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SOLVABILITY OF FRACTIONAL DIFFERENTIAL EQUATIONS WITH INTEGRAL BOUNDARY CONDITIONS AT RESONANCE

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Abstract. By using the coincidence degree theory due to Mawhin and constructing suitable operators, some sufficient conditions for the existence of solution for a class of fractional differential equations with integral boundary conditions at resonance are established, which are complement of previously known results. The interesting point is that we shall deal with the case $\dim \operatorname{Ker} L = 2$, which will cause some difficulties in constructing the projector Q. An example is given to illustrate our result.

1. Introduction

Boundary value problems (BVPs, for short) with integral boundary conditions constitute a very interesting and important class of problems. They include two, three, multi-point and nonlocal BVPs as special cases. The existence and multiplicity of positive solutions for such problems have received a great deal of attention.

Fractional derivatives are generalizations for derivative of integer order. There are several kinds of fractional derivatives, such as, Riemann–Liouville fractional derivative, Marchaud fractional derivative, Caputo's fractional derivative, Grünwald–Letnikov fractional derivative, etc. In the last few decades, fractional order

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models are found to be more adequate than integer order models for some real world problems. Fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. This is the main advantage of fractional differential equations in comparison with classical integer order models. In consequence, the subject of fractional differential equations is gaining much importance and attention. For details, see [3], [8], [9], [14], [20], [21], [32], [35], [36] and the references therein. In [1], [2], [31], [37]–[39], the authors have discussed the existence of solutions for BVP of nonlinear fractional differential equations. There are a large number of papers dealing with the solvability of nonlinear fractional differential equations. However, the theory of BVPs for nonlinear fractional differential equations is still in the initial stages and many aspects of this theory need to be explored. To the best of our knowledge, there is few paper to investigate the resonance case with $\dim \operatorname{Ker} L = 2$ on the integral boundary conditions.

On the finite interval $[0,1]$, the first-order, second-order and high-order multi-point BVPs at resonance have been studied by many authors (see, for example [4]–[7], [10]–[13], [16], [22]–[28], [30], [34]), where $\dim \operatorname{Ker} L = 1$. In [18], [40] the second-order multi-point BVPs at resonance have been discussed when $\dim \operatorname{Ker} L = 2$ on the finite interval $[0,1]$.

Recently, Zhang et al. [40] discussed the existence and uniqueness results for the following BVP with integral boundary conditions at resonance under the case $\dim \operatorname{Ker} L = 2$:

$$\begin{cases} x''(t) = f(t, x(t), x'(t)), & t \in (0,1), \\ x'(0) = \displaystyle\int_0^1 h(t) x'(t)\, dt, & x'(1) = \displaystyle\int_0^1 g(t) x'(t)\, dt, \end{cases}$$

where $h, g \in C([0,1],[0,+\infty))$ with

$$\int_0^1 h(t)\, dt = 1, \qquad \int_0^1 g(t)\, dt = 1,$$

and $f\colon [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous.

In [15], Jiang investigated the existence of solutions for the following BVP of fractional differential equations at resonance with $\dim \operatorname{Ker} L = 2$:

$$\begin{cases} D_{0+}^{\alpha} u(t) = f(t, u(t), D_{0+}^{\alpha-1} u(t)), \quad \text{a.e. } t \in [0,1], \\ u(0) = 0, \quad D_{0+}^{\alpha-1} u(0) = \displaystyle\sum_{i=1}^{m} a_i D_{0+}^{\alpha-1} u(\xi_i), \quad D_{0+}^{\alpha-2} u(1) = \displaystyle\sum_{j=1}^{n} b_j D_{0+}^{\alpha-2} u(\eta_j), \end{cases}$$

where $2 < \alpha < 3$, D_{0+}^{α} is the Riemann–Liouville fractional derivative, $0 < \xi_1 < \ldots < \xi_m < 1$, $0 < \eta_1 < \ldots < \eta_n < 1$,

$$\sum_{i=1}^{m} a_i = 1, \qquad \sum_{j=1}^{n} b_j = 1, \qquad \sum_{j=1}^{n} b_j \eta_j = 1,$$

with and $f\colon [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ satisfies Carathéodory condition.

Motivated by the result of [15], [40], in this paper, we consider the solvability of the following fractional differential equations with integral boundary conditions at resonance:

$$\begin{cases} {}^cD^\alpha x(t) = f(t, x(t), x'(t)) + e(t), \quad 1 < \alpha < 2,\ t \in (0,1), \\ x'(0) = \displaystyle\int_0^1 h(t)x'(t)\,dt, \qquad x'(1) = \displaystyle\int_0^1 g(t)x'(t)\,dt, \end{cases} \tag{1.1}$$

where ${}^cD^\alpha$ is the Caputo fractional derivative, $f\colon [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function and $e(\,\cdot\,) \in L^p[0,1]$.

BVP (1.1) is called a problem at resonance if $Lx := {}^cD^\alpha x(t) = 0$ has nontrivial solutions under the boundary condition, i.e. $\dim \operatorname{Ker} L \geq 1$.

The goal of this paper is to study the existence of solution for BVP (1.1) at resonance with $\dim \operatorname{Ker} L = 2$. To the best of our knowledge, the method of Mawhin's continuation theorem has not been developed for fractional differential equation with integral boundary conditions at resonance with $\dim \operatorname{Ker} L = 2$. So it is interesting and important to discuss the existence of solution for BVP (1.1) when $\dim \operatorname{Ker} L = 2$. Many difficulties occur when we deal with them. For example, the construction of the projector Q. So we need to introduce some new tools and methods to investigate the existence of solution for BVP (1.1).

This paper is organized as follows. In Section 2, we discuss the essentials of the theory of fractional differentiation, integration and briefly overview recent works in the area that are closely related to this work. In Section 3, we provide some necessary background. In particular, we shall introduce some lemmas and definitions associated with BVP (1.1). In obtaining a priori estimates, we rely on Hölder's inequality with a specific restriction on the conjugate exponents $p, q > 1$ due to the nature of singular kernels arising in related Hammerstein equations. In Section 4, the main results of BVP (1.1) will be given and proved. In Section 5, an example is given to illustrate our result.

2. Preliminaries

Let us recall some definitions and fundamental facts of fractional calculus theory, which can be found in [33], [17].

