, $\beta[\,\cdot\,]$ are linear functionals given by

$$\alpha[u] = \int_0^1 u(s)\,dA(s), \quad \beta[u] = \int_0^1 u(s)\,dB(s),$$

involving Riemann–Stieltjes integrals; A, B are functions of bounded variation and dA, dB are *positive* measures.

The assumptions above enable us to work in the cone

$$K = \{u \in PC_\tau[0,1],\ u \ge 0 : \min_{t \in [a,b]} u(t) \ge c\|u\|\},$$

where $[a,b]$ is arbitrary in $(\tau,1)$ and

$$c = \min\bigg\{c_1, c_2, c_3, \frac{c_2\|\gamma\| h_1}{\max\{h_2\|\gamma\|, h_3\|\delta\|\}}\bigg\}.$$

We make use of the following compactness criterion, which can be found, for example, in [1], [24] and is an extension of the classical Ascoli–Arzelà Theorem.

We recall that a set $S \subset PC_\tau[0,1]$ is said to be *quasi-equicontinuous* if for every $u \in S$ and for every $\varepsilon > 0$ there exists $\delta > 0$ such that $t_1, t_2 \in [0,\tau]$ (or $t_1, t_2 \in (\tau,1]$) and $|t_1 - t_2| < \delta$ implies $|u(t_1) - u(t_2)| < \varepsilon$.

LEMMA 3.2. *A set $S \subseteq PC_\tau[0,1]$ is relatively compact in $PC_\tau[0,1]$ if and only if S is bounded and quasi-equicontinuous.*

We now prove that T leaves the cone K invariant and is compact.

THEOREM 3.3. *Under the hypotheses above, $T\colon K \to K$ and T is compact.*

PROOF. It follows, as in [33], that for $u \in K$ and $t \in [a,b] \subset (\tau,1)$ we have

$$\Lambda u(t) \ge c_0\|\Lambda u\|,$$

where $c_0 = \min\{c_1,c_2,c_3\}$. For $t \in [0,\tau]$ we have $Gu(t) \le \|\delta\| h_3 u(\tau)$ and, for $t \in (\tau,1]$, $Gu(t) \le \|\gamma\| h_2 u(\tau)$. Therefore, for $t \in [0,1]$,

$$Gu(t) \le u(\tau)\max\{h_2\|\gamma\|, h_3\|\delta\|\}$$

and we obtain

$$\|Tu\| \le \|\Lambda u\| + u(\tau)\max\{h_2\|\gamma\|, h_3\|\delta\|\}. \tag{3.4}$$

For $t \in [a,b]$, we get

$$\begin{aligned} Tu(t) &\ge c_0\|\Lambda u\| + \gamma(t)(d_1 I + e_1 N)(u(\tau)) \\ &\ge c_0\|\Lambda u\| + \frac{c_2\|\gamma\| h_1}{\max\{h_2\|\gamma\|, h_3\|\delta\|\}} u(\tau)\max\{h_2\|\gamma\|, h_3\|\delta\|\}. \end{aligned}$$

Thus we obtain

$$\min_{t\in[a,b]} Tu(t) \ge c\|Tu\|.$$

Moreover, we have $Tu(t) \ge 0$ for $t \in [0,1]$. Hence $Tu \in K$ for every $u \in K$.

Now, we prove that the map T is compact. Firstly, we show that T sends bounded sets into bounded sets. Let $u \in K$ and $\|u\| \le r$. Then, for all $t \in [0,1]$, from (3.4) we have

$$\begin{aligned} \|Tu\| \le{}& r\|\gamma\| \int_0^1 dA(s) + r\|\delta\| \int_0^1 dB(s) \\ &+ \int_0^1 \Phi(s) g(s) \phi_r(s)\,ds + r\max\{h_2\|\gamma\|, h_3\|\delta\|\}. \end{aligned}$$

Secondly we show that T sends bounded sets into quasi-equicontinuous sets. For $t_1, t_2 \in [0,\tau]$, $t_1 < t_2$ and $u \in K$ such that $\|u\| \le r$, we have

$$|Tu(t_1) - Tu(t_2)| \le |\delta(t_1) - \delta(t_2)|(d_2 I + e_2 N)(u(\tau)) + |\Lambda u(t_1) - \Lambda u(t_2)|.$$

Then $|Tu(t_1) - Tu(t_2)| \to 0$ when $t_1 \to t_2$. A similar proof holds for $t_1, t_2 \in (\tau,1]$. By Lemma 3.2, T is a compact map. □

4. Fixed point index calculations

We now recall some basic facts regarding the classical fixed point index for compact maps, see for example the paper by H. Amann [2] and the books by D. Guo and V. Lakshmikantham [14] and by J. Andres and L. Górniewicz [3].

Let $\widetilde{K}$ be a cone in a Banach space X. If Ω is a bounded open subset of $\widetilde{K}$ (in the relative topology) we denote by $\overline{\Omega}$ and $\partial\Omega$ the closure and the boundary relative to $\widetilde{K}$. When Ω is an open bounded subset of X we write $\Omega_{\widetilde{K}} = \Omega \cap \widetilde{K}$, an open subset of $\widetilde{K}$. The following result (see for example Theorem 12.3 of [2]) is crucial for our existence results.

