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AN APPLICATION OF NONSMOOTH CRITICAL POINT THEORY

Zhouxin Li — Yaotian Shen — Yimin Zhang

Abstract. We consider a class of elliptic equation with natural growth. We obtain a region of the natural growth term with precise lower boundary less than zero.

1. Introduction and main results

Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ with smooth boundary. In this paper we consider the functional $I\colon W_0^{1,p}(\Omega) \to \mathbb{R}$, $2 \leq p < N$, given by

$$I(u) = \int_\Omega j(x,u,\nabla u) - \int_\Omega G(x,u). \tag{1.1}$$

Here $j(x,s,\xi)\colon \Omega \times \mathbb{R} \times \mathbb{R}^N$ is a function which is measurable with respect to x for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$, and of class C^1 with respect to (s,ξ) for almoast every $x \in \Omega$, $G(x,s) = \int_0^s g(x,t)\,dt$, where $g(x,s)$ is a Carathéodory function.

We are concerned with the existence and nonexistence of nontrivial critical points of the functional I. Let $j_s(x,s,\xi)$ and $j_\xi(x,s,\xi)$ denote the derivatives of $j(x,s,\xi)$ with respect to s and ξ respectively, we know that the Euler–Lagrange

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equation of the functional I is

$$\begin{cases} -\mathrm{div}(j_\xi(x,u,\nabla u)) + j_s(x,u,\nabla u) = g(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

As pointed out by D. Arcoya and L. Boccardo [1] (one can see also [6]) that since the function $j(x,u,\nabla u)$ depends on u, the functional I is not even Gâteaux differentiable on $W_0^{1,p}(\Omega)$ but only differentiable along directions in $W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$. For example, when $p=2$, if we set $j(x,u,\nabla u) = u^2|\nabla u|^2$, then $j_s(x,u,\nabla u) = 2u|\nabla u|^2$, it is easy to verify that $2u|\nabla u|^2$ not necessarily belong to $W^{-1,p'}(\Omega)$, the topological dual of $W_0^{1,p}(\Omega)$. $j_s(x,s,\xi)$ is called the natural growth term of Problem (1.2).

The study of Problem (1.2) arise from more concrete case as, for example, when $p=2$,

$$\begin{cases} -\displaystyle\sum_{ij=1}^N D_j(a_{ij}(x,u)D_iu) \\ \qquad +\displaystyle\frac{1}{2}\sum_{ij=1}^N \partial_s a_{ij}(x,u)D_iuD_ju = g(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Existence and multiplicity results for equations like (1.3) have been object of a very careful analysis since 1994 (see e.g. [1], [3], [4], [6], [13] and references therein). In these papers, the approaches are variational and the nontrivial critical points were obtained via the techniques of nonsmooth critical point theory.

In order to get the compactness result, one of the important assumptions in [1], [3], [4], [6] is

$$s\sum_{ij=1}^N \partial_s a_{ij}(x,s)\xi_i\xi_j \geq 0.$$

This sign condition plays an important part in the proof of the compactness for a Palais-Smale sequence. But if we consider another assumption, which says, there exist ν,σ with $2<\sigma<2N/(N-2)$, $\nu\in(0,\sigma-2)$ such that

$$s\sum_{ij=1}^N \partial_s a_{ij}(x,s)\xi_i\xi_j \leq \nu\sum_{ij=1}^N a_{ij}(x,s)\xi_i\xi_j,$$

we find that the region in which $s\sum_{ij=1}^N \partial_s a_{ij}(x,s)\xi_i\xi_j$ exists will vanish as the parameter σ tends to 2. It is interesting that the author in [13] studied the case $\sigma=2$, and assume that there exists $0<\alpha_1<1$ such that

$$-2\alpha_1\sum_{i,j}^N a_{ij}(x,s)\xi_i\xi_j \leq s\sum_{i,j}^N \partial_s a_{ij}(x,s)\xi_i\xi_j \leq 0.$$

Under this condition and other certain hypotheses, the author in [13] proved that there exists at least one weak solution of problem (1.3).

As to the general case $j(x,s,\xi)$, problem (1.1) was also studied by many authors, see for example [11] for $p=2$ and [1], [15], [16] for $1 \le p < N$. In these papers, the assumption on the natural growth term is $sj_s(x,s,\xi) \ge 0$. As mentioned above, this sign condition plays an important part in the proof of the compactness for a Palais–Smale sequence. On the other hand, the technique in [11], [15], [16] is variational via the nonsmooth critical point theory based on the notion of weak slope proposed by J. N. Corvellec, M. Degiovanni and M. Marzocchi in [7], [8], which is different to [1].

Let $a \ge 0$, $2 \le p < N$ and

$$j(x,s,\xi) = \frac{1}{p}\left(1 + \frac{1}{1+|s|^a}\right)|\xi|^p,$$

then by direct computation, one gets

$$sj_s(x,s,\xi) = -\frac{1}{p}\frac{a|s|^a}{(1+|s|^a)^2}|\xi|^p \le 0.$$

The equality holds if and only if $s=0$ or $\xi=0$. Thus, it remains an interesting question whether problem (1.1) has a nontrivial critical point when $sj_s(x,s,\xi) < 0$.

In this paper we discuss the general case of problem (1.1) for $j(x,s,\xi)$ and $2 \le p < N$. Motivated by [11], [13], we study the existence of nontrivial critical points for Problem (1.1) when the sign condition is dropped (see condition (j_3) in this section).

A crucial step in proving our main result is to show the compactness of the Palais–Smale sequence of the functional I when the sign condition is dropped. Under certain hypotheses on the functions j and g, we get the desire result. We use the nonsmooth critical point theory in [7], [8] to prove the existence of one or infinitely many nontrivial critical points of I. Moreover, we use the Pohozaev identity in [12] to prove that the corresponding Euler–Lagrange equation of problem (1.1), i.e. (1.2), has no weak solution in $C^2(\Omega) \cap C^1(\overline{\Omega})$ when Ω is star-shaped and condition (j_3) fails (see (j_6) in this section).

In this paper, we give the following assumptions on the functions j and g.

