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EXISTENCE OF NON-COLLISION PERIODIC SOLUTIONS FOR SECOND ORDER SINGULAR DYNAMICAL SYSTEMS

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Abstract. In this paper, we study the existence of non-collision periodic solutions for the second order singular dynamical systems. We consider the systems where the potential have a repulsive or attractive type behavior near the singularity. The proof is based on Schauder's fixed point theorem involving a new type of cone. The so-called strong force condition is not needed and the nonlinearity could have sign changing behavior. We allow that the Green function is non-negative, so the critical case for the repulsive case is covered. Recent results in the literature are generalized and improved.

1. Introduction

In this paper, we are concerned with the existence of non-collision T-periodic solution of the second order non-autonomous singular dynamical system

$$\ddot{x} + a(t)x = f(t,x) + e(t), \quad \text{(a repulsive singularity)} \tag{1.1}$$

or

$$-\ddot{x} + a(t)x = f(t,x) + e(t), \quad \text{(an attractive singularity)} \tag{1.2}$$

where we assume $a(t) \in C(\mathbb{R}/T\mathbb{Z},\mathbb{R})$ is a continuous T-periodic function. The nonlinearity $f(t,x) \in C((\mathbb{R}/T\mathbb{Z}) \times \mathbb{R}^N \setminus \{0\},\mathbb{R}^N)$ and $e(t) \in C(\mathbb{R}/T\mathbb{Z},\mathbb{R}^N)$ are

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vector-valued functions. As usual, by a non-collision T-periodic solution, we mean a function $x(t) \in C^2(\mathbb{R}/T\mathbb{Z}, \mathbb{R}^N)$ such that $x(t) \neq 0$ for all t and satisfies (1.1) and the periodic boundary condition

$$x(0) = x(T), \quad x'(0) = x'(T). \tag{1.3}$$

We are mainly interested in the systems with a singularity at $x = 0$, which means, there exists a vector $v \in \mathbb{R}^N$, $\|v\|_2 = 1$ such that

$$\lim_{x \to 0,\, x \in \mathcal{C}} \langle v, f(t,x) \rangle = \infty. \tag{1.4}$$

Given $x = (x_1, \ldots, x_N)$, $y = (y_1, \ldots, y_N) \in \mathbb{R}^N$, we use the usual scalar product and Euclidean norm

$$\langle x, y \rangle = \sum_{i=1}^{N} x_i y_i, \quad \|x\|_2 = \sqrt{\langle x, x \rangle}.$$

Here, we denote by $\mathcal{C} = \mathcal{C}_{\rho,v} = \{x \in \mathbb{R}^N : \langle v, x \rangle \geq \rho \|x\|_{\mathcal{N}}\}$ cone in $\mathbb{R}^N$, where $\rho \in (0,1]$ is some fixed number and $\|\cdot\|_{\mathcal{N}}$ is a norm in $\mathbb{R}^N$. In fact, $\mathcal{C}$ is just the cone $\overline{\mathbb{R}^N_+}$ if we take $v = (1/\sqrt{N}, 1/\sqrt{N}, \ldots, 1/\sqrt{N})$, $\rho = 1/\sqrt{N}$, $\|x\|_{\mathcal{N}} = \|x\|_1 = \sum_{i=1}^N |x_i|$.

Starting with the pioneering paper of Lazer and Solimini [15], the question of existence of non-collision periodic solution for singular scalar equations and dynamical systems has attracted so much attention [1], [2], [9], [14], [15], [17]–[19]. In the literature, the proof mainly based on variational approaches and topological methods. In particular, the method of upper and lower methods [3], nonlinear alternative principle of Leray–Schauder type [6], some fixed point theorems in cones for completely continuous operators [20] and degree theory [21] are the most relevant tools.