Definition 2.1. For a function $y\colon (0,\infty) \to \mathbb{R}$, the Riemann–Liouville fractional integral of order $\alpha > 0$ is defined as

$$I^\alpha y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{y(s)}{(t-s)^{1-\alpha}}\,ds, \quad \alpha > 0$$

provided the integral exists.

Definition 2.2. The Caputo derivative of fractional order $\alpha > 0$ is defined as

$$^{c}D^{\alpha}y(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{y^{(n)}(s)}{(t-s)^{\alpha-n+1}}\, ds, \quad n = [\alpha]+1,$$

provided the right-hand side is pointwise defined on $(0,\infty)$, where $[\alpha]$ denotes the integer part of the real number α.

The following are results for a fractional differential equation (see [19]).

Lemma 2.3 (in [19]). *Let $u \in C^n[0,1]$ and $n-1 < \alpha < n$, $n \in \mathbb{N}$ and $v \in C^1[0,1]$. Then, for $t \in [0,1]$,*

(a) $^{c}D^{\alpha}I^{\alpha}v(t) = v(t)$;

(b) $I^{\alpha\, c}D^{\alpha}u(t) = u(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!}u^{(k)}(0)$.

3. Preliminary lemmas

In this section, we present the main results in this paper, whose proofs will be done by using the following fixed point theorem due to Mawhin (see [29]).

Definition 3.1. Let Y and Z be real Banach spaces, $L\colon \operatorname{dom} L \subset Y \to Z$ is a linear operator, L is said to be a Fredholm operator of index zero provided that

(a) $\operatorname{Im} L$ is a closed subset of Z,

(b) $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < +\infty$.

Let Y and Z be real Banach spaces, $L\colon \operatorname{dom} L \subset Y \to Z$ be a Fredholm operator of index zero and $P\colon Y \to Y$, $Q\colon Z \to Z$ be continuous projectors such that

$$\operatorname{Im} P = \operatorname{Ker} L, \quad \operatorname{Ker} Q = \operatorname{Im} L, \quad Y = \operatorname{Ker} L \oplus \operatorname{Ker} P, \quad Z = \operatorname{Im} L \oplus \operatorname{Im} Q.$$

It follows that $L|_{\operatorname{dom} L \cap \operatorname{Ker} P} : \operatorname{dom} L \cap \operatorname{Ker} P \to \operatorname{Im} L$ is invertible. We denote the inverse of that map by K_P.

Definition 3.2. Let $L\colon \operatorname{dom} L \subset Y \to Z$ be a Fredholm operator of index zero. If Ω is an open bounded subset of Y such that $\operatorname{dom} L \cap \overline{\Omega} \neq \emptyset$, the map $N\colon Y \to Z$ will be called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and inverse $K_P(I-Q)N\colon \overline{\Omega} \to Y$ is compact.

The theorem we use is the Theorem 2.4 of [8] or the Theorem IV.13 of [29].

Theorem 3.3. *Let $L\colon \operatorname{dom} L \subset Y \to Z$ be a Fredholm operator of index zero and let $N\colon Y \to Z$ be L-compact on $\overline{\Omega}$. Assume that the following conditions are satisfied*:

(a) *$Lx \neq \lambda Nx$ for every $(x,\lambda) \in [(\operatorname{dom} L \setminus \operatorname{Ker} L) \cap \partial\Omega] \times (0,1)$*;

(b) *$Nx \notin \operatorname{Im} L$ for every $x \in \operatorname{Ker} L \cap \partial\Omega$;*

(c) *$\deg(QN|_{\operatorname{Ker} L}, \Omega \cap \operatorname{Ker} L, 0) \neq 0$, where $Q\colon Z \to Z$ is a projection given as above with $\operatorname{Im} L = \operatorname{Ker} Q$.*

Then the equation $Lx = Nx$ has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$.

Throughout this paper, suppose now that the function $f\colon [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ satisfies the S-Carathéodory conditions with respect to $L^p[0,1]$, $p \geq 1$, that is, the following hold:

(a) $f(t,\,\cdot\,)$ is continuous on $\mathbb{R}^2$ for almost every $t \in [0,1]$,

(b) $f(\,\cdot\,, z)$ is Lebesgue measurable on $[0,1]$, for each $z \in \mathbb{R}^2$,

(c) for each $r > 0$, there exists a function $\varphi_r \in L^p[0,1]$, $\varphi_r(t) \geq 0$, $t \in [0,1]$ such that

$$|f(t,z)| \leq \varphi_r(t), \quad \text{for a.e. } t \in [0,1],\ ||z|| < r.$$

Furthermore, from now on, we always assume the following conditions hold:

(C_1) $p > 1/(\alpha - 1)$ and $q = p/(p-1)$;

(C_2) $h, g \in C([0,1], [0,+\infty))$ with

$$\int_0^1 h(t)\,dt = 1, \qquad \int_0^1 g(t)\,dt = 1;$$

(C_3)

$$\triangle = \begin{vmatrix} \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^1 t^{\alpha-1}h(t)\,dt & \dfrac{1}{\Gamma(\alpha)}\left(1 - \displaystyle\int_0^1 t^{\alpha-1}g(t)\,dt\right) \\ \dfrac{1}{\Gamma(\alpha+1)}\displaystyle\int_0^1 t^{\alpha}h(t)\,dt & \dfrac{1}{\Gamma(\alpha+1)}\left(1 - \displaystyle\int_0^1 t^{\alpha}g(t)\,dt\right) \end{vmatrix} := \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0.$$

Let $Z = L^p[0,1]$ with norm

$$\|y\|_p = \left(\int_0^1 |y(s)|^p\,ds\right)^{1/p}.$$

Let $Y = C^1[0,1]$ with norm $\|x\| = \max\{\|x\|_\infty, \|x'\|_\infty\}$, where $\|x\|_\infty = \max_{t\in[0,1]} |x(t)|$. Then Y is a Banach space.

Define L to be the linear operator $\operatorname{dom} L \subset Y \to Z$ with $Lx = {}^cD^\alpha x(t)$, $x \in \operatorname{dom} L$, where

$$\operatorname{dom} L = \bigg\{x \in C^1[0,1] : {}^cD^\alpha x \in L^p[0,1],\ x'(0) = \int_0^1 h(t)x'(t)\,dt,\ x'(1) = \int_0^1 g(t)x'(t)\,dt\bigg\}.$$

Let the nonlinear operator $N\colon Y \to Z$ be defined by

$$Nx = f(t, x(t), x'(t)) + e(t), \quad t \in [0,1].$$

Then the BVP (1.1) can be written as $Lx = Nx$, $x \in \operatorname{dom} L$.

