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AVERAGING METHOD FOR NEUTRAL DIFFERENTIAL EQUATIONS IN FINITE DIMENSION

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Abstract. We prove in this paper a periodic existence theorem for neutral differential equations in finite dimension with high frequency terms. This study completes previous works about applications of averaging methods to periodic problems.

1. Introduction

The aim of this paper is to work out existence of periodic solutions for a class of differential equations with delay in $\mathbb{R}^N$.

More precisely we consider equations of neutral type

$$y'(\tau) = \Phi\bigg(\frac{\tau}{\varepsilon}, y(\tau - h(\varepsilon)), y'(\tau - h(\varepsilon)), \varepsilon\bigg), \quad \tau \in \mathbb{R}, \tag{1.1}$$

where Φ is a continuous map on $\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \times [0,1]$, T-periodic in its first variable, Lipschitz in its third variable and h is an arbitrary map from $[0,1]$ to $[0,\infty[$.

Really such equations with high periodic frequency term τ/ε concerns in physical applications the field of high frequency phenomenoms.

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By setting $t = \tau/\varepsilon$ and $x(t) = y(\varepsilon t)$ in the previous equation we are led to study the following equation

$$x'(t) = \varepsilon\Phi\bigg(t, x\bigg(t - \frac{h(\varepsilon)}{\varepsilon}\bigg), \frac{1}{\varepsilon}x'\bigg(t - \frac{h(\varepsilon)}{\varepsilon}\bigg), \varepsilon\bigg), \quad t \in \mathbb{R}. \tag{1.2}$$

From the classical works of N. N. Bogoljubov, M. N. Krylov (see for instance [10] and [3]) the existence problem of periodic solutions for ordinary differential equations with high frequency terms reduced to a form of averaging principle. It consists in replacing the right hand side of the considered equation by its average and then looking for equilibriums which are the first approximations of high frequency periodic solutions of the initial equation. In order to ground this method N. N. Bogoljubov and M. N. Krylov supposed that the averaging term at the equilibrium has a non degenerated derivative. In the sixties of the twentieth century J. Mawhin (see [11]) remarked that this condition may be replaced by assuming that the Poincaré topological index of this equilibrium is different from zero. In the seventies such a method has been applied to averaging principles for equations with delay (see V. V. Strygin [13], Perestuk A. M. Samoĭlenko [12]) and for neutral equations with deviating argument (see [1] and [2]). Other applications of the topological degree theory to the averaging method to the periodic problems for differential inclusions can be found in [4], [5].

About neutral equations which is the case studied in this article, let us recall the framework and the kind of results considered in [1] and [2]. These papers are concerned in the simpler case by equations of the form

$$x'(t) = \varepsilon\Phi(t, x(t-h), x'(t-h)), \tag{1.3}$$

with $\varepsilon > 0$ and where Φ is a continuous map on $\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N$ which is T-periodic in its first variable and Lipschitz in its third variable. It was assumed that

$$\Phi_0(y) = \frac{1}{T}\int_0^T \Phi(t, y, 0)\, dt$$

has an equilibrium y^*, namely $\Phi_0(y^*) = 0$ and that the condition $\deg(-\Phi_0, U) \neq 0$ holds, U being a neighbourhood of y^* which does not contain other equilibrium than y^* and deg being the topological degree of $-\Phi_0$ on U (see for instance [9]). In these conditions it was proved that (1.3) has at least a T-periodic solution $x(t,\varepsilon)$ satisfying $x(t,\varepsilon) \in U$ and $x(t,\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} y^*$ uniformly in $t \in \mathbb{R}$.

Now if we consider the following neutral equation

$$y'(\tau) = \varepsilon\Phi\bigg(\frac{\tau}{\varepsilon}, y(\tau - h), y'(\tau - h)\bigg) \tag{1.4}$$

containing high frequency terms like (1.1) the variable change $t = \tau/\varepsilon$ in (1.4) does not reduce to (1.3). Indeed we find the expression

$$\frac{1}{\varepsilon}x'\left(t - \frac{h}{\varepsilon}\right)$$

in the third variable of Φ like in (1.2) with a deviating argument. And we will give an example showing that in this situation the recalled results of [1] and [2] are no longer valid. We will need then an auxiliary condition.

