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POSITIVE SOLUTIONS OF SINGULARLY PERTURBED NONLINEAR ELLIPTIC PROBLEM ON RIEMANNIAN MANIFOLDS WITH BOUNDARY

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Abstract. Let (M,g) be a smooth connected compact Riemannian manifold of finite dimension $n \geq 2$ with a smooth boundary ∂M. We consider the problem

$$\begin{cases} -\varepsilon^2 \Delta_g u + u = |u|^{p-2}u, \quad u > 0 & \text{on } M, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial M, \end{cases}$$

where ν is an exterior normal to ∂M.

The number of solutions of this problem depends on the topological properties of the manifold. In particular we consider the Lusternik Schnirelmann category of the boundary.

1. Introduction

Let (M,g) be a smooth connected compact Riemannian manifold of finite dimension $n \geq 2$ with a smooth boundary ∂M, that is ∂M is the union of a finite number of connected, smooth, boundaryless, submanifold of M of dimension $n-1$. Here g denotes the Riemannian metric tensor. By Nash theorem we can consider (M,g) embedded as a regular submanifold embedded in $\mathbb{R}^N$. We are

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interested in finding solutions $u \in H^1_g(M)$ of the following singularly perturbed nonlinear elliptic problem

$$
\text{(P)} \qquad \begin{cases} -\varepsilon^2 \Delta_g u + u = |u|^{p-2}u, \quad u > 0, & \text{on } M, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial M, \end{cases}
$$

for $2 < p < 2^* = 2N/(N-2)$, where ν is the external normal to ∂M.

Here $H^1_g(M) = \{u \colon M \to \mathbb{R} : \int_M |\nabla_g u|^2 + u^2 \, d\mu_g < \infty\}$ where μ_g denotes the volume form on M associated to g.

Above type of equations have been extensively studied when M is a flat bounded domain $\Omega \subset \mathbb{R}^N$. We recall some classical result about the Neumann problem in Ω. In [16], [18], [19], C. S. Lin, W. M. Ni and I. Takagi established the existence of least-energy solution to (P) and showed that for ε small enough the least energy solution has a boundary spike. Later, in [11], [21] it was proved that for any stable critical point of the mean curvature of the boundary it is possible to construct single boundary spike layer solutions, while in [12], [15], [22] the authors construct multiple boundary spike solutions. Finally, in [9], [13] the authors proved that for any integer K there exists a boundary K-peaks solutions.

For which concerns the probem (P) on a manifold M, with boundary and without boundary, J. Byeon and J. Park [7] showed that the mountain pass solution u_ε has a spike layer.

A lot of works are devoted to show the influence of the topology of Ω on the number of solutions of the Dirichlet problem

$$
\begin{cases} -\varepsilon^2 \Delta_g u + u = |u|^{p-2}u, \quad u > 0 & \text{on } \Omega \subset \mathbb{R}^N; \\ u = 0 & \text{on } \partial\Omega, \end{cases}
$$

when Ω is a flat subset of $\mathbb{R}^N$. We limit to cite [1]–[3], [5]–[8].

Recently there have been some results on the effect of the topology of the manifold M on the number of solutions of the equation $-\varepsilon^2 \Delta_g u + u = |u|^{p-2}u$ on a manifold M without boundary. In [4] the authors proved that, if M has a rich topology, the equation has multiple solutions. More precisely they show that this equation has at least $\operatorname{cat}(M) + 1$ positive nontrivial solutions for ε small enough. Here $\operatorname{cat}(M)$ is the Lusternik–Schnirelmann category of M. In [20] there is the same result for a more general nonlinearity. Furthermore in [14] it was shown that, for some manifolds, the number of solution is influenced by the topology of a suitable subset of M depending on the geometry of M.

Our result concerns problem (P) on a manifold M with $\partial M \neq \emptyset$. In this case we show that the topology of the boundary ∂M influences the number of solutions, as follows.

Theorem 1.1. *For* ε *small enough the problem* (P) *has at least* $\operatorname{cat}(\partial M)+1$ *non constant distinct solutions.*

The paper is organized as follows. In Section 2 we introduce some notions and notations. In Section 3 we sketch the proof of the main result. The details of the proof are in Sections 4–7.

2. Preliminaries

We consider the C^2 functional defined on $H^1_g(M)$

$$J_\varepsilon(u) = \frac{1}{\varepsilon^N}\int_M \left(\frac{1}{2}\varepsilon^2|\nabla_g u|^2 + \frac{1}{2}|u|^2 - \frac{1}{p}|u^+|^p\right) d\mu_g.$$

where $u^+(x) = \max\{u(x),0\}$. It is well known that the critical points of $J_\varepsilon(u)$ constrained on the associated C^2 Nehari manifold

$$\mathcal{N}_\varepsilon = \{u \in H^1_g \setminus \{0\} : J'_\varepsilon(u)u = 0\}$$

are non trivial solution of problem (P).

Let $\mathbb{R}^n_+ = \{x = (\overline{x}, x_n) : \overline{x} \in \mathbb{R}^{n-1},\ x_n \geq 0\}$. It is known that there exists a least energy solution $V \in H^1(\mathbb{R}^n_+)$ of the equation

$$\begin{cases} -\Delta V + V = |V|^{p-2}V,\ V > 0 \quad \text{on } \mathbb{R}^n_+, \\ \left.\dfrac{\partial V}{\partial x_n}\right|_{(\overline{x},0)} = 0. \end{cases}$$

Moreover, V is radially symmetric and $|D^\alpha V(x)| \leq c\exp(-\mu|x|)$ with $|\alpha| \leq 2$, and c, μ positive constants.

If V is a solution, also $V(x+y)$ with $y = (\overline{y}, 0)$ is a solution, $V_\varepsilon(x) = V(x/\varepsilon)$ is a solution of

$$\begin{cases} -\varepsilon^2\Delta V_\varepsilon + V_\varepsilon = |V_\varepsilon|^{p-2}V_\varepsilon \quad \text{on } \mathbb{R}^n_+, \\ \left.\dfrac{\partial V_\varepsilon}{\partial x_n}\right|_{(\overline{x},0)} = 0. \end{cases}$$

We put

$$m_e^+ = \inf\{E^+(v) : v \in \mathcal{N}(E^+)\} \quad \text{and} \quad m_e = \inf\{E(v) : v \in \mathcal{N}(E)\},$$

where

$$\begin{aligned} E^+(v) &= \int_{\mathbb{R}^n_+} \frac{1}{2}|\nabla v|^2 + \frac{1}{2}|v|^2 - \frac{1}{p}|v^+|^p\,dx; \\ E(v) &= \int_{\mathbb{R}^n} \frac{1}{2}|\nabla v|^2 + \frac{1}{2}|v|^2 - \frac{1}{p}|v^+|^p\,dx \end{aligned}$$

and

$$\begin{aligned} \mathcal{N}(E^+) &= \{v \in H^1(\mathbb{R}^n_+) \setminus \{0\} : E^+(v)v = 0\}; \\ \mathcal{N}(E) &= \{v \in H^1(\mathbb{R}^n) \setminus \{0\} : E(v)v = 0\}. \end{aligned}$$

It holds

$$m_e = 2m_e^+ \quad \text{and} \quad m_e^+ = E^+(V) = \left(\frac{1}{2} - \frac{1}{p}\right)(S_e^+)^{p/(p-2)}$$

where $S_e^+ = \inf\{\|v\|^2_{H^1(\mathbb{R}^n_+)}/\|v\|^2_{L^p(\mathbb{R}^n_+)}, v \neq 0\}$.

Remark 2.1. On the tangent bundle of any compact Riemannian manifold $\mathcal{M}$ it is defined the exponential map $\exp : T\mathcal{M} \to \mathcal{M}$ which is of class C^∞. Moreover, there exists a constant $R > 0$ and a finite number of $x_i \in \mathcal{M}$ such that $\mathcal{M} = \bigcup_{i=1}^l B_g(x_i, R)$ and $\exp_{x_i}: B(0,R) \to B_g(x_i, R)$ is a diffeormophism for all i.