For convenience, we denote

$$T_1 y = \int_0^1 h(t) \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)\, ds\, dt, \qquad T_2 y = T_{21} y - T_{22} y,$$

where

$$T_{21} y = \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)\, ds, \qquad T_{22} y = \int_0^1 g(t) \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)\, ds\, dt.$$

LEMMA 3.4. *If conditions* (C_1)–(C_3) *hold, then* $L\colon \operatorname{dom} L \subset Y \to Z$ *is a Fredholm operator of index zero. Furthermore, the linear continuous projector operator* $Q\colon Z \to Z$ *can be defined by*

$$Qy = Q_1 y + (Q_2 y) \cdot t,$$

where $Q_1 y = (\triangle_{11} T_1 y + \triangle_{12} T_2 y)/\triangle$, $Q_2 y = (\triangle_{21} T_1 y + \triangle_{22} T_2 y)/\triangle$, $\triangle_{ij}$ *is the algebraic cofactor of* $a_{ij} (i,j = 1,2)$, *and the linear operator* $K_P\colon \operatorname{Im} L \to \operatorname{dom} L \cap \operatorname{Ker} P$ *can be written by*

$$(K_P y)(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\, ds, \quad y \in \operatorname{Im} L.$$

Moreover,

$$\|K_P y\| \le A \|y\|_p, \quad y \in \operatorname{Im} L, \tag{3.1}$$

where

$$A = \max\{A_1, A_2\},$$
$$A_1 = \frac{1}{\Gamma(\alpha)((\alpha-1)q+1)^{1/q}}, \qquad A_2 = \frac{1}{\Gamma(\alpha-1)((\alpha-2)q+1)^{1/q}}. \tag{3.2}$$

PROOF. It is clear that $\operatorname{Ker} L = \{a + bt : a, b \in \mathbb{R},\ t \in [0,1]\}$. Moreover, we have

$$\operatorname{Im} L = \bigg\{ y \in Z : T_1 y = \int_0^1 h(t) \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)\, ds\, dt = 0,$$
$$T_2 y = \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)\, ds - \int_0^1 g(t) \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)\, ds\, dt = 0 \bigg\}. \tag{3.3}$$

In fact, If $y \in \operatorname{Im} L$, then there exists $x \in \operatorname{dom} L$ such that ${}^cD^\alpha x(t) = y(t)$. Integrating it from 0 to t, we know

$$x'(t) = \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}\, y(s)\, ds + b_0. \tag{3.4}$$

Substituting boundary condition $x'(0) = \int_0^1 h(t) x'(t)\, dt$ into the (3.4), we have

$$x'(t) = \int_0^1 h(t) x'(t)\, dt + \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}\, y(s)\, ds.$$

Multiplying it with $h(t)$ and integrating from 0 to 1, we get

$$\int_0^1 h(t)x'(t)\,dt = \int_0^1 h(t)x'(t)\,dt \int_0^1 h(t)\,dt + \int_0^1 h(t) \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}\, y(s)\,ds\,dt.$$

By the condition $\int_0^1 h(t)\,dt = 1$, we obtain

$$T_1 y = \int_0^1 h(t) \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}\, y(s)\,ds\,dt = 0.$$

Substituting boundary condition $x'(1) = \int_0^1 g(t)x'(t)\,dt$ into the (3.4), we have

$$x'(t) = \int_0^1 g(t)x'(t)\,dt - \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\, y(s)\,ds + \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}\, y(s)\,ds.$$

Multiplying it with $g(t)$ and integrating from 0 to 1, we get

$$\begin{aligned}\int_0^1 g(t)x'(t)\,dt &= \int_0^1 g(t)x'(t)\,dt \int_0^1 g(t)\,dt \\ &\quad - \int_0^1 g(t) \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\, y(s)\,ds\,dt + \int_0^1 g(t) \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}\, y(s)\,ds\,dt.\end{aligned}$$

By the condition $\int_0^1 g(t)\,dt = 1$, we obtain

$$T_2 y = \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\, y(s)\,ds - \int_0^1 g(t) \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}\, y(s)\,ds\,dt = 0.$$

On the other hand, $y \in Z$ satisfies

$$\begin{aligned}T_1 y &= \int_0^1 h(t) \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}\, y(s)\,ds\,dt = 0, \\ T_2 y &= \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\, y(s)\,ds - \int_0^1 g(t) \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}\, y(s)\,ds\,dt = 0.\end{aligned}$$

Let

$$x(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\, y(s)\,ds,$$

then ${}^cD^\alpha x(t) = y(t)$, and

$$x'(t) = \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}\, y(s)\,ds.$$

Thus $x'(0) = 0 = \int_0^1 h(t)x'(t)\,dt$ and

$$x'(1) = \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\, y(s)\,ds = \int_0^1 g(t) \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}\, y(s)\,ds\,dt = \int_0^1 g(t)x'(t)\,dt.$$

Then $x \in \operatorname{dom} L$ and $Lx = y$, i.e. $y \in \operatorname{Im} L$. Therefore, (3.3) holds.

From the definitions of operator $Q\colon Z \to Z$ by

$$Qy = Q_1 y + (Q_2 y) \cdot t,$$

it is obvious that $\dim \operatorname{Im} Q = 2$. Again from

$$\begin{aligned} Q_1(Q_1y) &= \frac{1}{\triangle}(\triangle_{11}T_1(Q_1y) + \triangle_{12}T_2(Q_1y)) \\ &= \frac{1}{\triangle}(\triangle_{11}a_{11} + \triangle_{12}a_{12})(Q_1y) = Q_1y, \\ Q_1((Q_2y)\cdot t) &= \frac{1}{\triangle}(\triangle_{11}T_1((Q_2y)\cdot t) + \triangle_{12}T_2((Q_2y)\cdot t)) \\ &= \frac{1}{\triangle}(\triangle_{11}a_{21} + \triangle_{12}a_{22})(Q_2y) = 0, \\ Q_2(Q_1y) &= \frac{1}{\triangle}(\triangle_{21}T_1(Q_1y) + \triangle_{22}T_2(Q_1y)) \\ &= \frac{1}{\triangle}(\triangle_{21}a_{11} + \triangle_{22}a_{12})(Q_1y) = 0, \\ Q_2((Q_2y)\cdot t) &= \frac{1}{\triangle}(\triangle_{21}T_1((Q_2y)\cdot t) + \triangle_{22}T_2((Q_2y)\cdot t)) \\ &= \frac{1}{\triangle}(\triangle_{21}a_{21} + \triangle_{22}a_{22})(Q_2y) = Q_2y, \end{aligned}$$

