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ANISOTROPIC ELLIPTIC EQUATIONS IN $\mathbb{R}^N$: EXISTENCE AND REGULARITY RESULTS

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Abstract. We investigate a class of anisotropic elliptic equations in the whole $\mathbb{R}^N$. By a variational approach, we obtain existence and regularity of nontrivial solutions in the framework of anisotropic Sobolev spaces. In addition, when the data is assumed to be merely locally integrable, the existence of solutions is established for a subclass of equations.

1. Introduction

We are interested in the existence and regularity results of distributional solutions in an appropriate function space for nonlinear anisotropic elliptic equations. In this paper, first we consider an elliptic equation of the form

$$-\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^{N} \beta(x)|u|^{p_i-2}u = f(x)|u|^{s-1}u \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $N \geq 2$. We assume that β and f satisfy the following conditions: $\beta\colon \mathbb{R}^N \to \mathbb{R}$ is a continuous function satisfying

$$\beta(x) \geq \beta_0 \quad \text{a.e. in } \mathbb{R}^N \quad \text{and} \quad \lim_{|x|\to\infty} \beta(x) = \infty, \tag{1.2}$$

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the function $f\colon \mathbb{R}^N \to \mathbb{R}$ is nonnegative and satisfies

$$f \in L^{\omega}(\mathbb{R}^N) \cap L^{\omega/(1-\delta)}(\mathbb{R}^N), \quad \omega = \frac{\overline{p}^*}{\overline{p}^* - (s+1)}, \tag{1.3}$$

Herein, β_0 is a positive constant, $0 < \delta < 1$ is a small positive real,

$$\frac{1}{\overline{p}} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{p_i} \quad \text{and} \quad \overline{p}^* = \frac{N\overline{p}}{N-\overline{p}}$$

with $\overline{p} < N$. For (1.1) we assume that the exponents $p_1, \ldots, p_N$ and s are restricted as follows

$$\begin{cases} p_i > 1, \quad \displaystyle\sum_{i=1}^{N}\frac{1}{p_i} > 1, & i = 1, \ldots, N, \\ 0 < s < \overline{p}^* - 1, \quad \overline{p}^* := \dfrac{N}{\sum_{i=1}^{N}\frac{1}{p_i} - 1}, \\ p_{\max} = \max(p_1, \ldots, p_N) < \overline{p}^*. \end{cases} \tag{1.4}$$

Remark 1.1. Note that (1.3) gives more restrictive integrability condition on the function f. The function f is assumed to have optimal regularity conditions which ensure existence and regularity results of solutions. Since $p_{\max} < \overline{p}^*$, then $\overline{p}^*$ is the critical exponent associated to the operator:

$$\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\bigg(\bigg|\frac{\partial u}{\partial x_i}\bigg|^{p_i-2}\frac{\partial u}{\partial x_i}\bigg).$$

Second, the problem of the existence and regularity of solutions with integrable function will be studied for the following problem

$$-\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\bigg(\bigg|\frac{\partial u}{\partial x_i}\bigg|^{p_i-2}\frac{\partial u}{\partial x_i}\bigg) + \sum_{i=1}^{N}\beta(x)|u|^{r_i-1}u = f(x)|u|^{s-1}u + g(x) \tag{1.5}$$

in $\mathbb{R}^N$, where $g \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ and β satisfies (1.2). We strengthen a bit our condition on the data f of the problem (1.5). We require the nonnegative function f to satisfy

$$f \in L^{\omega}(\mathbb{R}^N) \cap L^{\omega/(1-\delta)}(\mathbb{R}^N), \quad \omega = \frac{\sigma}{\sigma - s}, \tag{1.6}$$

where, $0 < \delta < 1/p_i$ is a small positive real and

$$\sigma := \frac{(1-\delta)sp_{\min}}{1-\delta p_{\min}} \quad \text{for } i = 1, \ldots, N.$$

Herein, $p_{\min} = \min(p_1, \ldots, p_N)$.

For (1.5) we assume that the exponents $p_1, \dots, p_N$ and $r_1, \dots, r_N$ are restricted as follows:

$$
\begin{cases}
p_i > 1, \quad \displaystyle\sum_{i=1}^{N} \frac{1}{p_i} > 1, & i = 1, \dots, N, \\
\dfrac{\overline{p}(N-1)}{N(\overline{p}-1)} < p_i < \dfrac{\overline{p}(N-1)}{N-\overline{p}}, & i = 1, \dots, N, \\
r_i > p_{\max}, & i = 1, \dots, N.
\end{cases}
\tag{1.7}
$$

Remark 1.2. Note that the restriction on the exponents r_i, $i = 1, \dots, N$, in the list (1.7), is needed to obtain regularity of solution in Lemma 3.4 below. Observe that from the definition of

$$\sigma := \frac{(1-\delta)sp_{\min}}{1-\delta p_{\min}},$$

we deduce easily that $0 < s < \sigma$.

To the best of our knowledge, anisotropic equations with different orders of derivations in different directions involving critical exponents with unbounded nonlinearities were never studied before. In the isotropic case, we can refer the reader to the works by [10], [16] and [22] where existence and regularity results are obtained. In passing, we mention that in [12] the authors have studied another class of ansiotropic elliptic equations. Via an adaptation of the concentration-compactness lemma of P.-L. Lions to anisotropic operators, they have obtained the existence of multiple nonnegative solutions. Let us point out that in the case of bounded domains, more work in this direction can be found in [13] where the authors proved existence and nonexistence results for some anisotropic quasilinear elliptic equations.

Compared to [4], the main feature of the problem (1.5) is the combination of an anisotropic diffusion operator, a restrictive integrability conditions on f, a locally integrable right-hand side g, and an unbounded domain. In the case of the Dirichlet problem on a bounded domain, existence and regularity results for distributional solutions with L^1-data have been obtained in [6], [17] for a class of anisotropic elliptic and parabolic equations. For an anisotropic parabolic reaction-diffusion-advection system with a zero-flux boundary condition, still on a bounded domain, similar results are established in [3].

The remaining part of the paper is organized as follows: Our main "elliptic" results are stated in Section 2. Some preliminary results are given in Section 3. Main results are proved in Section 4 (for the problem (1.1)) and Section 5 (for the problem (1.5)).

2. Statement of main theorem

We let $1 \le p_1, \ldots, p_N < \infty$ be N real numbers. Denote by $\overline{p}$ the harmonic mean of these numbers, i.e.

$$\frac{1}{\overline{p}} = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{p_i},$$

and set $p_{\max} = \max(p_1, \ldots, p_N)$, $p_{\min} = \min(p_1, \ldots, p_N)$, $\overrightarrow{p} = (p_1, \ldots, p_N)$. We always have $p_{\min} \le \overline{p} \le p_{\max}$. The Sobolev conjugate of $\overline{p}$ is denoted by $\overline{p}^*$, i.e. $\overline{p}^* = (N\overline{p})/(N - \overline{p})$.

Anisotropic Sobolev spaces were introduced and studied by S. M. Nikol'skiĭ [21], L. N. Slobodeckiĭ [24], M. Troisi [25], and later by N. S. Trudinger [26] in the framework of Orlicz spaces.

Herein we need the anisotropic Sobolev space

$$\mathcal{D}^{\beta,\overrightarrow{p}}(\mathbb{R}^N) = \bigg\{ u \in W^{1,\overline{p}}(\mathbb{R}^N) : \frac{\partial u}{\partial x_i} \in L^{p_i}(\mathbb{R}^N), \\ \beta^{1/p_i} u \in L^{p_i}(\mathbb{R}^N),\ i = 1, \ldots, N \bigg\}.$$

It is a Banach space under the norm

$$\|u\| = \sum_{i=1}^{N} \|\beta^{1/p_i} u\|_{L^{p_i}(\mathbb{R}^N)} + \sum_{i=1}^{N} \bigg\| \frac{\partial u}{\partial x_i} \bigg\|_{L^{p_i}(\mathbb{R}^N)}.$$

Observe that in the case $\beta = 1$, we can replace $\mathcal{D}^{\beta,\overrightarrow{p}}(\mathbb{R}^N)$ by the standard anisotropic Sobolev space $W^{1,\overrightarrow{p}}(\mathbb{R}^N)$.

Now we define what we mean by weak solutions of the problems (1.1) and (1.5). We also supply our main existence results.

Definition 2.1. We say that $u \in \mathcal{D}^{\beta,\overrightarrow{p}}(\mathbb{R}^N)$ is a weak solution of (1.1) if

$$\sum_{i=1}^{N} \int_{\mathbb{R}^N} \bigg(\bigg| \frac{\partial u}{\partial x_i} \bigg|^{p_i - 2} \frac{\partial u}{\partial x_i} \frac{\partial \varphi}{\partial x_i} + \beta(x)|u|^{p_i-2} u\varphi \bigg) \, dx = \int_{\mathbb{R}^N} f(x)|u|^{s-1} u\varphi \, dx, \tag{2.1}$$

for all $\varphi \in \mathcal{D}^{\beta,\overrightarrow{p}}(\mathbb{R}^N)$.

Remark 2.2. Note that the assumptions (1.2) and (1.3) guarantee that the integrals given in (2.1) are well defined.

Now, we state the first main results of this paper.