This paper is organized as follows. In Section 2 we give in a suitable form some basic recalls about measures of noncompactness, condensing mappings (see [2]) and Degree Theory (see [6] or [8]). Assumptions and notations are detailed in Section 3. The fundamental existence theorem is stated and proved in Section 4. An example which show the necessity of the auxiliary condition mentioned above is given in Section 11.

2. Backgrounds

The main tool in our study is the topological degree theory applied to condensing operators. We will be satisfied here to recall basic facts in this theory put in an adapted form for the sequel. For more details we refer the reader to [2]. Let X be a Banach space.

The Hausdorff measure of noncompactness χ is defined by

$$\chi(\Omega) = \inf\{d > 0 : \Omega \text{ has a finite } d\text{-net in } X\}$$

for each bounded subset $\Omega \subset X$.

DEFINITION 2.1. Let U be an open bounded subset of X. The continuous operator $F\colon \overline{U} \to X$ is said to be *condensing* if the inequality

$$\chi(F(\Omega)) \geq \chi(\Omega)$$

for $\Omega \subseteq \overline{U}$, holds only if Ω is totally bounded in X.

DEFINITION 2.2. Let Λ be a metric space. The continuous operator $\mathcal{H}$: $\Lambda \times \overline{U} \to X$ is said to be *condensing* with respect to the couple of variables if the inequality

$$\chi(\mathcal{H}(\Lambda \times \Omega)) \geq \chi(\Omega)$$

for Ω bounded in X, holds only if Ω is totally bounded in X.

As proved in [2]

$$\mathcal{H}(\lambda, x) = \lambda F_0(x) + (1 - \lambda)F_1(x) \tag{2.1}$$

is condensing with respect to the couple of variables when F_0 and F_1 are condensing. Let U be a bounded open subset of X and F be a continuous condensing operator. Assume that we have $x \neq F(x)$ for $x \in \partial U$. Then it is possible to define an integer $\deg(I - F, U)$ which enjoys the following properties.

(I1) If $\deg(I - F, U) \neq 0$ then F has a fixed point in U, i.e. there is $x \in U$ satisfying

$$x = F(x).$$

(I2) If $U = \bigcup_{i=1}^{n} U_i$ with U_i pairwise disjoint bounded open subsets in X and $x \notin F(x)$ for $x \in \partial U_i$, $i = 1, \ldots, n$, then

$$\deg(I - F, U) = \sum_{i=1}^{n} \deg(I - F, U_i).$$

(I3) If $F \equiv I - x_0$ then it comes

$$\deg(I - F, U) = \begin{cases} 1 & \text{if } x_0 \in U, \\ 0 & \text{if } x_0 \notin U. \end{cases}$$

(I4) If $\mathcal{H}: [0,1] \times \overline{U} \to X$ is condensing continuous with respect to the couple of variables and such that $x \notin \mathcal{H}(\lambda, x)$ for $x \in \partial U$ and $\lambda \in [0,1]$, then $\deg(I - \mathcal{H}(\lambda, \cdot\,), U)$ does not depend upon λ.

In (I4) the maps $F_0(\,\cdot\,) = \mathcal{H}(0, \,\cdot\,)$ and $F_1(\,\cdot\,) = \mathcal{H}(1, \,\cdot\,)$ are said to be homotop. In the sequel we sometimes need the following definition: the condensing operators F_0 and F_1 are linearly homotop if formula (2.1) gives an homotopy. Let us end these recalls by a useful restriction degree theorem.

Theorem 2.3. *Let E be a Banach space and $\mathcal{U} \subset E$ a bounded open subset. Let $\mathcal{R} \subset E$ be a linear subspace of E. Suppose the map $F: \mathcal{U} \to \mathcal{R}$ satisfies $F(x) \neq x$ for all $x \in \partial\mathcal{U}$. Then we have*

$$\deg_E(I - F, \mathcal{U}) = \deg_{\mathcal{R}}((I - F)\,|_{\mathcal{R}}, \mathcal{U} \cap \mathcal{R}).$$

3. Assumptions and notations

Let $T > 0$. In the sequel C_T (respectively, C_T^1) stands for the space of continuous (respectively, continuously differentiable) T-periodic functions from $\mathbb{R}$ to $\mathbb{R}^N$. We will denote by $\|\cdot\|$ the supremum norm in C_T, and by $|\cdot|$ the euclidian norm in $\mathbb{R}^N$. The space C_T^1 is endowed with its usual norm, namely

$$\|x\|_{C_T^1} = |x(0)| + \|x'\|. \tag{3.1}$$

Now we make the following assumptions:

(A1) Let $h: [0,1] \to \mathbb{R}^+$ be an arbitrary map;
(A2) The map $\Phi: \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \times [0,1] \to \mathbb{R}^N$ is continuous;

(A3) The map Φ is T-periodic in its first variable, that is

$$\Phi(t+T,u,v,\varepsilon)=\Phi(t,u,v,\varepsilon)$$

for all $(t,u,v,\varepsilon)\in\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^N\times[0,1]$;

(A4) There is $k\in[0,1[$ such that

$$|\Phi(t,u,v_1,\varepsilon)-\Phi(t,u,v_2,\varepsilon)|\le k|v_1-v_2|$$

for all $(t,u,v_i,\varepsilon)\in\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^N\times[0,1]$, $i\in\{1,2\}$.

NOTATION. Let

$$\Phi_0(x_0)=\frac{1}{T}\int_0^T\Phi(s,x_0,0,0)\,ds,\quad \text{for all } x_0\in\mathbb{R}^N.$$

(A5) There is some $y^*\in\mathbb{R}^N$ and some bounded open subset $U\subset\mathbb{R}^N$ satisfying $\Phi_0(y^*)=0$, $y^*\in U$ and $\deg_{\mathbb{R}^n}(-\Phi_0,U)\ne 0$.

NOTATION. We will set

$$\begin{aligned}M&=\sup\{|\Phi(t,y,0,0)|:t\in\mathbb{R},y\in\partial U\},\\ \mathcal{M}&=\bigg\{y\in C_T:\|y\|\le\frac{1}{1-k}M\max\bigg(1,\frac{2}{T}\bigg)\text{ and }\int_0^T y(t)\,dt=0\bigg\}.\end{aligned}$$

Let us introduce the multivalued map $Z\colon\partial U\to R^N$ defined by

$$Z(x)=\bigg\{z\in R^N:z=\int_0^T\Phi(t,x,y(t),0)\,dt,\ y\in\mathcal{M}\bigg\}.$$

REMARK 3.1. The map Z is single valued if we have for instance

$$\Phi(t,u,v,\varepsilon)=\varphi(t,u,\varepsilon)+Bv,$$

with φ continuous and B linear, or if we have for instance

$$\Phi(t,u,v,\varepsilon)=\Psi(t,u,\varepsilon v,\varepsilon),$$

with Ψ continuous.

4. The existence theorem: statement and proof

We are in position to state our main result.

THEOREM 4.1. *Let* (A1)–(A5) *be fulfilled. In addition, suppose that we have*

$$0\notin\overline{\mathrm{co}}\,Z(x), \tag{4.1}$$

for all $x\in\partial U$. Then there is $\varepsilon_0\in]0,1[$ such that for all $\varepsilon\in]0,\varepsilon_0[$ equation (1.2) *has at least a periodic solution $t\mapsto x(t,\varepsilon)$ satisfying*

$$x(t,\varepsilon)\in U\quad\textit{and}\quad\lim_{\varepsilon\to 0}\|x'(\,\cdot\,,\varepsilon)\|=0.$$

Notation. In the proof we will need the following bounded open subset of C^1_T: $\mathcal{U} = \{x \in C^1_T : x(t) \in U,\ t \in [0,T],\ \|x'\| < 1\}$. Introduce also the operators F^1_ε and F^0_ε on C^1_T defined by

$$\begin{aligned} F^1_\varepsilon(x)(t) = {} & x(0) + \varepsilon \int_0^t \Phi\bigg(s, x\bigg(s - \frac{h(\varepsilon)}{\varepsilon}\bigg), \frac{1}{\varepsilon} x'\bigg(s - \frac{h(\varepsilon)}{\varepsilon}\bigg), \varepsilon\bigg)\, ds \\ & - \varepsilon\bigg(\frac{t}{T} - \frac{1}{2}\bigg) \int_0^T \Phi\bigg(s, x\bigg(s - \frac{h(\varepsilon)}{\varepsilon}\bigg), \frac{1}{\varepsilon} x'\bigg(s - \frac{h(\varepsilon)}{\varepsilon}\bigg), \varepsilon\bigg)\, ds, \end{aligned}$$

and $F^0_\varepsilon(x)(t) = x(0) + \varepsilon \Phi_0(x(0))$.