By choosing an orthogonal coordinate system $(y_1, \ldots, y_n)$ of $\mathbb{R}^n$ and identifying $T_{x_0}\mathcal{M}$ with $\mathbb{R}^n$ for $x_0 \in \mathcal{M}$ we can define by the exponential map the so called normal coordinates. For $x_0 \in \mathcal{M}$, g_{x_0} denotes the metric read through the normal coordinates. In particular, we have $g_{x_0}(0) = \mathrm{id}$. We set $|g_{x_0}(y)| = \det(g_{x_0}(y))_{ij}$ and $g^{ij}_{x_0}(y) = ((g_{x_0}(y))_{ij})^{-1}$.

Remark 2.2. If q belongs to the boundary ∂M, let $\overline{y} = (y_1, \ldots, y_{n-1})$ be Riemannian normal coordinates on the $n-1$ manifold ∂M at the point q. For a point $\xi \in M$ close to q, there exists a unique $\overline{\xi} \in \partial M$ such that $d_g(\xi, \partial M) = d_g(\xi, \overline{\xi})$. We set $\overline{y}(\xi) \in \mathbb{R}^{n-1}$ the normal coordinates for $\overline{\xi}$ and $y_n(\xi) = d_g(\xi, \partial M)$. Then we define a chart $\psi^\partial_q : \mathbb{R}^n_+ \to M$ such that $(\overline{y}(\xi), y_n(\xi)) = (\psi^\partial_q)^{-1}(\xi)$. These coordinates are called *Fermi coordinates* at $q \in \partial M$. The Riemannian metric $g_q(\overline{y}, y_n)$ read through the Fermi coordinates satisfies $g_q(0) = \mathrm{id}$.

In the following we choose $\rho > 0$ such that in the subset $(\partial M)_\rho := \{x \in M : d_g(x, \partial M) < \rho\}$ the Fermi coordinates are well defined. Moreover, we choose ρ small enough such that 3ρ is smaller than the radius $\rho(\partial M)$ of topological invariance of ∂M, defined below.

Definition 2.3. The radius of topological invariance $\rho(\mathcal{M})$ of $\mathcal{M} \subset \mathbb{R}^N$ is

$$\rho(\mathcal{M}) := \sup\{\rho > 0 : \mathrm{cat}\,((\mathcal{M})_\rho) = \mathrm{cat}\,(\mathcal{M})\}$$

where $(\mathcal{M})_\rho := \{x \in \mathbb{R}^N : d(x, \mathcal{M}) < \rho\}$

Fixed ρ, using Remark 2.1, we can choose R_M such that $\bigcup_{i=1}^l B_g(x_i, R_M)$ covers $M \setminus (\partial M)_\rho$, and $R_M < \rho$. We note by d^∂_g and $\exp^\partial$, respectively, the geodesic distance and the exponential map on by ∂M. By compactness of ∂M, there is an R^∂ and a finite number of points $q_i \in \partial M$, $i = 1, \ldots, k$ such that

$$I_{q_i}(R^\partial, \rho) := \{x \in M : d_g(x, \partial M) = d_g(x, \overline{\xi}) < \rho,\ d^\partial_g(q_i, \overline{\xi}) < R^\partial\}$$

form a covering of $(\partial M)_\rho$ and on every I_{q_i} the Fermi coordinates are well defined. In the following we can choose without loss of generality, $R = \min\{R^\partial, R_M\} < \rho$.

3. Main tools for the proof

Using the notation of the previous section we can state our main result more precisely.

Theorem 3.1. *There exists $\delta_0 \in (0, m_e^+)$ and $\varepsilon_0 > 0$ such that, for $\delta \in (0, \delta_0)$ and $\varepsilon \in (0, \varepsilon_0)$, where $\varepsilon = \varepsilon(\delta)$, the functional J_ε has at least* $\operatorname{cat}(\partial M)$ *critical points $u \in \mathcal{N}_\varepsilon \subset H_g^1(M)$ satisfying $J_\varepsilon(u) < m_e^+ + \delta$ and at least a non constant critical point with $m_e^+ + \delta \leq J_\varepsilon(u)$.*

We recall the definition of Lusternik–Schnirelmann category.

Definition 3.2. Let M a topological space and consider a closed subset $A \subset M$. We say that A has category k relative to M ($\operatorname{cat}_M A = k$) if A is covered by k closed sets A_j, $j = 1, \ldots, k$, which are contractible in M, and k is the minimum integer with this property.

Remark 3.3. Let M_1 and M_2 be topological spaces. If $g_1 \colon M_1 \to M_2$ and $g_2 \colon M_2 \to M_1$ are continuous operators such that $g_2 \circ g_1$ is homotopic to the identity on M_1, then $\operatorname{cat} M_1 \leq \operatorname{cat} M_2$. For the proof see [5].

We recall the following classical result (see for example [6]).

Theorem 3.4. *Let J be a $C^{1,1}$ real functional on a complete $C^{1,1}$ manifold $\mathcal{N}$. If J is bounded from below and satisfies the Palais–Smale condition then has at least* $\operatorname{cat}(J^d)$ *critical point in J^d where $J^d = \{u \in \mathcal{N} : J(u) < d\}$. Moreover, if $\mathcal{N}$ is contractible and* $\operatorname{cat} J^d > 1$, *there exists at least one critical point $u \notin J^d$.*

Applying the first claim of Theorem 3.4 to the functional J_ε on the manifold $\mathcal{N}_\varepsilon$ we obtain $\operatorname{cat} \mathcal{N}_\varepsilon \cap J_\varepsilon^{m_e^+ + \delta}$ critical points of J_ε. By the following Lemma we give an estimate of $\operatorname{cat} \mathcal{N}_\varepsilon \cap J_\varepsilon^{m_e^+ + \delta}$ through the topological properties of the boundary of M.

Lemma 3.5. *For δ and ε small enough we have* $\operatorname{cat}(\partial M) \leq \operatorname{cat} \mathcal{N}_\varepsilon \cap J_\varepsilon^{m_e^+ + \delta}$.

We are able to obtain the proof of this lemma building two suitable maps. To this aim we recall that by Nash embedding theorem [17] we may assume that M is embedded in a Euclidean space $\mathbb{R}^N$.

Hence the lemma follows by building a map $\Phi_\varepsilon \colon \partial M \to \mathcal{N}_\varepsilon \cap J_\varepsilon^{m_e^+ + \delta}$ and a map $\beta \colon \mathcal{N}_\varepsilon \cap J_\varepsilon^{m_e^+ + \delta} \to (\partial M)_\rho$ with $0 < \rho < \rho(\partial M)$ such that $\beta \circ \Phi_\varepsilon \colon \partial M \to (\partial M)_\rho$ is homotopic to the identity on ∂M (see Sections 4–6). Then by the properties of the category we get $\operatorname{cat}(\partial M) \leq \operatorname{cat} \mathcal{N}_\varepsilon \cap J_\varepsilon^{m_e^+ + \delta}$.

To finish the proof of Theorem 3.1 we build a set T_ε (Section 7) such that

$$\Phi_\varepsilon(\partial M) \subset T_\varepsilon \subset \mathcal{N}_\varepsilon \cap J_\varepsilon^{c_\varepsilon}$$

for a bounded constant $c_\varepsilon \leq c$, and such that T_ε is a contractible set in $\mathcal{N}_\varepsilon \cap J_\varepsilon^{c_\varepsilon}$ containing only positive functions. Since $1 < \operatorname{cat}(\partial M) \leq \operatorname{cat}(\Phi_\varepsilon(\partial M))$ by the same argument of Theorem 3.4 there exists a critical point $\overline{u}$ of J_ε in $\mathcal{N}_\varepsilon$ such that $m_e^+ + \delta \leq J_\varepsilon(\overline{u}) \leq c_\varepsilon$.

It remains to show that the critical points we have found are non-constant functions. This follows immediately from the fact that the only constant function on the Nehari manifold $\mathcal{N}_\varepsilon$ is the function $\overline{v}(x) \equiv 1$, for which

$$J_\varepsilon(\overline{v}) = \left(\frac{1}{2} - \frac{1}{p}\right)\frac{\mu_g(M)}{\varepsilon^n} \to \infty \quad \text{as } \varepsilon \to 0.$$

Hence the constant solution is excluded because c_ε is bounded.

