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ON SOME RESONANT BOUNDARY VALUE PROBLEM ON AN INFINITE INTERVAL

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Abstract. The existence of at least one solution to a nonlinear second order differential equation on the half-line with the boundary conditions $x'(0) = 0$ and with the first derivative vanishing at infinity is proved.

1. Introduction

In the paper the following asymptotic boundary value problem

$$x'' = f(t, x, x'), \quad x'(0) = 0, \quad \lim_{t\to\infty} x'(t) = 0, \tag{1.1}$$

where $f\colon \mathbb{R}_+ \times \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k$ is continuous and satisfies the appropriate growth conditions, is studied. Observe that the corresponding homogeneous linear problem, i.e.

$$x'' = 0, \quad x'(0) = 0, \quad \lim_{t\to\infty} x'(t) = 0,$$

has nontrivial constant solutions; hence we deal with a resonant situation.

The problem (1.1) has been already studied in [13]. In that paper, we have obtained the existence result in a completely different way than by using standard methods for resonant problems (by standard methods we mean methods considered, for instance, in the following papers: [1]–[4], [7], [10]–[12]). The

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method used in [13] enabled us to get existence under weak assumptions: a linear growth condition and a sign condition for the nonlinear term f. Similar assumptions appear also for other boundary value problems.

2. Preliminaries

First, we shall introduce notation and terminology.

By a space we mean a metric space. Given a space X with a metric d, a set $A \subset X$ and $\varepsilon > 0$, $B(A,\varepsilon) := \{x \in X \mid d_A(x) := \inf_{a\in A} d(x,a) < \varepsilon\}$ denotes the open ε-neighbourhood of A. Recall that a space X is an absolute neighbourhood retract (we write $X \in$ ANR) if, given a space Y and a homeomorphic embedding $i: X \to Y$ of X onto a closed subset $i(X) \subset Y$, $i(X)$ is a neighbourhood retract of Y, i.e. there is an open neighbourhood U of $i(X)$ in Y and a retraction $r: U \to i(X)$ (a map $r: U \to i(X)$ is a retraction provided that $r(y) = y$ for $y \in i(X)$).

We shall say that a nonempty space X is contractible provided there exist $x_0 \in X$ and a homotopy $h: X \times [0,1] \to X$ such that $h(x,0) = x$ and $h(x,1) = x_0$ for every $x \in X$.

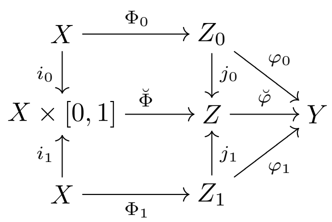
A compact (nonempty) space X is an R_δ-set (we write $X \in R_\delta$) if there is a decreasing sequence X_n of compact contractible spaces such that $X = \bigcap_{n\geq 1} X_n$.

Let X, Y be spaces. A set-valued map $\Phi: X \multimap Y$ is upper semicontinuous (written u.s.c.) if, given an open $V \subset Y$, the set $\{x \in X \mid \Phi(x) \subset V\}$ is open. We say that $\Phi: X \multimap Y$ is an R_δ-map if it is u.s.c. and, for each $x \in X$, $\Phi(x) \in R_\delta$.

By a decomposable map we mean a pair (D,F) consisting of a set-valued map $F: X \multimap Y$ and a diagram $D: X \overset{\Phi}{\multimap} Z \overset{\varphi}{\longrightarrow} Y$, where $Z \in$ ANR, $\Phi: X \multimap Z$ is an R_δ-map, and $\varphi: Z \to Y$ a single-valued continuous map, such that $F = \varphi \circ \Phi$.

A superposition of a set-valued map with compact values and a continuous function is an u.s.c. map, so any decomposable map is u.s.c.

