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NONTRIVIAL SOLUTIONS FOR NONVARIATIONAL QUASILINEAR NEUMANN PROBLEMS

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Abstract. We consider a nonlinear nonvariational Neumann problem with a nonsmooth potential. Using the spectrum of the assymptotic (as $|x| \to \infty$) differential operator and degree theoretic techniques based on the degree map of certain multivalued perturbations of $(S)_+$-operators, we establish the existence of at least one nontrivial smooth solution.

1. Introduction

Let $Z \subseteq \mathbb{R}^n$ be a bounded domain with a C^2-boundary ∂Z. We consider the following quasilinear Neumann problem with a nonsmooth potential (hemivariational inequality):

$$
\begin{cases} -\operatorname{div}(A(z, x(z))Dx(z)) \in \partial j(z, x(z)) & \text{a.e. on } Z, \\ \dfrac{\partial x}{\partial n} = 0 & \text{on } \partial Z. \end{cases} \tag{1.1}
$$

Here $A(z, x)$ is a bounded, $N \times N$-matrix valued Caratheodory function (i.e. it is measurable in $z \in Z$ and continuous in $x \in \mathbb{R}$) and $j(z, x)$ is a measurable potential function which is only locally Lipschitz and in general nonsmooth in the $x \in \mathbb{R}$ variable. By $\partial j(z, x)$ we denote the generalized (Clarke) subdifferential of $x \to j(z, x)$ (see Section 2).

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In the last decade, nonlinear elliptic problems driven by the p-Laplacian differential operator have attracted a lot of interest. Most of the works focused on the Dirichlet problem with a smooth potential (i.e. $j(z,\,\cdot\,) \in C^1(\mathbb{R})$). The study of the corresponding Neumann problem is lagging behind. In this direction we mention the works of G. Anello and G. Cordaro [1], D. Arcoya and L. Orsina [2], P. A. Binding, P. Drabek and Y. Huang [3], F. Faraci [7], T. Godoy, J. P. Gossez and S. Paczka [10], Y. Huang [14] (problems with a smooth potential) and M. Filippakis, L. Gasinski and N. S. Papageorgiou [8], S. Hu and N. S. Papageorgiou [13], S. Marano and D. Motreanu [15], D. Motreanu and N. S. Papageorgiou [16], F. Papalini [18], [19] (problems with a nonsmooth potential).

In all the aforementioned works the p-Laplacian differential operator is used. The p-Laplacian is $(p-1)$-homogeneous and so the Lusternik–Schnirelmann theory can be applied to determine its spectral properties. Moreover, the operator is variational and so the methods of critical point theory can be used to obtain solutions of the boundary value problems. For this reason, in all the above works the approach is variational. In contrast, in problem (1.1) the differential operator $x \to -\mathrm{div}(A(z,x)Dx)$ is neither homogeneous nor variational. So the minimax methods of critical point theory (smooth and nonsmooth alike) fail and we need to device new techniques in order to deal with problem (1.1). For this reason, we assume that for almost all $z \in Z$, the matrix-valued map $x \to A(z,x)$ has an asymptotic limit as $|x| \to \infty$. Then, using the spectrum of the corresponding asymptotic linear differential operator, we are able to overcome the lack of homogeneity of the original differential operator and provide conditions for the solvability of problem (1.1). We use the spectrum of the asymptotic differential operator together with degree theoretic methods based on the degree map for multivalued perturbations of $(S)_+$-operators due to S. Hu and N. S. Papageorgiou [11] (see also S. Hu and N. S. Papageorgiou [12]) and we are able to establish the existence of nontrivial smooth solutions.

Finally we mention that hemivariational inequalities are a useful tool in nonsmooth mechanics. Several such applications can be found in the book of Z. Naniewicz N. S. Panagiotopoulos [17].

2. Hypotheses and mathematical background

The hypotheses on the matrix-valued function $A(z,x)$ are the following:

H(A): $A\colon Z \times \mathbb{R} \to \mathbb{R}^{N\times N}$ is a map such that

(a) for all $x \in \mathbb{R}$, $z \mapsto A(z,x)$ is measurable;
(b) for almost all $z \in Z$, $x \mapsto A(z,x)$ is continuous;

(c) there exist constants $0 < c_0 < c_1$ such that

$$c_0\|\xi\| \le \|A(z,x)\xi\| \le c_1\|\xi\|$$

for almost all $z \in Z$, all $x \in \mathbb{R}$ and all $\xi \in \mathbb{R}^N$;

(d) there exists a constant $c_2 > 0$ such that

$$c_2\|\xi\|^2 \le (A(z,x)\xi,\xi)_{\mathbb{R}^N}$$

for almost all $z \in Z$, all $x \in \mathbb{R}$ and all $\xi \in \mathbb{R}^N$;

(e) there exists $\widehat{A} \in L^\infty(Z, \mathbb{R}^N \times \mathbb{R}^N)$ such that

$$A(z,x) \to \widehat{A}(z) \quad \text{for a.a. } z \in Z, \text{ as } |x| \to \infty.$$

Using the asymptotic limit function $\widehat{A}(z)$ of hypothesis H(A)(e), we consider the following linear Neumann eigenvalue problem:

$$\begin{cases} -\operatorname{div}(\widehat{A}(z)Dx(z)) = \lambda x(z) & \text{a.e. on } Z, \\ \dfrac{\partial x}{\partial n} = 0 & \text{on } \partial Z. \end{cases} \tag{2.1}$$

In what follows by $\langle\,\cdot\,,\,\cdot\,\rangle$ we denote the duality brackets for the pair $(H^1(Z), H^1(Z)^*)$. Then let $\widehat{V} \in \mathcal{L}(H^1(Z), H^1(Z)^*)$ be the continuous linear operator defined by

$$\langle \widehat{V}(x), y\rangle = \int_Z (\widehat{A}(z)Dx(z), Dy(z))_{\mathbb{R}^N}\,dz \quad \text{for all } x,y \in H^1(Z).$$

For every $\varepsilon > 0$ and every $x \in H^1(Z)$, we have:

$$\langle \widehat{V}(x), x\rangle + \varepsilon\|x\|_2^2 \ge c_2\|Dx\|_2^2 + \varepsilon\|x\|_2^2 \ge c_3\|x\|_2^2$$

with $c_3 = \min\{\varepsilon, c_2\}$. Then, by virtue of Corollary 7D of R. Showalter [20, p. 78], we know that problem (2.1) has a sequence of eigenvalues $\{\lambda_n\}_{n\ge 0}$, $\lambda_0 = 0 < \lambda_1 \le \ldots \le \lambda_n \to \infty$, with corresponding eigenfunctions which form an orthonormal basis in $L^2(Z)$ and an orthogonal basis in $H^1(Z)$.