Notation. We will use the following notation:

- $||u||_g = ||u||_{H_g^1} = \int_M |\nabla_g u|^2 + |u|^2 d\mu_g,\ \ |u|_{p,g}^p = \int_M |u|^p d\mu_g,$
- $|||u|||_\varepsilon = |||u|||_{\varepsilon,M} = \frac{1}{\varepsilon^n}\int_M \varepsilon^2 |\nabla_g u|^2 + |u|^2 d\mu_g,\ \ |u|_{p,\varepsilon}^p = \frac{1}{\varepsilon^n}\int_M |u|^p d\mu_g,$
- $|u|_p^p = \int_{\mathbb{R}^n} |u|^p\, dx,$
- If $A, B \subset \mathbb{R}^n$, then $A\Delta B := A \setminus B \cup B \setminus A$,
- d_g is the geodesic distance on M, and d_g^∂ is the geodesic distance on ∂M,
- $\exp^\partial$ is the exponential map on ∂M,
- $I_q(R,\rho) = \{\chi \in M : d_g(\chi, \partial M) < \rho,\ d_g^\partial(\overline{\chi}, q) < R\}$, where $\overline{\chi} \in \partial M$ is the unique point such that $d_g(\chi, \overline{\chi}) = d_g(\chi, \partial M)$,
- $B(x,R) \subset \mathbb{R}^n$ is the ball centered in x of radius R,
- $B_{n-1}(x,R) \subset \mathbb{R}^{n-1}$ is the $n-1$ ball centered in x of radius R.

4. The map Φ_ε

Let us define $\chi_R: \mathbb{R}^+ \to \mathbb{R}^+$ a smooth cut off function such that $\chi_R(t) \equiv 1$ if $0 \leq t \leq R/2$, $\chi_R(t) \equiv 0$ if $R \leq t$, and $|\chi_R'(t)| \leq 2/R$ for all t. Fixed a point $q \in \partial M$ and $\varepsilon > 0$, let us define on M the function $Z_{\varepsilon,q}(\xi)$ as

$$Z_{\varepsilon,q}(\xi) = \begin{cases} V_\varepsilon(y(\xi))\chi_R(|\overline{y}(\xi)|)\chi_\rho(y_n(\xi)) & \text{if } \xi \in I_q, \\ 0 & \text{otherwise,} \end{cases} \tag{4.1}$$

where

$$I_q(R,\rho) = I_q = \{\xi \in M : y_n = d_g(\xi, \partial M) < \rho \text{ and } |\overline{y}| = d_g^\partial(\exp_q^\partial(\overline{y}(\xi)), q) < R\}.$$

Here $y(\xi) = (\overline{y}(\xi), y_n(\xi)) = (\psi_q^\partial)^{-1}(\xi)$ are the Fermi coordinates at $q \in \partial M$ and $\exp_q^\partial: T_q(\partial M) \to \partial M$, is the exponential map on ∂M.

For each $\varepsilon > 0$ we can define a positive number $t_\varepsilon(Z_{\varepsilon,q})$ such that $t_\varepsilon(Z_{\varepsilon,q})Z_{\varepsilon,q}$ in $H_g^1(M) \cap \mathcal{N}_\varepsilon$. Namely, $t_\varepsilon(Z_{\varepsilon,q})$ turns out to verify

$$t_\varepsilon(Z_{\varepsilon,q}) = \left(\frac{|||Z_{\varepsilon,q}|||_\varepsilon^2}{|Z_{\varepsilon,q}|_{p,\varepsilon}^p}\right)^{1/(p-2)}.$$

Thus we can define a function $\Phi_\varepsilon\colon \partial M \to \mathcal{N}_\varepsilon$, $\Phi_\varepsilon(q) = t_\varepsilon(Z_{\varepsilon,q})Z_{\varepsilon,q}$.

PROPOSITION 4.1. *For any $\varepsilon > 0$ the application $\Phi_\varepsilon\colon \partial M \to \mathcal{N}_\varepsilon$ is continuous. Moreover, for any $\delta > 0$ there exists $\varepsilon_0 = \varepsilon_0(\delta) > 0$ such that, if $\varepsilon < \varepsilon_0$ then*

$$\Phi_\varepsilon(q) \in \mathcal{N}_\varepsilon \cap J_\varepsilon^{m_e^+ + \delta} \quad \textit{for all } q \in \partial M.$$

PROOF. Fixed $\varepsilon > 0$, by the continuity of $u \to t_\varepsilon(u)$ on $H_g^1(M)$ it is enough to prove that for any sequence $\{q_k\} \subset \partial M$ convergent to q we have

$$\lim_{k\to\infty} \|Z_{\varepsilon,q_k} - Z_{\varepsilon,q}\|_{H_g^1} = 0.$$

Since q_k converges to q, we have $\mu_g(I_{q_k} \Delta I_q) \to 0$ as $k \to \infty$, then we have

$$\int_{I_{q_k}\Delta I_q} |Z_{\varepsilon,q_k} - Z_{\varepsilon,q}|^2 \, d\mu_g \to 0 \quad \text{as } k \to \infty.$$

Now, setting $\eta_k(\overline{y}, y_n) = (\psi_{q_k}^\partial)^{-1}(\psi_q^\partial(\overline{y}, y_n))$ and $A_k = (\psi_q^\partial)^{-1}(I_{q_k} \cap I_q)$ we can write

$$\begin{aligned}
\int_{I_{q_k}\cap I_q} &|Z_{\varepsilon,q_k}(x) - Z_{\varepsilon,q}(x)|^2 \, d\mu_g \\
&= \int_{A_k} |V_\varepsilon(\eta_k(\overline{y}, y_n))\chi_R(|\pi_{\mathbb{R}^{n-1}}\eta_k(\overline{y}, y_n)|)\chi_\rho(d_g(q_k, \partial M)) \\
&\quad - V_\varepsilon(\overline{y}, y_n)\chi_R(|\overline{y}|)\chi_\rho(d_g(q, \partial M))|^2 |g_q(\overline{y}, y_n)|^{1/2} \, d\overline{y} \, dy_n \\
&\qquad \le c\int_{A_k} |\eta_k(\overline{y}, y_n) - (\overline{y}, y_n)|^2 d\overline{y} \, dy_n
\end{aligned}$$

for a suitable constant c coming from the mean value theorem applied to V_ε, χ_ρ, χ_R. By the definition of η_k and the smoothness of the exponential map we get

$$\|Z_{\varepsilon,q_k} - Z_{\varepsilon,q}\|_{L_g^2} \to 0 \quad \text{as } k \to \infty.$$

A similar argument can be used to show that $\|\nabla_g Z_{\varepsilon,q_k} - \nabla_g Z_{\varepsilon,q}\|_{L_g^2} \to 0$ as $k \to \infty$.