The function $j(x,s,\xi)\colon \Omega \times \mathbb{R} \times \mathbb{R}^N$ is measurable with respect to x for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$, and of class C^1 with respect to (s,ξ) for almost every $x \in \Omega$. We also assume that there exist α, β with $\beta \ge \alpha > 0$ and $\gamma > 0$ such that for almost every $x \in \Omega$ and every $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$

(j_1) $$\alpha|\xi|^p \le j_\xi(x,s,\xi)\xi \le \beta|\xi|^p,$$

(j_2) $$|j_s(x,s,\xi)| \le \gamma|\xi|^p.$$

Regarding the function $g(x,s)$, we assume that g is a Carathéodory function and that there exist $p < q < p^* := Np/(N-p)$ and $a(x) \in L^\infty(\Omega), b > 0$ such that

$$|g(x,s)| \le a(x) + b|s|^{q-1}. \tag{g$_1$}$$

We also assume that there exist $p < \sigma < p^*$, and $a_0(x) \in L^1(\Omega)$, $b_0(x) \in L^m(\Omega)$ with $m = p^*/(p^*-r), 1 < r < p$ such that

$$\sigma G(x,s) \le sg(x,s) + a_0(x) + b_0(x)|s|^r. \tag{g$_2$}$$

Let $\mu = (1 - \frac{\sigma}{p^*})$, we assume that there exist $\nu > 0$, $R > 0$ such that, for almost every $x \in \Omega$ and every $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$ with $|s| > R$,

$$-\mu j_\xi(x,s,\xi)\xi \le j_s(x,s,\xi)s, \tag{j$_3$}$$

$$j_\xi(x,s,\xi)\xi \le pj(x,s,\xi), \tag{j$_4$}$$

and

$$\sigma j(x,s,\xi) - j_\xi(x,s,\xi)\xi - j_s(x,s,\xi)s \ge \nu|\xi|^p, \tag{j$_5$}$$

where σ is given by (g$_2$).

We will prove different existence of critical points for the functional I in dependence on different growth rate of the function $g(x,s)$. First we study the nonsymmetric case. In this case, we assume that for almost every $x \in \Omega$,

$$\limsup_{|s|\to 0} \frac{g(x,s)}{|s|^{p-1}} < \alpha\lambda_1 \le \beta\lambda_1 < \liminf_{|s|\to\infty} \frac{g(x,s)}{|s|^{p-2}s}, \tag{g$_3$}$$

where λ_1 is the first eigenvalue of the p-Laplacian operator $-\Delta_p$.

Then we have the following result.

Theorem 1.1. *Assume conditions* (j$_1$)–(j$_5$) *and* (g$_1$)–(g$_3$) *hold, then there exists a nontrivial critical point* $u \in W_0^{1,p}(\Omega)$ *of problem* (1.1).

Next we study the symmetric case. In this case, we assume that for almost every $x \in \Omega$, $j(x,-s,-\xi) = j(x,s,\xi)$, $g(x,-s) = -g(x,s)$ and

$$\lim_{|s|\to\infty} \frac{g(x,s)}{|s|^{p-2}s} = \infty. \tag{g$_4$}$$

Then we have the following result.

Theorem 1.2. *Assume conditions* (j$_1$)–(j$_5$) *and* (g$_1$)–(g$_2$), (g$_4$) *hold, then there exist a sequence* $\{u_n\}$ *of nontrivial critical points of problem* (1.1) *in* $W_0^{1,p}(\Omega)$ *such that* $I(u_n) \to \infty$ *as* $n \to \infty$.

To get the nonexistence result, we assume, besides (j$_1$)–(j$_2$), the following (g$_1$)$'$ and (g$_2$)$'$ for simplicity.

We assume that $g(x,s) \equiv g(s)$ and there exist $p < q, \sigma < p^*$ and $b > 0$ such that

$$(\mathrm{g}_1)' \qquad |g(s)| \le b|s|^{q-1},$$

and

$$(\mathrm{g}_2)' \qquad 0 < \sigma G(x,s) \le s g(x,s).$$

For example, $g(x,s) = |s|^{q-2}s$ with $p < q < p^*$ and $\sigma = q$.

Then we have the following result.

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^N$ $(N \ge 3)$ be bounded and star-shaped, assume that j does not depend on x and that* (j_1)–(j_2) *and* $(\mathrm{g}_1)'$–$(\mathrm{g}_2)'$ *hold. Moreover, assume that for every $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$*

$$(\mathrm{j}_6) \qquad s j_s(s,\xi) < -\mu j_\xi(s,\xi)\xi,$$

where $\mu = (1 - \sigma/p^*)$, then (1.2) *has no nontrivial solution in $C^2(\Omega) \cap C^1(\overline{\Omega})$.*

The paper is arranged as follows. In Section 2, we set the abstract framework and specify its connections with our problem. In Section 3, we study the compactness of the Palais–Smale sequence. In Section 4, we prove the existence and nonexistence of nontrivial critical points.

Throughout this paper we denote by $\|\cdot\|$, $\|\cdot\|_q$ and $\|\cdot\|_{-1,p'}$ the standard norms of $W_0^{1,p}(\Omega)$, $L^q(\Omega)$ and $W^{-1,p'}(\Omega)$, respectively. "$\to$" ("$\rightharpoonup$") indicates the strong (weak) convergence in the corresponding function space.

2. Mathematical background

In this section we give some definitions and abstract critical point theories (for the proof, see [7], [8]) will be used in this paper. These definitions and theories also have been used in [6], [11], [15], [16].

Definition 2.1. Let X be a complete metric space endowed with the metric d, $f\colon X \to \mathbb{R}$ be a continuous function, and $u \in X$. We denote by $|df|(u)$ the supremum of the real numbers σ in $[0,\infty)$ such that there exist $\delta > 0$ and a continuous map

$$H\colon B(u;\delta) \times [0,\delta] \to X,$$

such that for every v in $B(u;\delta)$ and for every t in $[0,\delta]$ it results

$$d(H(v,t),v) \le t, \tag{2.1}$$

$$f(H(v,t)) \le f(v) - \sigma t. \tag{2.2}$$

where $B(u;\delta)$ is the open ball of center $u \in X$ and of radius δ. The extended real number $|df|(u)$ is called the weak slope of f at u.

If X is a Finsler manifold of class C^1, it turns out that $|df|(u) = \|f'(u)\|$.

Definition 2.2. Let X be a complete metric space, $f\colon X \to \mathbb{R}$ be a continuous function. A point $u \in X$ is a critical point of f if $|df|(u) = 0$. We say that $c \in \mathbb{R}$ is a critical value of f if there exists a critical point $u \in X$ of f with $f(u) = c$.