We say the two decomposable maps (D_0,F_0), (D_1,F_1) where $D_k: X \overset{\Phi_k}{\multimap} Z_k \overset{\varphi_k}{\longrightarrow} Y$, $k = 0,1$ are homotopic (we write $(D_0,F_0) \simeq (D_1,F_1)$) if there is a decomposable map $(\breve{D},\breve{F})$ with $\breve{D}: X \times [0,1] \overset{\breve{\Phi}}{\multimap} Z \overset{\breve{\varphi}}{\longrightarrow} Y$ and maps $j_k: Z_k \to Z$, $k = 0,1$ such that the diagram



where $i_k(x) = (x,k)$ for $x \in X$, $k = 0,1$, is commutative.

Theorem 2.1 ([8, p. 1797]). *If a decomposable map* $(D,F)\colon X \multimap X$, *where* X *is a compact* ANR *and is homotopic to identity* id_X, *i.e. there is a decomposable map* $(D',F')\colon X \multimap X$ *such that* $(D,F)\simeq(D',F')$ *and* $F'(x)=x$ *for* $x\in X$, *then*

$$\Lambda(D,F)=\lambda(\mathrm{id}_X)=\chi(X).$$

Hence, if $\chi(X)\neq 0$, *then* $\mathrm{Fix}(F)\neq\emptyset$.

The following simple corollary will be of crucial importance.

Corollary 2.2. *Let* Q *be a compact polyhedron with nontrivial Euler characteristic* $\chi(Q)\neq 0$. *If a decomposable map* $(D,F)\colon Q\multimap Q$ *is homotopic to identity, then* $\mathrm{Fix}(F)\neq\emptyset$.

Now, we shall present a result about the topological structure of the set of solutions of some nonlinear functional equation.

Theorem 2.3 ([6, p. 159]). *Let* X *be a space,* $(E,\|\cdot\|)$ *a Banach space and* $h\colon X\to E$ *a proper map, i.e.* h *is continuous and for every compact* $K\subset E$ *the set* $h^{-1}(K)$ *is compact. Assume further that for each* $\varepsilon>0$ *a proper map* $h_\varepsilon\colon X\to E$ *is given and the following two conditions are satisfied:*

(a) $\|h_\varepsilon(x)-h(x)\|<\varepsilon$, *for every* $x\in X$;

(b) *for any* $\varepsilon>0$ *and* $u\in E$ *such that* $\|u\|\le\varepsilon$, *the equation* $h_\varepsilon(x)=u$ *has exactly one solution.*

Then the set $S=h^{-1}(0)$ *is* R_δ.

Denote by $\mathrm{BC}(\mathbb{R}_+,\mathbb{R}^k)$ (we write BC) the Banach space of continuous and bounded functions with supremum norm and by $\mathrm{BCL}(\mathbb{R}_+,\mathbb{R}^k)$ (we write BCL) its closed subspace of continuous and bounded functions which have finite limits at $+\infty$.

The following theorem gives a sufficient condition for compactness in the space BC and, by the definition, in the space BCL as well.

Theorem 2.4 ([9]). *If* $B\subset\mathrm{BC}$ *satisfies following conditions:*

(a) *there exists* $L>0$, *that for every* $x\in B$ *and* $t\in[0,\infty)$ *we have* $|x(t)|\le L$,

(b) *for each* $t_0\ge 0$, *the family* B *is equicontinuous at* t_0,

(c) *for any* $\varepsilon>0$ *there exist* $T>0$ *and* $\delta>0$ *such that if* $|x(T)-y(T)|\le\delta$, *then* $|x(t)-y(t)|\le\varepsilon$ *for* $t\ge T$ *and all* $x,y\in B$.

Then B *is relatively compact in* BC.

3. The main result

Let us consider an asymptotic BVP

$$x''=f(t,x,x'),\quad x'(0)=0,\quad \lim_{t\to\infty}x'(t)=0, \tag{3.1}$$

where $f\colon \mathbb{R}_+ \times \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k$ is continuous.