In order to avoid collisions of the solution with the singularity, strong force condition is a common hypotheses. Consider the following system

$$\ddot{x} + a(t)x = \left(\sum_{i=1}^{n} x_i \right)^{-\alpha} + \left(\sum_{i=1}^{n} x_i \right)^{\beta} + e(t), \tag{1.5}$$

the strong force condition corresponds to the case that $\alpha \geq 1$, $\beta = 0$. Such a condition was first introduced with this name by W. B. Gordon in [13]. Since then, the strong force condition becomes standard in the related works [3], [7], [8], [10], [22]. In the recent year, there are also some works concerning the existence of periodic solutions under the presence of weak singularity, we refer [6], [11], [12].

Let us recall some recent works for systems (1.1), which motivate our study. In [12], D. Franco and J. R. L. Webb prove that (1.1) has at least one non-collision periodic solution assuming that $f(t,x) + e(t)$ satisfies suitable properties in one direction, which implies that $f(t,x) + e(t)$ neither needs to be positive nor to have constant sign behavior. In [6], J. Chu, P. J. Torres and M. Zhang get the existence result of positive solution of (1.1) when each component of $f(t,x)$ is superlinear at $x=+\infty$ and $e(t)\in\{e\in C(\mathbb{R}\setminus T\mathbb{Z}):\overline{e}\geq 0\}$. In [6], [12], we can find that the positivity of the Green function plays an important role, hence the critical case cannot be covered, such as $a(t)=k^2$. However, the case of scalar equation has been investigated in [5], [16].

In this paper, we will establish some existence results for the systems (1.1) and (1.2). The proof is based on Schauder fixed point theorem, combined with a new cone introduced by D. Franco and J. R. L. Webb in [12]. The strong force condition is not necessary and the components of the nonlinearity could have sign-changing behavior. Moreover, we only need that the Green function is non-negative, so the results are applicable to the critical case for the repulsive singularity. We also allow that some components of nonlinearity are nonsingular. Therefore some recent results have been generalized and improved.

The remaining part of the paper is organized as follows. In Section 2, some preliminary results will be given. In Section 3, the main results will be stated and proved. As an application, the study of (1.5) and other illustrating examples will be shown.

2. Preliminaries

Throughout this paper, we assume that Hill's equation

$$x''+a(t)x=0$$

associated with periodic boundary conditions (1.3) satisfies the following hypotheses:

(A) The associated Green function $G(t,s)$ is nonnegative for all $(t,s)\in[0,T]\times[0,T]$.

Under this assumption, we can define the function

$$x(t)=\int_0^T G(t,s)e(s)\,ds,\quad i=1,\dots,N,$$

which is the unique T-periodic solution of the linear equation

$$x''+a(t)x=e(t).$$

Now we give some remarks concerning condition (A). When $a(t)\equiv k^2$, condition (A) is equivalent to $0<k^2\leq\mu_1=(\pi/T)^2$. Note that μ_1 is the first eigenvalue of the linear problem with Dirichlet conditions $x(0)=x(T)=0$. When $a(t)$ is a non-constant function, there is a L^p-criterion in [19]. Let $K(q)$ denote the best Sobolev constant in the following inequality:

$$C\|u\|_q^2\leq\|u'\|_2^2,\quad \text{for all } u\in H_0^1(0,T).$$

The explicit formula for $K(q)$ is

$$K(q) = \begin{cases} \dfrac{2\pi}{q}\left(\dfrac{2}{2+q}\right)^{1-2/q}\left(\dfrac{\Gamma(1/q)}{\Gamma(1/2+1/q)}\right)^2 & \text{if } 1 \le q < \infty, \\ 4 & \text{if } q = \infty, \end{cases}$$

where Γ is the Gamma function.

From now on, we write $a \succ 0$ if $a \ge 0$ for almost every $t \in [0,T]$ and it is positive in a set of positive measure for a given function $a \in L^1[0,T]$. If the essential supremum and infimum of a exist, we denote them by a^* and a_* respectively. For an exponent $p \in [1,\infty]$, we denote the conjugate exponent of p by $\widetilde{p} = p/(p-1) \in [1,\infty]$.