Theorem 4.1. *Let $\widetilde{K}$ be a cone in a Banach space X and let Ω be an open bounded set with $\Omega_{\widetilde{K}} \neq \emptyset$ and $\overline{\Omega}_{\widetilde{K}} \neq \widetilde{K}$. Assume that $T\colon \overline{\Omega}_{\widetilde{K}} \to \widetilde{K}$ is a compact map such that $x \neq Tx$ for $x \in \partial\Omega_{\widetilde{K}}$. Then the fixed point index $i_{\widetilde{K}}(T, \Omega_{\widetilde{K}})$ has the following properties.*

- (a) *If there exists $x_0 \in \widetilde{K} \setminus \{0\}$ such that $x \neq Tx + \lambda x_0$ for all $x \in \partial\Omega_{\widetilde{K}}$ and all $\lambda \geq 0$, then $i_{\widetilde{K}}(T, \Omega_{\widetilde{K}}) = 0$.*
- (b) *If $Tx \neq \lambda x$ for all $x \in \partial\Omega_{\widetilde{K}}$ and all $\lambda \geq 1$, then $i_{\widetilde{K}}(T, \Omega_{\widetilde{K}}) = 1$.*
- (c) *Let Ω^1 be open in X with $\overline{\Omega^1_{\widetilde{K}}} \subset \Omega_{\widetilde{K}}$. If $i_{\widetilde{K}}(T, \Omega_{\widetilde{K}}) = 1$ and $i_{\widetilde{K}}(T, \Omega^1_{\widetilde{K}}) = 0$, then T has a fixed point in $\Omega_{\widetilde{K}} \setminus \overline{\Omega^1_{\widetilde{K}}}$. The same result holds if $i_{\widetilde{K}}(T, \Omega_{\widetilde{K}}) = 0$ and $i_{\widetilde{K}}(T, \Omega^1_{\widetilde{K}}) = 1$.*

We recall some useful facts valid for real 2×2 matrices.

Definition 4.2 ([33]). A 2×2 matrix $\mathcal{N}$ is said to be *order preserving* (or *positive*) if $p_1 \geq q_1$, $p_2 \geq q_2$ imply

$$\mathcal{N}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \geq \mathcal{N}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix},$$

in the sense of components.

Lemma 4.3 ([33]). *Let*

$$\mathcal{N} = \begin{pmatrix} p_{11} & -p_{12} \\ -p_{21} & p_{22} \end{pmatrix}$$

with $p_{11}, p_{12}, p_{21}, p_{22} \geq 0$ and $\det \mathcal{N} > 0$. Then $\mathcal{N}^{-1}$ is order preserving.

Lemma 4.4 ([33]). *Let $\mathcal{N}$ satisfy the hypotheses of Lemma 4.3. Suppose $p_1 \geq 0, p_2 \geq 0$ and*

$$\mathcal{N}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \quad \text{and} \quad \mathcal{M}\begin{pmatrix} x_\nu \\ y_\nu \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix},$$

where $\mathcal{M} = \nu I + \mathcal{N}$ with $\nu \geq 0$. Then $x_\nu \leq x$ and $y_\nu \leq y$.

We now introduce, in a similar way as in [18], the functionals

$$\begin{aligned}\alpha_1[u] &:= \alpha[u] + h_1 u(\tau) := \int_0^1 u(s)\,dA_1(s),\\ \alpha_2[u] &:= \alpha[u] + h_2 u(\tau) := \int_0^1 u(s)\,dA_2(s),\\ \beta_2[u] &:= \beta[u] + h_3 u(\tau) := \int_0^1 u(s)\,dB_2(s),\end{aligned}$$

and use the notations

$$f^{0,\rho} := \sup_{0\le u\le\rho,\ 0\le t\le 1} \frac{f(t,u)}{\rho}, \quad f_{\rho,\rho/c} := \inf_{\rho\le u\le\rho/c,\ a\le t\le b} \frac{f(t,u)}{\rho},$$

$$\frac{1}{m} := \sup_{t\in[0,1]} \int_0^1 k(t,s)g(s)\,ds, \quad \frac{1}{M} := \inf_{t\in[a,b]} \int_a^b k(t,s)g(s)\,ds,$$

$$\begin{aligned}D_1 &:= (1-\alpha_1[\gamma])(1-\beta[\delta]) - \alpha_1[\delta]\beta[\gamma],\\ D_2 &:= (1-\alpha_2[\gamma])(1-\beta_2[\delta]) - \alpha_2[\delta]\beta_2[\gamma],\end{aligned}$$

$$\mathcal{K}_C(s) := \int_0^1 k(t,s)\,dC(t),$$

where dC is one of the measures dA_1, dA_2, dB, dB_2. Note that $D_2 > 0$ implies $D_1 > 0$.

We assume from now on that

(C_5) $\alpha_2[\gamma] < 1$, $\beta_2[\delta] < 1$ and $D_2 > 0$.

We also make use of the following open bounded sets (relative to K):

$$K_\rho = \{u \in K : \|u\| < \rho\}, \quad V_\rho = \Big\{u \in K : \min_{a\le t\le b} u(t) < \rho\Big\}.$$

Note that the sets above can be nested, that is $K_\rho \subset V_\rho \subset K_{\rho/c}$.

We firstly prove that the index is 0 on the set V_ρ.

LEMMA 4.5. *Assume that there exists $\rho > 0$ such that*

$$\begin{aligned}f_{\rho,\rho/c}\bigg(\bigg(\frac{c_2\|\gamma\|}{D_1}(1-\beta[\delta]) + \frac{c_3\|\delta\|}{D_1}\beta[\gamma]\bigg)\int_a^b \mathcal{K}_{A_1}(s)g(s)\,ds\\ + \bigg(\frac{c_2\|\gamma\|}{D_1}\alpha_1[\delta] + \frac{c_3\|\delta\|}{D_1}(1-\alpha_1[\gamma])\bigg)\int_a^b \mathcal{K}_B(s)g(s)\,ds + \frac{1}{M}\bigg) > 1.\end{aligned} \tag{4.1}$$

Then the fixed point index $i_K(T, V_\rho)$ is equal to 0.