Moreover, these eigenvalues admit variational characterizations via the corresponding Rayleigh quotients. Using this spectrum, we can now state the hypotheses on the nonsmooth potential $j(z,x)$:

H(j): $j\colon Z \times \mathbb{R} \to \mathbb{R}$ is a function such that $j(z,0) = 0$ almost everywhere on Z and

(a) for all $x \in \mathbb{R}$, $z \mapsto j(z,x)$ is measurable;

(b) for almost all $z \in Z$, $x \mapsto j(z,x)$ is locally Lipschitz;

(c) for every $r > 0$, there exists $a_r \in L^\infty(Z)_+$ such that

$$|u| \le a_r(z) \quad \text{for a.a. } z \in Z, \text{ all } |x| \le r \text{ and all } u \in \partial j(z,x);$$

(d) there exist an integer $k \geq 0$ and functions $\widehat{\theta}, \theta \in L^\infty(Z)$ such that

$$\lambda_k \leq \widehat{\theta}(z) \leq \theta(z) \leq \lambda_{k+1} \quad \text{a.e. on } Z,$$

where the first and the third inequalities are strict on sets of positive Lebesgue measure and

$$\widehat{\theta}(z) \leq \liminf_{|x|\to\infty} \frac{u}{|x|^{p-2}x} \leq \limsup_{|x|\to\infty} \frac{u}{|x|^{p-2}x} \leq \theta(z)$$

uniformly for almost all $z \in Z$ and all $u \in \partial j(z,x)$;

(e) there exist functions $\widehat{\eta}, \eta \in L^\infty(Z)$ such that

$$\eta(z) \leq 0 \quad \text{a.e. on } Z,$$

where the inequality is strict on a set of positive Lebesgue measure and

$$\widehat{\eta}(z) \leq \liminf_{x\to 0} \frac{u}{|x|^{p-2}x} \leq \limsup_{x\to 0} \frac{u}{|x|^{p-2}x} \leq \eta(z)$$

uniformly for almost all $z \in Z$ and all $u \in \partial j(z,x)$.

Due to the nonsmoothness of the potential function $j(z,x)$, we will use some elements of the subdifferential theory for locally Lipschitz functions (see F. H. Clarke [6]). Also, due to the nonvariational character of our problem, we will use degree theoretic arguments based on the degree map for multivalued perturbations of $(S)_+$-operators (see S. Hu and N. S. Papageorgiou [11], [12]). So, in what follows, we present some basic definitions and facts from these two theories, which will be used in the sequel.

Let X be a Banach space, X^* its dual and denote by $\langle \cdot, \cdot \rangle$ the duality brackets for the pair (X, X^*). Given a locally Lipschitz function $\varphi: X \to \mathbb{R}$, we define the generalized directional derivative of φ at $x \in X$ in the direction $h \in X$, $\varphi^0(x;h)$, by

$$\varphi^0(x;h) = \limsup_{\substack{\lambda \downarrow 0 \\ x' \to x}} \frac{\varphi(x' + \lambda h) - \varphi(x')}{\lambda}.$$

The function $h \to \varphi^0(x;h)$ is sublinear, continuous and it is the support function of a nonempty, convex and w^*-compact set $\partial\varphi(x)$, defined by

$$\partial\varphi(x) = \{x^* \in X^* : \langle x^*, h \rangle \leq \varphi^0(x;h) \text{ for all } h \in X\}.$$

The multifunction $x \to \partial\varphi(x)$ is the "generalized (or Clarke) subdifferential" of φ. If $\varphi: X \to \mathbb{R}$ is continuous and convex, then it is locally Lipschitz and the generalized subdifferential of φ coincides with the subdifferential in the sense of convex analysis, $\partial_c\varphi(x)$, given by

$$\partial_c\varphi(x) = \{x^* \in X^* : \langle x^*, h \rangle \leq \varphi(x+h) - \varphi(x) \text{ for all } h \in X\}.$$

If $\varphi \in C^1(X)$, then φ is locally Lipschitz and $\partial\varphi(x) = \{\varphi'(x)\}$.

A multifunction $G: X \to 2^{X^*} \setminus \{\emptyset\}$ is said to be upper semicontinuous (u.s.c. for short) if, for every closed set $C \subseteq X^*$, we have that

$$G^-(C) = \{x \in X : G(x) \cap C \neq \emptyset\}$$

is closed in X. The generalized subdifferential multifunction $x \to \partial\varphi(x)$ is u.s.c. from X with the norm topology into X^* furnished with the w^*-topology.

We say that a multifunction $G: X \to 2^{X^*} \setminus \{\emptyset\}$ belongs in *class* (P), if it is u.s.c., for every $x \in X$, $G(x)$ is closed, convex and for every $A \subseteq X$ bounded, we have

$$G(A) = \bigcup_{x \in A} G(x)$$

is relatively compact in X^*.

From A. Cellina [5] (see also S. Hu and N. S. Papageorgiou [12, p. 106]), we know that if $G: D \subseteq X \to 2^{X^*} \setminus \{\emptyset\}$ is an u.s.c. multifunction with closed and convex values, then given $\varepsilon > 0$, we can find a continuous map $g_\varepsilon: D \to X^*$ such that

$$g_\varepsilon(x) \in G((x + B_\varepsilon) \cap D) + B_\varepsilon^*$$

for all $x \in D$ and $g_\varepsilon(D) \subseteq \overline{\operatorname{conv}}\, G(D)$. Here $B_\varepsilon = \{x \in X : \|x\| < \varepsilon\}$ and $B_\varepsilon^* = \{x^* \in X^* : \|x^*\| < \varepsilon\}$.

Note that, if the multifunction G belongs in class (P), then the continuous approximate selector g_ε is compact.

Now we can define the degree map that we shall use in the study of problem (1.1). Suppose X is a reflexive Banach space. Then, by the Troyanski renorming theorem (see L. Gasinski and N. S. Papageorgiou [9, p. 911]), we can equivalently renorm X so that both X and X^* are locally uniformly convex and with Fréchet differentiable norms. So, in what follows, we assume that both X and X^* are locally uniformly convex. Hence, if $\mathcal{F}: X \to X^*$ is the duality map defined by

$$\mathcal{F}(x) = \{x^* \in X^* : \langle x^*, x\rangle = \|x\|^2 = \|x^*\|^2\},$$

we have that $\mathcal{F}$ is a homeomorphism.

An operator $A: X \to X^*$, which is single-valued and everywhere defined, is said to be of type $(\mathrm{S})_+$, if for every sequence $\{x_n\}_{n\geq 1} \subseteq X$ such that $x_n \xrightarrow{w} x$ in X and $\limsup_{n\to\infty} \langle A(x_n), x_n - x\rangle \leq 0$, one has $x_n \to x$ in X.

Let U be a bounded open set in X and let $A: \overline{U} \to X^*$ be a demicontinuous operator of type $(\mathrm{S})_+$. Let $\{X_\alpha\}_{\alpha\in J}$ be the family of all finite dimensional subspaces of X and let A_α be the Galerkin approximation of A with respect to X_α, that is

$$\langle A_\alpha(x), y\rangle_{X_\alpha} = \langle A(x), y\rangle$$

for all $x \in \overline{U} \cap X_\alpha$ and all $y \in X_\alpha$.

By $\langle \cdot\,, \cdot \rangle_{X_\alpha}$ we denote the duality brackets for the pair (X_α, X_α^*). Then, for $x^* \notin A(\partial U)$, the degree map $d_{(S)_+}(A, U, x^*)$ is defined by

$$d_{(S)_+}(A, U, x^*) = d_B(A_\alpha, U \cap X_\alpha, x^*)$$

for X_α large enough (in the sense of inclusion). Here d_B stands for the classical Brouwer degree map. If X is separable and A is bounded (maps bounded sets to bounded ones), then we can use only a countable subfamily $\{X_n\}_{n\geq 1}$ of $\{X_\alpha\}_{\alpha\in J}$ such that

$$\overline{\bigcup_{n\geq 1} X_n} = X.$$

More details on the degree map $d_{(S)_+}$ can be found in F. Browder [4] and I. Skrypnik [21].