LEMMA 2.1 ([19]). *Assume that $a(t) \succ 0$ and $a \in L^p[0,T]$ for some $p \in [1,\infty]$. If $\|a\|_p < K(2\widetilde{p})$, then the standing hypothesis* (A) *holds.*

3. Main results

In this section, we will establish the existence of non-collision T-periodic solution for systems (1.1). The following is the main result in this section.

THEOREM 3.1. *Assume that $a(t)$ satisfies* (A)*, and there exists a vector $v \in \mathbb{R}^N$, $|v|_2 = 1$ such that* (1.4) *holds. Furthermore, assume that*

- (H_1) *$f(t,x) + e(t) \in \mathcal{C}$ for each $t \in \mathbb{R}$ and $x \in \mathcal{C}$.*
- (H_2) *for each $L > 0$, there exists a continuous function $\phi_L \succ 0$ such that*

$$\langle v, f(t,x)\rangle > \phi_L(t) \quad \textit{for all } (t,x) \in [0,T] \times [0,L].$$

- (H_3) *there exist continuous, non-negative functions $g(t)$, $h(t)$ and $k(t)$, such that*

$$0 \le \langle v, f(t,x)\rangle \le k(t)\{g(\|x\|_{\mathcal{N}}) + h(\|x\|_{\mathcal{N}})\} \quad \textit{for all } (t,x) \in [0,T] \times \mathbb{R}^N \setminus \{0\},$$

and $g(t) > 0$ is non-increasing and $h(t)/g(t)$ is non-decreasing for $t \in (0,\infty)$.

- (H_4) *there exists a positive constant $R > 0$ such that*

$$R \ge \Phi_{R*} + \gamma_* > 0 \quad \textit{and} \quad g(\sigma_1(\Phi_{R*} + \gamma_*))\left(1 + \frac{h(R/\rho)}{g(R/\rho)}\right)K^* + \gamma^* \le R.$$

Here, the constant σ_1 is from (3.1) *and*

$$\begin{aligned} \Phi_R(t) &= \int_0^T G(t,s)\phi_R(s)\,ds, & K(t) &= \int_0^T G(t,s)k(s)\,ds, \\ \gamma(t) &= \left\langle v, \int_0^T G(t,s)e(s)\,ds\right\rangle, & \gamma_i(t) &= \int_0^T G(t,s)e_i(s)\,ds. \end{aligned}$$

Then the system (1.1) *has at least one non-collision T-periodic solution.*

PROOF. Finding a periodic solution of system (1.1) is equivalent to finding a fixed point of the completely continuous operator

$$\mathcal{A}\colon \underbrace{C_T \times \ldots \times C_T}_{N} \to \underbrace{C_T \times \ldots \times C_T}_{N}$$

defined by

$$(\mathcal{A}x)(t) = \int_0^T G(t,s)[f(s,x(s)) + e(s)]\, ds,$$

where C_T denote the set of continuous T-periodic functions.

Let R be the positive constant satisfying (H_4) and $r = \Phi_{R*} + \gamma_*$. Then $R > r > 0$. We define the set

$$\Omega = \{x \in \mathcal{C} : r \leq \langle v, x(t)\rangle \leq R \text{ for all } t \in [0,T]\}.$$

Since $\|\cdot\|_2$ and $\|\cdot\|_{\mathcal{N}}$ are two norms of $\mathbb{R}^N$, there exist two positive constants σ_1, σ_2 such that

$$\sigma_1\|\cdot\|_2 \leq \|\cdot\|_{\mathcal{N}} \leq \sigma_2\|\cdot\|_2. \tag{3.1}$$

For each $x \in \Omega$, we know that $\sigma_1 r \leq \|x\|_{\mathcal{N}} \leq R/\rho$ and $0 \notin \Omega$. Hence we have successfully avoided the singularity for $\mathcal{A}$. On the other hand, it is easy to verify that Ω is convex and closed. Therefore, Ω is a bounded, closed and convex set.