Definition 2.3. Let X be a complete metric space, $f\colon X \to \mathbb{R}$ be a continuous function and $c \in \mathbb{R}$. We say that f satisfies the Palais–Smale condition at level c ($(\mathrm{PS})_c$ in short), if every sequence $\{u_n\}$ in X such that $|df|(u_n) \to 0$ and $f(u_n) \to c$ admits a subsequence $\{u_{n_k}\}$ converging in X.

Theorem 2.4. *Let X be a Banach space endowed with the norm $\|\cdot\|$ and $f\colon X \to \mathbb{R}$ a continuous function. First, suppose that there exist $w \in X$, $\eta > f(0)$ and $r > 0$ such that*

$$f(u) > \eta, \quad \text{for all } u \in X,\ \|u\| = r, \tag{2.3}$$

$$f(w) < \eta, \quad \|w\| > r. \tag{2.4}$$

We set $\Gamma = \{\gamma\colon [0,1] \to X$, is continuous and $\gamma(0) = 0,\ \gamma(1) = w\}$. Finally, suppose that f satisfies $(\mathrm{PS})_c$ condition at the level

$$c = \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} f(\gamma(t)) < \infty,$$

Then, there exists a nontrivial critical point u of f such that $f(u) = c$.

Theorem 2.5. *Let X be a Banach space, $f\colon X \to \mathbb{R}$ a continuous even functional. Assume that there exists a strictly increasing sequence $\{W_k\}$ of finite dimensional subspaces of X with the following properties:*

(a) *there exist $\rho > 0$, $\eta > f(0)$ and a subspace $V \subset X$ of finite codimension such that*

$$f(u) \geq \eta, \quad \text{for all } u \in V,\ \|u\| = \rho;$$

(b) *there exists a sequence $\{R_k\}$ in (ρ, ∞) such that*

$$f(u) \leq f(0), \quad \text{for all } u \in W_k,\ \|u\| \geq R_k;$$

(c) *f satisfies $(\mathrm{PS})_c$ condition for any $c \geq \eta$.*

Then there exists a sequence $\{u_k\}$ of critical points of f with

$$\lim_{k\to\infty} f(u_k) = \infty.$$

Definition 2.6. A sequence $\{u_n\} \subset W_0^{1,p}(\Omega)$ is a Concrete–Palais–Smale sequence at level c ($(\mathrm{CPS})_c$ in short) if there exists $y_n \in W^{-1,p'}(\Omega)$ with $y_n \to 0$ such that

$$I(u_n) \to c, \tag{2.5}$$

$$\langle I'(u_n),\varphi\rangle = \int_\Omega [j_\xi(x,u_n,\nabla u_n)\nabla\varphi + j_s(x,u_n,\nabla u_n)\varphi] - \int_\Omega g(x,u_n)\varphi = \langle y_n,\varphi\rangle, \tag{2.6}$$

for all $\varphi \in C_0^\infty(\Omega)$. Moreover, we say that I satisfies the $(\mathrm{CPS})_c$ condition if every $(\mathrm{CPS})_c$ sequence is strongly compact in $W_0^{1,p}(\Omega)$.

The next result connects the previous notions with abstract critical point theory.

Theorem 2.7. *The functional $I\colon W_0^{1,p}(\Omega)\to\mathbb{R}$ is continuous and*

$$|dI|(u) \geq \sup_{\varphi\in C_0^\infty(\Omega),\, \|\varphi\|=1} \bigg\{ \int_\Omega [j_\xi(x,u,\nabla u)\nabla\varphi + j_s(x,u,\nabla u)\varphi - g(x,u)\varphi] \bigg\},$$

for every $u\in W_0^{1,p}(\Omega)$. In particular, if $|dI|(u)<\infty$, then we have

$$|dI|(u) \geq \| -\mathrm{div}(j_\xi(x,u,\nabla u)) + j_s(x,u,\nabla u) - g(x,u)\|_{-1,p'}.$$

Proof. See [5, Theorem 2.1.3]. □

3. Compactness results

In this section, we prove that the functional I satisfies the $(\mathrm{CPS})_c$ condition in $W_0^{1,p}(\Omega)$, so does the $(\mathrm{PS})_c$ condition. Indeed, if $\{u_n\}\subset W_0^{1,p}(\Omega)$ is a $(\mathrm{PS})_c$ sequence of I, then by Theorem 2.7, it is also a $(\mathrm{CPS})_c$ sequence. Then, if I satisfies the $(\mathrm{CPS})_c$ condition, we can deduce that $\{u_n\}$ admits a convergent subsequence.

Proposition 3.1. *Assume* (j_1)–(j_5) *hold, let $u\in W_0^{1,p}(\Omega)$ and assume that there exists a $w\in W^{-1,p'}(\Omega)$ such that for every $v\in W_0^{1,p}\cap L^\infty(\Omega)$*

$$\int_\Omega j_\xi(x,u,\nabla u)\nabla v + \int_\Omega j_s(x,u,\nabla u)v = \langle w,v\rangle. \tag{3.1}$$

Then $j_\xi(x,u,\nabla u)\nabla u, j_s(x,u,\nabla u)u\in L^1(\Omega)$ and

$$\int_\Omega j_\xi(x,u,\nabla u)\nabla u + \int_\Omega j_s(x,u,\nabla u)u = \langle w,u\rangle. \tag{3.2}$$

Proof. Let $k\in\mathbb{R}^+$ be fixed, we define the following cutoff functions:

$$T_k(u) = \begin{cases} u & \text{if } |u|\leq k,\\ \mathrm{sgn} u\cdot k & \text{if } |u|>k, \end{cases} \qquad G_k(u) = u - T_k(u). \tag{3.3}$$

Then for every $v\in W_0^{1,p}(\Omega)$, we have $T_k(v)\in W_0^{1,p}\cap L^\infty(\Omega)$. Thus, we can take $T_k(u)$ as a test function in (3.1) and get

$$\int_\Omega j_\xi(x,u,\nabla u)\nabla T_k(u) + \int_\Omega j_s(x,u,\nabla u)T_k(u) = \langle w,T_k(u)\rangle. \tag{3.4}$$