If $G: X \to 2^{X^*} \setminus \{\emptyset\}$ is a multifunction belonging in class (P), then for every $x^* \notin (A + G)(\partial U)$, $\widehat{d}(A + G, U, x^*)$ is defined by

$$\widehat{d}(A + G, U, x^*) = d_{(S)_+}(A + g_\varepsilon, U, x^*)$$

for $\varepsilon > 0$ small, where g_ε is the continuous approximate selector of G mentioned earlier. Note that since G belongs in class (P), $g_\varepsilon: \overline{U} \to X^*$ is compact and so $x \mapsto A(x) + g_\varepsilon(x)$ is still of type $(S)_+$. More about the degree map $\widehat{d}$, can be found in S. Hu-N. S. Papageorgiou [11], [12].

One of the fundamental properties of a degree map is the homotopy invariance property. To formulate this property for the degree map $\widehat{d}$, we need to define the admissible homotopies for A and G.

Definition 2.1. (a) A one-parameter family $\{A_t\}_{t\in[0,1]}$ of maps from $\overline{U}$ into X^*, is said to be a *homotopy of class* $(S)_+$, if for any $\{x_n\}_{n\geq 1} \subseteq \overline{U}$ such that $x_n \xrightarrow{w} x$ and for any $\{t_n\}_{n\geq 1} \subseteq [0, 1]$ with $t_n \to t$ for which

$$\limsup_{n\to\infty} \langle A_{t_n}(x_n), x_n - x\rangle \leq 0,$$

one has $x_n \to x$ in X and $A_{t_n}(x_n) \xrightarrow{w} A_t(x_n)$ in X^* as $n \to \infty$.

(b) A one-parameter family $\{G_t\}_{t\in[0,1]}$ of multifunctions $G_t: \overline{U} \to 2^{X^*} \setminus \{\emptyset\}$ is said to be a *homotopy of class* (P), if $(t, x) \mapsto G(t, x)$ is u.s.c. from $[0, 1] \times X$ into $2^{X^*} \setminus \{\emptyset\}$, for every $(t, x) \in [0, 1] \times \overline{U}$ the set is closed, convex and

$$\overline{\bigcup \{G_t(x) : t \in [0, 1],\ x \in \overline{U}\}}$$

is compact in X^*.

With these admissible homotopies for A and G, the homotopy invariance property of $\widehat{d}$ can be formulated as follows:

If $\{A_t\}_{t\in[0,1]}$ *is a homotopy of class* $(\mathrm{S})_+$ *such that for every* $t \in [0,1]$, A_t *is bounded,* $\{G_t\}_{t\in[0,1]}$ *is a homotopy of class* (P) *and* $x^*\colon [0,1] \to X^*$ *is a continuous map such that*

$$x_t^* \notin (A_t + G_t)(\partial U) \quad \textit{for all } t \in [0,1],$$

then $\widehat{d}(A_t + G_t, U, x_t^*)$ *is independent of* $t \in [0,1]$.

Also the normalization property has the following form:

$$\widehat{d}(\mathcal{F}, U, x^*) = d_{(\mathrm{S})_+}(\mathcal{F}, U, x^*) = 1 \quad \text{for all } x^* \in \mathcal{F}(U).$$

Both degree maps $d_{(\mathrm{S})_+}$ and $\widehat{d}$ have all the usual properties such as normalization, homotopy invariance, solution property, additivity with respect to the domain, excision property, product property etc.

3. Existence of solutions

Let $V\colon H^1(Z) \to H^1(Z)^*$ be the nonlinear operator defined by

$$\langle V(x), y\rangle = \int_Z (A(z,x)Dx, Dy)_{\mathbb{R}^N}\, dz \quad \text{for all } x,\ y \in H^1(Z).$$

PROPOSITION 3.1. *If hypotheses* H(A) *hold, then* V *is an* $(\mathrm{S})_+$*-operator.*

PROOF. Suppose that $x_n \xrightarrow{w} x$ in $H^1(Z)$ and assume that

$$\limsup_{n\to\infty} \langle V(x_n), x_n - x\rangle \le 0. \tag{3.1}$$

By definition

$$\langle V(x_n), x_n - x\rangle = \int_Z (A(z,x_n)Dx_n, Dx_n - Dx)_{\mathbb{R}^N}\, dz.$$

We have (see H(A)(d))

$$\begin{aligned} \langle V(x_n), x_n - x\rangle &= \int_Z (A(z,x_n)Dx_n, Dx_n - Dx)_{\mathbb{R}^N}\, dz \\ &= \int_Z (A(z,x_n)Dx_n - A(z,x_n)Dx, Dx_n - Dx)_{\mathbb{R}^N}\, dz \\ &\quad + \int_Z (A(z,x_n)Dx, Dx_n - Dx)_{\mathbb{R}^N}\, dz \\ &\ge c_2\|Dx_n - Dx\|_2^2 + \int_Z (A(z,x_n)Dx, Dx_n - Dx)_{\mathbb{R}^N}\, dz \end{aligned} \tag{3.2}$$

Because $x_n \xrightarrow{w} x$ in $H^1(Z)$ and recalling that $H^1(Z)$ is embedded compactly in $L^2(Z)$, we can say that $x_n \to x$ in $L^2(Z)$. By passing to a subsequence, if necessary, we may also assume that

$$x_n(z) \to x(z) \quad \text{a.e. on } Z$$

and

$$|x_n(z)| \le h(z) \quad \text{for a.a. } z \in Z, \text{ all } n \ge 1 \text{ and with } h \in L^2(Z)_+.$$

Then

$$A(z, x_n(z))Dx(z) \to A(z, x(z))Dx(z) \quad \text{a.e. on } Z$$

(see H(A)(b)). This fact, together with H(A)(c) and the dominated convergence theorem, imply

$$A(\,\cdot\,, x_n(\,\cdot\,))Dx(\,\cdot\,) \to A(\,\cdot\,, x(\,\cdot\,))Dx(\,\cdot\,) \quad \text{in } L^2(Z, \mathbb{R}^N) \text{ as } n \to \infty.$$

Since $Dx_n \xrightarrow{w} Dx$ in $L^2(Z, \mathbb{R}^N)$, it follows that

$$\int_Z (A(z, x_n)Dx, Dx_n - Dx)_{\mathbb{R}^N}\, dz \to 0 \quad \text{as } n \to \infty.$$

Returning to (3.2), passing to the limit as $n \to \infty$ and using (3.1) and (3.3), we obtain

$$\|Dx_n \to Dx\|_2^2 \to 0 \quad \text{as } n \to \infty,$$

hence

$$x_n \to x \quad \text{in } H^1(Z) \text{ as } n \to \infty.$$

By Urysohn's criterion for convergent sequences, this convergence is true for the original sequence $\{x_n\}_{n\ge 1} \subseteq H^1(Z)$. □

Let $N: L^2(Z) \to 2^{L^2(Z)}$ be the multivalued Nemytskiĭ operator corresponding to the subdifferential multifunction $(z, x) \mapsto \partial j(z, x)$, i.e.

$$N(x) = S^2_{\partial j(\,\cdot\,, x(\,\cdot\,))} = \{u \in L^2(Z) : u(z) \in \partial j(z, x(z)) \text{ a.e. on } Z\}.$$

Proposition 3.2. *If hypotheses* H(j) *hold, then N has nonempty, weakly compact and convex values in $L^2(Z)$ and it is u.s.c. from $L^2(Z)$ endowed with the norm topology into $L^2(Z)$ with the weak topology* (*denoted by $L^2(Z)_w$*).

Proof. By virtue of hypotheses H(j)(c) and (d), we see that the values of N are $L^2(Z)$-bounded sets, which are easily seen to be closed and convex. Therefore for every $x \in L^2(Z)$, the set $N(x) \subseteq L^2(Z)$ is weakly compact and convex. We need to show that it is nonempty. For this purpose, let $\{s_n\}_{n\ge 1} \subseteq L^2(Z)$ be simple functions such that

$$s_n(z) \to x(z) \quad \text{a.e. on } Z \text{ and } |s_n(z)| \le |x(z)|$$

for almost all $z \in Z$ and all $n \ge 1$.