Next we shall prove $\mathcal{A}(\Omega) \subset \Omega$. Thus, as a consequence of the Schauder's fixed point theorem, it guarantees the existence of non-collision T-periodic solution.

Notice that $G(t,s) \geq 0$ for all $(t,s) \in [0,T] \times [0,T]$ and (H_1), it follows that

$$\begin{aligned}
a\langle v, \mathcal{A}x(t)\rangle &= \int_0^T G(t,s)\langle v, f(s,x(s)) + e(s)\rangle\, ds \\
&\geq \rho \int_0^T G(t,s)\|f(s,x(s)) + e(s)\|_{\mathcal{N}}\, ds \\
&\geq \rho\Big\|\int_0^T G(t,s)(f(s,x(s)) + e(s))ds\Big\|_{\mathcal{N}} \geq \rho\|\mathcal{A}x(t)\|_{\mathcal{N}}.
\end{aligned}$$

Hence $\mathcal{A}x(t) \in \mathcal{C}$ for all t and $\mathcal{A}$ is well defined.

By (H_2), we have

$$\begin{aligned}
\langle v, \mathcal{A}x(t)\rangle &= \int_0^T G(t,s)\langle v, f(s,x(s)) + e(s)\rangle\, ds \\
&= \int_0^T G(t,s)\langle v, f(s,x(s))\rangle\, ds + \left\langle v, \int_0^T G(t,s)e(s)\, ds\right\rangle \\
&\geq \int_0^T G(t,s)\phi_R(s)\, ds + \left\langle v, \int_0^T G(t,s)e(s)\, ds\right\rangle \geq \Phi_{R*} + \gamma_* = r.
\end{aligned}$$

On the other hand, it follows from (H_3) and (H_4) that

$$\begin{aligned}\langle v, \mathcal{A}x(t)\rangle &= \int_0^T G(t,s)\langle v, f(s,x(s)) + e(s)\rangle\, ds \\ &= \int_0^T G(t,s)\langle v, f(s,x(s))\rangle\, ds + \left\langle v, \int_0^T G(t,s)e(s)\, ds\right\rangle \\ &\le \int_0^T G(t,s)k(t)\{g(\|x\|_{\mathcal{N}}) + h(\|x\|_{\mathcal{N}})\}\, ds + \left\langle v, \int_0^T G(t,s)e(s)\, ds\right\rangle \\ &\le g(\sigma_1 r)\left(1 + \frac{h(R/\rho)}{g(R/\rho)}\right)K^* + \gamma^* \le R.\end{aligned}$$

Hence $\mathcal{A}(\Omega) \to \Omega$. By a direct application of Schauder fixed point theorem. $\square$

Remark 3.2. In [6], [12], [19], the positivity of the Green's function plays an important role in the application of fixed point theorems for completely continuous operators. The assumption (A) implies the Green's function is non-negative, therefore the result covers the critical case for systems with a repulsive singularity. We can compare the results we obtained with those in the above papers.

Remark 3.3. The components of nonlinearity could have sign changing behavior, assuming that the condition (H_2) is satisfied. Hence the recent results in [12] have been generalized.

Let us consider the case $\gamma_* = 0$, we have the following corollary.

Corollary 3.4. *Suppose $f(t,x)$ satisfies conditions* (H_1)–(H_3). *Furthermore, assume that*

(H_4') *there exists a positive constant $R > 0$ such that*

$$R \ge \Phi_{R*} > 0 \quad \text{and} \quad g(\sigma_1 \Phi_{R*})\left(1 + \frac{h(R/\rho)}{g(R/\rho)}\right)K^* + \gamma^* \le R.$$

If $\gamma_* = 0$, then the system (1.1) *has at least one non-collision T-periodic solution.*

As an application of Theorem 3.1 and Corollary 3.4, we get the following existence results for systems (1.5) assuming that there is no strong force condition.