PROOF. Let $u_0(t) \equiv 1$ for $t \in [0,1]$. Then $u_0 \in K$. We prove that

$$u \ne Tu + \lambda u_0 \quad \text{for } u \in \partial V_\rho \text{ and } \lambda \ge 0,$$

which ensures, by Theorem 4.1, that the index is 0 on the set V_ρ. In fact, if this is not so, there exist $u \in \partial V_\rho$ and $\lambda \geq 0$ such that $u = Tu + \lambda u_0$. Then we have

$$\begin{aligned} u(t) = {} & \gamma(t)(\alpha[u] + \chi_{(\tau,1]}(d_1 I + e_1 N)(u(\tau))) \\ & + \delta(t)(\beta[u] + \chi_{[0,\tau]}(d_2 I + e_2 N)(u(\tau))) + Fu(t) + \lambda, \end{aligned}$$

where F is given by (2.4). Since

$$\alpha_1[u] \leq \alpha[u] + (d_1 I + e_1 N)(u(\tau)),$$

we obtain for $t \in [a, b]$

$$u(t) \geq \gamma(t)\alpha_1[u] + \delta(t)\beta[u] + Fu(t) + \lambda. \tag{4.2}$$

Applying α_1 and β to both sides of (4.2) we get

$$\begin{aligned} \alpha_1[u] &\geq \alpha_1[\gamma]\alpha_1[u] + \alpha_1[\delta]\beta[u] + \alpha_1[Fu] + \lambda\alpha_1[u_0], \\ \beta[u] &\geq \beta[\gamma]\alpha_1[u] + \beta[\delta]\beta[u] + \beta[Fu] + \lambda\beta[u_0]. \end{aligned}$$

This can be written in the form

$$\begin{pmatrix} 1-\alpha_1[\gamma] & -\alpha_1[\delta] \\ -\beta[\gamma] & 1-\beta[\delta] \end{pmatrix} \begin{pmatrix} \alpha_1[u] \\ \beta[u] \end{pmatrix} \geq \begin{pmatrix} \alpha_1[Fu] + \lambda\alpha_1[u_0] \\ \beta[Fu] + \lambda\beta[u_0] \end{pmatrix} \geq \begin{pmatrix} \alpha_1[Fu] \\ \beta[Fu] \end{pmatrix}. \tag{4.3}$$

Let

$$\underline{\mathcal{M}} = \begin{pmatrix} 1-\alpha_1[\gamma] & -\alpha_1[\delta] \\ -\beta[\gamma] & 1-\beta[\delta] \end{pmatrix}.$$

Then

$$(\underline{\mathcal{M}})^{-1} = \frac{1}{D_1} \begin{pmatrix} 1-\beta[\delta] & \alpha_1[\delta] \\ \beta[\gamma] & 1-\alpha_1[\gamma] \end{pmatrix}.$$

By Lemma 4.3, $(\underline{\mathcal{M}})^{-1}$ is order preserving. Thus, if we apply $(\underline{\mathcal{M}})^{-1}$ to left and right-hand sides of the inequality (4.3) we obtain

$$\begin{pmatrix} \alpha_1[u] \\ \beta[u] \end{pmatrix} \geq \frac{1}{D_1} \begin{pmatrix} 1-\beta[\delta] & \alpha_1[\delta] \\ \beta[\gamma] & 1-\alpha_1[\gamma] \end{pmatrix} \begin{pmatrix} \alpha_1[Fu] \\ \beta[Fu] \end{pmatrix}$$

and therefore, for $t \in [a, b]$

$$\begin{aligned} u(t) \geq {} & \left(\frac{\gamma(t)}{D_1}(1-\beta[\delta]) + \frac{\delta(t)}{D_1}\beta[\gamma] \right) \alpha_1[Fu] \\ & + \left(\frac{\gamma(t)}{D_1}\alpha_1[\delta] + \frac{\delta(t)}{D_1}(1-\alpha_1[\gamma]) \right) \beta[Fu] + Fu(t) + \lambda. \end{aligned}$$

Then, for $t \in [a, b]$, we get

$$\begin{aligned} u(t) \geq {} & \left(\frac{\gamma(t)}{D_1}(1-\beta[\delta]) + \frac{\delta(t)}{D_1}\beta[\gamma] \right) \int_0^1 \mathcal{K}_{A_1}(s)g(s)f(s,u(s))\,ds \\ & + \left(\frac{\gamma(t)}{D_1}\alpha_1[\delta] + \frac{\delta(t)}{D_1}(1-\alpha_1[\gamma]) \right) \int_0^1 \mathcal{K}_B(s)g(s)f(s,u(s))\,ds \\ & + \int_0^1 k(t,s)g(s)f(s,u(s))\,ds + \lambda \end{aligned}$$

$$\geq \left(\frac{c_2\|\gamma\|}{D_1}(1-\beta[\delta]) + \frac{c_3\|\delta\|}{D_1}\beta[\gamma]\right)\int_a^b \mathcal{K}_{A_1}(s)g(s)f(s,u(s))\,ds$$
$$+ \left(\frac{c_2\|\gamma\|}{D_1}\alpha_1[\delta] + \frac{c_3\|\delta\|}{D_1}(1-\alpha_1[\gamma])\right)\int_a^b \mathcal{K}_B(s)g(s)f(s,u(s))\,ds$$
$$+ \int_a^b k(t,s)g(s)f(s,u(s))\,ds + \lambda.$$

From (4.1) we obtain $\min_{t\in[a,b]} u(t) > \rho + \lambda \geq \rho$, which contradicts the fact that $u \in \partial V_\rho$. □

Next, we prove that the index is 1 on the set K_ρ.

LEMMA 4.6. *Assume that there exists $\rho > 0$ such that*

$$f^{0,\rho}\left(\left(\frac{\|\gamma\|}{D_2}(1-\beta_2[\delta]) + \frac{\|\delta\|}{D_2}\beta_2[\gamma]\right)\int_0^1 \mathcal{K}_{A_2}(s)g(s)\,ds + \left(\frac{\|\gamma\|}{D_2}\alpha_2[\delta] + \frac{\|\delta\|}{D_2}(1-\alpha_2[\gamma])\right)\int_0^1 \mathcal{K}_{B_2}(s)g(s)\,ds + \frac{1}{m}\right) < 1. \tag{4.4}$$

Then the fixed point index $i_K(T, K_\rho)$ is equal to 1.