To prove the second statement of the theorem we first show that the following limits hold uniformly with respect to $q \in \partial M$.

$$\lim_{\varepsilon\to 0} \|Z_{\varepsilon,q}\|_{2,\varepsilon}^2 = \int_{\mathbb{R}_+^n} V^2(y) \, dy, \tag{4.2}$$

$$\lim_{\varepsilon\to 0} \|Z_{\varepsilon,q}\|_{p,\varepsilon}^p = \int_{\mathbb{R}_+^n} V^p(y) \, dy, \tag{4.3}$$

$$\lim_{\varepsilon\to 0}\varepsilon^2\|\nabla Z_{\varepsilon,q}\|^2_{2,\varepsilon} = \int_{\mathbb{R}^n_+}|\nabla V|^2(y)\,dy, \tag{4.4}$$

where $\|u\|_{q,\varepsilon}=(1/\varepsilon^n)\|u\|_{L^q}$. For (4.2) we have

$$\begin{aligned}
&\frac{1}{\varepsilon^n}\int_M |Z_{\varepsilon,q}(x)|^2\,d\mu_g\\
&=\frac{1}{\varepsilon^n}\int_{|\overline{y}|<R,\,0<y_n<\rho} V^2_\varepsilon(\overline{y},y_n)\chi^2_R(|\overline{y}|)\chi^2_\rho(y_n)|g_q(\overline{y},y_n)|^{1/2}d\overline{y}\,dy_n\\
&=\int_{|\overline{z}|<R/\varepsilon,\,0<z_n<\rho/\varepsilon} V^2(\overline{z},z_n)\chi^2_{R/\varepsilon}(|\overline{z}|)\chi^2_{\rho/\varepsilon}(z_n)|g_q(\varepsilon(\overline{z},z_n))|^{1/2}\,d\overline{z}\,dz_n\\
&=\int_{B_K} V^2(\overline{z},z_n)\chi^2_{R/\varepsilon}(|\overline{z}|)\chi^2_{\rho/\varepsilon}(z_n)|g_q(\varepsilon(\overline{z},z_n))|^{1/2}\,d\overline{z}\,dz_n\\
&\quad+\int_{\mathbb{R}^n_+\setminus B_K} V^2(\overline{z},z_n)\chi^2_{R/\varepsilon}(|\overline{z}|)\chi^2_{\rho/\varepsilon}(z_n)|g_q(\varepsilon(\overline{z},z_n))|^{1/2}\,d\overline{z}\,dz_n,
\end{aligned}$$

where $B_k=B(0,K)\cap\{z_n>0\}$. It is easy to see that the second addendum vanishes when $K\to\infty$. With respect to the first addendum, fixed K large enough, by compactness of manifold M and regularity of the exponential map and of the Riemannian metric g we have, for $\varepsilon\to 0$,

$$\int_{B_K} V^2(\overline{z},z_n)\chi^2_{R/\varepsilon}(|\overline{z}|)\chi^2_{\rho/\varepsilon}(z_n)|g_{\psi_q^\partial}(\varepsilon(\overline{z},z_n))|^{1/2}\,d\overline{z}\,dz_n \to \int_{B_K} V^2(y)\,dy$$

uniformly with respect to $q\in\partial M$. So we proved (4.2). In the same way we can prove (4.3) and (4.4).

At this point we observe that

$$J_\varepsilon(t_\varepsilon(Z_{\varepsilon,q})Z_{\varepsilon,q})=\left(\frac{1}{2}-\frac{1}{p}\right)[t_\varepsilon(Z_{\varepsilon,q})]^p\|Z_{\varepsilon,q}\|^p_{\varepsilon,p}.$$

By definition of $t_\varepsilon(Z_{\varepsilon,q})$ and by (4.2)–(4.4) we have that $t_\varepsilon(Z_{\varepsilon,q})\to 1$ as $\varepsilon\to 0$, uniformly with respect to $q\in\partial M$. Concluding we have

$$\lim_{\varepsilon\to 0}J_\varepsilon(t_\varepsilon(Z_{\varepsilon,q})Z_{\varepsilon,q})=\left(\frac{1}{2}-\frac{1}{p}\right)\int_{\mathbb{R}^n_+}V^p(y)\,dy=m_e^+$$

uniformly with respect to $q\in\partial M$. □

Remark 4.2. By Proposition 4.1, given δ, we have that $\mathcal{N}_\varepsilon\cap J_\varepsilon^{m_e^++\delta}\neq\emptyset$ for ε small enough. Moreover, let $m_\varepsilon:=\inf\{J_\varepsilon(u):u\in\mathcal{N}_\varepsilon\}$. At this point we have

$$\limsup_{\varepsilon\to 0} m_\varepsilon\le m_e^+.$$

5. Concentration properties

In this section we will show a property of concentration of the functions $u \in \mathcal{N}_\varepsilon \cap J_\varepsilon^{m_e^+ + \delta}$ when ε and δ are sufficiently small. This concentration property will be crucial to verify that the barycenter $\beta(u)$ (see Section 6) of the functions $u \in \mathcal{N}_\varepsilon \cap J_\varepsilon^{m_e^+ + \delta}$ is close to the boundary ∂M.

For any $\varepsilon > 0$ we can construct a finite closed partition $\mathcal{P}^\varepsilon = \{P_j^\varepsilon\}_{j \in \Lambda_\varepsilon}$ of M such that

- P_j^ε is closed for every j,
- $P_j^\varepsilon \cap P_k^\varepsilon \subset \partial P_j^\varepsilon \cap \partial P_k^\varepsilon$ for $j \neq k$,
- $K_1 \varepsilon \leq d_j^\varepsilon \leq K_2 \varepsilon$, where d_j^ε is the diameter of P_j^ε,
- $c_1 \varepsilon^n \leq \mu_g(P_j^\varepsilon) \leq c_2 \varepsilon^n$,
- for any j there exists an open set $I_j^\varepsilon \supset P_j^\varepsilon$ such that, if $P_j^\varepsilon \cap \partial M = \emptyset$, then $d_g(I_j^\varepsilon, \partial M) > K\varepsilon/2$, while, if $P_j^\varepsilon \cap \partial M \neq \emptyset$, then $I_j^\varepsilon \subset \{x \in M : d_g(x, \partial M) \leq (3/2)K\varepsilon\}$,
- there exists a finite number $\nu(M) \in \mathbb{N}$ such that every $x \in M$ is contained in at most $\nu(M)$ sets I_j^ε, where $\nu(M)$ does not depends on ε.

By compactness of M such a partition exists, at least for small ε. In the following we will choose always $\varepsilon_0(\delta)$ sufficiently small in order to have this partition.

Lemma 5.1. *There exists a constant $\gamma > 0$ such that, for any fixed $\delta > 0$ and for any $\varepsilon \in (0, \varepsilon_0(\delta))$, where $\varepsilon_0(\delta)$ is as in Proposition 4.1, given any partition $\mathcal{P}^\varepsilon$ of M as above, and any function $u \in \mathcal{N}_\varepsilon \cap J_\varepsilon^{m_e^+ + \delta}$, there exists a set $P_j^\varepsilon \subset \mathcal{P}^\varepsilon$ such that*

$$\frac{1}{\varepsilon^n} \int_{P_j^\varepsilon} |u^+|^p d\,\mu_g \geq \gamma > 0.$$

Proof. By Remark 4.1 we have that $\mathcal{N}_\varepsilon \cap J_\varepsilon^{m_e^+ + \delta} \neq \emptyset$. For any function $u \in \mathcal{N}_\varepsilon \cap J_\varepsilon^{m_e^+ + \delta}$ we denote by u_j^+ the restriction of u^+ to the set P_j^ε. Then we can write

$$\begin{aligned} \frac{1}{\varepsilon^n} \int_M (\varepsilon^2 |\nabla_g u|^2 + u^2)\, d\mu_g &= \frac{1}{\varepsilon^n} \int_M (u^+)^p\, d\mu_g = \frac{1}{\varepsilon^n} \sum_j \int_M (u_j^+)^p\, d\mu_g \\ &= \sum_j \frac{|u_j^+|_p^{p-2}}{\varepsilon^{n(p-2)/p}} \frac{|u_j^+|_p^2}{\varepsilon^{2n/p}} \leq \max_j \left\{ \frac{|u_j^+|_p^{p-2}}{\varepsilon^{n(p-2)/p}} \right\} \sum_j \frac{|u_j^+|_p^2}{\varepsilon^{2n/p}}. \end{aligned}$$