Example 3.5. Assume that $a(t)$ satisfies (A). The components of nonlinearity are given by

$$f_i(t,x) + e_i(t) = b(t)\left(\sum_{i=1}^n x_i\right)^{-\lambda} + e_i(t) \ge 0, \quad i = 1, \dots, N,$$

where $b(t) > 0$ is a continuous function, $0 < \lambda < 1$.

If $\gamma_* = 0$, then (1.1) has at least one non-collision T-periodic solution.

PROOF. We will apply Corollary 3.4. To this end, we take

$$\phi_L(t) = \frac{b(t)}{L^\lambda}, \quad g(t) = t^{-\lambda}, \quad h(t) = 0, \quad k(t) = b(t),$$
$$v = (1/\sqrt{N}, \dots, 1/\sqrt{N}), \quad \rho = 1/\sqrt{N}, \quad \|x\|_{\mathcal{N}} = \|x\|_1.$$

Then (H_1)–(H_3) are satisfied. Let $\beta(t) = \int_0^T G(t,s)b(s)\,ds$, the existence condition (H_4') becomes

$$\left(\frac{\sigma_1 R^\lambda}{\beta_*}\right)^\lambda \beta^* + \gamma^* \le R, \quad R > \frac{\beta_*}{R^\lambda} \tag{3.2}$$

for some R with $R > 0$. Notice that $\beta_* > 0$ as a consequence of (A). Since $0 < \lambda < 1$, we can choose $R > 0$ large enough such that (3.2) is satisfied and the proof is completed. □

The following example generalize the previous one when $b(t) = 1$.

EXAMPLE 3.6. Assume that $a(t)$ satisfies the condition (A). The components of nonlinearity are given by

$$f_i(t,x) + e_i(t) = \left(\sum_{i=1}^n x_i\right)^{-\alpha} + \mu\left(\sum_{i=1}^n x_i\right)^\beta + e_i(t) \ge 0, \quad i = 1, \dots, N,$$

where $0 < \alpha < 1$, $\beta \ge 0$, and μ is a non-negative parameter and for $e(t)$ with $\gamma_* = 0$.

(i) If $\alpha + \beta < 1 - \alpha^2$, then (1.1) has at least one non-collision T-periodic solution for each $\mu > 0$.

(ii) If $\alpha + \beta \ge 1 - \alpha^2$, then (1.1) has at least one non-collision T-periodic solution for each $0 \le \mu \le \mu_1$, where μ_1 is some positive constant.

PROOF. We will also apply Corollary 3.4. To this end, we take

$$\phi_L(t) = L^{-\alpha}, \quad g(t) = t^{-\alpha}, \quad h(t) = t^\beta, \quad k(t) = 1,$$
$$v = (1/\sqrt{N}, \dots, 1/\sqrt{N}), \quad \rho = 1/\sqrt{N}, \quad \|x\|_{\mathcal{N}} = \|x\|_1.$$

We know the conditions (H_1)–(H_3) are satisfied. Let $\omega(t) = \int_0^T G(t,s)\,ds$, the existence condition (H_4) becomes

$$\mu \le \frac{(\sigma_1\omega_*)^\alpha R^{1-\alpha^2} - (\sigma_1\omega_*)^\alpha \gamma^* R^{-\alpha^2} - 1}{\omega^* R^{\alpha+\beta}} \rho^{\alpha+\beta}$$

for some R with $R^{1+\alpha} > \omega^*$. So the system (1.1) has at least one non-collision T-periodic solution for

$$0 < \mu < \mu_1 = \sup_{R^{1+\alpha} > \omega^*} \frac{(\sigma_1\omega_*)^\alpha R^{1-\alpha^2} - (\sigma_1\omega_*)^\alpha \gamma^* R^{-\alpha^2} - 1}{\omega^* R^{\alpha+\beta}} \rho^{\alpha+\beta}.$$

Note that $\mu_1 = \infty$ if $\alpha + \beta < 1 - \alpha^2$ and $\mu_1 < \infty$ if $\alpha + \beta \geq 1 - \alpha^2$. Thus we have the desired results. □

In the following, let us consider the case when $\gamma_* > 0$.