Theorem 2.3. *Assume conditions* (1.2)–(1.3) *hold, and the corresponding exponents* $p_1, \ldots, p_N$ *and* s *are restricted as in* (1.4). *Then the problem* (1.1) *has at least one nontrivial weak solution in the sense of Definition* 2.1. *Moreover, the solution* u *satisfies*

$$u^\kappa \bigg| \frac{\partial u}{\partial x_i} \bigg|^{p_i} \in L^1(\mathbb{R}^N) \quad \text{for all } 0 < \kappa < \infty \text{ and } i = 1, \ldots, N.$$

Furthermore, if $p_i = p$ *for* $i = 1, \ldots, N$, *then* $u^\kappa \in L^1(\mathbb{R}^N)$ *with* $Np/(N-p) \le \kappa < \infty$.

REMARK 2.4. Remark that the condition $\sum_{i=1}^N 1/p_i > 1$ is indeed equivalent to $\overline{p} < N$. If $p_i = p$ for all i, then it is reduced to the isotropic case $p < N$. This condition is generally used for problems involving critical exponents in unbounded domains.

Next, we look for distributional solutions to (1.5) in the following sense:

DEFINITION 2.5. A distributional solution of (1.5) is a function $u: \mathbb{R}^N \to \mathbb{R}$ such that

$$u \in W^{1,\overrightarrow{q}}_{\mathrm{loc}}(\mathbb{R}^N) \cap L^{r_i}_{\mathrm{loc}}(\mathbb{R}^N), \quad f(x)u^{s-1}u \in L^1_{\mathrm{loc}}(\mathbb{R}^N),$$
$$\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i} \in L^1_{\mathrm{loc}}(\mathbb{R}^N), \quad \text{for all } 1 \le q_i < \frac{N(\overline{p}-1)}{\overline{p}(N-1)}p_i$$

with $i = 1, \ldots, N$ and

$$\sum_{i=1}^N \int_{\mathbb{R}^N} \left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\frac{\partial \varphi}{\partial x_i} + \beta(x)|u|^{r_i-1}u\varphi\right)dx = \int_{\mathbb{R}^N} f(x)u^{s-1}u\varphi\,dx + \int_{\mathbb{R}^N} g(x)\varphi\,dx, \tag{2.2}$$

for all $\varphi \in C_c^1(\mathbb{R}^N)$.

REMARK 2.6. Note that in Definition 2.5 all terms in (2.2) are well-defined.

Our second main results are collected in the following theorem:

THEOREM 2.7. *Assume* (1.2) *and* (1.6) *hold and the corresponding exponents* $p_1, \ldots, p_N$ *and* $r_1, \ldots, r_N$ *are restricted as in* (1.7). *Then* (1.5) *has at least one distributional solution* u *in the sense of Definition* 2.5. *If* $\overline{p} > N$, *then* $u \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$.

3. Mathematical preliminaries

3.1. Anisotropic Sobolev spaces. Later we will need the following anisotropic Sobolev embedding theorem.

THEOREM 3.1. *Let* Q *be a cube of* $\mathbb{R}^N$ *with faces parallel to the coordinate planes. Suppose* $u \in W^{1,\overrightarrow{p}}(Q)$, *and set*

$$\frac{1}{\overline{p}} = \frac{1}{N}\sum_{i=1}^N \frac{1}{p_i}, \quad r = \begin{cases} \overline{p}^* := \dfrac{N\overline{p}}{N-\overline{p}}, & \text{if } \overline{p}^* < N, \\ \text{any number from } [1,\infty), & \text{if } \overline{p}^* \ge N. \end{cases}$$

Then there exists a constant C, depending on N, $p_1, \ldots, p_N$ if $\overline{p} < N$ and also on r and meas(Q) *if $\overline{p} \geq N$, such that*

$$\|u\|_{L^r(Q)} \leq C \prod_{i=1}^{N} \bigg[\bigg\| \frac{\partial u}{\partial x_i} \bigg\|_{L^{p_i}(Q)} + \|u\|_{L^{p_i}(Q)} \bigg]^{1/N}.$$

Moreover, suppose $u \in W^{1,\overrightarrow{p}}(\mathbb{R}^N)$ and $\overline{p}^* < N$. Then there exists a constant $T_0 > 0$ depending only on N and $p_1, \ldots, p_N$ such that

$$T_0 \|u\|_{L^{\overline{p}^*}(\mathbb{R}^N)} \leq \prod_{i=1}^{N} \bigg\| \frac{\partial u}{\partial x_i} \bigg\|_{L^{p_i}(\mathbb{R}^N)}^{1/N}. \tag{3.2}$$

Remark 3.2. Theorem 3.1 is used to prove the "interpolation" lemma below, which is a technical result we will use later to obtain a priori estimates. A similar result is found in [6] with $W^{1,\overrightarrow{p}}(Q)$ replaced by $W_0^{1,\overrightarrow{p}}(\Omega)$ in the case of a Dirichlet boundary condition.

Remark 3.3. We can replace the geometric mean on the right-hand side of (3.2) by an arithmetic mean. Indeed, the inequality between geometric and arithmetic means implies

$$\|u\|_{L^{\overline{p}^*}(\mathbb{R}^N)} \leq \frac{1}{NT_0} \sum_{i=1}^{N} \bigg\| \frac{\partial u}{\partial x_i} \bigg\|_{L^{p_i}(\mathbb{R}^N)}.$$

3.2. Technical lemmas.

Lemma 3.4. *Let Q be a cube of $\mathbb{R}^N$ with faces parallel to the coordinate planes and $(u_\varepsilon)_{0<\varepsilon\leq 1}$ be a sequence in $W^{1,\overrightarrow{p}}(Q)$ with $\overline{p} \leq N$. Suppose that there exists a constant c, independent of ε, such that*

$$\|u_\varepsilon\|_{L^{p_i}(Q)} \leq c, \quad i = 1, \ldots, N, \quad \text{and} \quad \sup_{\gamma>0} \sum_{i=1}^{N} \int_{B_\gamma} \bigg| \frac{\partial u_\varepsilon}{\partial x_i} \bigg|^{p_i} dx \leq c,$$

where $B_\gamma = \{x \in Q : \gamma \leq |u_\varepsilon| \leq \gamma + 1\}$ for $\gamma > 0$, or

$$\sum_{i=1}^{N} \int_Q \frac{|\partial u_\varepsilon / \partial x_i|^{p_i}}{(1 + |u_\varepsilon|)^\gamma} \, dx \leq c.$$

Then for every q_i such that

$$1 \leq q_i < \frac{N(\overline{p} - 1)}{\overline{p}(N - 1)} p_i,$$

there exists a constant C, depending on Q, N, $p_1, \ldots, p_N$, $q_1, \ldots, q_N$, and c, but not ε, such that

$$\bigg\| \frac{\partial u_\varepsilon}{\partial x_i} \bigg\|_{L^{q_i}(Q)} \leq C, \quad i = 1, \ldots, N, \quad \text{and} \quad \|u_\varepsilon\|_{L^{\overline{q}}(Q)} \leq C, \quad \frac{1}{\overline{q}} = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{q_i}.$$

The proof of Lemma 3.4 is found in [4].

Lemma 3.5. *Let*

$$\mathcal{A} = \inf_{u\in\mathcal{D}^{\beta,\overrightarrow{p}}(\mathbb{R}^N),\|u\|_{L^{\overline{p}^*}(\mathbb{R}^N)}=1} \bigg\{\sum_{i=1}^N \frac{1}{p_i}\bigg\|\frac{\partial u}{\partial x_i}\bigg\|^{p_i}_{L^{p_i}(\mathbb{R}^N)}\bigg\}.$$

Then $\mathcal{A} > 0$.

Proof. By Remark 3.3, we obtain

$$\sum_{i=1}^N \bigg\|\frac{\partial u}{\partial x_i}\bigg\|_{L^{p_i}(\mathbb{R}^N)} \geq NT_0 > 0, \tag{3.3}$$

for $\|u\|_{L^{\overline{p}^*}(\mathbb{R}^N)} = 1$. Note that the minimum $\mathcal{A}_1$ of the function $h(x_1,\dots,x_N) = \sum_{i=1}^N x_i^{p_i}/p_i$ over the set $\{(x_1,\dots,x_N) \in \mathbb{R}^N : \sum_{i=1}^N x_i \geq NT_0, x_i \geq 0\}$ is achieved and $\mathcal{A}_1 > 0$. By (3.3), we conclude that $\mathcal{A} \geq \mathcal{A}_1 > 0$. □

3.3. Mountain Pass Theorem. To deal with the functional framework we apply the following basic theorem.

Theorem 3.6 (Mountain Pass [2]). *Let I be a C^1-differentiable functional on a Banach space E and satisfying the Palais–Smale condition* (PS)*, suppose that there exists a neighbourhood U of 0 in E and a positive constant α satisfying the following conditions:*

(a) $I(0) = 0$.
(b) $I(u) \geq \alpha$ *on the boundary of* U.
(c) *There exists an* $e \in E \setminus U$ *such that* $I(e) < \alpha$.

Then

$$c = \inf_{\gamma\in\Gamma} \sup_{y\in[0,1]} I(\gamma(y))$$

is a critical value of I with $\Gamma = \{g \in C([0,1]) : g(0) = 0,\ g(1) = e\}$.