Let us give now in three lemmas some useful immediate properties (see [2] for the proofs) of the above operators F^0_ε and F^1_ε.

Lemma 4.2. *The operator F^0_ε is compact in C^1_T.*

Lemma 4.3. *The fixed points of F^1_ε are the periodic solutions of* (1.2).

Lemma 4.4. *The operator F^1_ε is condensing with respect to the Hausdorff measure of noncompactness χ of C^1_T.*

The proof of Theorem 4.1 will be deduced from the following proposition:

Proposition 4.5. *There is $\varepsilon_0 \in\ ]0,1[$ such that F^1_ε and F^0_ε are linearly homotop on $\mathcal{U}$ for all $\varepsilon \in\ ]0, \varepsilon_0[$.*

Proof. By Lemmas 4.2 and 4.4 the maps F^1_ε and F^0_ε are condensing with respect to the Hausdorff measure of noncompactness of C^1_T endowed with the norm (3.1).

So according to the continuity of Φ (assumption (A2)) we have just to prove that, for all $\lambda \in [0,1]$ and all $\varepsilon > 0$ sufficiently small, the map $\lambda F^1_\varepsilon + (1-\lambda) F^0_\varepsilon$ has no fixed point on $\partial\mathcal{U}$. In this goal, by contradiction suppose the contrary. In other words suppose that there exist sequences $(\varepsilon_m)_m$, $(\lambda_m)_m$, $(x_m)_m$ satisfying

$$\begin{gathered} \varepsilon_m \to 0, \quad \varepsilon_m > 0, \qquad \lambda_m \in [0,1], \quad \lambda_m \to \lambda_0, \\ x_m \in \partial\mathcal{U} \quad \text{and} \quad (\lambda_m F^1_\varepsilon + (1-\lambda_m) F^0_\varepsilon)(x_m) = x_m. \end{gathered}$$

Then, we have

$$\begin{aligned} x_m(t) = {} & x_m(0) \\ & + \lambda_m \varepsilon_m \int_0^t \Phi\bigg(s, x_m\bigg(s - \frac{h(\varepsilon_m)}{\varepsilon_m}\bigg), \frac{1}{\varepsilon_m} x'_m\bigg(s - \frac{h(\varepsilon_m)}{\varepsilon_m}\bigg), \varepsilon_m\bigg)\, ds \\ & - \lambda_m \varepsilon_m \bigg(\frac{t}{T} - \frac{1}{2}\bigg) \\ & \cdot \int_0^T \Phi\bigg(s, x_m\bigg(s - \frac{h(\varepsilon_m)}{\varepsilon_m}\bigg), \frac{1}{\varepsilon_m} x'\bigg(s - \frac{h(\varepsilon_m)}{\varepsilon_m}\bigg), \varepsilon_m\bigg)\, ds \\ & + (1-\lambda_m)\varepsilon_m \Phi_0(x_m(0)). \end{aligned} \tag{4.2}$$

Putting $t = T$, using $x_m(0) = x_m(T)$ and dividing by ε_m in (4.2) we obtain:

$$\tag{4.3} \frac{1}{2}\lambda_m \int_0^T \Phi\bigg(s, x_m\bigg(s - \frac{h(\varepsilon_m)}{\varepsilon_m}\bigg), \frac{1}{\varepsilon_m} x'\bigg(s - \frac{h(\varepsilon_m)}{\varepsilon_m}\bigg), \varepsilon_m\bigg)\, ds = -(1-\lambda_m)\Phi_0(x_m(0)).$$

Since x_m is continuously derivable, relation (4.2) gives

$$\tag{4.4} \begin{aligned} x_m'(t) = {}& \lambda_m \varepsilon_m \Phi\bigg(t, x_m\bigg(t - \frac{h(\varepsilon_m)}{\varepsilon_m}\bigg), \frac{1}{\varepsilon} x_m'\bigg(t - \frac{h(\varepsilon_m)}{\varepsilon_m}\bigg), \varepsilon_m\bigg) \\ & - \lambda_m \varepsilon_m \frac{1}{T} \int_0^T \Phi\bigg(s, x_m\bigg(s - \frac{h(\varepsilon_m)}{\varepsilon_m}\bigg), \frac{1}{\varepsilon_m} x'\bigg(s - \frac{h(\varepsilon_m)}{\varepsilon_m}\bigg), \varepsilon_m\bigg)\, ds. \end{aligned}$$