Theorem 3.7. *Suppose that* $a(t)$ *satisfies* (A) *and the nonlinearity* $f(t, x) + e(t)$ *satisfies conditions* (H_1)–(H_3). *Furthermore, assume that*

(H_4'') *there exists a positive constant* $R > 0$ *such that*

$$g(\sigma_1 \gamma_*)\bigg(1 + \frac{h(R/\rho)}{g(R/\rho)}\bigg)K^* + \gamma^* \leq R.$$

If $\gamma_* > 0$, *then the system* (1.1) *has at least one non-collision* T*-periodic solution.*

Proof. Let R be the positive constant satisfying (H_4'') and $r = \gamma_*$, then $R > r > 0$ since $R > \gamma_*$. Following the same strategy and notation in the proof of Theorem 3.1. We only need to prove $\mathcal{A}(\Omega) \subset \Omega$.

For each $x \in \Omega$ and for all $t \in [0, T]$, by the non-negative sign of $G(t, s)$ and $f(t, x)$, we have

$$\begin{aligned}
\langle v, \mathcal{A}x(t)\rangle &= \int_0^T G(t,s)\langle v, f(s, x(s)) + e(s)\rangle\, ds \\
&= \int_0^T G(t,s)\langle v, f(s, x(s))\rangle\, ds + \bigg\langle v, \int_0^T G(t,s) e(s)\, ds \bigg\rangle \\
&\geq \int_0^T G(t,s)\phi_R(s)\, ds + \bigg\langle v, \int_0^T G(t,s) e(s)\, ds \bigg\rangle \\
&\geq \Phi_{R*} + \gamma_* \geq \gamma_* = r.
\end{aligned}$$

On the other hand, by (H_3) and (H_4), we have

$$\begin{aligned}
\langle v, \mathcal{A}x(t)\rangle &= \int_0^T G(t,s)\langle v, f(s, x(s)) + e(s)\rangle\, ds \\
&= \int_0^T G(t,s)\langle v, f(s, x(s))\rangle\, ds + \bigg\langle v, \int_0^T G(t,s) e(s)\, ds \bigg\rangle \\
&\leq \int_0^T G(t,s) k(t)\{g(\|x\|_{\mathcal{N}}) + h(\|x\|_{\mathcal{N}})\}\, ds + \bigg\langle v, \int_0^T G(t,s) e(s)\, ds \bigg\rangle \\
&\leq g(\sigma_1\gamma_*)\bigg(1 + \frac{h(R/\rho)}{g(R/\rho)}\bigg)K^* + \gamma^* \leq R.
\end{aligned}$$

We know $\mathcal{A}(\Omega) \subset \Omega$. Therefore we have the desired results by applying Schauder's fixed point theorem. □

Example 3.8. Assume that $a(t)$ satisfies the condition (A). The components of nonlinearity are given by

$$f_i(t, x) + e_i(t) = \bigg(\sum_{i=1}^n x_i\bigg)^{-\alpha} + \mu\bigg(\sum_{i=1}^n x_i\bigg)^{\beta} + e_i(t) \geq 0, \quad i = 1, \ldots, N,$$

where $0 < \alpha < 1$, $\beta \geq 0$, and μ is a non-negative parameter and for $e(t)$ with $\gamma_* > 0$.

(i) If $\alpha + \beta < 1$, then (1.1) has at least one non-collision T-periodic solution for each $\mu > 0$.

(ii) If $\alpha + \beta \geq 1$, then (1.1) has at least one non-collision T-periodic solution for each $0 \leq \mu \leq \mu_1$, where μ_1 is some positive constant.