PROOF. We show that $Tu \neq \lambda u$ for all $\lambda \geq 1$ when $u \in \partial K_\rho$; this ensures, by Theorem 4.1, that the index is 1 on K_ρ. In fact, if this is not so, then there exist $u \in K$ with $\|u\| = \rho$ and $\lambda \geq 1$ such that $\lambda u(t) = Tu(t)$. Then we have

$$\lambda u(t) = \gamma(t)(\alpha[u] + \chi_{(\tau,1]}(d_1 I + e_1 N)(u(\tau))) + \delta(t)(\beta[u] + \chi_{[0,\tau]}(d_2 I + e_2 N)(u(\tau))) + Fu(t).$$

Since

$$\alpha_2[u] \geq \alpha[u] + \chi_{(\tau,1]}(d_1 I + e_1 N)(u(\tau))$$

and

$$\beta_2[u] \geq \beta[u] + \chi_{[0,\tau]}(d_2 I + e_2 N)(u(\tau)),$$

we obtain

$$\lambda u(t) \leq \gamma(t)\alpha_2[u] + \delta(t)\beta_2[u] + Fu(t). \tag{4.5}$$

Applying α_2 and β_2 to both sides of (4.5) we obtain

$$\lambda\alpha_2[u] \leq \alpha_2[\gamma]\alpha_2[u] + \alpha_2[\delta]\beta_2[u] + \alpha_2[Fu],$$
$$\lambda\beta_2[u] \leq \beta_2[\gamma]\alpha_2[u] + \beta_2[\delta]\beta_2[u] + \beta_2[Fu].$$

Thus we have

$$\begin{pmatrix} \lambda - \alpha_2[\gamma] & -\alpha_2[\delta] \\ -\beta_2[\gamma] & \lambda - \beta_2[\delta] \end{pmatrix}\begin{pmatrix} \alpha_2[u] \\ \beta_2[u] \end{pmatrix} \leq \begin{pmatrix} \alpha_2[Fu] \\ \beta_2[Fu] \end{pmatrix}. \tag{4.6}$$

Setting

$$\overline{\mathcal{M}} = \begin{pmatrix} \lambda - \alpha_2[\gamma] & -\alpha_2[\delta] \\ -\beta_2[\gamma] & \lambda - \beta_2[\delta] \end{pmatrix},$$

we get

$$(\overline{\mathcal{M}})^{-1} = \frac{1}{\det(\overline{\mathcal{M}})} \begin{pmatrix} \lambda - \beta_2[\delta] & \alpha_2[\delta] \\ \beta_2[\gamma] & \lambda - \alpha_2[\gamma] \end{pmatrix},$$

where $\det(\overline{\mathcal{M}}) \geq D_2 > 0$.

By Lemma 4.3, $(\overline{\mathcal{M}})^{-1}$ is order preserving. Thus, if we apply $(\overline{\mathcal{M}})^{-1}$ to both sides of the inequality (4.6) we obtain

$$\begin{pmatrix} \alpha_2[u] \\ \beta_2[u] \end{pmatrix} \leq \frac{1}{\det(\overline{\mathcal{M}})} \begin{pmatrix} \lambda - \beta_2[\delta] & \alpha_2[\delta] \\ \beta_2[\gamma] & \lambda - \alpha_2[\gamma] \end{pmatrix} \begin{pmatrix} \alpha_2[Fu] \\ \beta_2[Fu] \end{pmatrix},$$

and by Lemma 4.4, we have

$$\begin{pmatrix} \alpha_2[u] \\ \beta_2[u] \end{pmatrix} \leq \frac{1}{D_2} \begin{pmatrix} 1 - \beta_2[\delta] & \alpha_2[\delta] \\ \beta_2[\gamma] & 1 - \alpha_2[\gamma] \end{pmatrix} \begin{pmatrix} \alpha_2[Fu] \\ \beta_2[Fu] \end{pmatrix}.$$

Hence we obtain

$$\begin{aligned} \lambda u(t) \leq{}& \frac{\gamma(t)}{D_2}[(1 - \beta_2[\delta])\alpha_2[Fu] + \alpha_2[\delta]\beta_2[Fu]] \\ &+ \frac{\delta(t)}{D_2}[(1 - \alpha_2[\gamma])\beta_2[Fu] + \beta_2[\gamma]\alpha_2[Fu]] + Fu(t). \end{aligned}$$

Using the inequality $f(s,u(s)) \leq \rho f^{0,\rho}$ and taking the supremum over $[0,1]$ gives

$$\begin{aligned} \lambda\rho \leq \rho f^{0,\rho} \bigg(& \bigg(\frac{\|\gamma\|}{D_2}(1 - \beta_2[\delta]) + \frac{\|\delta\|}{D_2}\beta_2[\gamma] \bigg) \int_0^1 \mathcal{K}_{A_2}(s)g(s)\,ds \\ &+ \bigg(\frac{\|\gamma\|}{D_2}\alpha_2[\delta] + \frac{\|\delta\|}{D_2}(1 - \alpha_2[\gamma]) \bigg) \int_0^1 \mathcal{K}_{B_2}(s)g(s)\,ds + \frac{1}{m} \bigg), \end{aligned}$$

contradicting (4.4). □

The two lemmas above lead to the following result valid for the impulsive integral equation

$$u(t) = \gamma(t)\alpha[u] + \delta(t)\beta[u] + \int_0^1 k(t,s)g(s)f(s,u(s))\,ds + Gu(t). \tag{4.7}$$

Theorem 4.7. *Equation* (4.7) *has at least one positive solution in K if either of the following conditions hold:*

- (S_1) *There exist ρ_1, ρ_2 with $\rho_1 < \rho_2$ such that* (4.4) *is satisfied for ρ_1 and* (4.1) *is satisfied for ρ_2.*
- (S_2) *There exist ρ_1, ρ_2 with $\rho_1 < c\rho_2$ such that* (4.1) *is satisfied for ρ_1 and* (4.4) *is satisfied for ρ_2.*

Equation (4.7) *has at least two positive solutions in K if one of the following conditions hold:*

- (H_1) *There exist ρ_1, ρ_2, ρ_3 with $\rho_1 < \rho_2 < c\rho_3$ such that* (4.4) *is satisfied for ρ_1,* (4.1) *is satisfied for ρ_2 and* (4.4) *is satisfied for ρ_3.*
- (H_2) *There exist ρ_1, ρ_2, ρ_3 with $\rho_1 < c\rho_2 < c\rho_3$ such that* (4.1) *is satisfied for ρ_1,* (4.4) *is satisfied for ρ_2 and* (4.1) *is satisfied for ρ_3.*

We omit the proof which follows simply from properties of fixed point index, for details of similar proofs see [19].