Accordingly to (4.3) relation (4.4) becomes

$$\tag{4.5} \begin{aligned} \frac{x_m'(t)}{\varepsilon_m} = {}& \lambda_m \Phi\bigg(t, x_m\bigg(t - \frac{h(\varepsilon_m)}{\varepsilon_m}\bigg), \frac{1}{\varepsilon} x_m'\bigg(t - \frac{h(\varepsilon_m)}{\varepsilon_m}\bigg), \varepsilon_m\bigg) \\ & + \frac{2}{T}(1-\lambda_m)\Phi_0(x_m(0)). \end{aligned}$$

Now we need the following lemma.

Lemma 4.6. *The sequence $(x_m'/\varepsilon_m)_m$ is relatively compact in C_T.*

Proof. First let us establish that $(x_m'/\varepsilon_m)_m$ is bounded in C_T. Since $x_m(t)$ belongs to the compact subset $\overline{U}$ of R^n and since Φ is continuous there is a constant $K > 0$ satisfying

$$\bigg|\Phi\bigg(t, x_m\bigg(t - \frac{h(\varepsilon_m)}{\varepsilon_m}\bigg), 0, \varepsilon_m\bigg)\bigg| \le K \quad \text{and} \quad |\Phi_0(x_m(0))| \le K,$$

for all t and m. For each m let $t_m \in \mathbb{R}$ be defined by

$$\bigg|\frac{1}{\varepsilon_m} x_m'(t_m)\bigg| = \sup_{t\in R}\bigg|\frac{1}{\varepsilon_m} x_m'(t)\bigg|.$$

Then from (4.5) and the triangle inequality we deduce

$$\begin{aligned} \bigg|\frac{1}{\varepsilon_m} x_m'(t_m)\bigg| \le {}& \lambda_m \bigg|\Phi\bigg(t_m, x_m\bigg(t_m - \frac{h(\varepsilon_m)}{\varepsilon_m}\bigg), \frac{1}{\varepsilon_m} x_m'\bigg(t_m - \frac{h(\varepsilon_m)}{\varepsilon_m}\bigg), \varepsilon_m\bigg) \\ & - \Phi\bigg(t_m, x_m\bigg(t_m - \frac{h(\varepsilon_m)}{\varepsilon_m}\bigg), 0, \varepsilon_m\bigg)\bigg| \\ & + \lambda_m \bigg|\Phi\bigg(t_m, x_m\bigg(t_m - \frac{h(\varepsilon_m)}{\varepsilon_m}\bigg), 0, \varepsilon_m\bigg)\bigg| \\ & + \frac{2}{T}(1-\lambda_m)|\Phi_0(x_m(0))| \\ \le {}& \lambda_m k \bigg|\frac{1}{\varepsilon_m} x_m'\bigg(t_m - \frac{h(\varepsilon_m)}{\varepsilon_m}\bigg)\bigg| + \frac{2}{T}K. \end{aligned}$$

Whence

$$\left|\frac{1}{\varepsilon_m}x'_m(t_m)\right| \le \frac{2K}{T(1-k)}. \tag{4.6}$$

Of course (4.6) involves

$$\|x'_m\| \le \frac{2K\varepsilon_m}{T(1-k)}. \tag{4.7}$$

What follows by the Ascoli–Arzelà Theorem that $(x_m)_m$ is relatively compact in C_T. Without loss of generality we can suppose $x_m \xrightarrow{C_T} x^0$. Clearly x^0 is a constant function with constant value in ∂U. So we will set in the sequel $x^0(t) = y \in \partial U$. From the previous conclusions and from (4.7) we see that we have $x_m \xrightarrow{C^1_T} x^0$, and that $(x_m)_m$ is relatively compact in C^1_T.

Now, for some $\rho > 0$, let $\{z_i : i = 1,\dots,p\}$ be a finite ρ-net in C_T of $(x'_m/\varepsilon_m)_m$. We are going to construct in C_T a relatively compact $k\rho$-net Ω of $(x'_m/\varepsilon_m)_m$, what implies the relative compactness in C_T of $(x'_m/\varepsilon_m)_m$ because $k < 1$. This next construction will end the proof of Lemma 4.6.