Because of hypothesis H(j)(a), for every $x \in \mathbb{R}$, the multifunction $z \mapsto \partial j(z, x)$ is graph measurable. So, by a straightforward application of the Yankov–von Neumann–Aumann selection theorem (see S. Hu and N. S. Papageorgiou [12, p. 158]), we can find a measurable function $f_n: Z \to \mathbb{R}$ such that

$$f_n(z) \in \partial j(z, x(z))$$

for almost all $z \in Z$, all $n \geq 1$. Hypotheses H(j)(c), (d) imply that

$$|f_n(z)| \leq c_3(1 + |x(z)|)$$

for almost all $z \in Z$, all $n \geq 1$ and some $c_3 > 0$, hence $\{f_n\}_{n\geq 1} \subseteq L^2(Z)$ is bounded. Therefore, we may assume (at least for a subsequence), that

$$f_n \xrightarrow{w} f \quad \text{in } L^2(Z).$$

Since the subdifferential multifunction has closed and convex values, by Mazur's lemma, we obtain

$$f(z) \in \partial j(z, x(z)) \quad \text{a.e. on } Z,$$

hence $f \in N(x)$, therefore $N(x) \neq \emptyset$.

Note that the weak topology on bounded subsets of $L^2(Z)$ is metrizable. Therefore, in order to show the upper semicontinuity of N from $L^2(Z)$ into $L^2(Z)_w$, it suffices to show that its graph

$$\operatorname{Gr} N = \{(x, u) \in L^2(Z) \times L^2(Z) : u \in N(x)\}$$

is sequentially closed in $L^2(Z) \times L^2(Z)_w$ (see S. Hu and N. S. Papageorgiou [12, p. 38]). So let $\{(x_n, u_n)\}_{n\geq 1} \subseteq \operatorname{Gr} N$ and assume that $x_n \to x$ in $L^2(Z)$ and $u_n \xrightarrow{w} u$ in $L^2(Z)$ as $n \to \infty$. We have

$$u_n(z) \in \partial j(z, x_n(z)) \quad \text{for a.a. } z \in Z, \text{ all } n \geq 1.$$

Invoking Proposition 3.9 of S. Hu and N. S. Papageorgiou [12, p. 694], in the limit as $n \to \infty$, we obtain

$$u(z) \in \partial j(z, x(z)) \quad \text{a.e. on } Z,$$

hence $u \in N(x)$. So $\operatorname{Gr} N$ is sequentially closed in $L^2(Z) \times L^2(Z)_w$ and from this we conclude that N is usc from $L^2(Z)$ into $L^2(Z)_w$. □

Recalling that $H^1(Z)$ is a separable Hilbert space, which is embedded compactly and densely in $L^2(Z)$, we have that $L^2(Z)^* = L^2(Z)$ is embedded compactly and densely in $H^1(Z)^*$. Therefore an immediate consequence of Proposition 3.2, is the following corollary:

Corollary 3.3. *If hypotheses* H(j) *hold, then* $N: H^1(Z) \to 2^{H^1(Z)^*} \setminus \{\emptyset\}$ *is a multifunction belonging in class* (P).

Proposition 3.1 and Corollary 3.3 permit the definition of the degree $\widehat{d}$ for the nonlinear multivalued operator $x \mapsto V(x) - N(x)$. To compute this degree for various sets we will need the following auxiliary result.

For every integer $k \geq 0$, let $E(\lambda_k)$ denote the eigenspace corresponding to the eigenvalue λ_k. Set

$$\overline{H}_k = \bigoplus_{i=0}^{k} E(\lambda_i) \quad \text{and} \quad \widehat{H}_k = \bigoplus_{i \geq k+1} E(\lambda_i).$$

Then we have the orthogonal direct sum decomposition

$$H^1(Z) = \overline{H}_k \oplus \widehat{H}_k.$$

Lemma 3.4.

(a) *If $\theta \in L^\infty(Z)_+$ and $\theta(z) \leq \lambda_{k+1}$ almost everywhere on Z with strict inequality on a set of positive Lebesgue measure, then there exists $\xi_1 > 0$ such that*

$$\psi_1(\widehat{x}) = \int_Z (\widehat{A}(z)D\widehat{x}, D\widehat{x})_{\mathbb{R}^N}\, dz - \int_Z \theta|\widehat{x}|^2\, dz \geq \xi_1\|\widehat{x}\|^2 \quad \textit{for all } \widehat{x} \in \widehat{H}_k.$$

(b) *If $\widehat{\theta} \in L^\infty(Z)_+$ and $\widehat{\theta}(z) \geq \lambda_k$ almost everywhere on Z with strict inequality on a set of positive Lebesgue measure, then there exists $\xi_2 > 0$ such that*

$$\psi_2(\overline{x}) = \int_Z \widehat{\theta}|\overline{x}|^2\, dz - \int_Z (\widehat{A}(z)D\overline{x}, D\overline{x})_{\mathbb{R}^N}\, dz \geq \xi_2\|\overline{x}\|^2 \quad \textit{for all } \overline{x} \in \overline{H}_k.$$

(c) *If $\eta \in L^\infty(Z)$ and $\eta(z) \leq 0$ almost everywhere on Z with strict inequality on a set of positive Lebesgue measure, then there exists $\xi_0 > 0$ such that*

$$\psi_0(x) = \int_Z (\widehat{A}(z)Dx, Dx)_{\mathbb{R}^N}\, dz - \int_Z \eta|x|^2\, dz \geq \xi_0\|x\|^2 \quad \textit{for all } x \in H^1(Z).$$

Proof. (a) From the variational characterization of λ_{k+1} we have $\psi_1 \geq 0$. Suppose that the result is not true. Exploiting the 2-homogeneity of ψ_1, we can find $\{\widehat{x}_n\}_{n\geq 1} \subseteq \widehat{H}_k$ such that

$$\|\widehat{x}_n\| = 1 \quad \text{for all } n \geq 1 \quad \textit{and} \quad \psi_1(\widehat{x}_n) \downarrow 0 \quad \text{as } n \to \infty.$$

We may assume that

$$\widehat{x}_n \xrightarrow{w} \widehat{x} \in \widehat{H}_k \quad \text{in } H^1(Z) \quad \text{and} \quad \widehat{x}_k \to \widehat{x} \quad \text{in } L^2(Z) \quad \text{as } n \to \infty.$$

Note that $\int_Z (\widehat{A}(z)Dx, Dx)_{\mathbb{R}^N}\, dz$ is equivalent to the usual $L^2(Z, \mathbb{R}^N)$ norm (see hypothesis H(A)(c)) and recall that the norm in a Banach space is w-lower semicontinuous. Hence

$$\int_Z (\widehat{A}(z)D\widehat{x}, D\widehat{x})_{\mathbb{R}^N}\, dz \leq \int_Z \theta|\widehat{x}|^2\, dz \leq \lambda_{k+1}\|\widehat{x}\|_2^2 \tag{3.3}$$

therefore

$$\int_Z (\widehat{A}(z)D\widehat{x}, D\widehat{x})_{\mathbb{R}^N}\, dz = \lambda_{k+1}\|\widehat{x}\|_2^2$$

(from the variational characterization of λ_{k+1}). Hence $\widehat{x} \in E(\lambda_{k+1})$. But the elements of $E(\lambda_{k+1})$ have the unique continuation property (see for example L. Gasinski and N. S. Papageorgiou [9]). So $x(z) \neq 0$ almost everywhere on Z. Then from (3.3) and using the hypothesis on θ, we have

$$\int_Z (\widehat{A}(z)Dx, Dx)_{\mathbb{R}^N}\,dz < \lambda_{k+1}\|\widehat{x}\|_2^2,$$

which contradicts the variational characterization of λ_{k+1}.