REMARK 4.8. It is possible to state results for three or more nontrivial solutions by expanding the lists in conditions (H_1), (H_2), we refer the reader to [20], [25] for the type of results that may be stated.

The following example illustrates our approach.

EXAMPLE 4.9. Consider the differential equation

$$u''(t)+\frac{t}{1+t}u'(t)-\frac{1}{1+t}u(t)+\frac{1+t}{e^t}f(t,u(t))=0,\quad t\in(0,1),\ t\neq\tau, \tag{4.8}$$

with impulsive effect

$$\Delta u|_{t=\tau}=I(u(\tau)),\quad \Delta u'|_{t=\tau}=N(u(\tau)),\quad \tau\in(0,1/2),$$

and BCs

$$u(0)=\alpha[u],\quad u'(1)=\beta[u],$$

where

$$I(w)=\begin{cases}\mu w, & 0\le w\le 1,\\ \dfrac{\mu}{2}w+\dfrac{\mu}{2}, & w\ge 1,\end{cases}\qquad N(w)=\begin{cases}-\sigma w, & 0\le w\le 1,\\ -\dfrac{\sigma}{2}w-\dfrac{\sigma}{2}, & w\ge 1.\end{cases}$$

Set $P(t)=\int_0^t s/(1+s)\,ds$. By multiplying by $e^{P(t)}$ we can rewrite (4.8) as

$$\left(u'(t)\frac{e^t}{1+t}\right)'-\frac{e^t}{(1+t)^2}u(t)+f(t,u(t))=0,\quad t\in(0,1),\ t\neq\tau.$$

In this case, we have

$$\gamma(t)=\frac{1}{e}t+e^{-t},\quad \delta(t)=t,$$

and for every fixed $[a,b]\subset(\tau,1)$ we obtain $c_2=b/e+e^{-b}$ and $c_3=a$.

Thus the Green's function is given by

$$k(t,s)=\frac{1}{W(0)}\begin{cases}s\left(\dfrac{1}{e}t+e^{-t}\right) & \text{if } s\le t,\\ t\left(\dfrac{1}{e}s+e^{-s}\right) & \text{if } s\ge t,\end{cases}$$

where $W(t)=\gamma(t)\delta'(t)-\delta(t)\gamma'(t)=e^{-t}(1+t)$ and therefore $W(0)=1$.

Upper and lower bounds for $k(t,s)$ are given by direct calculation as follows:

$$\Phi(s)=s\left(\frac{1}{e}s+e^{-s}\right),\quad c_1=\min\left\{\frac{1}{e}b+e^{-b},a\right\}.$$

Moreover, we have

$$\frac{1}{m}=\sup_{t\in[0,1]}\int_0^1 k(t,s)\,ds=\max_{t\in[0,1]}\left\{\frac{t^2}{2}e^{-t}+te^{-t}-\frac{1}{2e}t\right\}=\frac{1}{e},$$

and, for $[a,b]=[1/2,3/4]$,

$$\begin{aligned}\frac{1}{M} &= \inf_{t\in[1/2,3/4]}\int_{1/2}^{3/4} k(t,s)\,ds\\ &= \min_{t\in[1/2,3/4]}\left\{\frac{5}{32e}t+\frac{1}{2}e^{-t}t^2-\frac{1}{8}e^{-t}+te^{-t}-te^{-3/4}\right\}\\ &= \frac{5}{64e}+\frac{1}{2\sqrt{e}}-\frac{1}{2e^{3/4}}.\end{aligned}$$

We fix

$$\tau=1/5,\quad \mu=1/6,\quad \sigma=1/10,\quad \alpha[u]=u(1/10)/4,\quad \beta[u]=u(5/8)/5$$

and show that all the constants that appear in (4.4) can be computed, the ones appearing in (4.1) can be dealt with in a similar way. This choice leads to

$$h_1=1/8,\quad h_2=1/4,\quad h_3=1/3,$$

and (rounded to 3 decimal places)

$$\alpha_2[\gamma]=0.458,\quad \beta_2[\delta]=0.192,\quad D_2=0.404.$$

Therefore (C_5) is satisfied, $\int_0^1\mathcal{K}_{A_2}(s)\,ds=0.055$, $\int_0^1\mathcal{K}_{B_2}(s)\,ds=0.113$ and condition (4.4) reads $f^{0,\rho}<1.406$.

5. Others cases

In this Section we study three cases that do not fit directly in our theory but can be treated in a similar manner.

5.1. A critical case. We begin by observing that our method does not apply whenever one of the conditions in (C_5) fails. In the case of non-impulsive problems, when the associated linear operator is not invertible, the BVP is said to be at resonance.

J. R. L. Webb and M. Zima [37] studied six non-impulsive resonant BVPs subject to one nonlocal condition, using signed measures and improving the results of [21], where a completely different technique was used. The method utilized in [37] is based on earlier results of J. R. L. Webb and co-authors [34], [36] and on a technique similar to the one used by X. Han [15] for a three-point problem.

Here we present a modification of the approach of [37] to the setting of IBVPs, with two nonlocal BCs, by showing that a critical (for our method) IBVP can be transformed into an equivalent IBVP that fits our framework.

In particular, we study the IBVP

$$
\begin{aligned}
&u''(t) + h(t, u(t)) = 0, \quad t \in (0,1),\ t \neq \tau, \\
&\Delta u|_{t=\tau} = I(u(\tau)), \quad \Delta u'|_{t=\tau} = N(u(\tau)), \quad \tau \in (0,1), \\
&u(0) = \alpha[u], \quad u(1) = \beta[u],
\end{aligned} \tag{5.1}
$$

where

$$
\alpha[u] := \sum_{i=1}^{n} \overline{\alpha}_i u(\xi_i) + \int_0^1 \overline{\alpha}(t)u(t)\,dt, \quad \beta[u] := \sum_{j=1}^{m} \overline{\beta}_j u(\eta_j) + \int_0^1 \overline{\beta}(t)u(t)\,dt,
$$

ξ_i, η_j are distinct points in $(0,1)$, $\overline{\alpha}_i \geq 0$, $\overline{\beta}_j \geq 0$, $\overline{\alpha}, \overline{\beta}$ are non-negative continuous functions and the nonlinearity h is *not necessarily positive*.