we have

$$\begin{aligned} Q^2y &= Q(Q_1y + (Q_2y)\cdot t) \\ &= Q_1((Q_1y) + ((Q_2y)\cdot t)) + Q_2((Q_1y) + ((Q_2y)\cdot t))\cdot t \\ &= Q_1(Q_1y) + Q_1((Q_2y)\cdot t) + Q_2(Q_1y)\cdot t + Q_2((Q_2y)\cdot t)\cdot t \\ &= Q_1y + (Q_2y)\cdot t = Qy, \end{aligned}$$

which implies the operator Q is a linear projector. Obviously, Q is continuous.

Now, we will show that $\operatorname{Ker} Q = \operatorname{Im} L$. If $y \in \operatorname{Ker} Q$, from $Qy = 0$, we get

$$\begin{cases} \triangle_{11}T_1y + \triangle_{12}T_2y = 0, \\ \triangle_{21}T_1y + \triangle_{22}T_2y = 0. \end{cases}$$

Since

$$\begin{vmatrix} \triangle_{11} & \triangle_{12} \\ \triangle_{21} & \triangle_{22} \end{vmatrix} = \triangle \neq 0,$$

so $T_1y = T_2y = 0$, which yields $y \in \operatorname{Im} L$. On the other hand, if $y \in \operatorname{Im} L$, then $T_1y = T_2y = 0$, from the definitions of operator Q, it is obvious that $Qy = 0$, thus $y \in \operatorname{Ker} Q$. Hence, $\operatorname{Ker} Q = \operatorname{Im} L$.

For $y \in Z$, $y = (y - Qy) + Qy$, we have $Qy \in \operatorname{Im} Q$ and $Q(y - Qy) = 0$. It follows from $Q(y - Qy) = 0$, the definitions of Q, Q_1, Q_2 and condition (C_3), that $T_1(y - Qy) = T_2(y - Qy) = 0$, i.e. $y - Qy \in \operatorname{Im} L$. So, $Z = \operatorname{Im} L + \operatorname{Im} Q$. Take $y \in \operatorname{Im} L \cap \operatorname{Im} Q$, then $y = Qy = 0$, i.e. $Z = \operatorname{Im} L \oplus \operatorname{Im} Q$, So, we have $\dim \operatorname{Ker} L = \dim \operatorname{Im} Q = \operatorname{codim} \operatorname{Im} L = 2$, thus L is a Fredholm operator with index zero.

Define the continuous projection $P\colon Y \to \operatorname{Ker} L$ by

$$(Px)(t) = x(0) + x'(0)t, \quad t \in [0,1].$$

Then $Y = \operatorname{Ker} L \oplus \operatorname{Ker} P$.

Define the operator $K_P\colon \operatorname{Im} L \to \operatorname{dom} L \cap \operatorname{Ker} P$ by

$$K_P y = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\, y(s)\, ds.$$

Then K_P is the inverse operator of $L|_{\operatorname{dom} L \cap \operatorname{Ker} P}$ and $\|K_P y\| \le A\|y\|_p$.

In fact, for $x \in \operatorname{dom} L \cap \operatorname{Ker} P$,

$$(K_P L x)(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\, {}^cD^\alpha x(t)\, ds = x(t).$$

On the other hand, for $y \in \operatorname{Im} L$,

$$(L K_P y)(t) = {}^cD^\alpha \bigg(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\, y(t)\, ds \bigg) = y(t).$$

Under the assumption (C_1), since $-1 < (\alpha-2)q < 0$, by Hölder's inequality, for all $t \in [0,1]$, we have

$$\begin{aligned} |(K_P y)'(t)| &\le \frac{1}{\Gamma(\alpha-1)} \int_0^t |(t-s)^{\alpha-2} y(s)|\, ds \\ &\le \frac{1}{\Gamma(\alpha-1)} \bigg(\int_0^t (t-s)^{(\alpha-2)q}\, ds \bigg)^{1/q} \|y\|_p \le \frac{1}{\Gamma(\alpha-1)((\alpha-2)q+1)^{1/q}}\, \|y\|_p. \end{aligned}$$

Hence $\|(K_P y)'\|_\infty \le A_2 \|y\|_p$.

Similarly, for all $t \in [0,1]$, we obtain

$$\begin{aligned} |K_P y(t)| &\le \frac{1}{\Gamma(\alpha)} \int_0^t |(t-s)^{\alpha-1} y(s)|\, ds \\ &\le \frac{1}{\Gamma(\alpha)} \bigg(\int_0^t (t-s)^{(\alpha-1)q}\, ds \bigg)^{1/q} \|y\|_p \le \frac{1}{\Gamma(\alpha)((\alpha-1)q+1)^{1/q}} \|y\|_p. \end{aligned}$$

Hence $\|K_P y\|_\infty \le A_1 \|y\|_p$. Thus, we get (3.1). □

LEMMA 3.5. *Suppose that Ω is an open bounded subset of Y such that $\operatorname{dom} L \cap \overline{\Omega} \neq \emptyset$. Then N is L-compact on $\overline{\Omega}$.*

PROOF. Since Ω is bounded, there exists a constant $r > 0$ such that $\|x\| \le r$ for any $x \in \overline{\Omega}$.