We define the functions $\widetilde{u}_j$ by using a smooth real cutoff function $\chi_\varepsilon^j \colon M \to [0,1]$ such that $|\nabla_g \chi_\varepsilon^j| \leq K/\varepsilon$ for some constant K and, if $P_j^\varepsilon \cap \partial M = \emptyset$, then $\chi_\varepsilon^j = 1$ for $x \in P_j^\varepsilon$ and $\chi_\varepsilon^j = 0$ for $x \in M \setminus I_j^\varepsilon$, while if $P_j^\varepsilon \cap \partial M \neq \emptyset$, then $\chi_\varepsilon^j = 1$ for $x \in P_j^\varepsilon$ and $\chi_\varepsilon^j = 0$ for $M \setminus \overline{I}_j^\varepsilon$ and $x \in \partial I_j^\varepsilon \cap (M \setminus \partial M)$. So we define

$$\widetilde{u}_j(x) = u^+(x) \chi_\varepsilon^j(x).$$

It holds $\widetilde{u}_j \in H^1_g(M)$, hence using Sobolev inequalities there exists a positive constant C such that, for any j,

$$\frac{|u_j^+|_p^2}{\varepsilon^{2n/p}} \le \frac{|\widetilde{u}_j|_p^2}{\varepsilon^{2n/p}} \le C|||\widetilde{u}_j|||_\varepsilon^2 = C|||\widetilde{u}_j|||^2_{\varepsilon,P_j^\varepsilon} + C|||\widetilde{u}_j|||^2_{\varepsilon,I_j^\varepsilon\setminus P_j^\varepsilon}.$$

Moreover,

$$\int_{I_j^\varepsilon\setminus P_j^\varepsilon} |\widetilde{u}_j|^2\,d\mu_g \le \int_{I_j^\varepsilon\setminus P_j^\varepsilon} |u^+|^2\,d\mu_g,$$
$$\int_{I_j^\varepsilon\setminus P_j^\varepsilon} \varepsilon^2|\nabla\widetilde{u}_j|^2\,d\mu_g \le \int_{I_j^\varepsilon\setminus P_j^\varepsilon} (\varepsilon^2|\nabla u^+|^2 + K^2|u^+|^2)\,d\mu_g.$$

Hence we obtain

$$\begin{aligned}\sum_j \frac{|u_j^+|_p^2}{\varepsilon^{2n/p}} &\le C\sum_j |||u^+|||_\varepsilon^2 + C(K^2+1)\nu(M)|||u^+|||_\varepsilon^2 \\ &\le C(K^2+2)\nu(M)\frac{1}{\varepsilon^n}\int_M (\varepsilon^2|\nabla u|^2 + |u|^2)\,d\mu_g.\end{aligned}$$

We can conclude that

$$\max_j\left\{\left(\frac{1}{\varepsilon^n}\int_{P_j^\varepsilon}|u^+|^p\,d\mu_g\right)^{(p-2)/p}\right\} \ge \frac{1}{C(K^2+2)\nu(M)},$$

so the proof is complete. $\square$

Remark 5.2. Let δ and ε fixed. For any $u \in \mathcal{N}_\varepsilon \cap J_\varepsilon^{m_\varepsilon+2\delta}$ there exists $u_\delta \in \mathcal{N}_\varepsilon$ such that

$$J_\varepsilon(u_\delta) < J_\varepsilon(u), \quad |||u_\delta - u|||_\varepsilon < 4\sqrt{\delta}, \quad |(J_{\varepsilon|\mathcal{N}_\varepsilon})'(u_\delta)[\xi]| < \sqrt{\delta}|||\xi|||_\varepsilon.$$

This is simply the application of Ekeland variational principle (see [10]) to the functional J_ε on the manifold $\mathcal{N}_\varepsilon$.

Proposition 5.3. *For all $\eta \in (0,1)$ there exists a $\delta_0 < m_e^+$ such that for any $\delta \in (0,\delta_0)$ for any $\varepsilon \in (0,\varepsilon_0(\delta))$ (as in Proposition 4.1) and for any function $u \in \mathcal{N}_\varepsilon \cap J_\varepsilon^{m_e^+ + \delta}$ we can find a point $q = q(u) \in \partial M$ for which*

$$\left(\frac{1}{2} - \frac{1}{p}\right)\frac{1}{\varepsilon^n}\int_{I_q(\rho,R)} |u^+|^p\,d\mu_g \ge (1-\eta)m_e^+$$

where $I_q(\rho,R)$ is defined in the notation paragraph.

Proof. We prove this property for $u \in \mathcal{N}_\varepsilon \cap J_\varepsilon^{m_e^+ + \delta} \cap J_\varepsilon^{m_\varepsilon + 2\delta}$. From the thesis for these functions follows that

$$m_\varepsilon \ge (1-\eta)m_e^+. \tag{5.1}$$

By (5.1) and by Remark 4.2 we have that

$$\lim_{\varepsilon\to 0} m_\varepsilon = m_e^+.$$

Thus $J_\varepsilon^{m_e^+ +\delta} \subset J_\varepsilon^{m_\varepsilon+2\delta}$ for ε, δ small enough, and the general case is proved.

The proof is by contradiction. Hence we assume that there exists $\eta \in (0,1)$, two sequences of vanishing real numbers $\{\delta_k\}_k$ and $\{\varepsilon_k\}_k$ and a sequence of functions $\{u_k\}_k \subset \mathcal{N}_{\varepsilon_k} \cap J_{\varepsilon_k}^{m_e^+ +\delta_k} \cap J_{\varepsilon_k}^{m_{\varepsilon_k}+2\delta_k}$ such that, for any $q \in \partial M$ it holds

$$\left(\frac{1}{2}-\frac{1}{p}\right)\frac{1}{\varepsilon_k^n}\int_{I_q(\rho,R)} |u_k^+|^p\, d\mu_g < (1-\eta)m_e^+. \tag{5.2}$$

By Remark 5.2 and by definition of $\mathcal{N}_{\varepsilon_k}$ we can assume

$$J'_{\varepsilon_k}(u_k)[\varphi] \le \sqrt{\delta_k}|||\varphi|||_{\varepsilon_k} \quad \text{for all } \varphi \in H_g^1(M).$$

By Lemma 5.1 there exists a set $P_k^{\varepsilon_k} \in \mathcal{P}_{\varepsilon_k}$ such that

$$\frac{1}{\varepsilon_k^n}\int_{P_k^{\varepsilon_k}} |u_k^+|^p\, d\mu_g \ge \gamma > 0.$$

we have to examine two cases: either there exists a subsequence $P_{i_k}^{\varepsilon_{i_k}}$ such that $P_{i_k}^{\varepsilon_{i_k}} \cap \partial M \neq \emptyset$, or there exists a subsequence $P_{i_k}^{\varepsilon_{i_k}}$ such that $P_{i_k}^{\varepsilon_{i_k}} \cap \partial M = \emptyset$. For simplicity we write simply P_k for $P_{i_k}^{\varepsilon_{i_k}}$.

Case 1. $P_k \cap \partial M \neq \emptyset$. We choose a point q_k interior to $P_k \cap \partial M$. We have the Fermi coordinates $\psi_{q_k}^\partial : B_{n-1}(0,R)\times[0,\rho] \to M$, $\psi_{q_k}^\partial(\overline{y},y_n) = (\overline{x},x_n) = x$. We consider the function $w_k : \mathbb{R}_+^n \to \mathbb{R}$ defined by

$$u_k(\psi_{q_k}^\partial(\overline{y},y_n))\chi_R(|\overline{y}|)\chi_\rho(y_n) = u_k(\psi_{q_k}^\partial(\varepsilon_k\overline{z},\varepsilon z_n))\chi_R(|\varepsilon_k\overline{z}|)\chi_\rho(\varepsilon z_n) = w_k(\overline{z},z_n).$$

It is clear that $w_k \in H^1(\mathbb{R}_+^n)$ with $w_k(\overline{z},z_n)=0$ when $|\overline{z}| = 0, R/\varepsilon_k$ or $z_n = \rho/\varepsilon_k$. We now show some properties of the function w_k.

Step 1. *There exists a $w \in H^1(\mathbb{R}_+^n)$ such that the sequence w_k converges weakly in $H^1(\mathbb{R}_+^n)$ and strongly in $L^p_{\mathrm{loc}}(\mathbb{R}_+^n)$.*

We have the following inequality

$$\begin{aligned}\frac{1}{\varepsilon_k^n}&\int_M |u_k|^2 d\mu_g \\ &\ge \frac{1}{\varepsilon_k^n}\int_{B_{n-1}(0,R)\times[0,\rho]} |u_k(\psi_{q_k}^\partial(y))|^2\chi_R^2(|\overline{y}|)\chi_\rho^2((y_n))|g_{q_k}(y)|^{1/2}\,dy \\ &= \int_{B_{n-1}(0,R/\varepsilon_k)\times[0,\rho/\varepsilon_k]} |w_k|^2|g_{q_k}(\varepsilon z)|^{1/2}\,dz \ge c|w_k|^2_{L^2(\mathbb{R}_+^n)}\end{aligned} \tag{5.3}$$

where $z=\varepsilon y$ and $c>0$ is a suitable constant.