In this case we have $\gamma(t) = 1 - t$ and $\delta(t) = t$ and we assume that

$$
\begin{aligned}
\alpha_2[\gamma] &= \sum_{i=1}^{n} \overline{\alpha}_i (1 - \xi_i) + \int_0^1 \overline{\alpha}(t)(1-t)\,dt + h_2(1-\tau) < 1, \\
\beta_2[\delta] &= \sum_{j=1}^{m} \overline{\beta}_j \eta_j + \int_0^1 \overline{\beta}(t)t\,dt + h_3\tau < 1, \\
D_2 &= (1 - \alpha_2[\gamma])(1 - \beta_2[\delta]) - \alpha_2[\delta]\beta_2[\gamma] = 0.
\end{aligned} \tag{5.2}
$$

From (5.2) and $h_2 > 0$ we obtain $\alpha_2[\gamma], \alpha_2[\delta], \beta_2[\gamma] > 0$.

The problem (5.1) with no impulsive terms and only one nonlocal condition is a resonant problem studied in [37] and the functionals $\alpha[\,\cdot\,]$, $\beta[\,\cdot\,]$ in (5.2) are similar to one descripted in [13]. Even if our method can be applied to more general measures, we have chosen these particular functionals since they shed more light on our approach.

We can now consider an equivalent IBVP

$$
\begin{aligned}
&u''(t) - \omega^2 u(t) + f(t, u(t)) = 0, \quad t \in (0,1),\ t \neq \tau, \\
&\Delta u|_{t=\tau} = I(u(\tau)), \quad \Delta u'|_{t=\tau} = N(u(\tau)), \quad \tau \in (0,1), \\
&u(0) = \alpha[u], \quad u(1) = \beta[u],
\end{aligned}
$$

where $\omega > 0$ is such that $f(t,u) := h(t,u) + \omega^2 u \geq 0$.

We show that the condition (C_5) is fulfilled. The Green's function for the local problem

$$
u''(t) - \omega^2 u(t) + f(t, u(t)) = 0, \quad u(0) = 0, \quad u(1) = 0,
$$

is (see, for example, [37])

$$
\widehat{k}(t,s) = \frac{1}{\omega \sinh(\omega)} \begin{cases} \sinh(\omega s)\sinh(\omega(1-t)) & \text{for } s \leq t, \\ \sinh(\omega t)\sinh(\omega(1-s)) & \text{for } s > t, \end{cases}
$$

and we have
$$\widehat{\gamma}(t)=\frac{\sinh(\omega(1-t))}{\sinh(\omega)},\quad \widehat{\delta}(t)=\frac{\sinh(\omega t)}{\sinh(\omega)}.$$
Now, it can be shown that $\sinh(\omega x)/\sinh(\omega)<x$ for $x\in(0,1)$, that is,
$$\widehat{\gamma}(t)<\gamma(t)\quad\text{and}\quad \widehat{\delta}(t)<\delta(t),\quad \text{for } t\in(0,1).$$
This implies that
$$\alpha_2[\widehat{\gamma}]<\alpha_2[\gamma]<1\quad\text{and}\quad \beta_2[\widehat{\delta}]<\beta_2[\delta]<1$$
and
$$\alpha_2[\widehat{\delta}]<\alpha_2[\delta]\quad\text{and}\quad \beta_2[\widehat{\gamma}]<\beta_2[\gamma].$$
This ensures that
$$\widehat{D}_2:=(1-\alpha_2[\widehat{\gamma}])(1-\beta_2[\widehat{\delta}])-\alpha_2[\widehat{\delta}]\beta_2[\widehat{\gamma}]>0,$$
since
$$(1-\alpha_2[\widehat{\gamma}])(1-\beta_2[\widehat{\delta}])>(1-\alpha_2[\gamma])(1-\beta_2[\delta])=\alpha_2[\delta]\beta_2[\gamma]>\alpha_2[\widehat{\delta}]\beta_2[\widehat{\gamma}].$$
Thus (C_5) is satisfied. Furthermore one may use the function
$$\widehat{\Phi}(s)=\frac{\sinh(\omega s)\sinh(\omega(1-s))}{\omega\sinh(\omega)},$$
and for an arbitrary (but fixed) $[a,b]$ in $(\tau,1)$ we obtain
$$\widehat{c_1}=\min\bigg\{\frac{\sinh(\omega a)}{\sinh(\omega)},\frac{\sinh(\omega(1-b))}{\sinh(\omega)}\bigg\},\quad \widehat{c_2}=\frac{\sinh(\omega(1-b))}{\sinh(\omega)},\quad \widehat{c_3}=\frac{\sinh(\omega a)}{\sinh(\omega)}.$$
With suitable assumptions on the nonlinearity f we are now able to prove existence results for the IBVP (5.1) via Theorem 4.7.

5.2. A case with negative Wronskian. We illustrate an example where γ is negative and δ is positive. We proceed somewhat in a similar way as in Subsection 5.4 of [33]. The difference this time is that, due to the impulsive effect, we cannot study the fixed points of the operator (3.3), but we have to deal with an operator of different kind.