In order to construct a suitable $k\rho$-net introduce the following subset $W \subset C_T$ defined by

$$W = \{w_{i,m} : i = 1,\dots,p \text{ and } m \in N^*\}, \quad w_{i,m}(t) = z_i\left(t - \frac{h(\varepsilon_m)}{\varepsilon_m}\right).$$

It is obvious that W is relatively compact in C_T. Let

$$\begin{aligned}\omega_{i,m}(t) :=& \lambda_m\Phi\left(t, x_m\left(t_m - \frac{h(\varepsilon_m)}{\varepsilon_m}\right), z_i\left(t - \frac{h(\varepsilon_m)}{\varepsilon_m}\right), \varepsilon_m\right)\\ &+ \frac{2}{T}(1-\lambda_m)\Phi_0(x_m(0)),\\ \Omega =& \{\omega_{i,m} : i = 1,\dots,p \text{ and } m \in N^*\}.\end{aligned}$$

Then due to assumptions (A2)–(A4), the set Ω is relatively compact in C_T and from (A4) we have

$$\begin{aligned}&\left|\frac{1}{\varepsilon_m}x'_m(t) - \omega_{i,m}(t)\right|\\ &\quad\le \lambda_m\left|\Phi\left(t, x_m\left(t_m - \frac{h(\varepsilon_m)}{\varepsilon_m}\right), \frac{1}{\varepsilon_m}x'_m\left(t - \frac{h(\varepsilon_m)}{\varepsilon_m}\right), \varepsilon_m\right)\right.\\ &\qquad\left. - \Phi\left(t, x_m\left(t_m - \frac{h(\varepsilon_m)}{\varepsilon_m}\right), z_i\left(t - \frac{h(\varepsilon_m)}{\varepsilon_m}\right), \varepsilon_m\right)\right|\\ &\quad\le k\left|\frac{1}{\varepsilon_m}x'_m\left(t - \frac{h(\varepsilon_m)}{\varepsilon_m}\right) - z_i\left(t - \frac{h(\varepsilon_m)}{\varepsilon_m}\right)\right|.\end{aligned}$$

Since $\{z_i : i = 1,\dots,p\}$ is a ρ-net in C_T of $(x'_m/\varepsilon_m)_m$ it follows

$$\inf_{i=1,\dots,p}\left|\frac{1}{\varepsilon_m}x'_m(t) - \omega_{i,m}(t)\right| \le k\rho.$$

Consequently, Ω is a relatively compact $k\rho$-net of $(x'_m/\varepsilon_m)_m$ in C_T. The proof of Lemma 4.6 is now complete. $\square$

End of Proof of Proposition 4.5. Let w be a cluster point in C_T of $(x'_m/\varepsilon_m)_m$ and without loss of generality suppose

$$\frac{1}{\varepsilon_m}x'_m \xrightarrow{C_T} w, \text{ and } \operatorname{dist}\left(\frac{h(\varepsilon_m)}{\varepsilon_m}, h_0 + T\mathbb{N}\right) \xrightarrow{m\to\infty} 0, \quad h_0 \in [0,T].$$

Letting $m \to \infty$ in (4.5) we obtain

$$w(t) = \lambda_0 \Phi(t, y, w(t-h_0), 0) + \frac{2}{T}(1-\lambda_0)\Phi_0(y)$$

for all $t \in \mathbb{R}$. Now let $t_0 \in R$ be defined by

$$|w(t_0)| = \sup_{t\in R} |w(t)|.$$

Then from (4.5) and the triangle inequality we deduce

$$\begin{aligned} |w(t_0)| &\le \lambda_0 |\Phi(t_0, y, w(t_0-h_0), 0) - \Phi(t_0, y, 0, 0)| \\ &\quad + \lambda_0 |\Phi(t_0, y, 0, 0)| + \frac{2}{T}(1-\lambda_0)|\Phi_0(y)| \\ &\le \lambda_0 k |w(t_0-h_0)| + M \max(1, 2/T). \end{aligned}$$