The proofs of (b) and (c) are similar to those of (a) and are ommited. □

Using this lemma, we can compute the $\widehat{d}$-degree of $V - N$ for large balls.

PROPOSITION 3.5. *If hypotheses* H(j) *hold, then there exists* $R_0 > 0$ *such that*

$$\widehat{d}(V - N, B_R, 0) = (-1)^{\dim \overline{H}_k} \quad \textit{for all } R \geq R_0.$$

PROOF. Let $\widehat{g} \in L^\infty(Z)_+$ be such that $\lambda_k \leq \widehat{g}(z) \leq \lambda_{k+1}$ almost everywhere on Z with strict inequalities on sets (in general different) of positive Lebesgue measure. We consider the admissible homotopy $h_1\colon [0,1]\times H^1(Z) \to 2^{H^1(Z)^*}\setminus\{\emptyset\}$ defined by

$$h_1(t,x) = tV(x) + (1-t)\widehat{V}(x) - tN(x) - (1-t)\widehat{g}x.$$

CLAIM. We can find $R_0 > 0$ such that $0 \notin h_1(t,x)$ for all $t \in [0,1]$ and all $\|x\| = R \geq R_0$.

We argue indirectly. So suppose the Claim is not true. Then we can find $\{t_n\}_{n\geq 1} \subseteq [0,1]$ and $\{x_n\}_{n\geq 1} \subseteq H^1(Z)$ such that

$$t_n \to t \in [0,1], \quad \|x_n\| \to \infty \quad \text{and} \quad 0 \in h_1(t_n, x_n) \quad \text{for all } n \geq 1. \tag{3.4}$$

From the inclusion in (3.4), we know that for every $n \geq 1$, we can find $u_n \in N(x_n)$ such that

$$t_nV(x_n) + (1-t_n)\widehat{V}(x_n) = t_nu_n + (1-t_n)\widehat{g}x_n. \tag{3.5}$$

Let $y_n = x_n/\|x_n\|$, $n \geq 1$. Then $\|y_n\| = 1$ for all $n \geq 1$ and so we may assume that

$$y_n \xrightarrow{w} y \quad \text{in } H^1(Z) \quad \text{and} \quad y_n \to y \quad \text{in } L^2(Z) \text{ as } n \to \infty.$$

We divide (3.5) by $\|x_n\|$ and we have

$$t_n\frac{V(x_n)}{\|x_n\|} + (1-t_n)\widehat{V}(y_n) = t_n\frac{u_n}{\|x_n\|} + (1-t_n)\widehat{g}y_n \tag{3.6}$$

We take duality brackets with $y_n - y$. Hence

$$t_n\Big\langle \frac{V(x_n)}{\|x_n\|}, y_n - y\Big\rangle + (1-t_n)\Big\langle \widehat{V}(y_n), y_n - y\Big\rangle = t_n\int_Z \frac{u_n}{\|x_n\|}(y_n - y)\,dz + (1-t_n)\int_Z \widehat{g}y_n(y_n - y)\,dz.$$

From hypotheses H(j)(c) and (d), we know that

$$|u| \le c_3(1+|x|) \tag{3.7}$$

for almost all $z \in Z$, all $x \in \mathbb{R}$ and all $u \in \partial j(z,x)$. Because of (3.7) and since $y_n \to y$ in $L^2(Z)$, we have

$$\int_Z \frac{u_n}{\|x_n\|}(y_n - y)\,dz \to 0 \quad \text{and} \quad \int_Z \widehat{g} y_n (y_n - y)\,dz \to 0$$

as $n \to \infty$. Therefore it follows that

$$\lim_{n\to\infty}\left[t_n\left\langle \frac{V(x_n)}{\|x_n\|}, y_n - y\right\rangle + (1-t_n)\langle \widehat{V}(y_n), y_n - y\rangle\right] = 0. \tag{3.8}$$

We have

$$\begin{aligned} \left\langle \frac{V(x_n)}{\|x_n\|}, y_n - y\right\rangle &= \int_Z (A(z,x_n)Dy_n, Dy_n - Dy)_{\mathbb{R}^N}\,dz \\ &= \int_Z (A(z,x_n)(Dy_n - Dy), Dy_n - Dy)_{\mathbb{R}^N}\,dz \\ &\quad + \int_Z (A(z,x_n)Dy, Dy_n - Dy)_{\mathbb{R}^N}\,dz \\ &\ge c_2\|Dy_n - Dy\|_2^2 + \int_Z (A(z,x_n)Dy, Dy_n - Dy)_{\mathbb{R}^N}\,dz \end{aligned} \tag{3.9}$$

(see H(A)(d)). Note that $|x_n(z)| \to \infty$ almost everywhere on $\{y \ne 0\}$. Therefore by hypothesis H(A)(e)

$$A(z, x_n(z)) \to \widehat{A}(z) \quad \text{a.e. on } \{y \ne 0\} \text{ as } n \to \infty.$$

Also from Stampacchia's theorem we know that $Dy(z) = 0$ almost everywhere on $\{y = 0\}$. Therefore finally we can say that

$$A(z, x_n(z))Dy(z) \to \widehat{A}(z)Dy(z) \quad \text{a.e. on } Z.$$

From this convergence, hypothesis H(A)(c) and the dominated convergence theorem, it follows that

$$A(\,\cdot\,, x_n(\,\cdot\,))Dy(\,\cdot\,) \to \widehat{A}(\,\cdot\,)Dy(\,\cdot\,) \quad \text{in } L^2(Z, \mathbb{R}^N),$$

hence

$$\int_Z (A(z, x_n(z))Dy(z), Dy_n(z) - Dy(z))_{\mathbb{R}^N}\,dz \to 0 \quad \text{as } n \to \infty \tag{3.10}$$

Moreover,

$$\begin{aligned}(3.11)\qquad \langle \widehat{V}(y_n), y_n - y\rangle &= \int_Z (\widehat{A}(z)Dy_n, Dy_n - Dy)_{\mathbb{R}^N}\,dz \\ &= \int_Z (\widehat{A}(z)(Dy_n - Dy), Dy_n - Dy)_{\mathbb{R}^N}\,dz \\ &\quad + \int_Z (\widehat{A}(z)Dy, Dy_n - Dy)_{\mathbb{R}^N}\,dz \\ &\geq c_2\|Dy_n - Dy\|_2^2 + \int_Z (\widehat{A}(z)Dy, Dy_n - Dy)_{\mathbb{R}^N}\,dz\end{aligned}$$

(see H(A)(d) and (e)). Because $Dy_n \xrightarrow{w} Dy$ in $L^2(Z, \mathbb{R}^N)$, we have

$$(3.12)\qquad \int_Z (\widehat{A}(z)Dy, Dy_n - Dy)_{\mathbb{R}^N}\,dz \to 0 \quad \text{as } n \to \infty.$$

Returning to (3.8), using (3.9)–(3.12) and passing to the limit as $n \to \infty$, we obtain

$$\|Dy_n - Dy\|_2 \to 0 \quad \text{as } n \to \infty,$$

therefore

$$y_n \to y \quad \text{in } H^1(Z) \text{ as } n \to \infty.$$

So $\|y\| = 1$, hence $y \neq 0$.