Let us recall that the functional $I: E \to \mathbb{R}$ of class C^1 satisfies the Palais-Smale compactness condition (PS) if every sequence $(u_n)_{n=1}^\infty \subset E$ for which there exists $M > 0$ such that: $I(u_n) \leq M$ and $I'(u_n) \to 0$ strongly in E^* as n goes to infinity (called a (PS) sequence), has a convergent subsequence. Here, E^* denotes the dual of E.

Remark 3.7. The Moutain pass theorem is a fundamental tool where it is used to prove existence results for variational problems of a general class of elliptic equations utilizing the topological min–max approach.

3.4. The variational formulation. Let us consider the functional

$$I: \mathcal{D}^{\beta,\overrightarrow{p}}(\mathbb{R}^N) \to \mathbb{R}$$

given by

$$I(u)=\sum_{i=1}^{N}\frac{s+1}{p_i}\int_{\mathbb{R}^N}\bigg(\bigg|\frac{\partial u}{\partial x_i}\bigg|^{p_i}+\beta(x)|u|^{p_i}\bigg)\,dx-\int_{\mathbb{R}^N}f(x)|u|^{s+1}\,dx,$$

for all $u\in\mathcal{D}^{\beta,\overrightarrow{p}}(\mathbb{R}^N)$. By assumption (1.3) and Sobolev's inequality, we can see that the functional K defined by

$$K(u)=\int_{\mathbb{R}^N}f(x)|u|^{s+1}\,dx$$

is indeed well defined and of class $\mathcal{C}^1$ on the space $\mathcal{D}^{\beta,\overrightarrow{p}}(\mathbb{R}^N)$ with

$$\langle K'(u);\varphi\rangle=(s+1)\int_{\mathbb{R}^N}f(x)|u|^{s-1}u\varphi\,dx,$$

for all $u,\varphi\in\mathcal{D}^{\beta,\overrightarrow{p}}(\mathbb{R}^N)$. Herein, $\langle\,\cdot\,;\,\cdot\,\rangle$ denotes the duality pairing between $\mathcal{D}^{\beta,\overrightarrow{p}}(\mathbb{R}^N)$ and $(\mathcal{D}^{\beta,\overrightarrow{p}}(\mathbb{R}^N))^*$. Therefore a weak solution of a problem (1.1) is a critical point u of I, i.e.

$$\langle I'(u);\varphi\rangle=0\quad\text{for all }\varphi\in\mathcal{D}^{\beta,\overrightarrow{p}}(\mathbb{R}^N).$$

Herein, $(\mathcal{D}^{\beta,\overrightarrow{p}}(\mathbb{R}^N))^*$ is the dual of $\mathcal{D}^{\beta,\overrightarrow{p}}(\mathbb{R}^N)$.

The following lemma is crucial to prove Theorem 2.3, it has basic topology properties.

Lemma 3.8. *Assume* (1.2) *and* (1.3) *hold, then*

(a) $\mathcal{D}^{\beta,\overrightarrow{p}}(\mathbb{R}^N)$ *is compactly embedded in* $L^{\overline{p}}(\mathbb{R}^N)$.
(b) K' *is a compact map from* $\mathcal{D}^{\beta,\overrightarrow{p}}(\mathbb{R}^N)$ *to* $(\mathcal{D}^{\beta,\overrightarrow{p}}(\mathbb{R}^N))^*$.

Proof. (a) Without loss more of generality, we will show that $u_n\to 0$ strongly in $L^{\overline{p}}(\mathbb{R}^N)$ for such sequence $u_n\in\mathcal{D}^{\beta,\overrightarrow{p}}(\mathbb{R}^N)$ which converges weakly to 0.

Indeed, we have $\|u_n\|\le C$ for some constant $C>0$. From (1.2), for a given $\varepsilon>0$ and $R>0$ such that

$$\beta(x)\ge 2\frac{C^{\overline{p}}}{\varepsilon}\quad\text{for all }|x|\ge R,$$

we have

$$u_n\rightharpoonup 0\quad\text{weakly in }W^{1,\overline{p}}(B_R),$$

where B_R is the Ball of radius R centered at origin. By using the compact imbedding $W^{1,\overline{p}}(B_R)\hookrightarrow L^{\overline{p}}(B_R)$, we get

$$\int_{B_R}|u_n|^{\overline{p}}\,dx\le\frac{\varepsilon}{2}\quad\text{for all }n\ge n_0,\tag{3.4}$$

for some $n_0\in\mathbb{N}$.

Since $p_{\min} < \overline{p} < p_{\max}$, there exists $0 < \alpha < 1$ such that

$$\frac{1}{\overline{p}} = \frac{\alpha}{p_{\min}} + \frac{1-\alpha}{p_{\max}}.$$

Then applying the Hölder inequality gives

$$\begin{aligned}(3.5)\quad &\left(\int_{\mathbb{R}^N} \beta(x)|u|^{\overline{p}}\,dx\right)^{1/\overline{p}} \\ &\le \left(\int_{\mathbb{R}^N} \beta(x)|u|^{p_{\min}}\,dx\right)^{\alpha/p_{\min}} \left(\int_{\mathbb{R}^N} \beta(x)|u|^{p_{\max}}\,dx\right)^{(1-\alpha)/p_{\max}}\end{aligned}$$

since

$$\beta(x)|u|^{\overline{p}} = (\beta(x))^{\alpha\overline{p}/p_{\min}}|u|^{\alpha\overline{p}}(\beta(x))^{(1-\alpha)\overline{p}/p_{\max}}|u|^{(1-\alpha)\overline{p}}.$$

An application of Young's inequality, we deduce from (3.5)

$$\begin{aligned}&\left(\int_{\mathbb{R}^N} \beta(x)|u|^{\overline{p}}\,dx\right)^{1/\overline{p}} \\ &\quad\le \alpha\left(\int_{\mathbb{R}^N} \beta(x)|u|^{p_{\min}}\,dx\right)^{1/p_{\min}} + (1-\alpha)\left(\int_{\mathbb{R}^N} \beta(x)|u|^{p_{\max}}\,dx\right)^{1/p_{\max}} \\ &\quad\le \left(\int_{\mathbb{R}^N} \beta(x)|u|^{p_{\min}}\,dx\right)^{1/p_{\min}} + \left(\int_{\mathbb{R}^N} \beta(x)|u|^{p_{\max}}\,dx\right)^{1/p_{\max}}.\end{aligned}$$

This implies

$$(3.6)\quad \left(\int_{\mathbb{R}^N} \beta(x)|u|^{\overline{p}}\,dx\right)^{1/\overline{p}} \le \sum_{i=1}^{N} \|\beta^{1/p_i}u\|_{L^{p_i}(\mathbb{R}^N)} + \sum_{i=1}^{N}\left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p_i}(\mathbb{R}^N)} = \|u\|.$$

Finally we deduce from (3.6)

$$(3.7)\qquad \frac{2}{\varepsilon}\int_{\mathbb{R}^N\setminus B_R} |u_n|^{\overline{p}}\,dx \le \int_{\mathbb{R}^N\setminus B_R} \frac{\beta(x)}{C^{\overline{p}}}|u_n|^{\overline{p}}\,dx \le 1,$$

Combining (3.4) and (3.7), we get

$$\int_{\mathbb{R}^N} |u_n|^{\overline{p}}\,dx \le \varepsilon, \quad \text{for all } n \ge n_0.$$

(b) Let u_n be a sequence in $\mathcal{D}^{\beta,\overrightarrow{p}}(\mathbb{R}^N)$ which converges weakly to u_0. The compactness of K' follows from the estimate

$$\langle K'(u_n) - K'(u_0); \varphi\rangle = J,$$

where

$$J = \int_{\mathbb{R}^N} f(x)(|u_n|^{s-1}u_n - |u_0|^{s-1}u_0)\varphi\,dx.$$

The objective is to prove that $J \to 0$. On one hand, by choosing δ sufficiently small such that $\delta \le (\omega/\overline{p})s(1-\overline{p}/\overline{p}^*)$, we obtain $\overline{p} \le sx < \overline{p}^*$ with $x = 1/(s/\overline{p}^* +$

$\delta/\omega) > 1$. Then in view of (1.3) and Hölder inequality, we get the following estimate

$$J \leq \|f\|_{L^{\omega/(1-\delta)}(\mathbb{R}^N)} \||u_n|^{s-1}u_n - |u_0|^{s-1}u_0\|_{L^x(\mathbb{R}^N)} \|\varphi\|_{L^{\overline{p}^*}(\mathbb{R}^N)}.$$

On the other hand, since the imbedding $\mathcal{D}^{\beta,\overrightarrow{p}}(\mathbb{R}^N) \hookrightarrow L^{\overline{p}}(\mathbb{R}^N)$ is compact, it follows from the interpolation inequality i.e.

$$\|u\|_{L^t(\mathbb{R}^N)} \leq \|u\|^{\sigma}_{L^{\overline{p}}(\mathbb{R}^N)} \|u\|^{1-\sigma}_{L^{\overline{p}^*}(\mathbb{R}^N)}, \quad \text{for all } u \in L^{\overline{p}}(\mathbb{R}^N) \cap L^{\overline{p}^*}(\mathbb{R}^N),$$

where $1/t = \sigma/\overline{p} + (1-\sigma)/\overline{p}^*$, that the imbedding $\mathcal{D}^{\beta,\overrightarrow{p}}(\mathbb{R}^N) \hookrightarrow L^{p_1}(\mathbb{R}^N)$ is compact for $\overline{p} \leq p_1 < \overline{p}^*$. Hence, we get $J \to 0$ (strongly) as n goes to infinity, since $\overline{p} \leq sx < \overline{p}^*$. Therefore

$$K'(u_n) \to K'(u_0) \text{ strongly in } (\mathcal{D}^{\beta,\overrightarrow{p}}(\mathbb{R}^N))^*.$$

as n tends to infinity. This ends the proof of Lemma 3.8. □

Remark 3.9. In Lemma 3.8 the function f is supposed to be not in $L^\infty(\mathbb{R}^N)$, we consider more restrictions on the regularity of f for optimal values of δ.