Since from (j_1), we can deduce that

$$\begin{aligned} j(x,s,\xi) &= \int_0^1 j_\xi(x,s,t\xi)\xi\,dt \geq \int_0^1 \alpha|t\xi|^p t^{-1}\,dt = \frac{\alpha}{p}|\xi|^p, \\ j(x,s,\xi) &= \int_0^1 j_\xi(x,s,t\xi)\xi\,dt \leq \int_0^1 \beta|t\xi|^p t^{-1}\,dt = \frac{\beta}{p}|\xi|^p. \end{aligned}$$

This means that

$$\frac{\alpha}{p}|\xi|^p \leq j(x,s,\xi) \leq \frac{\beta}{p}|\xi|^p \tag{3.5}$$

for almost every $x \in \Omega$ and every $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$. Let R be given by (j_3), we denote

$$A_R := \{x \in \Omega : |u| > R\}, \quad B_R := \Omega \setminus A_R. \tag{3.6}$$

Then by (j_2), we have

$$|j_s(x,u,\nabla u)u| \leq R\gamma|\nabla u|^p \quad on B_R. \tag{3.7}$$

Denote

$$\Omega^+ := \{x \in \Omega : 0 \leq j_s(x,u,\nabla u)u\}, \tag{3.8}$$

$$\Omega^- := \{x \in \Omega : -\mu j_\xi(x,u,\nabla u)\nabla u \leq j_s(x,u,\nabla u)u \leq 0\}. \tag{3.9}$$

Then, by (j_5) and (3.5), we have

$$|j_s(x,u,\nabla u)u| \leq \sigma j(x,u,\nabla u) \leq \frac{\sigma\beta}{p}|\nabla u|^p \quad \text{on } A_R \cap \Omega^+, \tag{3.10}$$

by (j_3) and (j_1), we have

$$|j_s(x,u,\nabla u)u| \leq \mu j_\xi(x,u,\nabla u)\nabla u \leq \mu\beta|\nabla u|^p \quad \text{on } A_R \cap \Omega^-. \tag{3.11}$$

Combining (3.7), (3.10) and (3.11), we have

$$|j_s(x,u,\nabla u)u| \leq \left[R\gamma + \left(\frac{\sigma}{p} + \mu\right)\beta\right]|\nabla u|^p.$$

This means that $j_s(x,u,\nabla u)u \in L^1(\Omega)$. Thus, we can use Lebesgue Dominated Convergence Theorem to pass the limit in (3.4) and get (3.2). □

Proposition 3.2. *Assume* (j_1)–(j_5) *and* (g_1)–(g_2) *hold, then every Concrete–Palais–Smale sequence* $\{u_n\}$ *is bounded in* $W_0^{1,p}(\Omega)$.

Proof. Assume that $\{u_n\} \subset W_0^{1,p}(\Omega)$ such that (2.5) and (2.6) hold. Let us fix $\varepsilon > 0$ and consider the function $\vartheta_\varepsilon\colon \mathbb{R} \to \mathbb{R}$ defined by

$$\vartheta_\varepsilon(s) = \begin{cases} 0 & \text{for } 0 \leq s \leq R, \\ (1+\varepsilon)(s-R) & \text{for } R \leq s \leq R_\varepsilon, \\ s & \text{for } R_\varepsilon \leq s, \\ -\vartheta_\varepsilon(-s) & \text{for } s \leq 0, \end{cases}$$

where R is given in (j_3) and $R_\varepsilon = (1+\varepsilon)R/\varepsilon$. Then, for every $u \in W_0^{1,p}(\Omega)$, it results

$$|\nabla \vartheta_\varepsilon(u)| \leq (1+\varepsilon)|\nabla u|. \tag{3.12}$$

Moreover, $\vartheta_\varepsilon(u)$ has the same sign of u. By Proposition 3.1, we can take u_n as test functions in (2.6) and get

$$\begin{aligned} \int_{A_{R,n}} [j_\xi(x,u_n,\nabla u_n)\nabla \vartheta_\varepsilon(u_n) + j_s(x,u_n,\nabla u_n)\vartheta_\varepsilon(u_n)] \\ = \int_{A_{R,n}} g(x,u_n)u_n + \int_{A_{R,n}} g(x,u_n)(\vartheta_\varepsilon(u_n)-u_n) + \langle y_n, \vartheta_\varepsilon(u_n)\rangle. \end{aligned} \tag{3.13}$$

where $A_{R,n}$ is defined as in (3.6). Since $\vartheta_\varepsilon(u_n)$ has the same sign of u_n, by (j_5) and (3.12), we can deduce that

$$\begin{aligned} &\int_{A_{R,n}\cap\Omega_n^+} [\sigma j(x,u_n,\nabla u_n) - j_\xi(x,u_n,\nabla u_n)\nabla\vartheta_\varepsilon(u_n) - j_s(x,u_n,\nabla u_n)\vartheta_\varepsilon(u_n)] \\ &\geq \int_{A_{R,n}\cap\Omega_n^+} [\sigma j(x,u_n,\nabla u_n) - (1+\varepsilon) j_\xi(x,u_n,\nabla u_n)\nabla u_n - j_s(x,u_n,\nabla u_n)u_n] \\ &\geq \nu \int_{A_{R,n}\cap\Omega_n^+} |\nabla u_n|^p - \varepsilon\beta \int_\Omega |\nabla u_n|^p, \end{aligned}$$

and by (j_4), (3.5),

$$\begin{aligned} &\int_{A_{R,n}\cap\Omega_n^-} [\sigma j(x,u_n,\nabla u_n) - j_\xi(x,u_n,\nabla u_n)\nabla\vartheta_\varepsilon(u_n) - j_s(x,u_n,\nabla u_n)\vartheta_\varepsilon(u_n)] \\ &\qquad \geq \int_{A_{R,n}\cap\Omega_n^-} [\sigma j(x,u_n,\nabla u_n) - (1+\varepsilon) j_\xi(x,u_n,\nabla u_n)\nabla u_n] \\ &\qquad \geq \left(\frac{\sigma-p}{p}\right)\alpha \int_{A_{R,n}\cap\Omega_n^-} |\nabla u_n|^p - \varepsilon\beta \int_\Omega |\nabla u_n|^p, \end{aligned}$$

where Ω_n^+ and Ω_n^- are defined as in (3.8) and (3.9). Thus we get

$$\begin{aligned} \min\left\{\nu, \frac{\sigma-p}{p}\alpha\right\} \int_{A_{R,n}} |\nabla u_n|^p - 2\varepsilon\beta \int_\Omega |\nabla u_n|^p \\ \leq \int_{A_{R,n}} [\sigma j(x,u_n,\nabla u_n) - j_\xi(x,u_n,\nabla u_n)\nabla\vartheta_\varepsilon(u_n) - j_s(x,u_n,\nabla u_n)\vartheta_\varepsilon(u_n)]. \end{aligned} \tag{3.14}$$