For simplicity we set $\widetilde{\chi}(y) = \chi_R(\overline{y})\chi_\rho(y_n)$. We have

$$\begin{aligned}\int_{\mathbb{R}_+^n}|\nabla w_k|^2\,dx &\le 2\int_{\mathbb{R}_+^n}\sum_i\left(\frac{\partial u_k}{\partial z_i}(\varepsilon_k z)\right)^2\widetilde{\chi}^2(\varepsilon_k z)\,dz \\ &\quad + 2\int_{\mathbb{R}_+^n}\sum_i u_k^2(\varepsilon_k z)\left(\frac{\partial\widetilde{\chi}}{\partial z_i}(\varepsilon_k z)\right)^2 dz = I_1 + I_2.\end{aligned}$$

By definition of $\widetilde{\chi}$ and w_k we have

$$(5.4)\quad \frac{\varepsilon_k^2}{\varepsilon_k^n}\int_M |\nabla_g u_k|^2\,d\mu_g \geq \frac{\varepsilon_k^2}{\varepsilon_k^n}\int_{\psi^\partial_{q_k}(B_{n-1}(0,R)\times[0,\rho])} |\nabla_g u_k|^2\,d\mu_g$$
$$= \int_{B_{n-1}(0,R/\varepsilon_k)\times[0,\rho/\varepsilon_k]} \sum_{ij} g^{ij}_{q_k}\frac{\partial u_k}{\partial z_i}(\varepsilon_k z)\frac{\partial u_k}{\partial z_j}(\varepsilon_k z)|g_{q_k}(\varepsilon z)|^{1/2}\,dz \geq cI_1.$$

where c depends only on the Riemannian manifold M. In a similar way we have

$$(5.5)\qquad I_2 \leq \frac{c\varepsilon_k^2}{R^2\rho^2\varepsilon_k^n}\int_M |u_k|^2\,d\mu_g.$$

By (5.3)–(5.5) we get that $\|w_k\|_{H^1(\mathbb{R}^n_+)}$ is bounded. Then we have the claim.

Step 2. *The limit function w is a weak solution of*

$$\begin{cases} -\Delta w + w = (w^+)^{p-1} & \textit{in } \mathbb{R}^n_+,\\ \dfrac{\partial w}{\partial \nu} = 0 & \textit{for } y = (\overline{y},0). \end{cases}$$

Firstly for any $\varphi \in C_0^\infty(\mathbb{R}^n_+)$ we define on the manifold M the function $\widetilde{\varphi}_k(x) := \varphi((1/\varepsilon_k)(\psi^\partial_{q_k})^{-1}(x))$. We have that

$$(5.6)\quad |||\widetilde{\varphi}_k|||_{\varepsilon_k} = \int_{\mathbb{R}^n_+}\bigg[\sum_{ij} g^{ij}_{q_k}(\varepsilon_k z)\frac{\partial\varphi}{\partial z_i}(z)\frac{\partial\varphi}{\partial z_j}(z) + |\varphi(z)|^2\bigg]|g_{q_k}(\varepsilon_k z)|^{1/2}\,dz$$
$$\leq c||\varphi||^2_{H^1(\mathbb{R}^n_+)}$$

where c depends only on M.

We set

$$F_{\varepsilon_k}(v) = \int_{\mathbb{R}^n_+}\bigg[\sum_{ij}\frac{g^{ij}_{q_k}(\varepsilon_k z)}{2}\frac{\partial v}{\partial z_i}(z)\frac{\partial v}{\partial z_j}(z) + \frac{v^2(z)}{2} - \frac{|w_k^+(z)|^p}{p}\bigg]|g_{q_k}(\varepsilon_k z)|^{1/2}\,dz$$

so

$$|F'_{\varepsilon_k}(w_k)[\varphi]| = \int_{\operatorname{supp}\varphi}\bigg[\sum_{ij} g^{ij}_{q_k}(\varepsilon_k z)\frac{\partial w_k}{\partial z_i}(z)\frac{\partial\varphi}{\partial z_j}(z)$$
$$+ (w_k(z) - (w_k^+(z))^{p-1})\varphi(z)\bigg]|g_{q_k}(\varepsilon_k z)|^{1/2}.$$

It is easy to verify that, for $k = k(\varphi)$ large enough,

$$|F'_{\varepsilon_k}(w_k)[\varphi]| = |J'_{\varepsilon_k}(u_k)[\widetilde{\varphi}_k]|.$$

By Ekeland principle (Remark 5.2) and by (5.6) we have that

$$|F'_{\varepsilon_k}(w_k)[\varphi]| = |J'_{\varepsilon_k}(u_k)[\widetilde{\varphi}_k]| \leq \sqrt{\delta_k}|||\widetilde{\varphi}_k|||_{\varepsilon_k} \to 0 \quad \text{as } k\to\infty.$$

At this point to get the claim it is sufficient to show that

$$(5.7)\qquad F'_{\varepsilon_k}(w_k)[\varphi] \to (E^+)'(w)[\varphi].$$

In fact we have

$$|F'_{\varepsilon_k}(w_k)[\varphi] - (E^+)'(w)[\varphi]| \le I_1 + I_2 + I_3,$$

where

$$\begin{aligned}
I_1 &= \int_{\operatorname{supp}\varphi} \bigg(\sum_{ij} g^{ij}_{q_k}(\varepsilon_k z) \frac{\partial w_k}{\partial z_i}(z) \frac{\partial \varphi}{\partial z_j}(z) |g_{q_k}(\varepsilon_k z)|^{1/2} - \delta_{ij} \frac{\partial w}{\partial z_i}(z) \frac{\partial \varphi}{\partial z_j}(z) \bigg)\, dz, \\
I_2 &= \int_{\operatorname{supp}\varphi} |\varphi(z)||g_{q_k}(\varepsilon_k z)|^{1/2} |w_k(z) - w(z)|\, dz, \\
I_3 &= \int_{\operatorname{supp}\varphi} |\varphi(z)||g_{q_k}(\varepsilon_k z)|^{1/2} |(w_k^+(z))^{p-1} - (w(z))^{p-1}|\, dz.
\end{aligned}$$

Because $\operatorname{supp}\varphi$ is a compact set, $|g^{ij}_{q_k}(\varepsilon_k z) - \delta_{ij}| \le c\varepsilon_k |z|^2$ and by Step 1 we get (5.7).

Step 3. *The limit function w is a least energy solution of*

$$\begin{cases} -\Delta w + w = (w^+)^{p-1} & \textit{in } \mathbb{R}^n_+, \\ \dfrac{\partial w}{\partial \nu} = 0 & \textit{for } y = (\overline{y}, 0). \end{cases}$$

We will show that $w \ne 0$. We are in the case $P_k \cap \partial M \ne \emptyset$. We can choose $T > 0$ such that

$$P_k \subset I_{q_k}(\varepsilon_k T, \varepsilon_k T) \quad \text{for } k \text{ large enough}$$

where q_k is a point in P_k. By definition of w_k and by Lemma 5.1 there exist a q_k such that, for k large enough,

$$\begin{aligned}
&\|w_k^+\|_{L^p(B_{n-1}(0,T)\times[0,T])} \\
&\quad = \int_{B_{n-1}(0,T)\times[0,T]} |\chi_R(\varepsilon_k|\overline{z}|)\chi_\rho(\varepsilon_k z_n) u_k^+(\psi^\partial_{q_k}(\varepsilon_k z))|^p\, dz \\
&\quad = \frac{1}{\varepsilon_k^n} \int_{B_{n-1}(0,\varepsilon_k T)\times[0,\varepsilon_k T]} |u_k^+(\psi^\partial_{q_k}(y))|^p\, dy \\
&\quad \ge \frac{c}{\varepsilon_k^n} \int_{B_{n-1}(0,\varepsilon_k T)\times[0,\varepsilon_k T]} |u_k^+(\psi^\partial_{q_k}(y))|^p |g_{q_k}(y)|^{1/2}\, dy \\
&\quad \ge \frac{c}{\varepsilon_k^n} \int_{I_{q_k}(\varepsilon_k T,\varepsilon_k T)} |u_k^+|^p d\mu_g \ge c\gamma > 0.
\end{aligned}$$

Since w_k converge strongly to w in $L^p(B_{n-1}(0,T)\times[0,T])$, we have $w \ne 0$.