Let us see that the assumption (1.3) gives a compact imbedding result which is used to prove that the functional I satisfies a compactness condition, that is, the Palais–Smale sequence obtained by Mountain Pass type argument converges to a weak nontrivial solution.

In order to prove that a Palais–Smale sequence converges to a solution of the problem (1.1), we need to establish the following lemma.

Lemma 3.10. *Suppose $p_{\max} < s+1$, let $(u_n)_{n=0}^\infty$ be a Palais–Smale sequence. Then $(u_n)_{n=0}^\infty$ possesses a subsequence which converges strongly in $\mathcal{D}^{\beta,\overrightarrow{p}}(\mathbb{R}^N)$.*

Proof. Let $(u_n)_{n=1}^\infty \in \mathcal{D}^{\beta,\overrightarrow{p}}(\mathbb{R}^N)$ be a Palais–Smale sequence. We have

$$\begin{aligned}
I(u_n) &- \frac{1}{p_{\max}}\langle I'(u_n); u_n\rangle \\
&= \sum_{i=1}^N \frac{s+1}{p_i} \int_{\mathbb{R}^N} \bigg(\bigg|\frac{\partial u_n}{\partial x_i}\bigg|^{p_i} + \beta(x)|u_n|^{p_i}\bigg)\,dx - \int_{\mathbb{R}^N} f(x)|u_n|^{s+1}\,dx \\
&\quad - \frac{1}{p_{\max}} \sum_{i=1}^N (s+1) \int_{\mathbb{R}^N} \bigg(\bigg|\frac{\partial u_n}{\partial x_i}\bigg|^{p_i} + \beta(x)|u_n|^{p_i}\bigg)\,dx \\
&\quad + \frac{s+1}{p_{\max}} \int_{\mathbb{R}^N} f(x)|u_n|^{s+1}\,dx \\
&= \sum_{i=1}^N (s+1)\bigg(\frac{1}{p_i} - \frac{1}{p_{\max}}\bigg) \int_{\mathbb{R}^N} \bigg(\bigg|\frac{\partial u_n}{\partial x_i}\bigg|^{p_i} + \beta(x)|u_n|^{p_i}\bigg)\,dx \\
&\quad + \bigg(\frac{s+1}{p_{\max}} - 1\bigg) \int_{\mathbb{R}^N} f(x)|u_n|^{s+1}\,dx,
\end{aligned} \tag{3.8}$$

Since $p_{\max} < s+1$, we deduce from (3.8)

$$\sum_{i=1}^{N}(s+1)\left(\frac{1}{p_i}-\frac{1}{p_{\max}}\right)\int_{\mathbb{R}^N}\left(\left|\frac{\partial u_n}{\partial x_i}\right|^{p_i}+\beta(x)|u_n|^{p_i}\right)dx \\ \leq M-\frac{1}{p_{\max}}\langle I'(u_n);u_n\rangle,$$

where M is the constant of Palais–Smale sequence. From this inequality, we easily deduce that u_n is a bounded sequence in $\mathcal{D}^{\beta,\overrightarrow{p}}(\mathbb{R}^N)$. Consequently, there exists a subsequence still denoted by u_n such that it converges weakly in $\mathcal{D}^{\beta,\overrightarrow{p}}(\mathbb{R}^N)$.

Now, we claim that u_n converges strongly in $\mathcal{D}^{\beta,\overrightarrow{p}}(\mathbb{R}^N)$. Indeed, for any pair integer (n,m) we have

$$\sum_{i=1}^{N}\int_{\mathbb{R}^N}\left(\left|\frac{\partial u_n}{\partial x_i}\right|^{p_i-2}\frac{\partial u_n}{\partial x_i}-\left|\frac{\partial u_m}{\partial x_i}\right|^{p_i-2}\frac{\partial u_m}{\partial x_i}\right)\left(\frac{\partial u_n}{\partial x_i}-\frac{\partial u_m}{\partial x_i}\right) \\ = \langle I'(u_n)-I'(u_m);(u_n-u_m)\rangle \\ + \int_{\mathbb{R}^N} f(x)((|u_n|^{s-1}u_n-|u_m|^{s-1}u_m)(u_n-u_m))\,dx.$$

By Palais–Smale condition, it is easy to see that $\langle I'(u_n)-I'(u_m);(u_n-u_m)\rangle \to 0$ as n and m tend to infinity.

From Lemma 3.8 (K' is compact), we have

$$\int_{\mathbb{R}^N} f(x)(|u_n|^{s-1}u_n-|u_m|^{s-1}u_m)(u_n-u_m)\,dx \to 0,$$

as n and m tend to infinity. Finally, in virtue of the following algebraic relation

$$|\xi_1-\xi_2|^r \leq ((|\xi_1|^{r-2}\xi_1-|\xi_2|^{r-2}\xi_2)(\xi_1-\xi_2))^{\rho/2}(|\xi_1|^r+|\xi_2|^r)^{1-\rho/2},$$

with $\rho=r$ if $1<r\leq 2$ and $\rho=2$ if $2<r$, we deduce that $(u_n)_{n=0}^{\infty}$ is a Cauchy sequence in $\mathcal{D}^{\beta,\overrightarrow{p}}(\mathbb{R}^N)$, therefore it converges strongly. This concludes the proof of Lemma 3.10. □

4. Proof of Theorem 2.3

For the proof of the existence result, we apply Mountain Pass Theorem 3.6 and local minimization to find nontrivial solutions. For that, we will study the cases when $s \notin [p_{\min}-1, p_{\max}-1]$. On the other hand, to prove our regularity result due to Proposition 4.2 below, we construct an effective iteration scheme to bound the maximal norm of the solution with it's partial derivative. First, we need the following Lemma to show that the functional I satisfies the geometric conditions of Theorem 3.6.

Lemma 4.1. *Suppose* (1.2), (1.3) *and* $p_{\max} < s+1$, *then*

(a) *There exist constants* α *and* ρ, *such that* $I(u) \geq \alpha$ *for* $\|u\| = \rho$.

(b) $I(tu) \to -\infty$ *as* $t \to \infty$.

PROOF. (a) From Theorem 3.1 and Remark 3.2, we obtain

$$I(u) \geq \sum_{i=1}^{N} \frac{s+1}{p_i} \int_{\mathbb{R}^N} \bigg(\bigg| \frac{\partial u}{\partial x_i} \bigg|^{p_i} + \beta(x)|u|^{p_i} \bigg) \, dx - C\|f\|_{L^\omega(\mathbb{R}^N)} \|u\|^{s+1}_{L^{\overline{p}^*}(\mathbb{R}^N)},$$

which implies, for small $\|u\|$, that

$$I(u) \geq \frac{s+1}{p_{\max}} \|u\|^{p_{\max}} - C'\|u\|^{s+1},$$

for some constants $C, C' > 0$. Herein, we have used that $\|u\|_{L^{\overline{p}^*}(\mathbb{R}^N)} \leq D\|u\|$ for some constant $D > 0$. Therefore, there exist α and ρ small enough positive constants such that $I(u) \geq \alpha > 0$ for all $\|u\| = \rho$.

(b) From the expression

$$I(t^{1/p_{\max}} u) = \sum_{i=1}^{N} \frac{t^{p_i/p_{\max}}(s+1)}{p_i} \int_{\mathbb{R}^N} \bigg(\bigg| \frac{\partial u}{\partial x_i} \bigg|^{p_i} + \beta(x)|u|^{p_i} \bigg) \, dx - t^{(s+1)/p_{\max}} \int_{\mathbb{R}^N} f(x)|u|^{s+1} \, dx,$$

and the fact that $p_{\max} < s+1$, we deduce that $I(t^{1/p_{\max}} u) \to -\infty$ as $t \to \infty$. □

In view of Lemmas 3.10 and 4.1, we can apply the Mountain Pass Theorem (c.f. [2]) which garantees the existence of nontrivial weak solutions of (1.1).