On the other hand, by (3.5), we have

$$\frac{\alpha}{p} \int_{B_{R,n}} |\nabla u_n|^p \leq \int_{B_{R,n}} j(x,u_n,\nabla u_n), \tag{3.15}$$

where $B_{R,n}$ is defined as in (3.6). Now let

$$\nu_0 := \min\left\{\nu, \frac{\sigma-p}{p}\alpha, \frac{\sigma}{p}\alpha\right\} > 0$$

and compute $\sigma I(u_n) - \langle y_n, \vartheta_\varepsilon(u_n)\rangle$. By (3.13)–(3.15), we can deduce that

$$
\begin{aligned}
(3.16)\quad \nu_0 \int_\Omega |\nabla u_n|^p - 2\varepsilon\beta \int_\Omega |\nabla u_n|^p &\le \sigma \int_\Omega j(x, u_n, \nabla u_n) \\
&\quad - \int_{A_{R,n}} [j_\xi(x, u_n, \nabla u_n)\vartheta_\varepsilon(u_n) + j_s(x, u_n, \nabla u_n)\vartheta_\varepsilon(u_n)] \\
&= \sigma I(u_n) + \int_\Omega G(x, u_n) - \langle y_n, \vartheta_\varepsilon(u_n)\rangle - \int_\Omega g(x, u_n)\vartheta_\varepsilon(u_n) \\
&\le \sigma I(u_n) + \bigg|\int_{B_{R,n}} G(x, u_n)\bigg| + \int_{A_{R,n}} [\sigma G(x, u_n) - g(x, u_n)u_n] \\
&\quad + |\langle y_n, \vartheta_\varepsilon(u_n)\rangle| + \bigg|\int_{A_{R,n}} g(x, u_n)(\vartheta_\varepsilon(u_n) - u_n)\bigg|.
\end{aligned}
$$

Note that

$$
\begin{aligned}
(3.17)\quad &\bigg|\int_{A_{R,n}} g(x, u_n)(\vartheta_\varepsilon(u_n) - u_n)\bigg| \\
&\qquad = \bigg|\int_{\{x\in\Omega : R<|u_n|<R_\varepsilon\}} g(x, u_n)(\vartheta_\varepsilon(u_n) - u_n)\bigg| \le C(R, \varepsilon),
\end{aligned}
$$

and by (g_1), we have

$$
(3.18)\qquad \bigg|\int_{B_{R,n}} G(x, u_n)\bigg| \le \int_{B_{R,n}} (|a(x)||u_n| + b|u_n|^q) \le C(R).
$$

Now choose $\varepsilon = \nu_0/4\beta$, combine (3.16)–(3.18), by (g_2) and Sobolev inequality, we get

$$
(3.19)\qquad \frac{\nu_0}{2}\|u_n\|^p \le C + \left(1 + \frac{\nu_0}{4\beta}\right)\|y_n\|_{-1,p'}\|u_n\| + \|b_0\|_m\|u_n\|^r,
$$

where $C = C(R, \varepsilon, c)$, c is given by (2.5). Note that $1 < r < p$ and $\|y_n\|_{-1,p'} \to 0$, (3.19) yields the conclusion. □

Proposition 3.3. *Assume* (j_1)–(j_5) *and* (g_1)–(g_2) *hold, then every bounded Concrete–Palais–Smale sequence* $\{u_n\}$ *converges strongly to* $u \in W_0^{1,p}(\Omega)$.

Proof. By assumptions, there exists a u in $W_0^{1,p}(\Omega)$ such that up to a subsequence, u_n converges weakly to u in $W_0^{1,p}(\Omega)$, u_n converges strongly to u in $L^q(\Omega)$, $1 < q < p^*$ and u_n converges to u almost everywhere in Ω.

For $k \in \mathbb{R}^+$ be fixed, without lost of generality, we assume that $k > R$, where R is given by (j_3), we denote

$$
\begin{aligned}
A_{k,n} &= \{x \in \Omega : |u_n| > k\}, & B_{k,n} &= \Omega \setminus A_{k,n}, \\
A^+_{k,n} &= \{x \in \Omega : u_n > k\}, & A^-_{k,n} &= \{x \in \Omega : u_n < -k\}.
\end{aligned}
$$

Let T_k and G_k as defined in (3.3). Without lost of generality, we assume that $|\nabla G_k(u_n)| \ne 0$ and $|\nabla T_k(u_n)| \ne 0$ for every n. We divide the proof in two steps.

Step 1. We prove that for any $\varepsilon > 0$, there exists $k > 0$ large enough such that $\lim_{n\to\infty} \|G_k(u_n)\| \le \varepsilon$. Because $\nabla G_k(u_n) = \nabla u_n$ in $A_{k,n}$ and $\nabla G_k(u_n) = 0$ in $B_{k,n}$, Proposition 3.1 implies that we can take $\varphi = G_k(u_n)$ as test functions in (2.6) and get

$$\tag{3.20} \begin{aligned} \int_{A_{k,n}} j_\xi(x,u_n,\nabla u_n)\nabla G_k(u_n) + \int_{A_{k,n}} j_s(x,u_n,\nabla u_n)G_k(u_n) \\ = \int_{A_{k,n}} g(x,u_n)G_k(u_n) + \langle y_n, G_k(u_n)\rangle. \end{aligned}$$