For $x \in \overline{\Omega}$, since f is a S-Carathéodory function, by the definitions of Q_1 and (C_3), we get

$$\begin{aligned}|Q_1Nx(s)| &= \left|\frac{1}{\triangle}(\triangle_{11}T_1Nx(s) + \triangle_{12}T_2Nx(s))\right| \\ &\le \frac{1}{\triangle}[a_{22}|T_1Nx(s)| + a_{21}|T_2Nx(s)|] \\ &\le \frac{1}{\triangle}[a_{22}T_1(\varphi_r(s) + |e(s)|) \\ &\quad + a_{21}(T_{21}(\varphi_r(s) + |e(s)|) + T_{22}(\varphi_r(s) + |e(s)|))] \le l_1.\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}|Q_2Nx(s)| &= \left|\frac{1}{\triangle}(\triangle_{21}T_1Nx(s) + \triangle_{22}T_2Nx(s))\right| \\ &\le \frac{1}{\triangle}[a_{12}|T_1Nx(s)| + a_{11}|T_2Nx(s)|] \\ &\le \frac{1}{\triangle}[a_{12}T_1(\varphi_r(s) + |e(s)|) \\ &\quad + a_{11}(T_{21}(\varphi_r(s) + |e(s)|) + T_{22}(\varphi_r(s) + |e(s)|))] \le l_2.\end{aligned}$$

Thus,

$$\begin{aligned}(3.5)\quad \|QNx\|_p &= \left(\int_0^1 |QNx(s)|^p\,ds\right)^{1/p} \\ &\le \left(\int_0^1 (|Q_1Nx(s)| + |Q_2Nx(s)|)^p\,ds\right)^{1/p} \le l_1 + l_2.\end{aligned}$$

So, $QN(\overline{\Omega})$ is bounded.

Now, we will prove that $K_P(I-Q)N(\overline{\Omega})$ is compact.

(a) Obviously, $K_P(I-Q)N\colon \overline{\Omega} \to Y$ is continuous. For $x \in \overline{\Omega}$, since

$$\begin{aligned}(3.6)\quad \|Nx\|_p &= \left(\int_0^1 |f(s, x(s), x'(s)) + e(s)|^p\,ds\right)^{1/p} \\ &\le \left(\int_0^1 (\varphi_r(s) + |e(s)|)^p\,ds\right)^{1/p} := l_3,\end{aligned}$$

we have

$$|K_P(I-Q)Nx(t)| = \left|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(I-Q)Nx(s)\,ds\right|$$

$$\le \frac{1}{\Gamma(\alpha)}\int_0^t |(t-s)^{\alpha-1}(I-Q)Nx(s)|\,ds \le \frac{1}{\Gamma(\alpha)((\alpha-1)q+1)^{1/q}}(l_1 + l_2 + l_3).$$

Similarly, we obtain

$$|[K_P(I-Q)Nx]'(t)| = \left|\frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}(I-Q)Nx(s)\,ds\right|$$

$$\leq \frac{1}{\Gamma(\alpha-1)} \int_0^t |(t-s)^{\alpha-2}(I-Q)Nx(s)|\, ds$$
$$\leq \frac{1}{\Gamma(\alpha-1)((\alpha-2)q+1)^{1/q}} (l_1+l_2+l_3).$$

Since l_1, l_2 and l_3 are constants, by (3.5) and (3.6), we get that $K_P(I-Q)N(\overline{\Omega})$ is bounded.

(b) We will prove that functions belonging to $K_P(I-Q)N(\overline{\Omega})$ are equicontinuous on $[0,1]$.

For all $\varepsilon > 0$, let $\delta = \min\{\delta_1, \delta_2\}$, where

$$\delta_1 = ((\alpha-1)q+1)\biggl(\frac{\Gamma(\alpha)\varepsilon}{(((\alpha-1)q+1)^{1/q}+1)(l_1+l_2+l_3)}\biggr)^q,$$
$$\delta_2 = \biggl(\frac{\Gamma(\alpha-1)((\alpha-2)q+1)^{1/q}\varepsilon}{2(l_1+l_2+l_3)}\biggr)^{q/((\alpha-2)q+1)}.$$

First, we prove that for any $t_1, t_2 \in [0,1]$ such that $0 < t_2 - t_1 < \delta_1$, we have

$$|K_P(I-Q)Nx(t_2) - K_P(I-Q)Nx(t_1)| < \varepsilon. \tag{3.7}$$

In fact,

$$\begin{aligned}
&|K_P(I-Q)Nx(t_2) - K_P(I-Q)Nx(t_1)| \\
&\quad = \biggl|\frac{1}{\Gamma(\alpha)} \int_0^{t_2} (t_2-s)^{\alpha-1}(I-Q)Nx(s)\, ds \\
&\qquad - \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1-s)^{\alpha-1}(I-Q)Nx(s)\, ds\biggr| \\
&\quad \leq \frac{1}{\Gamma(\alpha)} \int_0^{t_1} |[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}](I-Q)Nx(s)|\, ds \\
&\qquad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} |(t_2-s)^{\alpha-1}(I-Q)Nx(s)|\, ds \\
&\quad \leq \frac{(\|Nx\|_p + \|QNx\|_p)}{\Gamma(\alpha)} \biggl(\biggl(\int_0^{t_1} |(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}|^q\, ds\biggr)^{1/q} \\
&\qquad + \biggl(\int_{t_1}^{t_2} |t_2-s|^{(\alpha-1)q}\, ds\biggr)^{1/q}\biggr) \\
&\quad \leq \frac{(l_1+l_2+l_3)}{\Gamma(\alpha)} \biggl(\biggl(\int_0^{t_1} [(t_2-s)^{(\alpha-1)q} - (t_1-s)^{(\alpha-1)q}]\, ds\biggr)^{1/q} \\
&\qquad + \biggl(\int_{t_1}^{t_2} |t_2-s|^{(\alpha-1)q}\, ds\biggr)^{1/q}\biggr) \\
&\quad = \frac{(l_1+l_2+l_3)}{\Gamma(\alpha)((\alpha-1)q+1)^{1/q}} ((t_2^{(\alpha-1)q+1} - (t_2-t_1)^{(\alpha-1)q+1} - t_1^{(\alpha-1)q+1})^{1/q} \\
&\qquad + ((t_2-t_1)^{(\alpha-1)q+1})^{1/q})
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{(l_1+l_2+l_3)}{\Gamma(\alpha)((\alpha-1)q+1)^{1/q}}((t_2^{(\alpha-1)q+1}-t_1^{(\alpha-1)q+1})^{1/q}+(t_2-t_1)^{(\alpha-1)+1/q})\\
&\leq \frac{(l_1+l_2+l_3)}{\Gamma(\alpha)((\alpha-1)q+1)^{1/q}}([(\alpha-1)q+1]^{1/q}(t_2-t_1)^{1/q}+(t_2-t_1)^{1/q})\\
&\leq \frac{(l_1+l_2+l_3)}{\Gamma(\alpha)((\alpha-1)q+1)^{1/q}}([(\alpha-1)q+1]^{1/q}+1)(t_2-t_1)^{1/q}<\varepsilon.
\end{aligned}$$