To prove the existence result in the case $s+1 < p_{\min}$, we may use the local minimization of the functional I. Indeed, by hypothesis (1.3), the functional I is weakly lower semi continuous differentiable. Moreover, I is bounded below. In fact, we have

$$I(u) \geq \sum_{i=1}^{N} \frac{s+1}{p_i} \int_{\mathbb{R}^N} \bigg(\bigg| \frac{\partial u}{\partial x_i} \bigg|^{p_i} + \beta(x)|u|^{p_i} \bigg) \, dx - C\|f\|_{L^\omega(\mathbb{R}^N)} \|u\|^{s+1}_{L^{\overline{p}^*}(\mathbb{R}^N)},$$

which implies that

$$I(u) \geq \frac{s+1}{p_{\max}} \|u\| - C'\|f\|_\omega \|u\|^{s+1},$$

for some constants $C, C' > 0$. This implies that I is bounded below. Thus I has a critical point u

$$I(u) = \inf\{I(v) : v \in \mathcal{D}^{\beta, \overrightarrow{p}}(\mathbb{R}^N)\},$$

which is solution of the problem (1.1). Note that u must be nontrivial since

$$I(s\varphi) = \sum_{i=1}^{N} \frac{s^{p_i}(s+1)}{p_i} \int_{\mathbb{R}^N} \bigg| \frac{\partial u}{\partial x_i} \bigg|^{p_i} dx - s^{s+1} \int_{\mathbb{R}^N} f(x)|\varphi|^{s+1} \, dx$$

for some $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$. Hence, since $s+1 < p_{\min}$, we get $I(s\varphi) < 0$ for small s.

To complete the proof of Theorem 2.3, we need the following result.

PROPOSITION 4.2. *Let u be a solution of* (1.1). *Then $u^\kappa|\partial u/\partial x_i|^{p_i} \in L^1(\mathbb{R}^N)$ for all $0 < \kappa < \infty$ and $i = 1, \dots, N$. Moreover, if $p_i = p$ for $i = 1, \dots, N$, then $u^\kappa \in L^1(\mathbb{R}^N)$ with $Np/(N-p) \le \kappa < \infty$.*

PROOF. In this proof, we may choose $u \ge 0$ since we can show that argument developed here is true for u^+ and u^- where $u^+ = \max(u, 0)$ and $u^- = \min(u, 0)$. We set $u_M(x) = \min\{u(x), M\}$, $M \in \mathbb{N}$. Observe that $(u_M)^j \in D^{\beta,\overrightarrow{p}}(\mathbb{R}^N)$ for any real $j \ge 1$. We have

$$\sum_{i=1}^N \int_{\mathbb{R}^N} \bigg(\bigg|\frac{\partial u}{\partial x_i}\bigg|^{p_i-2} \frac{\partial u}{\partial x_i}\frac{\partial \varphi}{\partial x_i} + \beta(x)|u|^{p_i-2}u\varphi \bigg)\, dx = \int_{\mathbb{R}^N} f(x)|u|^{s-1}u\varphi\, dx \tag{4.1}$$

for all $\varphi \in D^{\beta,\overrightarrow{p}}(\mathbb{R}^N)$. Inserting $\varphi = u_M^j$ into (4.1), gives

$$\sum_{i=1}^N j \int_{\mathbb{R}^N} u_M^{j-1} \bigg|\frac{\partial u}{\partial x_i}\bigg|^{p_i} dx \le \int_{\mathbb{R}^N} f(x)|u|^{s+j}\, dx, \tag{4.2}$$

for any $j \ge 1$. First, we set $j_0 = 1 + \overline{p}^*\delta/\omega$ and $t_0 = \overline{p}^*\delta/\omega$. Using Hölder's inequality with $(1-\delta)/\omega + (s+j_0)/\overline{p}^* = 1$, taking $j = j_0$ and sending $M \to \infty$ in (4.2), we get by Fatou's lemma

$$u^{t_0}\bigg|\frac{\partial u}{\partial x_i}\bigg|^{p_i} \in L^1(\mathbb{R}^N), \quad i = 1, \dots, N. \tag{4.3}$$

Second, we set $t_1 = \overline{p}^*\frac{\delta}{\omega} + p_i\overline{p}^*\frac{\delta}{\omega}$ and $j_1 = 1 + \overline{p}^*\delta/\omega + p_i\overline{p}^*\delta/\omega = j_0 + p_i\overline{p}^*\delta/\omega$ for $i = 1, \dots, N$. Observe that $(s+j_1)/\overline{p}^* + ((1-\delta)/\omega - p_i\delta/\omega) = 1$ and $f \in L^{\omega/(1-\delta(1+p_i))}(\mathbb{R}^N)$ for δ small enough and $i = 1, \dots, N$. Repeating the same argument as (4.3) to deduce

$$u^{t_1}\bigg|\frac{\partial u}{\partial x_i}\bigg|^{p_i} \in L^1(\mathbb{R}^N), \quad i = 1, \dots, N.$$

Iterating this process gives

$$u^{t_m}\bigg|\frac{\partial u}{\partial x_i}\bigg|^{p_i} \in L^1(\mathbb{R}^N), \quad i = 1, \dots, N,$$

where $t_m = \overline{p}^*(\delta/\omega)(1 + p_i + \dots + {p_i}^m)$ for $i = 1, \dots, N$. Hence, it follows

$$u^{\kappa}\bigg|\frac{\partial u}{\partial x_i}\bigg|^{p_i} \in L^1(\mathbb{R}^N), \quad i = 1, \dots, N,$$

for all $0 < \kappa < \infty$.

Now we prove the second part of Proposition 4.2. Let $p_i = p$ for $i = 1, \dots, N$. Since

$$(u_M)^{j-1}\bigg|\frac{\partial u_M}{\partial x_i}\bigg|^p = \bigg(\frac{p}{j+p-1}\bigg)^p \bigg|\frac{\partial (u_M)^{(j+p-1)/p}}{\partial x_i}\bigg|^p,$$

we deduce from Sobolev's inequality $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N)$ and (4.2)

$$\bigg(\int_{\mathbb{R}^N} (u_M)^{N(j+p-1)/(N-p)}\,dx\bigg)^{(N-p)/N} \le C\int_{\mathbb{R}^N} f(x)|u|^{s+j}\,dx, \tag{4.4}$$

for some constant $C > 0$.

We set $j_0 = 1 + p^*\delta/\omega$, $\tau_0 = N(j_0 + p - 1)/(N - p) = N(p + p^*\delta/\omega)/(N - p)$ and we let $M \to \infty$ in (4.4). The result is

$$u^{\tau_0} \in L^1(\mathbb{R}^N), \tag{4.5}$$

where we have used $j = j_0$ in (4.4) and $(1 - \delta)/\omega + (s + j_0)/p^* = 1$. Next, we set

$$j_1 = 1 + p^*\frac{\delta}{\omega} + \frac{N}{N-p}p^*\frac{\delta}{\omega} = j_0 + \frac{N}{N-p}p^*\frac{\delta}{\omega}.$$

Observe that

$$\frac{s+j_1}{p^*} + \bigg(\frac{1-\delta}{\omega} - \frac{N}{N-p}\frac{\delta}{\omega}\bigg) = 1$$

and $f \in L^{\omega/(1-\delta(1+N/(N-p))}(\mathbb{R}^N)$ for δ small enough. Taking $j = j_1$ in (4.4) and repeating the same argument as (4.5) to deduce

$$u^{\tau_1} \in L^1(\mathbb{R}^N) \quad \text{where } \tau_1 = \frac{N}{N-p}(j_1 + p - 1).$$

By iteration, we get

$$u^{\tau_m} \in L^1(\mathbb{R}^N) \quad \text{where } \tau_m = \frac{N}{N-p}(j_m + p - 1),$$

with

$$j_m = 1 + p^*\frac{\delta}{\omega} + \frac{N}{N-p}p^*\frac{\delta}{\omega} + \ldots + \bigg(\frac{N}{N-p}\bigg)^m p^*\frac{\delta}{\omega}.$$

Hence, it follows that

$$u^\kappa \in L^1(\mathbb{R}^N) \quad \text{for all } \frac{Np}{N-p} \le \kappa < \infty. \qquad \square$$

This concludes the proof of Theorem 2.3.

Remark 4.3. We mention that a similar result for (1.1) can be obtained for the boundary value problem

$$-\sum_{i=1}^N \frac{\partial}{\partial x_i}\bigg(\bigg|\frac{\partial u}{\partial x_i}\bigg|^{p_i-2}\frac{\partial u}{\partial x_i}\bigg) + \sum_{i=1}^N \beta(x)|u|^{p_i-2}u = f(x)|u|^{s-1}u \quad \text{in } \Omega$$

where Ω is an exterior domain with $\mathcal{C}^{1,\eta}$ boundary, $0 < \eta < 1$.

5. Proof of Theorem 2.7

For any $R > 0$, let $B_R = \{x \in \mathbb{R}^N : |x| < R\}$. In what follows, it is always understood that ε takes values in a sequence tending to zero. Let $(g_\varepsilon)_{0<\varepsilon<1} \subset C_c^\infty(\Omega)$ be a sequence of smooth approximations of g such that

$$\begin{cases} |g_\varepsilon| \le \dfrac{1}{\varepsilon} \quad \text{and} \quad |g_\varepsilon| \le |g|; \\ g_\varepsilon \to g \text{ in } L^1_{\mathrm{loc}}(\mathbb{R}^N) \quad \text{as } \varepsilon \to 0. \end{cases}$$

From classical results, see, e.g. [20], [18], [15], we can produce sequences

$$(u_\varepsilon)_{0<\varepsilon\le 1} \subset W_0^{1,\overrightarrow{p}}(B_{1/\varepsilon}) \cap \bigcap_{i=1}^N L^{r_i}(B_{1/\varepsilon}),$$

satisfying the weak formulation

$$\sum_{i=1}^N \int_{B_{1/\varepsilon}} \bigg(\bigg| \frac{\partial u_\varepsilon}{\partial x_i} \bigg|^{p_i - 2} \frac{\partial u_\varepsilon}{\partial x_i} \frac{\partial \varphi}{\partial x_i} + \beta(x) |u_\varepsilon|^{r_i - 1} u_\varepsilon \varphi \bigg)\, dx = \int_{B_{1/\varepsilon}} f(x) |u_\varepsilon|^{s-1} u_\varepsilon \varphi + \int_{B_{1/\varepsilon}} g_\varepsilon \varphi \, dx, \tag{5.1}$$

for all $\varphi \in W_0^{1,\overrightarrow{p}}(B_{1/\varepsilon}) \cap L^\infty(B_{1/\varepsilon})$, where $W_0^{1,\overrightarrow{p}}(B_{1/\varepsilon}) = \{u \in W^{1,\overrightarrow{p}}(B_{1/\varepsilon}) : u = 0 \text{ on } \partial B_{1/\varepsilon}\}$.