Consider the IBVP
$$\begin{gathered}u''(t)+g(t)f(t,u(t))=0,\quad t\in(0,1),\ t\neq\tau,\\ \Delta u|_{t=\tau}=I(u(\tau)),\quad \Delta u'|_{t=\tau}=N(u(\tau)),\quad \tau\in(0,1),\\ u'(0)=\alpha[u],\quad u(1)=\beta[u].\end{gathered}\tag{5.3}$$

In this case we have $\gamma(t)=t-1$ and $\delta(t)=1$ on $[0,1]$, that is, γ is negative and non-decreasing and δ is positive on $[0,1]$ and non-decreasing. Furthermore the Wronskian $W(t)$ is less than 0 on $[0,1]$ and
$$k(t,s)=\frac{1}{W(0)}\begin{cases}t-1 & \text{for } s\le t,\\ s-1 & \text{for } s>t,\end{cases}=\begin{cases}1-t & \text{for } s\le t,\\ 1-s & \text{for } s>t,\end{cases}$$

is non-negative on $[0,1]\times[0,1]$. For an arbitrary $[a,b]\subset(\tau,1)$ we have

$$\begin{aligned} k(t,s) &\le 1-s && \text{for } t\in[0,1] \text{ and } s\in[0,1],\\ k(t,s) &\ge (1-b)(1-s) && \text{for } t\in[a,b] \text{ and } s\in[0,1]. \end{aligned}$$

We associate to the IBVP (5.3) the operator

$$\begin{aligned} \widetilde{T}u(t) := {}&(1-t)\widetilde{\alpha}[u] + t\beta[u] + (1-t)\chi_{(\tau,1]}(-N)(u(\tau))\\ &+ \chi_{[0,\tau]}(-I+(\tau-1)N)(u(\tau)) + Fu(t), \end{aligned}$$

where $\widetilde{\alpha}[u] := \beta[u]-\alpha[u]$.

Under the new assumptions

- $I\colon[0,\infty)\to\mathbb{R}$ and $N\colon[0,\infty)\to\mathbb{R}$ are continuous functions and there exist $h_1,h_2>0$ and $h_3\ge 0$ such that, for $w\in[0,\infty)$,

$$h_1 w \le -N(w) \le h_2 w \quad\text{and}\quad 0\le(-I+(\tau-1)N)(w)\le h_3 w,$$

- $\widetilde{\alpha}[u]$ is a positive functional,

the operator $\widetilde{T}$ is compact and leaves invariant the cone

$$K=\{u\in PC_\tau[0,1] : u\ge 0,\ \min_{t\in[a,b]} u(t)\ge c\|u\|\},$$

where

$$c=\min\left\{a,\frac{(1-b)h_1}{\max\{h_2,h_3\}}\right\}.$$

We introduce the measures

$$\alpha_1[u]=\widetilde{\alpha}[u]+h_1u(\tau),\quad \alpha_2[u]=\widetilde{\alpha}[u]+h_2u(\tau)\quad\text{and}\quad \beta_2[u]=\beta[u]+h_3u(\tau)$$

and we assume that

$$(5.4)\quad 1-\alpha_2[-\gamma]>0,\quad 1-\beta_2[1]>0,\quad (1-\alpha_2[-\gamma])(1-\beta_2[1])-\alpha_2[1]\beta_2[-\gamma]>0.$$

We sketch the proof of the fact that the index is 1 on the set K_ρ. Suppose there exist $u\in K$ with $\|u\|=\rho$ and $\lambda\ge 1$ such that $\lambda u(t)=\widetilde{T}u(t)$. Then we have

$$\begin{aligned} \lambda u(t) = {}&-\gamma(t)(\widetilde{\alpha}[u]+\chi_{(\tau,1]}(-N)(u(\tau)))\\ &+t\beta[u]+\chi_{[0,\tau]}(-I+(\tau-1)N)(u(\tau))+Fu(t)\\ \le {}&-\gamma(t)(\widetilde{\alpha}[u]+\chi_{(\tau,1]}(-N)(u(\tau)))\\ &+\beta[u]+\chi_{[0,\tau]}(-I+(\tau-1)N)(u(\tau))+Fu(t). \end{aligned}$$

Since

$$\alpha_2[u]\ge\widetilde{\alpha}[u]+\chi_{(\tau,1]}(-N)(u(\tau))\quad\text{and}\quad \beta_2[u]\ge\beta[u]+\chi_{[0,\tau]}(-I+(\tau-1)N)(u(\tau)),$$

we obtain

$$\lambda u(t) \leq -\gamma(t)\alpha_2[u] + \beta_2[u] + Fu(t). \tag{5.5}$$

Applying α_2 and β_2 to both sides of (5.5) we obtain

$$\begin{aligned}\lambda\alpha_2[u] &\leq \alpha_2[-\gamma]\alpha_2[u] + \alpha_2[1]\beta_2[u] + \alpha_2[Fu],\\ \lambda\beta_2[u] &\leq \beta_2[-\gamma]\alpha_2[u] + \beta_2[1]\beta_2[u] + \beta_2[Fu].\end{aligned}$$

Thus we have

$$\begin{pmatrix} \lambda - \alpha_2[-\gamma] & -\alpha_2[1] \\ -\beta_2[-\gamma] & \lambda - \beta_2[1] \end{pmatrix} \begin{pmatrix} \alpha_2[u] \\ \beta_2[u] \end{pmatrix} \leq \begin{pmatrix} \alpha_2[Fu] \\ \beta_2[Fu] \end{pmatrix}.$$

Then, by suitably modifying (4.4), we can proceed as in Lemma 4.6.

We now sketch the proof of the fact that the index is zero on V_ρ. Instead of (4.1) we assume that there exist $\rho > 0$ such that

$$\begin{aligned} f_{\rho,\rho/c}\bigg(&\bigg(\frac{1-b}{D_a}(1-a\beta[\delta]) + \frac{a}{D_a}\beta[-\gamma]\bigg)\int_a^b \mathcal{K}_{A_1}(s)g(s)\,ds \\ &+ \bigg(\frac{a(1-b)}{D_a}\alpha_1[\delta] + \frac{a}{D_a}(1-\alpha_1[-\gamma])\bigg)\int_a^b \mathcal{K}_B(s)g(s)\,ds + \frac{1}{M}\bigg) > 1, \end{aligned}$$

where $D_a := (1-\alpha_1[-\gamma])(1-a\beta[\delta]) - a\alpha_1[\delta]\beta[-\gamma]$ is positive from (5.4).