Next, we will prove that if $t_1, t_2 \in [0,1]$ are such that $0 < t_2 - t_1 < \delta_2$, then

$$|[K_P(I-Q)Nx]'(t_2)-[K_P(I-Q)Nx]'(t_1)|<\varepsilon. \tag{3.8}$$

In fact,

$$\begin{aligned}
&|[K_P(I-Q)Nx]'(t_2)-[K_P(I-Q)Nx]'(t_1)|\\
&\quad=\bigg|\frac{1}{\Gamma(\alpha-1)}\int_0^{t_2}(t_2-s)^{\alpha-2}(I-Q)Nx(s)\,ds\\
&\qquad-\frac{1}{\Gamma(\alpha-1)}\int_0^{t_1}(t_1-s)^{\alpha-2}(I-Q)Nx(s)\,ds\bigg|\\
&\quad\leq\frac{1}{\Gamma(\alpha-1)}\int_0^{t_1}\big|[(t_2-s)^{\alpha-2}-(t_1-s)^{\alpha-2}](I-Q)Nx(s)\big|\,ds\\
&\qquad+\frac{1}{\Gamma(\alpha-1)}\int_{t_1}^{t_2}|(t_2-s)^{\alpha-2}(I-Q)Nx(s)|\,ds\\
&\quad\leq\frac{(\|Nx\|_p+\|QNx\|_p)}{\Gamma(\alpha-1)}\bigg(\bigg(\int_0^{t_1}|(t_1-s)^{\alpha-2}-(t_2-s)^{\alpha-2}|^q\,ds\bigg)^{1/q}\\
&\qquad+\bigg(\int_{t_1}^{t_2}|t_2-s|^{(\alpha-2)q}\,ds\bigg)^{1/q}\bigg)\\
&\quad\leq\frac{(l_1+l_2+l_3)}{\Gamma(\alpha-1)}\bigg(\bigg(\int_0^{t_1}[(t_1-s)^{(\alpha-2)q}-(t_2-s)^{(\alpha-2)q}]\,ds\bigg)^{1/q}\\
&\qquad+\bigg(\int_{t_1}^{t_2}|t_2-s|^{(\alpha-2)q}\,ds\bigg)^{1/q}\bigg)\\
&\quad=\frac{(l_1+l_2+l_3)}{\Gamma(\alpha-1)((\alpha-2)q+1)^{1/q}}((t_1^{(\alpha-2)q+1}+(t_2-t_1)^{(\alpha-2)q+1}\\
&\qquad-t_2^{(\alpha-2)q+1})^{1/q}+((t_2-t_1)^{(\alpha-2)q+1})^{1/q})\\
&\quad\leq\frac{(l_1+l_2+l_3)}{\Gamma(\alpha-1)((\alpha-2)q+1)^{1/q}}\times 2\delta_2^{(\alpha-2)+1/q}<\varepsilon.
\end{aligned}$$

By (3.7) and (3.8), we get that functions from $K_P(I-Q)N(\overline{\Omega})$ are equi-continuous on $[0,1]$. The Arzela–Ascoli Theorem implies that N is L-compact on $\overline{\Omega}$. $\square$

4. Main results

Theorem 4.1. *Let $f\colon [0,1]\times\mathbb{R}^2\to\mathbb{R}$ be continuous, and assume that*

(H_1) *There exist functions $\gamma_1,\gamma_2,\gamma_3,r\in L^p[0,1]$, and constant $\theta\in(0,1)$ such that for all $(u,v)\in\mathbb{R}^2$, $t\in[0,1]$ either*

$$|f(t,u,v)|\le\gamma_1(t)|u|+\gamma_2(t)|v|+\gamma_3(t)|v|^\theta+r(t)\tag{4.1}$$

or else

$$|f(t,u,v)|\le\gamma_1(t)|u|+\gamma_2(t)|v|+\gamma_3(t)|u|^\theta+r(t).\tag{4.2}$$

(H_2) *There exist constants $c_1,c_2>0$ such that*

$$Q_1Nx\neq0\quad\text{and}\quad Q_2Nx\neq0$$

hold for $x\in\operatorname{dom}L\setminus\operatorname{Ker}L$ with $|x(t)|\ge c_1$, $|x'(t)|\ge c_2$, for all $t\in[0,1]$.

(H_3) *There exist constants $M_1>0,M_2>0$ such that either*

$$aQ_1N(a+bt)>0,\qquad bQ_2N(a+bt)>0$$

or

$$aQ_1N(a+bt)<0,\qquad bQ_2N(a+bt)<0$$

hold for $a,b\in\mathbb{R}$ with $|a|>M_1$, $|b|>M_2$.

Then BVP (1.1) *with $\int_0^1 h(t)\,dt=1$, $\int_0^1 g(t)\,dt=1$ has at least one solution in $C^1[0,1]$ provided that $(A+2A_2)(\|\gamma_1\|_p+\|\gamma_2\|_p)<1$, where A, A_2 is defined by* (3.2).

Proof. We divide the proof into the following three steps.

Step 1. Let $\Omega_1=\{x\in\operatorname{dom}L\setminus\operatorname{Ker}L : Lx=\lambda Nx,\ \text{for some }\lambda\in[0,1]\}$. Then Ω_1 is bounded.