Let us indicate the main steps of the proof of Theorem 2.7: First, we prove ε-uniform local a priori estimates for u_ε, which imply almost every convergence of u_ε. Second, we prove strong L^1_{loc} convergence of the nonlinear terms in (5.1). Finally, we complete the proof of Theorem 2.7 by passing to the limit in (5.1) as $\varepsilon \to 0$.

Later we will use C, C_1, C_2, etc. to denote constants that are independent of ε.

5.1. A priori estimates. In this subsection we set $R := 1/\varepsilon$ and let ρ be any number such that $0 < \rho < R/2$.

PROPOSITION 5.1. *Assume that* (1.2), (1.6) *hold and the exponents* $p_1, \ldots,$ p_N *and* $r_1, \ldots, r_N$ *are restricted as in* (1.7). *Then, there exist a constant* C, *not depending on* ε, *such that*

$$\|u_\varepsilon\|_{L^{r_i}(B_\rho)} \le C, \quad i = 1, \ldots, N, \tag{5.2}$$

$$\|f(x) |u_\varepsilon|^{s-1} u_\varepsilon\|_{L^1(B_\rho)} \le C. \tag{5.3}$$

Moreover, for every $1 \le q_i < N(\overline{p}-1)p_i/(\overline{p}(N-1))$ *there exists a constant* C, *depending on* B_ρ, N, $p_1,\dots,p_N$, $q_1,\dots,q_N$, $\|g\|_{L^1(B_{2\rho})}$ *but not* ε, *such that*

$$\left\|\frac{\partial u_\varepsilon}{\partial x_i}\right\|_{L^{q_i}(B_{\rho'})} \le C, \quad i=1,\dots,N, \tag{5.4}$$

$$\|u_\varepsilon\|_{L^{\overline{q}}(B_{\rho'})} \le C, \quad \frac{1}{\overline{q}} := \frac{1}{N}\sum_{i=1}^N \frac{1}{q_i}, \tag{5.5}$$

for any ρ' *such that* $0<\rho'<\rho$.

Proof. The proof borrows ideas from [4], [7]. We introduce for $\gamma > 1$ the function

$$\varphi_\gamma(\sigma) = \begin{cases} (\gamma-1)\displaystyle\int_0^\sigma \frac{1}{(1+t)^\gamma}\,dt = 1 - \frac{1}{(1+\sigma)^{\gamma-1}} & \text{for } \sigma \ge 0, \\ -\varphi_\gamma(-\sigma) & \text{for } \sigma < 0, \end{cases} \tag{5.6}$$

and a smooth cut-off function $\theta = \theta(x)$ that is supported in the ball $B_{2\rho}$ such that

$$0 \le \theta \le 1 \quad (\text{recall } 0<\rho<R/2),$$
$$\theta(x) = 1 \quad \text{for } |x| \le \rho \text{ and } |\nabla\theta| \le 2/\rho.$$

Note that $|\varphi_\gamma| \le 1$ and, by assuming $\rho \ge 2$, there holds $|\nabla\theta| \le 1$.

Let $\alpha > 1$. We take $\varphi = \varphi_\gamma(u_\varepsilon)\theta^\alpha$ in (5.1), we get

$$\begin{aligned} &\int_{B_R}\sum_{i=1}^N \left|\frac{\partial u_\varepsilon}{\partial x_i}\right|^{p_i}\varphi_\gamma'(u_\varepsilon)\theta^\alpha\,dx + \int_{B_R}\sum_{i=1}^N \beta(x)|u_\varepsilon|^{r_i-1}u_\varepsilon\varphi_\gamma(u_\varepsilon)\theta^\alpha\,dx \\ &\qquad + \alpha\int_{B_R}\sum_{i=1}^N \left|\frac{\partial u_\varepsilon}{\partial x_i}\right|^{p_i-2}\frac{\partial u_\varepsilon}{\partial x_i}\frac{\partial \theta}{\partial x_i}\varphi_\gamma(u_\varepsilon)\theta^{\alpha-1}\,dx \\ &\qquad = \int_{B_R} f(x)|u_\varepsilon|^{s-1}u_\varepsilon\varphi_\gamma(u_\varepsilon)\theta^\alpha\,dx + \int_{B_R} g_\varepsilon\varphi_\gamma(u_\varepsilon)\theta^\alpha\,dx. \end{aligned} \tag{5.7}$$

Now we choose γ and α so that

$$1 < \gamma < \frac{r_i}{p_i-1}, \qquad \alpha > \frac{p_i r_i}{r_i - \gamma(p_i-1)}, \quad \text{for } i=1,\dots,N.$$

We use the definitions of θ and φ_γ along with (5.7). The result is

$$\begin{aligned} &\int_{B_R}\sum_{i=1}^N \left|\frac{\partial u_\varepsilon}{\partial x_i}\right|^{p_i}\varphi_\gamma'(u_\varepsilon)\theta^\alpha\,dx + \int_{B_R}\sum_{i=1}^N \beta(x)|u_\varepsilon|^{r_i-1}u_\varepsilon\varphi_\gamma(u_\varepsilon)\theta^\alpha\,dx \\ &\qquad \le \int_{B_R} f(x)|u_\varepsilon|^{s-1}u_\varepsilon\varphi_\gamma(u_\varepsilon)\theta^\alpha\,dx \\ &\qquad\qquad + \int_{B_{2\rho}}|g|\,dx + C_1\int_{B_R}\sum_{i=1}^N \left|\frac{\partial u_\varepsilon}{\partial x_i}\right|^{p_i-1}\theta^{\alpha-1}\,dx. \end{aligned} \tag{5.8}$$

Using Young's inequality, we estimate as follows

$$\begin{aligned}(5.9)\qquad &\left|\frac{\partial u_\varepsilon}{\partial x_i}\right|^{p_i-1}\theta^{\alpha-1}\\ &= \left|\frac{\partial u_\varepsilon}{\partial x_i}\right|^{p_i-1}(\varphi'_\gamma(u_\varepsilon))^{(p_i-1)/p_i}\theta^{\alpha(p_i-1)/p_i}(\varphi'_\gamma(u_\varepsilon))^{(1-p_i)/p_i}\theta^{(\alpha-p_i)/p_i}\\ &\le \frac{1}{2C_1}\left|\frac{\partial u_\varepsilon}{\partial x_i}\right|^{p_i}\varphi'_\gamma(u_\varepsilon)\theta^\alpha + C_2\frac{\theta^{\alpha-p_i}}{\varphi'_\gamma(u_\varepsilon)^{p_i-1}}\\ &= \frac{1}{2C_1}\left|\frac{\partial u_\varepsilon}{\partial x_i}\right|^{p_i}\varphi'_\gamma(u_\varepsilon)\theta^\alpha + C_3(1+|u_\varepsilon|)^{\gamma(p_i-1)}\theta^{\alpha-p_i}\\ &= \frac{1}{2C_1}\left|\frac{\partial u_\varepsilon}{\partial x_i}\right|^{p_i}\varphi'_\gamma(u_\varepsilon)\theta^\alpha + C_4|u_\varepsilon|^{\gamma(p_i-1)}\theta^{\alpha-p_i} + C_{12}\theta^{\alpha-p_i}.\end{aligned}$$

Similary, we can estimate the last term in (5.9):

$$\begin{aligned}C_4|u_\varepsilon|^{\gamma(p_i-1)}\theta^{\alpha-p_i} &= C_4|u_\varepsilon|^{\gamma(p_i-1)}\theta^{\alpha\gamma(p_i-1)/r_i}\theta^{\alpha(r_i-\gamma(p_i-1))/r_i-p_i}\\ &\le \frac{\varphi_\gamma(1)}{4}\beta_0|u_\varepsilon|^{r_i}\theta^\alpha + C_5\theta^{\alpha-p_ir_i/(r_i-\gamma(p_i-1))}.\end{aligned}$$

By another application of Young's inequality, we deduce

$$\begin{aligned}(5.10)\qquad \int_{B_R} f(x)|u_\varepsilon|^{s-1}u_\varepsilon\varphi_\gamma(u_\varepsilon)\theta^\alpha\,dx &\le C(\delta)\int_{B_R}(f(x))^{w/(1-\delta)}\varphi_\gamma(u_\varepsilon)\theta^\alpha\,dx\\ &\quad + C(\delta)\int_{B_R}|u_\varepsilon|^{\sigma/(s+\delta(\sigma-s))}\varphi_\gamma(u_\varepsilon)\theta^\alpha\,dx,\end{aligned}$$

where w is defined in (1.6). Now we fixed arbitrary $k = 1,\dots,N$. Observe that

$$\begin{aligned}&|t|^{s-1}t\varphi_\gamma(t) \ge |t|^s\varphi_\gamma(1), \qquad \text{for } t \ge 1 \text{ and a.e. } x \in \mathbb{R}^N,\\ &\frac{\sigma}{s+\delta(\sigma-s)} = p_{\min} < r_k, \quad k = 1,\dots,N.\end{aligned}$$