We now show that

$$\bigg(\frac{1}{2} - \frac{1}{p}\bigg)|w^+|_p^p \le m_e^+.$$

Since $u_k \in \mathcal{N}_{\varepsilon_k} \cap J_{\varepsilon_k}^{m_e^+ + \delta_k}$, it holds

$$\begin{aligned}\frac{m_e^+ + \delta_k}{1/2 - 1/p} &\geq \frac{1}{1/2-1/p} J_{\varepsilon_k}(u_k) = \frac{1}{\varepsilon_k^n} \int_M |u_k^+|^p \, d\mu_g \\ &\geq \frac{1}{\varepsilon_k^n} \int_{B_{n-1}(q_k, R/2)\times[0,\rho/2]} |u_k^+(\psi_{q_k}^\partial(y))|^p |g_{q_k}(y)|^{1/2} \, dy \\ &= \int_{B_{n-1}(q_k, R/2\varepsilon_k)\times[0,\rho/2\varepsilon_k]} |u_k^+(\psi_{q_k}^\partial(\varepsilon_k z))|^p |g_{q_k}(\varepsilon_k z)|^{1/2} \, dz.\end{aligned}$$

We set

$$f_k(z) = u_k^+(\psi_{q_k}^\partial(\varepsilon_k z)) |g_{q_k}(\varepsilon_k z)|^{1/2} \zeta_k(z)$$

where ζ_k is the characteristic function of the set $B_{n-1}(q_k, R/\varepsilon_k) \times [0, \rho/\varepsilon_k]$. The sequence of function f_k is bounded in $L^p(\mathbb{R}^n_+)$, hence, up to subsequence, converges weakly to some $f \in L^p(\mathbb{R}^n_+)$. We get, for any $\varphi \in C_0^\infty(\mathbb{R}^n_+)$,

$$\int_{\mathbb{R}^n_+} f_k(z)\varphi(z)\,dz \to \int_{\mathbb{R}^n_+} w^+(z)\varphi(z)\,dz \quad \text{as } k\to\infty.$$

Hence f is equal to the positive function $w^+ = w \neq 0$. Moreover, we have

$$\left(\frac{1}{2} - \frac{1}{p}\right)|w|_p^p \leq \liminf_{k\to\infty} \left(\frac{1}{2}-\frac{1}{p}\right) \int_{\mathbb{R}^n_+} |f_k(z)|^p \, dz \leq m_e^+.$$

Concluding $w \in \mathcal{N}^+$ and $E^+(w) \leq m_e^+$, so w is a least energy solution.

Conclusion of the Case 1. *At this point we can show that, for any $T > 0$, it holds, for k large enough,*

$$\left(\frac{1}{2}-\frac{1}{p}\right)|w_k|^p_{L^p(B_{n-1}(0,T)\times[0,T])} \leq \frac{2}{3}(1-\eta)m_e^+.$$

In fact we recall that for any $q \in \partial M$ the Riemannian metric $g_q(y)$ read through the Fermi coordinates is such that $g_q(\varepsilon_k z) = 1 + O(\varepsilon_k|z|)$. Hence fixed T

$$|g_q(\varepsilon_k z)|^{-1/2} \leq \frac{2}{3} \quad \text{for } k \text{ big enough and for } z \in B_{n-1}(0,T)\times[0,T].$$

By this fact, using the definition of w_k and (5.2) we have, for k large,

(5.8)
$$\begin{aligned}|w_k^+|^p_{L^p(B_{n-1}(0,T)\times[0,T])} &\leq \int_{B_{n-1}(0,T)\times[0,T]} |u_k^+(\psi_{q_k}^\partial(\varepsilon_k z))|^p |g_{q_k}(\varepsilon_k z)|^{1/2} \frac{2}{3}\,dz \\ &= \frac{2}{3}\frac{1}{\varepsilon_k^n}\int_{I(q_k,\varepsilon_k T,\varepsilon_k T)} |u_k^+|^p\,d\mu_g \leq \frac{2}{3}(1-\eta)\frac{m_e^+}{(1/2-1/p)}.\end{aligned}$$

On the other side, by Step 3, we have that

$$E^+(w) = \left(\frac{1}{2}-\frac{1}{p}\right)|w|_p^p = m_e^+.$$

Now, by Step 1, there exists $T > 0$ such that, for k big enough, we have

$$\frac{2}{3}(1-\eta)\frac{m_e^+}{(1/2-1/p)} < |w_k^+|^p_{L^p(B_{n-1}(0,T)\times[0,T])}. \tag{5.9}$$

By (5.8) and (5.9) we have a contradiction.

Case 2. $P_k^\varepsilon \cap \partial M = \emptyset$. We choose a point q_k interior to P_k^ε and we consider the normal coordinates at q_k. We set $w_k(z)$ as

$$u_k(x)\chi_R(\exp_{q_k}^{-1}(x)) = u_k(\exp_{q_k}(y))\chi_R(y) = u_k(\exp_{q_k}(\varepsilon_k z))\chi_R(\varepsilon_k z) = w_k(z).$$

Then $w_k \in H_0^1(B(0, R/\varepsilon_k)) \subset H^1(\mathbb{R}^n)$. Arguing as in the previous step, we can establish some properties of the function w_k. We omit the proof of single steps.

Step 1. w_k is bounded in H^1 and converge to some $w \in H^1$ weakly L^p_{loc} in and strongly in H^1.

Step 2. w is a weak solution of $-\Delta w + w = (w^+)^{p-1}$ in $\mathbb{R}^n$.

Step 3. w is strictly positive, and it is a least energy solution of $-\Delta w + w = |w|^{p-1}w$, that is

$$\left(\frac{1}{2}-\frac{1}{p}\right)|w|_p^p = E(w) = m_e = 2m_e^+. \tag{5.10}$$

Conclusion of the Case 2. By (5.10) and (5.2) we have the contradiction. This concludes the proof. □

Remark 5.4. We point out that in the proof of Proposition 5.3, by Remark 4.2 and (5.1) we showed that

$$\lim_{\varepsilon\to 0} m_\varepsilon = m_e^+.$$

6. The map β

For any $u \in \mathcal{N}_\varepsilon$ we can define its center of mass as a point $\beta(u) \in \mathbb{R}^N$ by

$$\beta(u) = \frac{\displaystyle\int_M x|u^+(x)|^p \, d\mu_g}{\displaystyle\int_M |u^+(x)|^p \, d\mu_g}.$$

The application is well defined on $\mathcal{N}_\varepsilon$, since $u \in \mathcal{N}_\varepsilon$ implies $u^+ \neq 0$. In the following we will show that if $u \in \mathcal{N}_\varepsilon \cap J^{m_e^+ + \delta}$ then $\beta(u) \in (\partial M)_{3\rho}$,using the concentration property (Proposition 5.3) of the function $u \in \mathcal{N}_\varepsilon \cap J^{m_e^+ + \delta}$ if ε and δ are sufficiently small.