In fact, if $x\in\Omega_1$, $Lx=\lambda Nx$, thus $\lambda\neq0$, $Nx\in\operatorname{Im}L=\operatorname{Ker}Q$, i.e. $QNx=0$, by the definition of Q, we have $Q_1Nx=Q_2Nx=0$. Thus, from (H_2), there exist $t_0,t_1\in[0,1]$ such that $|x(t_0)|\le c_1$, $|x'(t_1)|\le c_2$. Since

$$|x(0)|=\left|x(t_0)-\int_0^{t_0}x'(t)\,dt\right|\le c_1+\|x'\|_\infty,\tag{4.3}$$

and x' is absolutely continuous for all $t\in[0,1]$,

$$x'(t)=x'(t_1)+\int_{t_1}^t\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}{}^cD^\alpha x(s)\,ds,$$

which implies

$$\begin{aligned}\|x'\|_\infty&\le|x'(t_1)|+\left|\int_{t_1}^t\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}{}^cD^\alpha x(s)\,ds\right|\\&\le c_2+\frac{1}{\Gamma(\alpha-1)((\alpha-2)q+1)^{1/q}}\|{}^cD^\alpha x(t)\|_p\\&=c_2+A_2\|Lx\|_p\le c_2+A_2\|Nx\|_p.\end{aligned}\tag{4.4}$$

Thus, from this, by (4.3) and (4.4), we obtain

$$(4.5)\quad \|Px\| = \max\Big\{\max_{t\in[0,1]}|x(0)+x'(0)t|, |x'(0)|\Big\} \le |x(0)| + |x'(0)| \le c_1 + 2c_2 + 2A_2\|Nx\|_p.$$

Again from all $x \in \Omega_1$, $(I-P)x \in \operatorname{dom} L \cap \operatorname{Ker} P$, $LPx = 0$, thus from Lemma 3.4, we get

$$(4.6)\quad \|(I-P)x\| = \|K_P L(I-P)x\| \le A\|L(I-P)x\|_p = A\|Lx\|_p \le A\|Nx\|_p.$$

Hence, from (4.5) and (4.6), we have

$$(4.7)\qquad \|x\| \le \|Px\| + \|(I-P)x\| \le c_1 + 2c_2 + (A+2A_2)\|Nx\|_p.$$

If (H_1) holds, then from (4.1) and (4.7), we get

$$(4.8)\quad \|x\| \le (A+2A_2)(\|\gamma_1\|_p\|x\|_\infty + \|\gamma_2\|_p\|x'\|_\infty + \|y_3\|_p\|x'\|_\infty^\theta + \|r\|_p + \|e\|_p) + c_1 + 2c_2.$$

Thus, from $\|x\|_\infty \le \|x\|$ and (4.8), we obtain

$$(4.9)\quad \|x\|_\infty \le \frac{A+2A_2}{1-(A+2A_2)\|\gamma_1\|_p} \cdot \bigg(\|\gamma_2\|_p\|x'\|_\infty + \|\gamma_3\|_p\|x'\|_\infty^\theta + \|r\|_p + \|e\|_p + \frac{c_1+2c_2}{A+2A_2}\bigg).$$

Again from (4.8), (4.9), one has

$$(4.10)\quad \|x'\|_\infty \le \frac{(A+2A_2)\|\gamma_3\|_p}{1-(A+2A_2)(\|\gamma_1\|_p+\|\gamma_2\|_p)}\|x'\|_\infty^\theta + \frac{A+2A_2}{1-(A+2A_2)(\|\gamma_1\|_p+\|\gamma_2\|_p)}\bigg(\|r\|_p + \|e\|_p + \frac{c_1+2c_2}{A+2A_2}\bigg).$$

Since $\theta \in (0,1)$, from the above last inequality, there exists a constant $K_1 > 0$ such that

$$(4.11)\qquad \|x'\|_\infty \le K_1,$$

thus from (4.9) and (4.11), there exists a constant $K_2 > 0$ such that $\|x\|_\infty \le K_2$, hence $\|x\| = \max\{\|x\|_\infty, \|x'\|_\infty\} \le \max\{K_1, K_2\}$. Therefore Ω_1 is bounded.

If (4.2) holds, similar to the above argument, we can prove that Ω_1 is bounded too. The proof of Step 1 is finished.

Step 2. Let $\Omega_2 = \{x \in \operatorname{Ker} L : Nx \in \operatorname{Im} L\}$. Now we show that Ω_2 is bounded.

In fact, $x \in \Omega_2$ implies $x = a + bt$, $a, b \in \mathbb{R}$ and $QNx = 0$. Thus, $Q_1N(a + bt) = Q_2N(a+bt) = 0$. By ($H_2$), there exist $t_0, t_1 \in [0,1]$ such that $|x(t_0)| \le c_1$, $|x'(t_1)| \le c_2$, then

$$\|x'\|_\infty = |b| \le c_2.$$

Moreover,

$$\|x\|_\infty \le \|x'\|_\infty + c_1 \le c_1 + c_2.$$

So, $\|x\| \le c_1 + c_2$ is bounded. The proof of Step 2 is complete.

Step 3. Considering the first part of the condition (H_3), let

$$\Omega_3 = \{x \in \operatorname{Ker} L : H(x,\lambda) = \lambda x + (1-\lambda)QNx = 0, \text{ for some } \lambda \in [0,1]\},$$

where $J\colon \operatorname{Ker} L \to \operatorname{Im} Q$ is the linear isomorphism given by

$$J(a+bt) = a+bt, \quad a,b \in \mathbb{R},\ t \in [0,1].$$

Then Ω_3 is bounded.

In fact, $x = a+bt \in \Omega_3$ then $\lambda(a+bt) + (1-\lambda)QN(a+bt) = 0$. By the definition of Q we have

$$\lambda a + (1-\lambda)Q_1 N(a+bt) = 0, \qquad \lambda b + (1-\lambda)Q_2 N(a+bt) = 0.$$

If $\lambda = 1$, then $a = b = 0$. In this case, it is clear that Ω_3 is bounded.

If $\lambda \ne 1$, and $|a| > M_1$ or $|b| > M_2$, from the first part of (H_3), we know

$$\lambda a^2 = -(1-\lambda)aQ_1N(a+bt) < 0, \qquad \lambda b^2 = -(1-\lambda)bQ_2N(a+bt) < 0,$$

which contradicts with $\lambda a^2 > 0$, $\lambda b^2 > 0$. It follows that $|a| \le M_1$, $|b| \le M_2$. Then $\|x\| \le |a| + |b| \le M_1 + M_2$. The proof of Step 3 is complete.

On the other hand, if the second part of the condition (H_3) holds, then let

$$\Omega_3 = \{x \in \operatorname{Ker} L : H(x,\lambda) = -\lambda x + (1-\lambda)QNx = 0, \text{ for some } \lambda \in [0,1]\}.$$

By the similar method, we can prove Ω_3 is bounded.