The following assumptions will be needed throughout the paper:

- (i) $|f(t,x,y)| \le a(t)|y| + b(t)$, where $\int_0^\infty a(s)\,ds < \infty$, $\int_0^\infty b(s)\,ds < \infty$;
- (ii) there exists $M > 0$ such that $x_i f_i(t,x,y) > 0$ for $t \ge 0$, $y \in \mathbb{R}^k$, $x \in \mathbb{R}^k$ and $|x_i| \ge M$, $i = 1, \dots, k$.

Definition 3.1. A function $x\colon \mathbb{R}_+ \to \mathbb{R}^k$ is called a solution of (3.1) if the following holds:

- (a) $x \in C^2(\mathbb{R}_+, \mathbb{R}^k)$;
- (b) $x''(t) = f(t, x(t), x'(t))$ for every $t \in \mathbb{R}_+$;
- (c) $x'(0) = 0$, $\lim_{t\to\infty} x'(t) = 0$.

Now, we can formulate our main result.

Theorem 3.2. *Under assumptions* (i) *and* (ii), *problem* (3.1) *has at least one solution.*

The proof will be divided into a sequence of lemmas.

Given $c \in \mathbb{R}^k$ and $x \in \mathrm{BCL}$ let

$$A(c,x)(t) = \int_0^t f\bigg(s, c + \int_0^s x(u)\,du, x(s)\bigg)\,ds, \quad t \ge 0.$$

It is clear that $A(c,x)\colon [0,\infty) \to \mathbb{R}^k$ is continuous. For $t \ge 0$,

$$|A(c,x)(t)| \le \int_0^t (a(s)|y_c(s)| + b(s))\,ds \le M_1 \|x\|_{\mathrm{BC}} + M_2,$$

where

$$M_1 := \int_0^\infty a(s)\,ds, \quad M_2 := \int_0^\infty b(s)\,ds.$$

Hence

$$\|A(c,x)(t)\|_{\mathrm{BC}} \le M_1\, \|x\|_{\mathrm{BC}} + M_2. \tag{3.2}$$

Therefore $A(c,x) \in \mathrm{BC}$.

Moreover, observe that the function $[0,\infty) \ni t \mapsto f(t, c + \int_0^t y_c(u)\,du, y_c(t))$ is integrable. Hence, in particular, $\lim_{t\to\infty} A(c,x)(t)$ exists, i.e. $A(c,x) \in \mathrm{BCL}$. It follows that the operator $A\colon \mathbb{R}^k \times \mathrm{BCL} \to \mathrm{BCL}$ is well-defined.

Lemma 3.3. *Under assumption* (i) *the operator* $A\colon \mathbb{R}^k \times \mathrm{BCL} \to \mathrm{BCL}$ *is completely continuous.*

Proof. The continuity of A is an easy consequence of the Lebesque Dominated Convergence Theorem. It order to prove the complete continuity let us consider the set $B := \{y = A(c,x) \mid c \in \mathbb{R}^k,\ \|x\| \le R\}$, where $R > 0$. We shall see that B is relatively compact in BCL. To this reason we use Theorem 2.4.

First observe that B is bonded (see (3.2)): for any $y \in B$,

$$\|y\|_{\mathrm{BC}} \leq M_1 R + M_2.$$

Hence the condition (a) of the Theorem 2.4 holds true.

We shall now show that the family B is equicontinuos, i.e. given $t_0 \geq 0$ and $\varepsilon > 0$, there is $\delta > 0$ such that if $t \geq 0$ and $|t - t_0| < \delta$, then $|y(t) - y(t_0)| < \varepsilon$ for any $c \in \mathbb{R}^k$ and $y \in B$. Let us choose an arbitrary $\varepsilon > 0$. By (i) , there exist $\delta_1, \delta_2 > 0$ such that