Proposition 6.1. *For any $u \in \mathcal{N}_\varepsilon \cap J^{m_e^+ + \delta}$, with ε and δ small enough, it holds*

$$\beta(u) \in (\partial M)_{3\rho}.$$

Proof. Since $m_\varepsilon \to m_e^+$ and by Proposition 5.3 we get that for any $u \in \mathcal{N}_\varepsilon \cap J^{m_e^+ + \delta}$ there exists $q \in \partial M$ such that

$$(1-\eta)m_e^+ \leq \left(\frac{1}{2} - \frac{1}{p}\right)\frac{1}{\varepsilon^n}|u^+|^p_{L^p(I_q(\rho,R))}. \tag{6.1}$$

Since $u \in \mathcal{N}_\varepsilon \cap J^{m_e^+ + \delta}$ we have

$$\left(\frac{1}{2} - \frac{1}{p}\right)\frac{1}{\varepsilon^n}|u^+|^p_{p,g} < m_e^+ + \delta. \tag{6.2}$$

Then by (6.1) and (6.2) we get

$$\int_{I_q(\rho,R)} \frac{|u^+|^p}{|u^+|^p_{p,g}}\, d\mu_g \geq \frac{1-\eta}{1+\delta/m_e^+}.$$

By definition of β we have

$$\begin{aligned}|\beta(u) - q| &\leq \left|\int_{I_q(\rho,R)} (x-q)\frac{|u^+|^p}{|u^+|^p_{p,g}}\, d\mu_g\right| + \left|\int_{M\setminus I_q(\rho,R)} (x-q)\frac{|u^+|^p}{|u^+|^p_{p,g}}\, d\mu_g\right| \\ &\leq 2\rho + D\left(1 - \frac{1-\eta}{1+\delta/m_e^+}\right),\end{aligned}$$

where D is the diameter of the manifold M as a subset of $\mathbb{R}^n$. Choosing η and δ small enough we get the claim. □

Proposition 6.2. *The composition*

$$\beta \circ \Phi_\varepsilon \colon \partial M \to (\partial M)_{3\rho} \subset \mathbb{R}^n$$

is well defined and homotopic to the identity of ∂M.

Proof. By Propositions 6.1 and 4.1 the map $\beta \circ \Phi_\varepsilon \colon \partial M \to (\partial M)_{\rho(\partial M)}$ is well defined.

To prove that $\beta \circ \Phi_\varepsilon \colon \partial M \to (\partial M)_{3\rho}$ is homotopic to the identity it is enough to evaluate the map

$$\begin{aligned}\beta(\Phi_\varepsilon(q)) - q &= \frac{\int_{B_{n-1}(0,R)\times[0,\rho]} y|V_\varepsilon(y)\chi_R(|\overline{y}|)\chi_\rho(y_n)|^p\, dy}{\int_{B_{n-1}(0,R)\times[0,\rho]} |V_\varepsilon(y)\chi_R(|\overline{y}|)\chi_\rho(y_n)|^p\, dy} \\ &= \frac{\varepsilon\int_{B_{n-1}(0,R/\varepsilon)\times[0,\rho/\varepsilon]} z|V(z)\chi_R(|\varepsilon\overline{z}|)\chi_\rho(\varepsilon z_n)|^p\, dz}{\int_{B_{n-1}(0,R/\varepsilon)\times[0,\rho/\varepsilon]} |V(z)\chi_R(|\varepsilon\overline{z}|)\chi_\rho(\varepsilon z_n)|^p\, dz}.\end{aligned}$$

By the exponential decay of V we get $|\beta(\Phi_\varepsilon(q)) - q| < c\varepsilon$, where c is a constant not depending on q. □

7. The set T_ε

To finish the proof of Theorem 3.1, it remains to show that there exists a critical point $\overline{u}$ of J_ε in $\mathcal{N}_\varepsilon$ with $m_e^+ + \delta < J_\varepsilon(\overline{u}) < c_\varepsilon$, for bounded constants c_ε. As explained in Section 3, this is achieved by constructing a set T_ε which contains only positive functions, is contractible in $\mathcal{N}_\varepsilon \cap J_\varepsilon^{c_\varepsilon}$ and contains $\Phi_\varepsilon(\partial M)$. The process of building the set T_ε is analogous to the process of Section 6 of [4]; for clearness we prefer to show it.

To define the set T_ε we use the functions $Z_{\varepsilon,q}(x)$ as defined in (4.1). We recall that $Z_{\varepsilon,q}(x) \in H_g^1(M)$ are positive functions. Let $W(x) \in H^1(\mathbb{R}_+^n)$ be any positive function and denote as usual $W_\varepsilon(x) = W(x/\varepsilon)$. For $q_0 \in \partial M$ a fixed point on the boundary of M we introduce the functions

$$v_\varepsilon(x) := \begin{cases} W_\varepsilon(y(x))\widetilde{\chi}(y(x)) & \text{if } x \in I_{q_0}(R,\rho), \\ 0 & \text{otherwise}, \end{cases}$$

where $y(x) = (\psi_{q_0}^\partial)^{-1}(x)$ and $\widetilde{\chi}(y) = \chi_R(\overline{y})\chi_\rho(y_n)$ as in the previous part of the paper.

We define the cone

$$C_\varepsilon := \{u(x) := \theta v_\varepsilon(x) + (1-\theta) Z_{\varepsilon,q}(x) : \theta \in [0,1],\ q \in \partial M\} \subset H_g^1(M).$$

By the properties of the map Φ_ε proved in Proposition 4.1, we have that C_ε is compact and contractible in $H_g^1(M)$. We now project it on the Nehari manifold $\mathcal{N}_\varepsilon$ by the factor $t_\varepsilon(u)$ to obtain

$$T_\varepsilon := \left\{ t_\varepsilon(u)u : u \in C_\varepsilon,\ t_\varepsilon^{p-2}(u) = \frac{|||u|||_\varepsilon^2}{|u|_{p,g}^p/\varepsilon^n} \right\} \subset \mathcal{N}_\varepsilon.$$

We get that $\Phi_\varepsilon(\partial M) \subset T_\varepsilon$, that T_ε contains only positive functions and that it is compact and contractible in $\mathcal{N}_\varepsilon$. Hence if we define

$$c_\varepsilon := \max_{u \in C_\varepsilon} J_\varepsilon(t_\varepsilon(u)u)$$

we get that $T_\varepsilon \subset \mathcal{N}_\varepsilon \cap J_\varepsilon^{c_\varepsilon}$. The last step is to prove the following proposition.

Proposition 7.1. *There exists a constant $c > 0$ such that for ε small enough it holds $c_\varepsilon < c$.*

Proof. By the definition of the Nehari manifold, we recall that for $u \in C_\varepsilon$ it holds

$$J_\varepsilon(t_\varepsilon(u)u) = \left(\frac{1}{2} - \frac{1}{p}\right) t_\varepsilon^2(u) |||u|||_\varepsilon^2 = \left(\frac{1}{2} - \frac{1}{p}\right) \frac{|||u|||_\varepsilon^{2p/(p-2)}}{(|u|_{p,g}^p/\varepsilon^n)^{2/(p-2)}}. \tag{7.1}$$

Arguing as (4.2)–(4.4) for v_ε and W instead of $Z_{\varepsilon,q}$ and V, we find that there exists a constant $k_1 > 0$ such that

$$|||u|||_\varepsilon^2 \leq ||W||_{H^1}^2 + ||V||_{H^1}^2 + k_1 \tag{7.2}$$

for ε small enough. Moreover, for ε small enough, we find constants $k_2 > 0$ and $k_3 > 0$ such that

$$\frac{1}{\varepsilon^n}|v_\varepsilon|^p_{p,g} \geq |W|^p_p - k_2 > 0, \qquad \frac{1}{\varepsilon^n}|Z_{\varepsilon,q}|^p_{p,g} \geq |V|^p_p - k_3 > 0.$$

Hence, since v_ε and $Z_{\varepsilon,q}$ are positive functions and $\theta \in [0,1]$, there exists k_4 such that

$$\frac{1}{\varepsilon^n}|u|^p_{p,g} \geq \frac{1}{\varepsilon^n}\max\{|\theta v_\varepsilon|^p_{p,g}, |(1-\theta)Z_{\varepsilon,q}|^p_{p,g}\} \geq k_4 \tag{7.3}$$

for ε small enough. Putting together (7.1)–(7.3) we get the thesis. □

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