By virtue of (3.7), $\{u_n/\|x_n\|\}_{n\geq 1} \subseteq L^2(Z)$ is bounded. So we may assume that

$$\frac{u_n}{\|x_n\|} \xrightarrow{w} h \quad \text{in } L^2(Z) \text{ as } n \to \infty.$$

Given $\varepsilon > 0$ and $n \geq 1$, we introduce the following two sets

$$C_{\varepsilon,n}^{+} = \left\{ z \in Z : x_n(z) > 0,\ \widehat{\theta}(z) - \varepsilon \leq \frac{u_n(z)}{x_n(z)} \leq \theta(z) + \varepsilon \right\}$$

and

$$C_{\varepsilon,n}^{-} = \left\{ z \in Z : x_n(z) < 0,\ \widehat{\theta}(z) - \varepsilon \leq \frac{u_n(z)}{x_n(z)} \leq \theta(z) + \varepsilon \right\}.$$

Note that

$$x_n(z) \to \infty \quad \text{for a.a. } z \in \{y > 0\} \quad \text{and} \quad x_n(z) \to -\infty \quad \text{for a.a. } z \in \{y < 0\}.$$

Then hypothesis H(j)(d) implies that

$$\chi_{C_{\varepsilon,n}^{+}}(z) \to 1 \quad \text{a.e. on } \{y > 0\} \quad \text{and} \quad \chi_{C_{\varepsilon,n}^{-}}(z) \to 1 \quad \text{a.e. on } \{y < 0\}.$$

Via the dominated convergence theorem, we have

$$\left\| (1 - \chi_{C_{\varepsilon,n}^{+}}) \frac{u_n}{\|x_n\|} \right\|_{L^1(\{y>0\})} \to 0 \quad \text{and} \quad \left\| (1 - \chi_{C_{\varepsilon,n}^{-}}) \frac{u_n}{\|x_n\|} \right\|_{L^1(\{y<0\})} \to 0$$

as $n \to \infty$. From the definition of the set $C^+_{\varepsilon,n}$, we have

$$\begin{aligned}\chi_{C^+_{\varepsilon,n}}(z)(\theta(z)-\varepsilon)y_n(z) &\leq \chi_{C^+_{\varepsilon,n}}(z)\frac{u_n(z)}{\|x_n\|} \\ &= \chi_{C^+_{\varepsilon,n}}(z)\frac{u_n(z)}{x_n(z)}y_n(z) \leq \chi_{C^+_{\varepsilon,n}}(z)(\theta(z)+\varepsilon)y_n(z)\end{aligned}$$

almost everywhere on Z. Passing to the limit as $n \to \infty$ and using Mazur's lemma, we obtain

$$(\widehat{\theta}(z)-\varepsilon)y(z) \leq h(z) \leq (\theta(z)+\varepsilon)y(z) \quad \text{a.e. on } \{y>0\}.$$

Because $\varepsilon > 0$ was arbitrary, we let $\varepsilon \downarrow 0$ and have

$$\widehat{\theta}(z)y(z) \leq h(z) \leq \theta(z)y(z) \quad \text{a.e. on } \{y>0\}. \tag{3.13}$$

Similarly working with the set $C^-_{\varepsilon,n}$, we obtain

$$\theta(z)y(z) \leq h(z) \leq \widehat{\theta}(z)y(z) \quad \text{a.e. on } \{y<0\}. \tag{3.14}$$

Finally, it is clear from (3.7) that

$$h(z) = 0 \quad \text{a.e. on } \{y=0\}. \tag{3.15}$$

From (3.13)–(3.15) it follows that there exists $g_\infty \in L^\infty(Z)_+$ such that

$$\widehat{\theta}(z) \leq g_\infty(z) \leq \theta(z) \quad \text{a.e. on } Z \quad \text{and} \quad h(z) = g_\infty(z)y(z) \quad \text{a.e. on } Z.$$

For every $v \in L^\infty(Z)$, we have

$$\left\langle \frac{V(x_n)}{\|x_n\|}, v \right\rangle = \int_Z (A(z,x_n)Dy_n, Dv)_{\mathbb{R}^N}\,dz.$$

Recall that $y_n \to y$ in $H^1(Z)$. So we may assume that

$$Dy_n(z) \to Dy(z) \quad \text{a.e. on } Z.$$

Since $|x_n(z)| \to \infty$ almost everywhere on $\{y \neq 0\}$, hypothesis H(A)(e) implies that

$$A(z,x_n) \to \widehat{A}(z) \quad \text{a.e. on } \{y \neq 0\}.$$

Also $Dy(z) = 0$ almost everywhere on $\{y=0\}$. Hence

$$A(z,x_n)Dy_n(z) \to 0 \quad \text{a.e. on } \{y=0\}.$$

Therefore

$$A(z,x_n(z))Dy_n(z) \to \widehat{A}(z)Dy(z) \quad \text{a.e. on } Z \text{ as } n \to \infty.$$

From this convergence, hypothesis H(A)(c) and the dominated convergence theorem, we infer that

$$A(\,\cdot\,,x_n(\,\cdot\,))Dy_n(\,\cdot\,) \to \widehat{A}(\,\cdot\,)Dy(\,\cdot\,) \quad \text{in } L^2(Z,\mathbb{R}^N) \text{ as } n \to \infty.$$

Hence

$$\int_Z (A(z,x_n)Dy_n, Dv)_{\mathbb{R}^N}\,dz \to \int_Z (\widehat{A}(z)Dy, Dv)_{\mathbb{R}^N}\,dz,$$

so

$$\left\langle \frac{V(x_n)}{\|x_n\|}, v\right\rangle \to \langle \widehat{V}(y), v\rangle \quad \text{for all } v \in H^1(Z),$$

therefore

$$\frac{V(x_n)}{\|x_n\|} \to \widehat{V}(y) \quad \text{in } H^1(Z)^*.$$

Returning to (3.6) and passing to the limit as $n \to \infty$, we obtain

$$\widehat{V}(y) = (tg_\infty + (1-t)\widehat{g})y = gy, \tag{3.16}$$

with $g = tg_\infty + (1-t)\widehat{g} \in L^\infty(Z)_+$, $\widehat{\theta}(z) \le g(z) \le \theta(z)$ almost everywhere on Z. From (3.16) we have

$$\begin{cases} -\operatorname{div}(\widehat{A}(z)Dy(z)) = g(z)y(z) & \text{a.e. on } Z, \\ \dfrac{\partial y}{\partial n} = 0 & \text{on } \partial Z. \end{cases} \tag{3.17}$$

Exploiting the monotonicity of the eigenvalues on the weight function (see for example L. Gasinski and N. S. Papageorgiou [9]), we have

$$1 = \widehat{\lambda}_k(\lambda_k) > \widehat{\lambda}_k(g) \quad \text{and} \quad 1 = \widehat{\lambda}_{k+1}(\lambda_{k+1}) < \widehat{\lambda}_{k+1}(g).$$

Using this in (3.17), we infer that $y = 0$, a contradiction to the fact that $\|y\| = 1$. This proves the Claim.