Step 4. Now we shall prove that all the conditions of Theorem 3.3 are satisfied. By Lemma 3.5, we have $K_P(I-Q)N\colon \overline{\Omega} \to Y$ is compact, then N is L-compact. Let $\Omega \supset \bigcup_{i=1}^{3} \Omega_i$ be open bounded set. By Steps 1–3, we obtain:

(1) $Lx \ne \lambda Nx$, for all $(x,\lambda) \in [(\operatorname{dom} L \setminus \operatorname{Ker} L) \cap \partial\Omega] \times (0,1)$;
(2) $Nx \notin \operatorname{Im} L$, for all $x \in \operatorname{Ker} L \cap \partial\Omega$;
(3) let $H(x,\lambda) = \pm\lambda x + (1-\lambda)QNx = 0$, $\lambda \in [0,1]$.

According to the above argument, we know $H(x,\lambda) \ne 0$ for $x \in \operatorname{Ker} L \cap \partial\Omega$. Thus, by the homotopy property of degree, we get

$$\begin{aligned}\deg(JQN|_{\operatorname{Ker} L}, \Omega \cap \operatorname{Ker} L, 0) &= \deg(H(\,\cdot\,,0), \Omega \cap \operatorname{Ker} L, 0)\\ &= \deg(H(\,\cdot\,,1), \Omega \cap \operatorname{Ker} L, 0)\\ &= \deg(\pm J, \Omega \cap \operatorname{Ker} L, 0) = \pm 1 \ne 0.\end{aligned}$$

By Theorem 3.3, we have that $Lx = Nx$ has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$, i.e. (1.1) has at least one solution in Y. □

5. Example

Example 5.1. Consider the following BVP

$$\begin{cases} {}^cD^{3/2}x(t) = m(t)\bigg[t^2 + \dfrac{\sin x(t)}{12} + \dfrac{(1+t)x'(t)}{14} \\ \qquad\qquad + 3\sin(x'(t))^{1/3} + 5 + \cos^2 t\bigg], \qquad 0 < t < 1, \\ x'(0) = \displaystyle\int_0^1 x'(t)\,dt, \quad x'(1) = \int_0^1 x'(t)\,dt, \end{cases} \tag{5.1}$$

where

$$m(t) = \begin{cases} -1, & t \in [0, 1/3], \\ 3t - 2, & t \in [1/3, 2/3], \\ 0, & t \in [2/3, 4/5], \\ 5t - 4, & t \in [4/5, 1]. \end{cases} \tag{5.2}$$

Let $\alpha = 3/2$, $p = 3$, $q = 3/2$, $h(t) = 1$, $g(t) = 1$, and

$$\begin{aligned} f(t, x(t), x'(t)) &= m(t)\omega(t, x(t), x'(t)) \\ &= m(t)\bigg[t^2 + 4 + \frac{\sin x(t)}{12} + \frac{(1+t)x'(t)}{14} + 3\sin(x'(t))^{\frac{1}{3}}\bigg], \\ e(t) &= m(t)\tau(t) = m(t)[1 + \cos^2 t]. \end{aligned}$$

It is not difficult to see that

$$\int_0^1 h(t)\,dt = 1, \qquad \int_0^1 g(t)\,dt = 1, \qquad A = \frac{2\sqrt{2}}{\sqrt{\pi}}$$

and

$$\triangle = \begin{vmatrix} \dfrac{1}{\Gamma(\frac{3}{2})}\displaystyle\int_0^1 t^{3/2-1}\,dt & \dfrac{1}{\Gamma(\frac{3}{2})}\bigg(1 - \displaystyle\int_0^1 t^{3/2-1}\,dt\bigg) \\ \dfrac{1}{\Gamma(\frac{3}{2}+1)}\displaystyle\int_0^1 t^{3/2}\,dt & \dfrac{1}{\Gamma(\frac{3}{2}+1)}\bigg(1 - \displaystyle\int_0^1 t^{3/2}\,dt\bigg) \end{vmatrix} = \begin{vmatrix} \dfrac{4}{3\sqrt{\pi}} & \dfrac{2}{3\sqrt{\pi}} \\ \dfrac{8}{15\sqrt{\pi}} & \dfrac{4}{5\sqrt{\pi}} \end{vmatrix} \neq 0.$$

Now we prove (H_1)–(H_3) are satisfied.

Let $\gamma_1(t) = 1/12$, $\gamma_2(t) = 1/7$, $\gamma_3(t) = 3$, $r(t) = 5$, $\theta = 1/3$. Then we have

$$(A + 2A_2)(\|\gamma_1\|_p + \|\gamma_2\|_p) = \frac{19}{14}\sqrt{\frac{2}{\pi}} < 1$$

and

$$\begin{aligned} |f(t,u,v)| = |m(t)| \cdot |\omega(t)| &= |m(t)| \cdot \bigg|t^2 + 4 + \frac{\sin u}{12} + \frac{(1+t)v}{14} + 3\sin(v^{1/3})\bigg| \\ &\le \gamma_1(t)|u| + \gamma_2(t)|v| + \gamma_3(t)|v|^\theta + r(t). \end{aligned}$$

Thus (H_1) is satisfied.

Taking $c_1 = 24$, $c_2 = 140$. So, as $|x(t)| \geq c_1$, $|x'(t)| \geq c_2$, we have $\omega(t, x(t), x'(t)) + \tau(t) > 0$ or $\omega(t, x(t), x'(t)) + \tau(t) < 0$. Therefore,

$$\begin{aligned} Q_1 Nx &= \frac{1}{\triangle}(\triangle_{11} T_1 Nx + \triangle_{12} T_2 Nx) \\ &= \frac{1}{\triangle} \frac{8}{15\sqrt{\pi}} \int_0^1 \frac{4-5s}{(1-s)^{1/2}} [f(s, x(s), x'(s)) + e(s)]\, ds \neq 0, \end{aligned} \tag{5.3}$$

and

$$\begin{aligned} Q_2 Nx &= \frac{1}{\triangle}(\triangle_{21} T_1 Nx + \triangle_{22} T_2 Nx) \\ &= \frac{1}{\triangle} \frac{4}{3\sqrt{\pi}} \int_0^1 \frac{3s-2}{(1-s)^{1/2}} [f(s, x(s), x'(s)) + e(s)]\, ds \neq 0. \end{aligned} \tag{5.4}$$

From (5.3) and (5.4), we obtain that the condition (H_2) holds.

Let $M_1 = 108$, $M_2 = 140$. Then, as $|a| > M_1$, $|b| > M_2$, we have that (H_3) holds. Thus, Theorem 4.1 implies that BVP (5.1) has at least one solution in $C^1[0, 1]$.

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