Using Young's inequality and (1.7), we deduce from (5.10)

$$\begin{aligned}(5.11)\qquad \int_{B_R} f(x)|u_\varepsilon|^{s-1}u_\varepsilon\varphi_\gamma(u_\varepsilon)\theta^\alpha\,dx &\le C_6 + \frac{\beta_0}{4}\int_{B_R}|u_\varepsilon|^{r_k}\varphi_\gamma(u_\varepsilon)\theta^\alpha\,dx\\ &\le C_6 + \frac{\beta_0}{4}\int_{B_R}\sum_{i=1}^N|u_\varepsilon|^{r_i}\varphi_\gamma(u_\varepsilon)\theta^\alpha\,dx.\end{aligned}$$

Summarizing from (5.8) we get

$$\begin{aligned}(5.12)\qquad \frac{1}{2}\int_{B_R}\sum_{i=1}^N\left|\frac{\partial u_\varepsilon}{\partial x_i}\right|^{p_i}\varphi'_\gamma(u_\varepsilon)\theta^\alpha\,dx + \frac{\varphi_\gamma(1)}{2}\beta_0\int_{B_R}\sum_{i=1}^N|u_\varepsilon|^{r_i}\theta^\alpha\,dx&\\ \le \int_{B_{2\rho}}|g|\,dx + C_7\mathrm{meas}(B_{2\rho}).&\end{aligned}$$

We then exploit the definitions of φ_γ and θ to obtain from (5.12) that

$$\int_{B_\rho} |u_\varepsilon|^{r_i}\,dx \le C_8, \quad i = 1,\dots,N, \tag{5.13}$$

which proves (5.2) and, via (5.11), also (5.3). Moreover, it follows that

$$\sum_{i=1}^{N} \int_{B_\rho} \frac{|\partial u_\varepsilon/\partial x_i|^{p_i}}{(1+|u_\varepsilon|)^\gamma}\,dx \le C_9. \tag{5.14}$$

We let now $0 < \rho' < \rho$. We cover $\overline{B}_{\rho'}$ with a finite number of cubes well contained in B_ρ with edges parallel to the coordinate axes, and let Q be any of them. From (5.13) and (5.14) we deduce

$$\int_{Q} |u_\varepsilon|^{r_i}\,dx \le C_{10}, \quad i = 1,\dots,N, \tag{5.15}$$

and

$$\sum_{i=1}^{N} \int_{Q} \frac{|\partial u_\varepsilon/\partial x_i|^{p_i}}{(1+|u_\varepsilon|)^\gamma}\,dx \le C_{11}. \tag{5.16}$$

Finally, we remark that the estimates (5.4) and (5.5) are direct consequences of (5.15), (5.16) and Lemma 3.4. □

5.2. Strong convergence. In this section, we let

$$q_{\min} := \min_{1\le l\le N} q_i, \tag{5.17}$$

where $q_1,\dots,q_N$ are restricted as in Proposition 5.1. We will denote $B_{\rho'}$ by B_ρ. Given any $\rho > 0$, let ε be such that $1/\varepsilon > 2\rho$. In view of Proposition 5.1, u_ε is uniformly (in ε) bounded in $W^{1,q_{\min}}(B_\rho)$. Without loss of generality, we can therefore assume that

$$\begin{cases} u_\varepsilon \to u & \text{strongly in } L^{q_{\min}}(B_\rho) \\ & \text{and a.e. in } B_\rho, \\ f(x)|u_\varepsilon|^{s-1}u_\varepsilon \to f(x)|u|^{s-1}u & \text{a.e. in } B_\rho, \\ \beta(x)|u_\varepsilon|^{r_i-2}u_\varepsilon \to \beta(x)|u|^{r_i-2}u & \text{a.e. in } B_\rho, \end{cases} \tag{5.18}$$

for $i = 1,\dots,N$. By a standard diagonal process, we can in fact assume that

$$\begin{cases} u_\varepsilon \to u & \text{in } L^1_{\rm loc}(\mathbb{R}^N) \text{ and a.e. in } \mathbb{R}^N, \\ u_\varepsilon \to u & \text{weakly in } W^{1,q_{\min}}_{\rm loc}(\mathbb{R}^N), \\ \text{and} & \\ f(x)|u_\varepsilon|^{s-1}u_\varepsilon \to f(x)|u|^{s-1}u & \text{a.e. in } \mathbb{R}^N, \\ \beta(x)|u_\varepsilon|^{r_i-1}u_\varepsilon \to \beta(x)|u|^{r_i-1}u & \text{a.e. in } \mathbb{R}^N, \end{cases}$$

for $i = 1, \dots, N$. Now, we are interested in the strong convergence in $L^1(B_\rho)$ of the sequences $(f(x)|u_\varepsilon|^{s-1}u_\varepsilon)_{0<\varepsilon\le 1}$, $\big(\sum_{i=1}^N \beta(x)|u_\varepsilon|^{r_i-2}u_\varepsilon\big)_{0<\varepsilon\le 1}$ to respectively $f(x)|u|^{s-1}u$, $\sum_{i=1}^N \beta(x)|u|^{r_i-2}u$ for $i = 1, \dots, N$.

PROPOSITION 5.2. *Assume* (1.2) *and* (1.6) *hold and that the corresponding exponents* $p_1, \dots, p_N$ *and* $r_1, \dots, r_N$ *are restricted as in* (1.7). *Then the sequences* $\big(\sum_{i=1}^N \beta(x)|u_\varepsilon|^{r_i-1}u_\varepsilon\big)_{0<\varepsilon\le 1}$ *and* $(f(x)|u_\varepsilon|^{s-1}u_\varepsilon)_{0<\varepsilon\le 1}$ *converge to respectively* $\sum_{i=1}^N \beta(x)|u|^{r_i-1}u$ *and* $f(x)|u|^{s-1}u$ *almost everywhere in* $\mathbb{R}^N$ *and strongly in* $L^1(B_\rho)$ *for any* $\rho > 0$.

PROOF. By exploiting (1.6), Young inequality and the convergence proof just given, we deduce easily that $f(x)|u_\varepsilon|^{s-1}u_\varepsilon$ converges to $f(x)|u|^{s-1}u$ almost everywhere in $\mathbb{R}^N$ and strongly in $L^1(B_\rho)$ for any $\rho > 0$. In view of (5.18) and a theorem of Vitali (see, e.g. [11]), it is sufficient to establish the equi-integrability of $\big(\sum_{i=1}^N \beta(x)|u_\varepsilon|^{r_i-1}u_\varepsilon\big)_{0<\varepsilon\le 1}$ on B_ρ. To this end, we follow [4], [7] and introduce for $\gamma, l > 1$ the test function $\varphi_{\gamma,l}$ defined by

$$\varphi_{\gamma,l}(t) = \begin{cases} \varphi_\gamma(t-l) & \text{if } t \ge l, \\ 0 & \text{if } |t| < l, \\ -\varphi_{\gamma,l}(-t) & \text{if } t \le -l, \end{cases} \tag{5.19}$$

where φ_γ is defined in (5.6). Let $\alpha > 1$. Inserting $\varphi = \varphi_{\gamma,l}(u_\varepsilon)\theta^\alpha$ into (5.1) and proceeding more or less as we did up to (5.12), we deduce

$$\begin{aligned} \frac{1}{2}\int_{B_R} \sum_{i=1}^N \left|\frac{\partial u_\varepsilon}{\partial x_i}\right|^{p_i} \varphi'_{\gamma,l}(u_\varepsilon)\theta^\alpha\,dx + \frac{1}{2}\int_{B_R} \sum_{i=1}^N \beta(x)|u_\varepsilon|^{r_i-1}u_\varepsilon\varphi_{\gamma,l}(u_\varepsilon)\theta^\alpha\,dx \\ \le \int_{B_{2\rho}\cap\{|u_\varepsilon|\ge l\}} |g|\,dx + C_1 \text{meas}(B_{2\rho}\cap\{|u_\varepsilon|\ge l\}). \end{aligned} \tag{5.20}$$

Next, since $g \in L^1(B_{2\rho})$ and u_ε is bounded in $L^1(B_{2\rho})$ uniformly with respect to ε,

$$\int_{B_{2\rho}\cap\{|u_\varepsilon|\ge l\}} |g|\,dx + \text{meas}(B_{2\rho}\cap\{|u_\varepsilon|\ge l\}) \to 0, \quad \text{as } l \to \infty. \tag{5.21}$$

We finally obtain from (5.19)–(5.21)

$$\begin{aligned} &\int_{B_\rho\cap\{|u_\varepsilon|\ge l+1\}} \left|\sum_{i=1}^N \beta(x)|u_\varepsilon|^{r_i-1}u_\varepsilon\right| dx \\ &\qquad \le C\int_{B_R} \sum_{i=1}^N \beta(x)|u_\varepsilon|^{r_i-1}u_\varepsilon\varphi_{\gamma,l}(u_\varepsilon)\theta^\alpha\,dx \overset{l\to\infty}{\longrightarrow} 0 \quad (\text{uniformly in } \varepsilon). \end{aligned}$$

This implies the desired equi-integrability of $\big(\sum_{i=1}^N \beta(x)|u_\varepsilon|^{r_i-1}u_\varepsilon\big)_{0<\varepsilon\le 1}$. □

Proposition 5.3. *Assume* (1.2) *and* (1.6) *hold and that the corresponding exponents* $p_1,\dots,p_N$ *and* $r_1,\dots,r_N$ *are restricted as in* (1.7). *Then the sequence*

$$\bigg(\sum_{i=1}^N\bigg|\frac{\partial u_\varepsilon}{\partial x_i}\bigg|^{p_i-2}\frac{\partial u_\varepsilon}{\partial x_i}\bigg)_{0<\varepsilon\le 1}$$

converges to

$$\sum_{i=1}^N\bigg|\frac{\partial u}{\partial x_i}\bigg|^{p_i-2}\frac{\partial u}{\partial x_i}\quad a.e.\ in\ \mathbb{R}^N$$

and strongly in $L^1(B_\rho)$ *for any* $\rho>0$.