$$\begin{aligned}
&\text{if} \quad |t - t_0| < \delta_1, \quad \text{then} \quad \int_{\min\{t_0,t\}}^{\max\{t_0,t\}} a(s)\,ds < \frac{\varepsilon}{2R},\\
&\text{if} \quad |t - t_0| < \delta_2, \quad \text{then} \quad \int_{\min\{t_0,t\}}^{\max\{t_0,t\}} b(s)\,ds < \frac{\varepsilon}{2}.
\end{aligned}$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then, for $|t - t_0| < \delta$, we get

$$\begin{aligned}
|y(t) - y(t_0)| &\leq \int_{\min\{t_0,t\}}^{\max\{t_0,t\}} \left| f\Big(s, c + \int_0^s x(u)\,du, x(s)\Big) \right| ds\\
&\leq R \int_{\min\{t_0,t\}}^{\max\{t_0,t\}} a(s)\,ds + \int_{\min\{t_0,t\}}^{\max\{t_0,t\}} b(s)\,ds < R\tfrac{\varepsilon}{2R} + \tfrac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

It remains to prove condition (c) of Theorem 2.4, i.e. we shall show that given $\varepsilon > 0$, there are $T > 0$ and $\delta > 0$ such that for any $y, z \in B$ if $|y(T) - z(T)| < \delta$, then $|y(t) - z(t)| < \varepsilon$ for any $t \geq T$. There is $T > 0$ such that

$$\int_T^\infty a(s)\,ds < \frac{\varepsilon}{6R}, \quad \int_T^\infty b(s)\,ds < \frac{\varepsilon}{6}.$$

Let $\delta := \varepsilon/3$. If $|y(T) - z(T)| \leq \delta$, then for $t \geq T$ we get

$$\begin{aligned}
|y(t) - z(t)| &\leq |y(T) - z(T)| + 2R \int_T^\infty a(s)\,ds + 2 \int_T^\infty b(s)\,ds\\
&\leq \frac{\varepsilon}{3} + 2R\frac{\varepsilon}{6R} + 2\frac{\varepsilon}{6} = \varepsilon,
\end{aligned}$$

and the proof is complete. □

Given $c \in \mathbb{R}^k$, let $x \in \mathrm{BCL}$ and $x = \lambda A(c, x)$ for some $\lambda \in [0, 1]$. Then

$$x(t) = \lambda \int_0^t f\left(s, c + \int_0^s x(u)\,du, x(s)\right) ds.$$

The Gronwall inequality implies that

$$|x(t)| \leq M_2 e^{M_1}. \tag{3.3}$$

Therefore, the Leray–Schauder Alternative implies that for each $c \in \mathbb{R}^k$ the set $\mathrm{Fix}(A(c, \cdot))$ of fixed points of $A(c, \cdot)\colon \mathrm{BCL} \to \mathrm{BCL}$ is nonempty.

Lemma 3.4. *Let assumption* (i) *hold and let* $\Phi: \mathbb{R}^k \multimap \mathrm{BCL}$ *be given by* $\Phi(c) := \mathrm{Fix}(A(c, \cdot\,))$. *The set-valued map* Φ *is upper semicontinuous with compact values.*

Proof. The set-valued map Φ is upper semicontinuous with compact values if given a sequence (c_n) in $\mathbb{R}^k$, $c_n \to c_0$ and $(x_n) \in \Phi(c_n)$, (x_n) has a converging subsequence to some $x_0 \in \Phi(x_0)$. Taking any sequence (c_n), $c_n \to c_0$ and $(x_n) \in \Phi(c_n)$ we have

$$x_n = A(c_n, x_n). \tag{3.4}$$

By (3.3), we get that the fixed points of $A(c, \cdot\,)$ are equibounded for any c. Hence both sequences (x_n) and (c_n) are bounded. Lemma 3.3 yields that the operator A is completely continuous. Then, by (3.4), (x_n) is relatively compact. Hence, passing to a subsequence if necessary, we may assume that $x_n \to x_0$ in BCL. The continuity of A implies that $x_0 = A(c_0, x_0)$. Hence, $x_0 \in \Phi(c_0)$ and the proof is complete. □