The Claim permits the use of the homotopy invariance property of the degree map $\widehat{d}$. So

$$\widehat{d}(V - N, B_R, 0) = d_{(\mathrm{S})_+}(\widehat{V} - \widehat{g}I, B_R, 0) \quad \text{for all } R \ge R_0. \tag{3.18}$$

We have to compute $d_{(\mathrm{S})_+}(\widehat{V} - \widehat{g}I, B_R, 0)$. To this end we consider the orthogonal direct sum decomposition

$$H^1(Z) = \overline{H}_k \oplus \widehat{H}_k.$$

Let $\overline{p}_k$ and $\widehat{p}_k$ be the orthogonal projections on the component spaces $\overline{H}_k$ and $\widehat{H}_k$, respectively. Also let $\mathcal{F}: H^1(Z) \to H^1(Z)^*$ be the duality map for the Sobolev space $H^1(Z)$. We consider the $(\mathrm{S})_+$-homotopy $h_2: [0,1] \times H^1(Z) \to H^1(Z)^*$ defined by

$$h_2(t,x) = t(\widehat{p}_k^*(\mathcal{F}(\widehat{x})) - \overline{x}) + (1-t)(\widehat{V} - \widehat{g}I)(x),$$

where for every $x \in H^1(Z)$, we have $x = \overline{x} + \widehat{x}$ with $\overline{x} = \overline{p}_k(x) \in \overline{H}_k$, $\widehat{x} = \widehat{p}_k(x) \in \widehat{H}_k$.

Next we show that $h_2(t,x) \neq 0$ for all $t \in [0,1]$ and all $x \neq 0$. Indeed, since on the finite dimensional space $\overline{H}_k$ all norms are equivalent, we have that

$$\begin{aligned}\langle h_2(t,x), \widehat{x} - \overline{x}\rangle &\geq t\langle \mathcal{F}(\widehat{x}), \widehat{x}\rangle + tc_4\|\overline{x}\|^2 + (1-t)\langle \widehat{V}(x) - \widehat{g}x, \widehat{x} - \overline{x}\rangle \\ &\geq tc_4\|x\|^2 + (1-t)\bigg[\int_Z (\widehat{A}(z)D\widehat{x}, D\widehat{x})_{\mathbb{R}^N}\,dz - \int_Z \widehat{g}\widehat{x}^2\,dz \\ &\qquad + \int_Z \widehat{g}\overline{x}^2\,dz - \int_Z (\widehat{A}(z)D\overline{x}, D\overline{x})_{\mathbb{R}^N}\,dz\bigg],\end{aligned}$$

for some $c_4 \in (0,1)$. Here we have used the orthogonality of the component spaces.

Using Lemma 3.4(a) and (b) we obtain

$$\langle h_2(t,x), \widehat{x} - \overline{x}\rangle \geq tc_4\|x\|^2 + (1-t)\widehat{\xi}\|x\|^2 \geq c_5\|x\|^2,$$

with $\widehat{\xi} = \min\{\xi_1, \xi_2\}$, for for some $c_5 \in (0,1)$, hence

$$h_2(t,x) \neq 0 \quad \text{for all } t \in [0,1] \text{ and all } x \neq 0.$$

Invoking, once again, the homotopy invariance property of the degree map $\widehat{d}$, we have

$$d_{(\mathrm{S})_+}(\widehat{V} - \widehat{g}I, B_r, 0) = d_{(\mathrm{S})_+}(\widehat{p}_k^* \circ \mathcal{F} \circ \widehat{p}_k - \overline{p}_k, B_r, 0) \quad \text{for all } r > 0. \tag{3.19}$$

Set

$$B_{r/2}^{\widehat{H}_k} = \{\widehat{x} \in \widehat{H}_k : \|\widehat{x}\| < r/2\} \quad \text{and} \quad B_{r/2}^{\overline{H}_k} = \{\overline{x} \in \overline{H}_k : \|\overline{x}\| < r/2\}.$$

Then from the excision and product properties of the degree, we have

$$\begin{aligned} &d_{(\mathrm{S})_+}(\widehat{p}_k^* \circ \mathcal{F} \circ \widehat{p}_k - \overline{p}_k, B_r, 0) \\ &\qquad = d_{(\mathrm{S})_+}(\mathcal{F}\,|_{\widehat{H}_k}, B_{r/2}^{\widehat{H}_k}, 0).d_B(-I, B_{r/2}^{\overline{H}_k}, 0) = 1.(-1)^{\dim \overline{H}_k}\end{aligned} \tag{3.20}$$

From (3.18)–(3.20), we conclude that

$$\widehat{d}(V - N, B_R, 0) = (-1)^{\dim \overline{H}_k} \quad \text{for all } R \geq R_0. \qquad \square$$

Next we conduct a similar computation for small balls.

Proposition 3.6. *If hypotheses* H(j) *hold, then there exists* $\rho_0 > 0$ *such that*

$$\widehat{d}(V - N, B_\rho, 0) = 1 \quad \textit{for all } 0 < \rho \leq \rho_0.$$

Proof. Let $A_0 \in L^\infty(Z, \mathbb{R}^{N \times N})$ be defined by $A_0(z) = A(z,0)$. We introduce the continuous linear operator $V_0 \in \mathcal{L}(H^1(Z), H^1(Z)^*)$ defined by

$$\langle V_0(x), y\rangle = \int_Z (A_0(z)Dx, Dy)_{\mathbb{R}^N}\,dz \quad \text{for all } x,\ y \in H^1(Z).$$

We consider the admissible homotopy $h_3\colon [0,1] \times H^1(Z) \to H^1(Z)^*$ defined by

$$h_3(t,x) = tV(x) + (1-t)V_0(x) - tN(x) - (1-t)\widehat{h}x,$$

with $\widehat{h} \in L^\infty(Z)$ satisfying $\widehat{\eta}(z) \le \widehat{h}(z) \le \eta(z)$ almost everywhere on Z.

CLAIM *There exists $\rho_0 > 0$ such that $0 \notin h_3(t,x)$ for all $t \in [0,1]$ and all $0 < \|x\| = \rho \le \rho_0$.*

Again we argue by contradiction. So suppose that the Claim is not true. Then we can find $\{t_n\}_{n\ge1} \subseteq [0,1]$ and $\{x_n\}_{n\ge1} \subseteq H^1(Z)$ such that

$$t_n \to t \in [0,1], \quad \|x_n\| \to 0 \quad \text{and} \quad 0 \in h_3(t_n, x_n) \quad \text{for all } n \ge 1 \tag{3.21}$$

Note that hypothesis H(j)(e) implies that we can find $\delta > 0$ such that $|u| \le c_5|x|$ for almost all $z \in Z$, all $|x| \le \delta$, all $u \in \partial j(z,x)$ and some $c_5 > 0$.

On the other hand from (3.7), we see that there exists $c_6 = c_6(\delta) > 0$ such that $|u| \le c_6|x|$ for almost all $z \in Z$, all $|x| > \delta$ and all $u \in \partial j(z,x)$. Hence we can say that

$$|u| \le c_7|x| \tag{3.22}$$

for almost all $z \in Z$, all $x \in \mathbb{R}$ and all $u \in \partial j(z,x)$, with $c_7 = \max\{c_5, c_6\}$.