By (j_3), (j_1) and note that $G_k(u_n)$ has the same sign of u_n, we have

$$\tag{3.21} \begin{aligned} \int_{A^+_{k,n}} j_s(x,u_n,\nabla u_n)G_k(u_n) &= \int_{A^+_{k,n}} j_s(x,u_n,\nabla u_n)(u_n-k) \\ &= \int_{A^+_{k,n}\cap\Omega^+_n} \Big(1-\frac{k}{u_n}\Big) j_s(x,u_n,\nabla u_n)u_n \\ &\quad + \int_{A^+_{k,n}\cap\Omega^-_n} \Big(1-\frac{k}{u_n}\Big) j_s(x,u_n,\nabla u_n)u_n \\ &\ge \int_{A^+_{k,n}\cap\Omega^-_n} j_s(x,u_n,\nabla u_n)u_n \\ &\ge -\mu \int_{A^+_{k,n}\cap\Omega^-_n} j_\xi(x,u_n,\nabla u_n)\nabla u_n \ge -\mu \int_{A^+_{k,n}} j_\xi(x,u_n,\nabla u_n)\nabla u_n, \end{aligned}$$

where Ω^+_n and Ω^-_n are defined as in (3.8) and (3.9). Analogously,

$$\tag{3.22} \int_{A^-_{k,n}} j_s(x,u_n,\nabla u_n)G_k(u_n) \ge -\mu \int_{A^-_{k,n}} j_\xi(x,u_n,\nabla u_n)\nabla u_n,$$

Therefore, combining (3.21) and (3.22), we get

$$\tag{3.23} \int_{A_{k,n}} j_s(x,u_n,\nabla u_n)G_k(u_n) \ge -\mu \int_{A_{k,n}} j_\xi(x,u_n,\nabla u_n)\nabla u_n.$$

Note that $\nabla u_n = \nabla G_k(u_n)$ in $A_{k,n}$, from (3.20), (3.23) and according to (j_1), we get

$$\alpha(1-\mu)\|G_k(u_n)\|^2 \le \int_{A_{k,n}} |g(x,u_n)||G_k(u_n)|\,dx + \|y_n\|_{-1}\|G_k(u_n)\|.$$

Since $g(x,u_n) \to g(x,u)$ and $y_n \to 0$ in $W^{-1,p'}(\Omega)$, respectively, we get the conclusion.

Step 2. We prove that for a fixed k large enough, $\|T_k(u_n) - T_k(u)\| \to 0$ as n tends to infinity. Let $v_n = T_k(u_n) - T_k(u)$ and $\varphi(t) = te^{\eta t^2}$. Since $\varphi(v_n) \in$

$W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, we can take $\varphi = \varphi(v_n)$ as test functions in (2.6) and get

$$\begin{aligned}(3.24)\qquad &\int_\Omega g(x,u_n)\varphi(v_n)\,dx + \langle y_n, \varphi(v_n)\rangle \\ &= \int_\Omega \varphi'(v_n) j_\xi(x,u_n,\nabla u_n)\nabla v_n + \int_\Omega \varphi(v_n) j_s(x,u_n,\nabla u_n) := \mathrm{I} + \mathrm{II}.\end{aligned}$$

Firstly, by the definitions of T_k and G_k,

$$\begin{aligned}(3.25)\qquad \mathrm{I} = &\int_{A_{k,n}} \varphi'(v_n) j_\xi(x,u_n,\nabla G_k(u_n))\nabla v_n \\ &+ \int_{B_{k,n}} \varphi'(v_n) j_\xi(x,u_n,\nabla T_k(u_n))\nabla v_n := \mathrm{III} + \mathrm{IV}.\end{aligned}$$

According to (j_1) and by Hölder inequality, we get

(3.26)
$$\begin{aligned}|\mathrm{III}| &= \bigg|\int_{A_{k,n}} \varphi'(v_n) j_\xi(x,u_n,\nabla G_k(u_n))\nabla G_k(u_n)\nabla G_k(u_n)\nabla v_n|\nabla G_k(u_n)|^{-2}\bigg| \\ &\leq \beta\varphi'(2k)\int_{A_{k,n}} |\nabla G_k(u_n)|^{p-1}|\nabla v_n| \\ &\leq \beta\varphi'(2k)\|G_k(u_n)\|^{p-1}\|\nabla v_n\|_{L^p(A_{k,n})} \leq \varepsilon_n.\end{aligned}$$

Here and in the following, we use ε_n to denote a quantity which tends to zero as n tends to infinity. Since $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$, thus $v_n \rightharpoonup 0$ in $W_0^{1,p}(\Omega)$, by (j_1), we get

$$(3.27)\qquad \mathrm{IV} \geq \alpha\int_{B_{k,n}} \varphi'(v_n)|\nabla T_k(u_n)|^{p-2}\nabla T_k(u_n)\nabla v_n.$$

Recall that for $p \geq 2$ and all $x, y \in \mathbb{R}^N$ [10, Lemma 4.1, p. 5709],

$$(3.28)\qquad (|x|^{p-2}x - |y|^{p-2}y, x-y) \geq C_p|x-y|^p,$$

where $C_p > 0$ is a constant. Note that

$$\int_{B_{k,n}} \varphi'(v_n)|\nabla T_k(u)|^{p-2}(\nabla T_k(u))\nabla v_n \to 0.$$

Let $x = \nabla T_k(u_n), y = \nabla T_k(u)$ in (3.28), by direct computation, we get

$$(3.29)\qquad \mathrm{IV} \geq C_1\int_{B_{k,n}} \varphi'(v_n)|\nabla v_n|^p - \varepsilon_n,$$

where $C_1 > 0$ is a constant. Combining (3.26) and (3.29), we get

$$(3.30)\qquad \mathrm{I} \geq C_1\int_{B_{k,n}} \varphi'(v_n)|\nabla v_n|^p - \varepsilon_n.$$

Secondly, we consider II in (3.24). By (j_2), we have

$$\begin{aligned} \text{II} &\le \gamma\Big(\int_{A_{k,n}} |\varphi(v_n)||\nabla G_k(u_n)|^p + \int_{B_{k,n}} |\varphi(v_n)||\nabla T_k(u_n)|^p\Big) \\ &\le \gamma\Big(\int_{A_{k,n}} |\varphi(v_n)||\nabla G_k(u_n)|^p\Big) + 2^{p-1}\gamma\Big(\int_{B_{k,n}} |\varphi(v_n)||\nabla T_k(u)|^p\Big) \\ &\quad + 2^{p-1}\gamma\Big(\int_{B_{k,n}} |\varphi(v_n)||\nabla v_n|^p\Big) := \text{V} + \text{VI} + \text{VII}. \end{aligned} \tag{3.31}$$