Lemma 3.5. *If assumption* (i) *holds, then* Φ *is an* R_δ*-map.*

Proof. Since the map Φ is u.s.c., it remains to show that for any $c \in \mathbb{R}^k$ the set $\Phi(c)$ is R_δ. Let $X = \{x \in \mathrm{BCL} \mid \|x\| \le L\}$, where $L := M_2 e^{M_1}$ is taken from (3.3). We will show that if $A(c, \cdot\,): X \to \mathrm{BCL}$ is a compact map (it is easy to see that A_c is compact) and $h: X \to \mathrm{BCL}$ is a compact vector field associated with $A(c, \cdot\,)$, i.e. $h(x) = x - A(c, x)$, then there exists a sequence $h_n: X \to \mathrm{BCL}$ of continuous proper mappings satisfying conditions (a) and (b) of Theorem 2.3 with respect to h.

First, notice that $A(c, x) = 0$ for every $x \in X$. Moreover, for every $T \in (0, \infty)$ and for every $x, y \in \mathrm{BCL}$, if $x(t) = y(t)$ for each $t \in [0, T]$, then $A(c, x)(t) = A(c, y)(t)$ for each $t \in [0, T]$.

For the proof it is sufficient to define a sequence $A^n(c, \cdot): X \to \mathrm{BCL}$ of compact maps such that $A(c, x) = \lim_{n\to\infty} A^n(c, x)$ uniformly in X and show that $h_n(x) = x - A^n(c, x)$ is a one-to-one map. To do this we define auxiliary mappings $r_n: \mathbb{R}_+ \to \mathbb{R}_+$ by

$$r_n(t) := \begin{cases} 0 & \text{for } t \in [0, 1/n], \\ t - 1/n & \text{for } t \in (1/n, \infty). \end{cases}$$

Now we are able to define the sequence $(A^n(c, \cdot\,))$ as follows

$$A^n(c, x) = A(c, x)(r_n(t)), \quad \text{for } x \in X,\ n \in \mathbb{N}. \tag{3.5}$$

It is easy to see that A_c^n are continuous and compact. Since $|r_n(t) - t| \le 1/n$, we deduce from the compactness of $A(c, \cdot\,)$ and (3.5) that $A^n(c, x) \to A(c, x)$ uniformly in X.

Now, we shall prove that h_n is a one-to-one map. Assume that fore some $x, y \in X$ we have $h_n(x) = h_n(y)$. This implies that

$$x - y = A^n(c, x) - A^n(c, y).$$

If $t \in [0, 1/n]$, then we have

$$x(t) - y(t) = A(c, x)(r_n(t)) - A(c, y)(r_n(t)) = A(c, x)(0) - A(c, y)(0) = 0.$$

Thus, we obtain $x(t) = y(t)$ for every $t \in [0, 1/n]$.

If $t \in [1/n, 2/n]$, then we have that $0 < r_n(t) \leq 1/n$. Hence, by the property of operator $A(c, \dots)$ mentioned above, we get $x(t) = y(t)$ for $t \in [0, 2/n]$. Finally, by repeating the procedure infinitely many times we infer that $x(t) = y(t)$ for every $t \in [0, \infty)$. Therefore h_n is a one-to-one map. Hence the assumptions of Theorem 2.3 hold and $h^{-1}(0) = \mathrm{Fix} A(c, \cdot\,)$ is an R_δ-set. □

Remark 3.6. For a different treatment of Lemma 3.5, see [5].