From the inclusion in (3.21), we have

$$t_nV(x_n) + (1-t_n)V_0(x_n) = t_nu_n + (1-t_n)\widehat{h}x_n \tag{3.23}$$

with $u_n \in N(x_n)$. We set

$$y_n = \frac{x_n}{\|x_n\|}, \quad n \ge 1.$$

We may assume that

$$y_n \xrightarrow{w} y \quad \text{in } H^1(Z) \quad \text{and} \quad y_n \to y \quad \text{in } L^2(Z) \text{ as } n \to \infty.$$

We divide (3.23) with $\|x_n\|$ and obtain

$$t_n\frac{V(x_n)}{\|x_n\|} + (1-t_n)V_0(y_n) = t_n\frac{u_n}{\|x_n\|} + (1-t_n)\widehat{h}y_n. \tag{3.24}$$

Taking duality brackets with $y_n - y$, we have

$$\begin{aligned} t_n\Big\langle \frac{V(x_n)}{\|x_n\|}, y_n - y\Big\rangle &+ (1-t_n)\langle V_0(y_n), y_n - y\rangle \\ &= t_n\int_Z \frac{u_n}{\|x_n\|}(y_n - y)\,dz + (1-t_n)\int_Z \widehat{h}y_n(y_n-y)\,dz. \end{aligned}$$

Note that

$$\int_Z \frac{u_n}{\|x_n\|}(y_n - y)\,dz \to 0$$

(see (3.22)) and

$$\int_Z \widehat{h}y_n(y_n - y)\,dz \to 0 \quad \text{as } n \to \infty.$$

Since $\|x_n\| \to 0$, we may assume that $x_n(z) \to 0$ almost everywhere on Z and so

$$A(z, x_n(z)) \to A_0(z) \quad \text{a.e. on } Z$$

(see H(A)(b)). Then arguing as in the proof of Proposition 3.5, we show that

$$y_n \to y \text{ in } H^1(Z), \text{ hence } \|y\| = 1, \text{ i.e. } y \neq 0.$$

In addition, for every $v \in H^1(Z)$, we have

$$\left\langle \frac{V(x_n)}{\|x_n\|}, v \right\rangle = \int_Z (A(z, x_n) Dy_n, Dv)_{\mathbb{R}^N} \, dz \\ \to \int_Z (A_0(z) Dy, Dv)_{\mathbb{R}^N} \, dz = \langle V_0(y), v \rangle,$$

hence

$$\frac{V(x_n)}{\|x_n\|} \xrightarrow{w} V_0(y) \quad \text{in } H^1(Z)^*.$$

From (3.22), we see that $\{u_n/\|x_n\|\}_{n\geq 1} \subseteq L^2(Z)$ is bounded. So we may assume that

$$\frac{u_n}{\|x_n\|} \xrightarrow{w} \beta \quad \text{in } L^2(Z) \text{ as } n \to \infty.$$

Given $\varepsilon > 0$ and $n \geq 1$, we introduce the sets

$$D^+_{\varepsilon,n} = \left\{ z \in Z : x_n(z) > 0,\ \widehat{\eta}(z) - \varepsilon \leq \frac{u_n(z)}{x_n(z)} \leq \eta(z) + \varepsilon \right\}$$

and

$$D^-_{\varepsilon,n} = \left\{ z \in Z : x_n(z) < 0,\ \widehat{\eta}(z) - \varepsilon \leq \frac{u_n(z)}{x_n(z)} \leq \eta(z) + \varepsilon \right\}.$$

Since $\|x_n\| \to 0$, we may assume that $x_n(z) \to 0$ almost everywhere on Z. Hence by virtue of hypothesis H(j)(e), we have

$$\chi_{D^+_{\varepsilon,n}}(z) \to 1 \quad \text{a.e. on } \{y > 0\} \quad \text{and} \quad \chi_{D^-_{\varepsilon,n}}(z) \to 1 \quad \text{a.e. on } \{y < 0\}.$$

Arguing as in the proof of Proposition 3.5, we infer that $\beta = h_0 y$, with $h_0 \in L^\infty(Z)$,

$$\widehat{\eta}(z) \leq h_0(z) \leq \eta(z) \quad \text{a.e. on } Z.$$

We pass to the limit as $n \to \infty$ in (3.24) and obtain $V_0(y) = hy$, with $h = th_0 + (1-t)\widehat{h} \in L^\infty(Z)$, $\widehat{\eta}(z) \leq h(z) \leq \eta(z)$ almost everywhere on Z. We take duality brackets with y. So

$$\int_Z (A_0(z) Dy, Dy)_{\mathbb{R}^N} \, dz = \int_Z hy^2 \, dz \leq 0$$

therefore

$$\|Dy\|_2 = 0, \quad \text{i.e. } y = \widehat{c} \in \mathbb{R}$$

(see H(A)(d)). Note that $\widehat{c} \neq 0$ (since $\|y\| = 1$). Hence

$$0 = \int_Z (A_0(z) Dy, Dy)_{\mathbb{R}^N} \, dz = |\widehat{c}|^2 \int_Z h \, dz < 0,$$

a contradiction. Therefore the Claim is true.

The Claim permits the use of the homotopy invariance property and we have

$$\widehat{d}(V-N,B_\rho,0)=d_{(\mathrm{S})_+}(V_0-\widehat{h}I,B_\rho,0)\quad\text{for all } 0<\rho\le\rho_0. \tag{3.25}$$

To compute $d_{(\mathrm{S})_+}(V_0-\widehat{h}I,B_\rho,0)$, we consider the $(\mathrm{S})_+$-homotopy

$$h_4(t,x)=t(V_0-\widehat{h}I)(x)+(1-t)\mathcal{F}(x).$$

Then, for every $t\in[0,1]$ and $x\neq 0$, we have

$$\begin{aligned}\langle h_4(t,x),x\rangle &= t\bigg[\int_Z (A_0(z)Dx,Dx)_{\mathbb{R}^N}\,dz-\int_Z \widehat{h}x^2\,dz\bigg]+(1-t)\|x\|^2\\ &\ge t\xi_0\|x\|^2+(1-t)\|x\|^2>0.\end{aligned}$$

(see Lemma 3.4(c)). Therefore, once again, the homotopy invariance property implies

$$d_{(\mathrm{S})_+}(V_0-\widehat{h}I,B_\rho,0)=d_{(\mathrm{S})_+}(\mathcal{F},B_\rho,0)=1\quad\text{for all } 0<\rho,$$

hence

$$d(V-N,B_\rho,0)=1\quad\text{for all } 0<\rho\le\rho_0$$

(see (3.25)). □

Now, we are ready for the existence result concerning problem (1.1).

Theorem 3.7. *If hypotheses* H(A) *and* H(j) *hold and* $\dim\overline{H}_k$ *is odd, then problem* (1.1) *has a nontrivial solution* $x\in C^1(\overline{Z})$.

Proof. We may assume that $\rho_0<R_0$ and let $0<\rho\le\rho_0$ and $R_0\le R$. Then from the additivity and excision properties of the degree map $\widehat{d}$, we have

$$\widehat{d}(V-N,B_R,0)=\widehat{d}(V-N,B_\rho,0)+\widehat{d}(V-N,B_R\setminus\overline{B}_\rho,0),$$

hence

$$(-1)^{\dim\overline{H}_k}=1+\widehat{d}(V-N,B_R\setminus\overline{B}_\rho,0)$$

(see Propositions 3.5 and 3.6), so

$$\widehat{d}(V-N,B_R\setminus\overline{B}_\rho,0)=-2.$$

So from the solution property, we know that we can find $x\in B_R\setminus\overline{B}_\rho$ such that $0\in V(x)-N(x)$, hence $0=V(x)-u$ with $u\in N(x)$, so

$$\begin{cases}-\mathrm{div}(A(z,x(z))Dx(z))=u(z)\in\partial j(z,x(z)) & \text{a.e. on } Z,\\ \dfrac{\partial x}{\partial n}=0 & \text{on } \partial Z.\end{cases}$$

Moreover, from standard regularity theory we have $x\in C^1(\overline{Z})$ (see for example L. Gasinski and N. S. Papageorgiou [9]). So $x\in C^1(\overline{Z})$ is a nontrivial solution of (1.1) and note that the Neumann boundary condition is understood pointwise. □

Remark 3.8. If $k=0$, then $\overline{H}_k=\mathbb{R}$ and so the Theorem 3.7 applies.

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