Let $u_0 \in K \setminus \{0\}$. Suppose that there exist $u \in \partial V_\rho$ and $\lambda \geq 0$ such that $u = \widetilde{T}u + \lambda u_0$. Then for $t \in [a,b]$ we get

$$\begin{aligned} u(t) &= -\gamma(t)(\widetilde{\alpha}[u] - N(u(\tau))) + t\beta[u] + Fu(t) + \lambda u_0 \\ &\geq -\gamma(t)\alpha_1[u] + t\beta[u] + Fu(t) + \lambda u_0 \geq -\gamma(t)\alpha_1[u] + a\beta[u] + Fu(t) + \lambda u_0. \end{aligned}$$

Applying α_1 and β to left and right-hand sides of the above inequality gives

$$\begin{aligned} \alpha_1[u] &\geq \alpha_1[-\gamma]\alpha_1[u] + a\alpha_1[\delta]\beta[u] + \alpha_1[Fu] + \lambda\alpha_1[u_0],\\ \beta[u] &\geq \beta[-\gamma]\alpha_1[u] + a\beta[\delta]\beta[u] + \beta[Fu] + \lambda\beta[u_0]. \end{aligned}$$

This can be rewritten in the form

$$\begin{aligned} &\begin{pmatrix} 1-\alpha_1[-\gamma] & -a\alpha_1[\delta] \\ -\beta[-\gamma] & 1-a\beta[\delta] \end{pmatrix} \begin{pmatrix} \alpha_1[u] \\ \beta[u] \end{pmatrix} \\ &\qquad \geq \begin{pmatrix} \alpha_1[Fu] + \lambda\alpha_1[u_0] \\ \beta[Fu] + \lambda\beta[u_0] \end{pmatrix} \geq \begin{pmatrix} \alpha_1[Fu] \\ \beta[Fu] \end{pmatrix}. \end{aligned} \tag{5.6}$$

If we set

$$\underline{\mathcal{M}}_a = \begin{pmatrix} 1-\alpha_1[-\gamma] & -a\alpha_1[\delta] \\ -\beta[-\gamma] & 1-a\beta[\delta] \end{pmatrix}$$

and apply

$$(\underline{\mathcal{M}}_a)^{-1} = \frac{1}{D_a}\begin{pmatrix} 1-a\beta[\delta] & a\alpha_1[\delta] \\ \beta[-\gamma] & 1-\alpha_1[-\gamma] \end{pmatrix}$$

to left and right-hand sides of the inequality (5.6), we obtain

$$\begin{pmatrix} \alpha_1[u] \\ \beta[u] \end{pmatrix} \geq \frac{1}{D_a} \begin{pmatrix} 1 - a\beta[\delta] & a\alpha_1[\delta] \\ \beta[-\gamma] & 1 - \alpha_1[-\gamma] \end{pmatrix} \begin{pmatrix} \alpha_1[Fu] \\ \beta[Fu] \end{pmatrix}.$$

The rest of the proof follows as in Lemma 4.5.

5.3. A case with a signed measure. We illustrate an example where β is a functional given by a signed measure and α is given by a positive measure; in particular we utilize as β the continuously distributed signed measure given in Example 5.4 of [33].

Consider the IBVP

$$\begin{gathered} u''(t) + g(t)f(t,u(t)) = 0, \quad t \in (0,1),\ t \neq \tau, \\ \Delta u|_{t=\tau} = I(u(\tau)), \quad \Delta u'|_{t=\tau} = N(u(\tau)), \quad \tau \in (0,1), \\ u(0) = \alpha[u], \quad u'(1) = \beta[u], \end{gathered}$$

where $\beta[u] := \int_0^1 u(s)\beta(s)\,ds$ with $\beta(t) = -\cos(\pi t)$.

In this case we have $\gamma(t) = 1$, $\delta(t) = t$ and

$$k(t,s) = \begin{cases} s & \text{for } s \leq t, \\ t & \text{for } s > t, \end{cases}$$

and consider the operator

$$\widetilde{T}u(t) := (1-t)\alpha[u] + t\widetilde{\beta}[u] + \chi_{(\tau,1]}(I - \tau N)(u(\tau)) + t\chi_{[0,\tau]}(-N)(u(\tau)) + Fu(t),$$

where $\widetilde{\beta}[u] := \beta[u] + \alpha[u]$.

We assume that $\widetilde{\beta}[u]$ is a positive functional. We introduce the measures

$$\begin{aligned} \alpha_1[u] &:= (1-b)(\alpha[u] + h_1 u(\tau)), \\ \alpha_2[u] &:= \alpha[u] + h_2 u(\tau), \\ \beta_2[u] &:= \widetilde{\beta}[u] + h_3 u(\tau). \end{aligned}$$

The indices can be computed with the same assumptions as in Lemmas 4.6 and 4.5, but the proofs are slightly different. We sketch the proofs.

Suppose that there exist $u \in K$ with $\|u\| = \rho$ and $\lambda \geq 1$ such that $\lambda u(t) = \widetilde{T}u(t)$. Then we have

$$\begin{aligned} \lambda u(t) &= (1-t)\alpha[u] + t\widetilde{\beta}[u] + \chi_{(\tau,1]}(I - \tau N)(u(\tau)) + t\chi_{[0,\tau]}(-N)(u(\tau)) + Fu(t) \\ &\leq \alpha[u] + (I - \tau N)(u(\tau)) + t(\widetilde{\beta}[u] - N(u(\tau))) + Fu(t) \\ &\leq \alpha_2[u] + t\beta_2[u] + Fu(t) = \gamma(t)\alpha_2[u] + \delta(t)\beta_2[u] + Fu(t). \end{aligned}$$

The rest of the proof follows as in Lemma 4.6.

Now, let $u_0 \in K \setminus \{0\}$. Suppose that there exist $u \in \partial V_\rho$ and $\lambda \geq 0$ such that $u = \widetilde{T}u + \lambda u_0$. Then for $t \in [a,b]$ we get

$$\begin{aligned} u(t) &= (1-t)\alpha[u] + t\widetilde{\beta}[u] + (I - \tau N)(u(\tau)) + Fu(t) + \lambda u_0 \\ &\geq (1-b)\alpha[u] + (1-b)h_1 u(\tau) + t\widetilde{\beta}[u] + Fu(t) + \lambda u_0 \\ &= \gamma(t)\alpha_1[u] + \delta(t)\widetilde{\beta}[u] + Fu(t) + \lambda u_0. \end{aligned}$$

The rest of the proof follows as in Lemma 4.5. We omit further details.

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