Let $\varphi \colon \mathrm{BCL} \to \mathbb{R}^k$ be given by $\varphi(y) = \lim_{t\to\infty} y(t)$. It is easily seen that φ is continuous. Hence the map $g = \varphi \circ \Phi$ is decomposable with a decomposition

$$\mathbb{R}^k \xrightarrow{\Phi}\!\circ\ \mathrm{BCL} \xrightarrow{\varphi} \mathbb{R}^k.$$

If, for some $c \in \mathbb{R}^k$, $0 \in g(c)$, then there is $y \in \Phi(c)$ (in other words $y'(t) = f(t, c + \int_0^t y(s)\,ds, y(t)))$ such that $0 = \lim_{t\to\infty} y(t)$. Putting $x(t) := c + \int_0^t y(s)\,ds$, we see that

$$x''(t) = f(t, x(t), x'(t)), \quad x'(0) = 0 = \lim_{t\to\infty} x'(t),$$

i.e. x is a solution to the initial equation (3.1).

Now, set $\widehat{M} := M + 1$, where M is as in (ii).

Lemma 3.7. *Let $Q := [-\widehat{M}, \widehat{M}]^k$. There is $\widetilde{c} \in Q$ such that $0 \in g(\widetilde{c})$.*

Proof. Let $c_i = \widehat{M}$ and $y \in \Phi(c)$. First, we shall show that $y_i(t) \geq 0$ for $t \geq 0$. We have $y_i(0) = 0$. Assume that for some t we have $y_i(t) < 0$. Then there exists $t_* := \inf\{t \mid y_i(t) < 0\}$ such that, $y_i(t_*) = 0$ and $y_i(t) \geq 0$ for $t < t_*$. Since $y_i(t)$ is continuous there exists $t_1 > t_*$ such that $\int_{t_*}^{t_1} |y_i(t)|\,dt \leq 1$. Hence, we get

$$x_i(t) = c_i + \int_{t_*}^t y_i(s)\,ds \geq M + 1 + \int_{t_*}^t y_i(s)\,ds \geq M \quad \text{for } t \in [t_*, t_1].$$

Now, by condition (ii) we get $x_i(t) f_i(t, x(t), y(t)) = x_i(t) y_i'(t) > 0$. Hence $y_i'(t) > 0$ for $t \in [t_*, t_1]$. It means that $y_i(t)$ is increasing on $[t_*, t_1]$. Since $y_i(t_*) = 0$, we get a contradiction. Hence $y_i(t) \geq 0$ for $t \geq 0$.

Moreover, by the above arguments, $\lim_{t\to\infty} y_i(t) > 0$.

Let $d = (d_1, \dots, d_k) \in \mathbb{R}^k$. By the definition of g, for $i = 1, \dots, k$, we get

$$\text{if} \quad d \in g(c_1, \dots, c_{i-1}, \widehat{M}, c_{i+1}, \dots, c_k), \quad \text{then} \quad d_i > 0. \tag{3.6}$$

We can proceed analogously to prove that, for every $i = 1, \dots, k$,

$$\text{(3.7)} \qquad \text{if} \quad d \in g(c_1, \dots, c_{i-1}, -\widehat{M}, c_{i+1}, \dots, c_k), \quad \text{then} \quad d_i < 0.$$

Let $g_i = \mathrm{P}_i g$ for $i = 1, \dots, k$, where $\mathrm{P}_i: \mathbb{R}^k \to \mathbb{R}$ is the projection onto the i-th axis. By (3.6) and (3.7), for $i = 1, \dots, k$, we have

$$g_i(c_1, \dots, c_{i-1}, \widehat{M}, c_{i+1}, \dots, c_k) \subset (0, \infty),$$
$$g_i(c_1, \dots, c_{i-1}, -\widehat{M}, c_{i+1}, \dots, c_k) \subset (-\infty, 0).$$

It is easy to see that g_i is u.s.c. map. By (3.6) and the fact that g_i is u.s.c. there exists $\gamma_i > 0$ such that for any $c \in Q$, where $c_i \in (\widehat{M} - \gamma_i, \widehat{M}]$, we get $g_i(c) \subset (0, \infty)$, for every $i = 1, \dots, k$. Similarly, by (3.7) and the fact that g_i is u.s.c. there exists $\beta_i > 0$ such that for any $c \in Q$, where $c_i \in [-\widehat{M}, -\widehat{M} + \beta_i)$, we have $g_i(c) \subset (-\infty, 0)$, for every $i = 1, \dots, k$.