Since $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$, thus $v_n \rightharpoonup 0$ in $W_0^{1,p}(\Omega)$, we have

$$\text{VI} = 2^{p-1}\gamma \int_{B_{k,n}} |\varphi(v_n)||\nabla T_k(u)|^p \to 0. \tag{3.32}$$

Moreover, by Step 1 and note that $|\varphi(v_n)| \le \varphi(2k)$, we have

$$\text{V} = \gamma \int_{A_{k,n}} |\varphi(v_n)||\nabla G_k(u_n)|^p \le \varepsilon_n. \tag{3.33}$$

Combining (3.31)–(3.33), we get

$$\text{II} \le C_2 \int_{B_{k,n}} |\varphi(v_n)||\nabla v_n|^p + \varepsilon_n, \tag{3.34}$$

where $C_2 > 0$ is a constant. According to Lemma 1.2 in [5], for $a, b > 0$, we have $a\varphi'(t) - b|\varphi(t)| \ge a/$ for every $t \in \mathbb{R}$ with $\eta > (b/2a)^2$. Taking $a = C_1$, $b = C_2$ in φ, and combining (3.24), (3.30) and (3.34), we get

$$\frac{C_1}{2} \int_{B_{k,n}} |\nabla v_n|^p \le \int_\Omega g(x, u_n)\varphi(v_n)\, dx + \varepsilon_n \to 0. \tag{3.35}$$

On the other hand, since $v_n = T_k(u_n) - T_k(u) = \operatorname{sign} u_n \cdot k - T_k(u)$ in $A_{k,n}$, we have

$$\int_{A_{k,n}} |\nabla v_n|^p = \int_{A_{k,n}} |\nabla T_k(u)|^p \to 0. \tag{3.36}$$

Thus (3.35) and (3.36) imply that $\|T_k(u_n) - T_k(u)\| \to 0$.

Finally, since for any fixed $k \in \mathbb{R}^+$,

$$\|u_n - u\| \le \|T_k(u_n) - T_k(u)\| + \|G_k(u_n)\| + \|G_k(u)\|.$$

We get u_n converges strongly to u. This completes the proof. □

Now let us recall the modified compactness condition introduced by Cerami which allows rather general minimax results.

Definition 3.4. Let X be a Banach space, a functional $J \in C(X,\mathbb{R})$ is said to satisfy the Cerami condition if for all $c \in \mathbb{R}$

(a) every bounded sequence $\{u_j\} \subset X$ such that $\{J(u_j)\}$ is bounded and $|dJ|(u_j) \to 0$ possesses a convergent subsequence, and
(b) there exit $\delta, R, \beta > 0$ such that for all $u \in J^{-1}[c-\delta, c+\delta]$ with $\|u\| \geq R$, $|dJ|(u) \cdot \|u\| \geq \beta$.

Proposition 3.5. *Assume conditions* (j_1)–(j_5) *and* (g_1)–(g_2) *hold, then* I *satisfies the Cerami condition.*

Proof. Firstly, according to Theorem 2.7 and Propositions 3.2 and 3.3, (a) is obvious.

Secondly, we prove that I satisfies (b). Suppose by contradiction. Let $c \in \mathbb{R}$ and assume that, up to a subsequence, $\{u_n\} \subset W_0^{1,p}(\Omega)$ such that $I(u_n) \to c$ and $|dI|(u_n) \cdot \|u_n\| \to 0$ with $\|u_n\| \to \infty$. By Theorem 2.7, we have

$$\|I'(u_n)\|_{-1,p'} \leq |df|(u_n). \tag{3.37}$$

On the other hand, by Proposition 3.1, we can take $\varphi = u_n$ as test functions in (2.6). By (3.37), we get $\langle I'(u_n), u_n\rangle \to 0$. Thus, we can argue as for (3.19) and get a contradiction. This completes the proof. $\square$

4. Proof of main theorems

In this section, we will use the compactness results in the previous section to prove our main results. Since we have proved that the functional I satisfies the $(\mathrm{PS})_c$ condition, it is trivial to prove Theorems 1.1 and 1.2. For the sake of completeness, we give the proof here.

Proof of Theorem 1.1. Firstly, Proposition 3.5 show that I satisfies $(\mathrm{PS})_c$ condition. Secondly, we prove that I satisfies the geometrical conditions of Theorem 2.4.

In fact, (j_1) implies that for every $u \in W_0^{1,p}(\Omega)$,

$$\frac{1}{p}\alpha\|u\|^p - \int_\Omega G(x,u) \leq I(u) \leq \beta\|u\|^p - \int_\Omega G(x,u).$$

By (g_3) and the definition of λ_1, when $u \in W_0^{1,p}(\Omega)$ small enough, we have

$$\int_\Omega G(x,u) < \frac{1}{p}\alpha\|u\|^p.$$

Note that $I(0) = 0$, thus (2.1) of Theorem 2.4 is satisfied. Now for $\varphi_1 \in W_0^{1,p}(\Omega)$, the first eigenfunction of $-\Delta_p$ operator, $\varphi_1 > 0$, $\|\varphi_1\| = 1$ and $t \in \mathbb{R}^+$, by (g_3) and the definition of λ_1, we have

$$I(t\varphi_1) \leq \beta t^p \int_\Omega |\nabla\varphi_1|^p - \int_\Omega G(x, t\varphi_1) \to -\infty.$$

Thus, we have $I(t\varphi_1) < 0$ when $t > 0$ large enough and the condition (2.2) of Theorem 2.4 is satisfied. Therefore, Theorem 2.4 yields the conclusion. □

Proof of Theorem 1.2. Note that the Sobolev space is a separable Banach space with infinite dimension, by [14, Theorem 7.7], there exist two sequences $\{v_n\} \subset W_0^{1,p}(\Omega)$ and $\{\varphi_n\} \subset W^{-1,p'}(\Omega)$ such that

(i) $< \varphi_n, v_m >= \delta_n^m$, where $\delta_n^m = 1$ when $m = n$ and $\delta_n^m = 0$, else

(ii) $W_0^{1,p}(\Omega) = \overline{\text{span}}\{v_m : m \in \mathbb{N}\}$ and $W^{-1,p'}(\Omega) = \overline{\text{span}}\{\varphi_n : n \in \mathbb{N}\}$.