Thus

$$|w(t_0)| \le \frac{1}{1-k} M \max(1, 2/T).$$

Moreover, we have

$$\int_0^T w(t)\,dt = 0,$$

since $\int_0^T (x'_m/\varepsilon_m)(t)\,dt = 0$, for all m. Consequently, it comes $w \in \mathcal{M}$, and thus $w(\,\cdot\, - h_0) \in \mathcal{M}$. Then, we deduce

$$z_0 := \int_0^T \Phi(t, y, w(t-h_0), 0)\,dt \in Z(y). \tag{4.8}$$

Passing to the limit (4.3) yields

$$\lambda_0 \frac{1}{2} z_0 + (1-\lambda_0)\Phi_0(y) = 0. \tag{4.9}$$

But (4.9) contains a contradiction with (4.1) as we are going to show now. Indeed, let $\psi\colon \mathbb{R}^N \to \mathbb{R}$ be a linear functional strictly positive on the closed convex subset $\overline{\text{co}}\, Z(y)$ which does not contain zero. Our definitions and relation (4.8) yield

$$\Phi_0(y) \in Z(y) \quad \text{and} \quad \lambda_0 z_0 + (1-\lambda_0)\Phi_0(y) \in \overline{\text{co}}\, Z(y).$$

Then, using (4.9), we obtain the following contradiction

$$\begin{aligned} 0 &= \psi\bigg(\lambda_0 \frac{1}{2} z_0 + (1-\lambda_0)\Phi_0(y)\bigg) \\ &= \frac{1}{2}\psi(\lambda_0 z_0 + (1-\lambda_0)\Phi_0(y)) + \frac{1}{2}(1-\lambda_0)\psi(\Phi_0(y)) > 0. \end{aligned}$$

The proof is now complete. □

End of Proof of Theorem 2.3. Since, by Proposition 4.5, F_ε^1 and F_ε^0 are homotop for all $\varepsilon \in \,]0, \varepsilon_0[$, the Degree Theory gives

$$\deg_{C_T^1}(I - F_\varepsilon^1, \mathcal{U}) = \deg_{C_T^1}(I - F_\varepsilon^0, \mathcal{U}),$$

for each $\varepsilon \in \,]0, \varepsilon_0[$. Now remarking that

$$(I - F_\varepsilon^0)(x_0) = -\varepsilon \Phi_0(x_0(0)),$$

for all constant function x_0, and using the restriction theorem we conclude

$$\deg_{C_T^1}(I - F_\varepsilon^1, \mathcal{U}) = \deg_{C_T^1}(I - F_\varepsilon^0, \mathcal{U}) = \deg_{\mathbb{R}^N}(-\varepsilon\Phi_0, U)$$

Since, from (A5), $-\varepsilon\Phi_0$ and $-\Phi_0$ are homotop it comes

$$\deg_{\mathbb{R}^N}(-\varepsilon\Phi_0, U) = \deg_{\mathbb{R}^N}(-\Phi_0, U)$$

and thus (using again (A5))

$$\deg_{C_T^1}(I - F_\varepsilon^1, \mathcal{U}) = \deg_{\mathbb{R}^N}(-\Phi_0, U) \neq 0,$$

for each $\varepsilon \in \,]0, \varepsilon_0[$. Consequently, from the degree property, F_ε^1 has at least one fixed point $x(\,\cdot\,, \varepsilon) \in \mathcal{U}$ for $\varepsilon \in \,]0, \varepsilon_0[$. In other words (1.2) has periodic solution $x(\,\cdot\,, \varepsilon)$ for $\varepsilon \in \,]0, \varepsilon_0[$, and $x(t, \varepsilon) \in U$ for all $t \in R$. Moreover, likewise for (4.6), we can show

$$|x'(t, \varepsilon)| \leq \frac{2L\varepsilon}{T(1-k)}$$

for all $t \in R$, where we have set

$$L := \sup\{|\Phi(t, u, 0, \varepsilon)| : t \in R,\ u \in \overline{U},\ \varepsilon \in [0, 1]\}. \qquad \square$$

5. A counterexample

Let $\beta \in \,]0, 1]$, $q \in [0, 1/2[$ and $\varphi: \mathbb{R} \to \mathbb{R}$ be the continuous map defined by

$$\varphi(x) = \begin{cases} qx^2 & \text{if } x \in [-1, 1], \\ q & \text{if } x \notin [-1, 1]. \end{cases}$$