The image of g is compact, hence $\widehat{g} := \sup\{|d| \mid d \in g_i(c), c \in Q,\ i = 1, \dots, k\} < \infty$.

Let $\delta := \min\left\{\beta_1, \dots, \beta_k, \gamma_1, \dots, \gamma_k, \widehat{M}\right\}$ and set $\varepsilon := \delta/\widehat{g}$. Considering the set-valued mapping given by $F_i(c) = c_i - \varepsilon g_i(c)$ we get the following inequality

$$-\widehat{M} \le c_i - \varepsilon y \le \widehat{M}, \quad \text{for any } c_i \in [-\widehat{M}, \widehat{M}] \text{ and } y \in g_i(c).$$

Now, let us consider the multi-valued mapping $F(c) = c - \varepsilon g(c)$, where $c \in Q$. By the above, we get that F maps the hypercube Q into itself.

Let us define a pair (D, F) consisting of a set-valued map $F: Q \multimap Q$ and the diagram

$$D: Q \xrightarrow{\Phi_0} \mathrm{BCL} \xrightarrow{\varphi} Q,$$

where $F = \varphi \circ \Phi_0$ and $\Phi_0(c) := \{x \in \mathrm{BCL} \mid x(t) = c - \varepsilon y(t),\ t \in \mathbb{R}_+,\ y \in \Phi(c)\}$.

Notice, that BCL, as a Banach space, is ANR. Moreover, Φ_0 is an R_δ-map. Hence (D, F) is a decomposable map.

Now, to apply Corollary 2.2, it is sufficient to show that the decomposable map (D, F) is homotopic to the identity id_Q, which means that there exists a decomposable map $(D', F'): Q \multimap Q$ such that $(D, F) \simeq (D', F')$ and $F'(c) = c$ for $c \in Q$.

Let $D': Q \xrightarrow{\Phi_1} \mathrm{BCL} \xrightarrow{\varphi} Q$, where $\Phi_1: Q \ni c \to x(t) \equiv c \in \mathrm{BCL}$, then $F': Q \to Q$ and $F'(c) = c$ for every $c \in Q$.

Now, let us put $X, Y = Q$, $Z = Z_0 = Z_1 = \mathrm{BCL}$, $\varphi = \varphi_0 = \varphi_1$ and consider the following decomposable map $(\breve{D}, \breve{F})$ with $\breve{D}: Q \times [0,1] \xrightarrow{\breve{\Phi}} \mathrm{BCL} \xrightarrow{\varphi} Q$, where $\breve{\Phi}(c, \lambda) := \{x \in \mathrm{BCL} \mid x(t) = (1-\lambda) y(t) + \lambda z(t),\ t \in \mathbb{R}_+,\ y \in \Phi_0(c),\ z \in \Phi_1(c)\}$. It is immediate to see that $\breve{\Phi}$ is an R_δ-map. Moreover, one can see that the appropriate diagram is commutative. Hence, (D, F) is homotopic to the identity.

The Euler characteristic of Q satisfies $\chi(Q) = 1$. Thus, by Corollary 2.2, $\mathrm{Fix}(F) \neq \emptyset$ and hence there exists $\widetilde{c} \in Q$ such that $\widetilde{c} \in F(\widetilde{c})$.

On the other hand $F(\widetilde{c}) = \widetilde{c} - \varepsilon g(\widetilde{c})$. Thus $0 \in F(\widetilde{c}) - \widetilde{c} = -\varepsilon g(\widetilde{c})$, and from this $0 \in g(\widetilde{c})$. □

This ends the proof of Theorem 3.2 and completes the paper.

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