Without lost of generality, we assume that v_m is a normalized sequence, that is $\|v_m\| = 1$, $m = 1, 2, \ldots$ and $v_k \perp v_l, k \neq l$. Denote $V_m = \overline{\text{span}}\{v_l : l \geq m\}$ and $V_m^\perp$ the topological complementary subspace of V_m in $W_0^{1,p}(\Omega)$ and hence $W_0^{1,p}(\Omega) = V_m \oplus V_m^\perp$. It is obviously that $V_1 = W_0^{1,p}(\Omega)$, $V_1^\perp = \phi$. Denote

$$\lambda_{q,m} = \inf_{u \in V_m} \frac{\|u\|}{\|u\|_q},$$

where $1 < q < p^*$. We have $\lambda_{q,m} \to +\infty$ as $m \to \infty$.

Firstly, note that $C_c^\infty(\Omega)$ is dense in $L^{p^{*\prime}}(\Omega)$, then for every $\varepsilon > 0$, there exist $a_c(x) \in C_c^\infty(\Omega)$ and $a_\varepsilon(x) \in L^{p^{*\prime}}(\Omega)$ with $\|a_\varepsilon\|_{p^{*\prime}} \leq \varepsilon$ such that $a(x) = a_c(x) + a_\varepsilon(x)$ and condition (g_1) implies that

$$|g(x,u)| \leq a_c(x) + a_\varepsilon(x) + b|u|^{q-1}.$$

Now choose $u \in V_m$ with $\|u\| = 1$, from condition (j_1), we have

$$I(u) \geq \frac{1}{p}\alpha\|u\|^2 - (\|a_c\|_2\|u\|_2 + \|a_\varepsilon\|_{p^{*\prime}}\|u\|_{p^*} + b\|u\|_q^q) \geq \frac{1}{p}\alpha - \left(\frac{c_1}{\lambda_{2,m}} + c_2\varepsilon + \frac{b}{\lambda_{q,m}^q}\right).$$

We can choose ε small enough and m large enough to such that

$$\frac{c_1}{\lambda_{2,m}} + c_2\varepsilon + \frac{b}{\lambda_{q,m}^q} < \frac{\alpha}{p},$$

this implies the geometrical condition (a) of Theorem 2.5 is satisfied.

Secondly, because $V_m^\perp$ is a finite dimensional subspace, since all norms in a finite dimensional space are all equivalent, we know that there exists a $C_2 > 0$ such that for every $u \in V_m^\perp$, $\|u\| \leq C_2\|u\|_p$. From conditions (j_1) and (g_4), we have

$$I(u) \leq \beta \int_\Omega |\nabla u|^p - \int_\Omega G(x,u) \to -\infty,$$

when $\|u\| \to \infty$, this implies the geometrical condition (b) of Theorem 2.5 is satisfied. Therefore, there exist a sequence $\{u_n\}$ of critical points of I such that $I(u_n) \to \infty$. This completes the proof. □

Before we prove Theorem 1.3, we give the Pohozaev identity in [12]. Let $F(x,u,r) \colon \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N$ be a functional of class C^1, we consider the following equation

$$\text{(F)} \qquad \operatorname{div}\{F_r(x,u,Du)\} = F_u(x,u,Du).$$

Here we write $Du = (\partial u/\partial x_1, \ldots, \partial u/\partial x_N)$, $F_n = \partial F/\partial u$, $F_{x_i} = \partial F/\partial x_i$ and $F_{r_i} = \partial F/\partial r_i$, $r = (r_1, \ldots, r_N)$. Assume that $F(x,0,0) = 0$. Let $a(x)$ and $h(x)$ be two functions of class $C^1(\Omega) \cap C(\overline{\Omega})$ and $u \in C^1(\Omega) \cap C(\overline{\Omega})$ be a solution of problem (F), then we have the following Pohozaev identity.

$$\begin{aligned}\int_{\partial\Omega} (F(x,0,Du) - D_i u F_{r_i}(x,0,Du))(h\cdot\nu)\,ds &= \int_\Omega F(x,u,Du)\operatorname{div} h \\ &+ \int_\Omega h_i F_{x_i}(x,u,Du) - \int_\Omega (D_j u D_i h_j + u D_i a(x)) F_{r_i}(x,u,Du) \\ &- \int_\Omega a(x)(D_i u F_{r_i}(x,u,Du) + u F_u(x,u,Du)).\end{aligned}$$

We refer also to [9], where the above variational relation is proved for C^1 solutions.

Proof of Theorem 1.3. Assume on the contrary, $u \in C^1(\Omega) \cap C(\overline{\Omega})$ is a weak solution of equation (1.2), let

$$F(x,u,Du) = j(u,\nabla u) - G(u)$$

and let a be independent on x, $h = x$. We get

$$\begin{aligned}-\int_{\partial\Omega} j(0,\nabla u)(x\cdot\nu)\,ds = \bigg(\frac{n-p}{p} + a\bigg)\int_\Omega j_\xi(u,\nabla u)\nabla u \\ + \int_\Omega j_s(u,\nabla u)u - \int_\Omega (nG(u) - aug(u)).\end{aligned}$$

We take $a = -N/\sigma$, note that Ω is a star shape region, by $(\mathrm{g}_2)'$, we get

$$\bigg(1 - \frac{\sigma}{p^*}\bigg)\int_\Omega j_\xi(u,\nabla u)\nabla u + \int_\Omega j_s(u,\nabla u)u \geq 0. \tag{4.1}$$

Therefore if u satisfies (j_6), then (4.1) implies that $u \equiv 0$. This completes the proof. □

Example 4.1. Let $\Omega \subset \mathbb{R}^N (N \geq 3)$ be a open bounded domain, and $a \geq 0$, $\lambda > 0$, $2 \leq p < N$, $p < q < p^* = Np/(N-p)$. Let

$$J(u) = \frac{1}{p}\int_\Omega \bigg(1 + \frac{1}{1+|u|^a}\bigg)|\nabla u|^p - \frac{\lambda}{q}\int_\Omega |u|^q.$$

Then by Theorem 1.2, there exist a sequence $\{u_n\} \subset W_0^{1,p}(\Omega)$ of nontrivial critical points of J such that $J(u_n) \to \infty$ as $n \to \infty$.

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Zhouxin Li
School of Mathematical Sciences
Peking University
Beijing, 100871, P.R. China

E-mail address: lzx@math.pku.edu.cn

Yaotian Shen and Yimin Zhang
Department of Mathematics
South China Univveristy of Tech.
Guangzhou, 510640, P.R. China