Proof. The proof of Proposition 5.3 is more or less similar to the proof found in [4], but we conclude it for the convenience of the reader. It suffices to show that $(\nabla u_\varepsilon)_{0<\varepsilon\le 1}$ is a Cauchy sequence in measure on B_ρ, i.e. for any $\mu>0$,

$$\text{meas}(\{x\in B_\rho:|(\nabla u_{\varepsilon'}-\nabla u_\varepsilon)(x)|\ge\mu\})\to 0,\quad \text{as } \varepsilon,\varepsilon'\to 0.$$

For any $\gamma,\lambda>0$, we have

$$\{x\in B_\rho:|(\nabla u_{\varepsilon'}-\nabla u_\varepsilon)(x)|\ge\mu\}\subset L_1\cup L_2\cup L_3\cup L_4,$$

where $L_1=\{x\in B_\rho:|\nabla u_\varepsilon(x)|\ge\gamma\}$, $L_2=\{x\in B_\rho:|\nabla u_{\varepsilon'}(x)|\ge\gamma\}$,

$$L_3=\{x\in B_\rho:|(u_\varepsilon-u_{\varepsilon'})(x)|\ge\lambda\},$$

and

$$\begin{aligned}L_4=\{x\in B_\rho:|(\nabla u_\varepsilon-\nabla u_{\varepsilon'})(x)|\ge\mu,\ |\nabla u_\varepsilon(x)|\le\gamma,\\ |\nabla u_{\varepsilon'}(x)|\le\gamma,\ |(u_\varepsilon-u_{\varepsilon'})(x)|\le\lambda\}.\end{aligned}$$

In view of Proposition 5.1, by choosing γ large we can make meas(L_1) and meas(L_2) arbitrarily small. Since $(u_\varepsilon)_{0<\varepsilon\le 1}$ is a Cauchy sequence in $L^1(B_\rho)$, then, for $\lambda>0$ fixed, meas(L_3) tends to 0 as $\varepsilon,\varepsilon'\to 0$. It remains to control meas(L_4). Since the set of (ξ_1,ξ_2) such that $|\xi_1|\le\gamma$, $|\xi_2|\le\gamma$, and $|\xi_1-\xi_2|\le\mu$ is a compact set and $\xi\mapsto A(x,\xi)$ is continuous for almost every $x\in B_\rho$, the quantity

$$\sum_{i=1}^N[|\xi_1|^{p_i-2}\xi_1-|\xi_2|^{p_i-2}\xi_2][\xi_1-\xi_2]$$

reaches its minimum value on this compact set, and we will denote it by $q(x)$. It is not hard to verify that $q(x)>0$ almost everywhere in B_ρ. Consequently, for any $\eta>0$ there exists $\eta'>0$ such that

$$\int_{L_4}q(x)\,dx<\eta'\Rightarrow\text{meas}(L_4)\le\eta. \tag{5.22}$$

Hence, it is sufficient to show that for any given $\beta' > 0$, one can produce a small enough $\lambda > 0$ such that

$$\int_{L_4} q(x)\,dx < \beta'. \tag{5.23}$$

For any $\lambda > 0$, define $T_\lambda(z) = \min(\lambda, \max(z, -\lambda))$. Note that T_λ is a Lipschitz continuous function satisfying $0 \le |T_\lambda(z)| \le \lambda$. By the definitions of $q(x)$ and L_4, we have

$$\begin{aligned}\int_{L_4} q(x)\,dx \le \int_{L_4} \sum_{i=1}^N &\left[\left|\frac{\partial u_\varepsilon}{\partial x_i}\right|^{p_i-2}\frac{\partial u_\varepsilon}{\partial x_i} - \left|\frac{\partial u_{\varepsilon'}}{\partial x_i}\right|^{p_i-2}\frac{\partial u_{\varepsilon'}}{\partial x_i}\right] \\ &\times \left[\frac{\partial u_\varepsilon}{\partial x_i} - \frac{\partial u_{\varepsilon'}}{\partial x_i}\right] \times \mathbf{1}_{\{|u_\varepsilon - u_{\varepsilon'}| \le \lambda\}}\,dx \\ = \int_{L_4} \sum_{i=1}^N &\left[\left|\frac{\partial u_\varepsilon}{\partial x_i}\right|^{p_i-2}\frac{\partial u_\varepsilon}{\partial x_i} - \left|\frac{\partial u_{\varepsilon'}}{\partial x_i}\right|^{p_i-2}\frac{\partial u_{\varepsilon'}}{\partial x_i}\right]\frac{\partial T_\lambda(u_\varepsilon - u_{\varepsilon'})}{\partial_{x_i}}\,dx.\end{aligned} \tag{5.24}$$

Let θ be the cut-off function used in the proof of Proposition 5.1 and let $q_{\min}$ be the number defined in (5.17). Thanks to Proposition 5.1, we can find a $q \in [p_{\min} - 1, q_{\min})$ such that $\|\partial u_\varepsilon/\partial x_i\|_{L^q(B_{2\rho})}$ is bounded independently of ε for all $i = 1, \dots, N$. Specifying $T_\lambda(u_\varepsilon - u_{\varepsilon'})\theta$ as test function in the weak formulations for u_ε and $u_{\varepsilon'}$ and then subtracting the results, we find

$$\begin{aligned}&\int_{B_\rho} \sum_{i=1}^N \left[\left|\frac{\partial u_\varepsilon}{\partial x_i}\right|^{p_i-2}\frac{\partial u_\varepsilon}{\partial x_i} - \left|\frac{\partial u_{\varepsilon'}}{\partial x_i}\right|^{p_i-2}\frac{\partial u_{\varepsilon'}}{\partial x_i}\right]\frac{\partial T_\lambda(u_\varepsilon - u_{\varepsilon'})}{\partial_{x_i}}\,dx \\ &\le 2\lambda\left[C_1 + C_2\int_{B_{2\rho}}\sum_{i=1}^N \left|\frac{\partial u_\varepsilon}{\partial x_i}\right|^{p_i-1}dx + C_3\|u_\varepsilon\|_{L^s(B_{2\rho})} + \|g\|_{L^1(B_{2\rho})}\right] \\ &\le 2\lambda\left[C_1 + C_4\int_{B_{2\rho}}\left|\frac{\partial u_\varepsilon}{\partial x_i}\right|^{q}dx + C_3\|u_\varepsilon\|_{L^s(B_{2\rho})} + \|g\|_{L^1(B_{2\rho})}\right] \xrightarrow{\lambda\to 0} 0\end{aligned} \tag{5.25}$$

(uniformly in ε and ε'). For λ small enough, we have from (5.24) and (5.25) that (5.23) holds, and, by (5.22), also that $\operatorname{meas}(L_4) \le \beta$. Thus, we have the convergence of $(\nabla u_\varepsilon)_{0<\varepsilon\le 1}$ to ∇u in measure. Then we can finally conclude that along a subsequence

$$\sum_{i=1}^N \left|\frac{\partial u_\varepsilon}{\partial x_i}\right|^{p_i-2}\frac{\partial u_\varepsilon}{\partial x_i} \to \sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i} \quad \text{strongly in } L^1(B_\rho). \qquad \square$$

In view of the previous results, we can indeed send $\varepsilon \to 0$ in the weak formulation (5.1) with $\varphi \in C_c^1(\mathbb{R}^N)$, thereby obtaining the existence of a distributional solution (in the sense of Definition 2.5) to (1.5). The L^∞_{loc}-bound for u_ε is proved by replacing $\overline{q}^*$ in the proof Lemma 3.4 by any number $r \in [1, \infty)$ and using (3.1).

Remark 5.4. The existence result obtained in the present section also applies to the following Dirichlet problem on a open bounded domain in $\mathbb{R}^N$

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\bigg(\bigg|\frac{\partial u}{\partial x_i}\bigg|^{p_i-2}\frac{\partial u}{\partial x_i}\bigg) \\ \qquad\qquad + \displaystyle\sum_{i=1}^{N} \beta(x)|u|^{r_i-1}u = f(x)|u|^{s-1}u + g(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

where β, f satisfy the conditions stated in (1.2) and (1.6).

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