Put $U = \,]-\rho, \rho[$ for some $\rho > 0$. Consider the following neutral differential equation

$$x'(t) = \varepsilon\varphi\bigg(\frac{1}{\varepsilon} x'\bigg(t - \frac{\pi}{2}\bigg)\bigg) - \varepsilon x\bigg(t - \frac{\pi}{2}\bigg) - \varepsilon\beta \sin t. \tag{5.1}$$

Here we have:

$$\Phi(t,x,y,\varepsilon)=\varphi(y)-x+\beta\sin t,$$
$$T=2\pi,\quad k=2q,\quad \Phi_0(x)=-x,\quad \deg(-\Phi_0,U)=1,$$
$$M=\sup\{|\Phi(t,x,0,0)|:t\in R,\ x\in\partial U\}=\rho+\beta.$$

Therefore, in this example, (A1)–(A5) are obviously fulfilled. But we will prove the following proposition:

Proposition 5.1. *The equation* (5.1) *has no* 2π*-periodic solution* $x(\,\cdot\,,\varepsilon)$ *satisfying* $x(\,\cdot\,,\varepsilon)\to 0$ *in* $C_{2\pi}$ *as* $\varepsilon\to 0,\varepsilon>0$.

In view of Theorem 4.1 this example shows that condition (4.1) of this theorem is not satisfied here. Indeed Proposition 5.1 implies that the conclusion of Theorem 4.1 does not hold on $U=\,]-\rho,\rho[$ for each $\rho>0$. The additional condition (4.1) is thus necessary in Theorem 4.1.

Proof of Proposition 5.1. By contradiction, assume that equation (5.1) has 2π-periodic solution $x(\,\cdot\,,\varepsilon)$ satisfying $x(\,\cdot\,,\varepsilon)\to 0$ in $C_{2\pi}$ as $\varepsilon\to 0$. Then, it comes

$$\int_0^{2\pi}x(t,\varepsilon)\,dt=\int_0^{2\pi}\varphi\left(\frac{1}{\varepsilon}x'\left(t-\frac{\pi}{2},\varepsilon\right)\right)dt,$$

and consequently ,

$$\int_0^{2\pi}\varphi\left(\frac{1}{\varepsilon}x'(t,\varepsilon)\right)dt=\int_0^{2\pi}\varphi\left(\frac{1}{\varepsilon}x'\left(t-\frac{\pi}{2},\varepsilon\right)\right)dt\overset{\varepsilon\to 0}{\longrightarrow}0.\tag{5.2}$$

Because we have $\varphi\ge 0$ relation (5.1) involves

$$\frac{1}{\varepsilon}x'(t,\varepsilon)\ge\beta\frac{\sqrt{2}}{2}+\delta(t,\varepsilon),\quad t\in\left[\frac{3\pi}{4},\pi\right]=J$$

with $\sup_{t\in J}|\delta(t,\varepsilon)|\overset{\varepsilon\to 0}{\longrightarrow}0$. Then, using again $\varphi\ge 0$, we obtain

$$\int_0^{2\pi}\varphi\left(\frac{1}{\varepsilon}x'(t,\varepsilon)\right)dt\ge\int_{3\pi/4}^{\pi}\varphi\left(\frac{1}{\varepsilon}x'(t,\varepsilon)\right)dt\ge\frac{\pi}{4}\varphi\left(\beta\frac{\sqrt{2}}{2}\right)+o_\varepsilon(1).$$

This last inequality contradicts (5.2). □

Really we can see directly that condition (4.1) fails for $\rho>0$ sufficiently small if we take for instance

$$\beta\ge 1-2q.\tag{5.3}$$

Let us choose ρ such that we have

$$\sqrt{\frac{2\rho}{q}}\le 1.$$

Such a choice is clearly possible by taking for instance $\rho > 0$ sufficiently small. Then, by setting

$$y(t) = \sqrt{\frac{2\rho}{q}} \cos t,$$

owing to (5.3), we easily check $y \in \mathcal{M}$ and $\varphi(y(t)) = 2\rho \cos^2 t$. So it follows

$$0 = \int_0^{2\pi} \Phi(t, \rho, y(t), 0)\, dt \in Z(\rho).$$

Therefore, condition (4.1) does not hold.

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