Proof. We will apply Theorem 3.7. So we take $\phi_L(t)$, $g(t)$, $h(t)$, $k(t)$ and $\omega(t)$ as in the proof of Example 3.6, then conditions (H_1)–(H_3) are satisfied. The existence condition (H_4'') becomes

$$\mu \leq \frac{(\sigma_1\gamma_*)^\alpha R - (\sigma_1\gamma_*)^\alpha \gamma^* - \omega^*}{\omega^* R^{\alpha+\beta}} \rho^{\alpha+\beta}$$

for some $R > 0$. Hence the system (1.1) has at least one positive periodic solution for

$$0 < \mu < \mu_2 = \sup_{R>0} \frac{(\sigma_1\gamma_*)^\alpha R - (\sigma_1\gamma_*)^\alpha \gamma^* - \omega^*}{\omega^* R^{\alpha+\beta}} \rho^{\alpha+\beta}.$$

Note that $\mu_2 = \infty$ if $\alpha + \beta < 1$ and $\mu_2 < \infty$ if $\alpha + \beta \geq 1$. We get the desired results. □

Remark 3.9. In Corollary 3.4 and Theorem 3.7, we only need that $\gamma_* > 0$ or $\gamma_* = 0$, which means that the components γ_i of γ could have sign changing behavior. It is interesting to comparing the results that we obtained with those in [5].

In the previous part, we have studied the existence of non-collision T-periodic solution for the system (1.1), on condition that the strong force condition is not satisfied. Moreover, the results are applicable to the case that the components of the nonlinearity have sign changing behavior. The following example show that we still get the existence of non-collision T-periodic for system (1.1), when some components of the nonlinearity are singular and others are nonsingular. Let us consider the following system

$$\begin{cases} \ddot{x}_1 + a(t)x_1 = (x_1 + x_2)^{-\alpha} + e_1(t), \\ \ddot{x}_2 + a(t)x_2 = \mu(x_1 + x_2)^{\beta} + e_2(t), \end{cases} \tag{3.3}$$

where $0 < \alpha < 1$, $\beta \geq 0$, and μ is a non-negative parameter.

Example 3.10. Assume that $a(t)$ satisfies the condition (A). For continuous functions $e_1(t)$, $e_2(t) \in C(\mathbb{R}/T\mathbb{Z}, \mathbb{R})$ with $\gamma_* = 0$, we have:

(i) If $\alpha + \beta < 1 - \alpha^2$, then (3.3) has at least one non-collision T-periodic solution for each $\mu > 0$.

(ii) If $\alpha + \beta \geq 1 - \alpha^2$, then (3.3) has at least one non-collision T-periodic solution for each $0 \leq \mu \leq \mu_1$, where μ_1 is some positive constant.

PROOF. We still apply Corollary 3.4. We still take $\phi_L(t)$, $g(t)$, $h(t)$, $k(t)$ and $\omega(t)$ as in the proof of Example 3.6, then conditions (H_1)–(H_3) are satisfied. The rest of proof is similar to Example 3.6. We omit the details. □

REMARK 3.11. We remark that the methods of Section 3 can be applied to the dynamical systems with an attractive singularity (1.2)

$$-\ddot{x} + a(t)x = f(t,x) + e(t),$$

when the Green function is non-negative. For example, when $a(t) \equiv k^2$, $k > 0$, the Green function of (1.2) is

$$G(t,s) = \begin{cases} \dfrac{e^{-k(s-t)} + e^{k(T+s-t)}}{2k(e^k - 1)}, & 0 \le s \le t \le T, \\[2mm] \dfrac{e^{-k(t-s)} + e^{k(T+t-s)}}{2k(e^k - 1)}, & 0 \le t \le s \le T. \end{cases}$$

It is easy to see that the Green function is positive and

$$\frac{e^{kT/2}}{ke^{kT} - 1} \le G(t,s) \le \frac{e^{kT}}{2ke^{kT} - 1}.$$

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