NOTE ON PERIODIC SOLUTIONS
OF RELATIVISTIC PENDULUM TYPE SYSTEMS

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ABSTRACT. We establish multiplicity results of periodic solutions for relativistic pendulum type systems of ordinary differential equations. We provide a different approach to the problems and answer some questions raised in [6], [7] by Brezis and Mawhin recently.

1. Background and main result

In a series of interesting papers ([1]–[3], [6], [7], [24], [25]) the problem on the existence and multiplicity of periodic solutions for relativistic pendulum equations and related systems of similar type have been studied in recent years. This has been done mainly by fixed point theorem method. In [6], [7] variational arguments have been explored by using minimization methods in convex sets of Banach spaces. The current paper is inspired by these work in particular by [6] and [7] of Brezis and Mawhin. In [6], [7] some open questions were raised concerning multiplicity of periodic solutions for these type of systems of equations. In this paper we propose a different approach mainly in variational nature and our new approach enable us to give positive answers to some questions raised in [6], [7] and also to generalize the existing results to a broader class of systems.

To motivate the discussion let us consider the following equation

$$(1.1) \qquad (\phi(u'))' + A\sin u = h(x)$$

where $\phi\colon(-a,a)\to\mathbb{R}$ is an increasing homeomorphism satisfying $\phi(0)=0$, A is a constant, $h\in L_T^1(\mathbb{R})$ (i.e. T-periodic) and $\int_0^{2\pi} h(x)\,dx = 0$. This contains the so-called relativistic pendulum equation coming from models for dynamics of special relativity when $\phi(t)=t/\sqrt{1-t^2}$ and $a=1$:

$$(1.2) \qquad \left(\frac{u'}{\sqrt{1-(u')^2}}\right)' + A\sin u = h(x).$$

This class of equations has received much attention in recent years starting from papers of Torres [24], [25]. This belongs to a family of equations of relativistic type equations of the form

$$(1.3) \qquad (\phi(u'))' - g(x,u) = h(x),$$

when $\phi\colon(-a,a)\to\mathbb{R}$ is an increasing homeomorphism, g is T-periodic in x and 2π-periodic in u, and h is T-periodic and has mean value zero, module some smoothness conditions. For the following relativistic equation with continuous periodic forcing h and arbitrary dissipation f

$$(\phi(u'))' + f(u)u' + A\sin u = h(x).$$

it was proved in [24], [25] that the existence of at least two T-periodic solutions when

$$(1.4) \qquad aT < 2\sqrt{3} \quad\text{and}\quad |\overline{h}| < A\left(1 - \frac{aT}{2\sqrt{3}}\right),$$

and of at least one T-periodic solution when

$$(1.5) \qquad aT = 2\sqrt{3} \quad\text{and}\quad \overline{h} = 0.$$

A Schauder fixed-point theorem approach was used in [24], [25]. The assumptions have been improved in [3] by using a Leray–Schauder degree argument. Another multiplicity result was given in [3] using a upper and lower solution method.

In a different direction, in recent papers of [6], [7] Brezis and Mawhin employed variational arguments to study the existence of periodic solutions of (1.3) and they established as corollary the existence of a T-periodic solution for (1.2) under the conditions that $A\in\mathbb{R}$ and $\overline{h}=0$. Thus the conditions (1.4), (1.5) are removed, and this was done by a minimization arguments in closed convex subsets of a Banach space.

More precisely, the following conditions are assumed in [6]:

(H$_\Phi$1) Φ is continuous on $[-a,a]$, of class C^1 on $(-a,a)$, strictly convex, and $\phi := \Phi'\colon(-a,a)\to\mathbb{R}$ is a homeomorphism such that $\phi(0)=0$.

(H$_g$) g is a Carathéodory funciton, bounded on $\mathbb{R}^2$, $g(\,\cdot\,,u)$ is T-periodic for any $u\in\mathbb{R}$ and some $T>0$, $g(x,\,\cdot\,)$ is 2π-periodic for almost every $x\in\mathbb{R}$, $G(x,u):=\int_0^u g(x,s)\,ds$ is bounded on $\mathbb{R}^2$, and $G(x,\,\cdot\,)$ is 2π-periodic for almost every $x\in\mathbb{R}$.

THEOREM 1.1 ([6]). *Under conditions* $(H_\Phi 1)$ *and* (H_g), (1.3) *has a classical periodic solution.*

In fact this solution can be formulated as a minimizer of a minimization problem over a closed convex set.

In [7] the variational methods were further explored to treat the corresponding relativistic type systems of equations. Consider

$$(1.6) \qquad (\phi(u'))' = \nabla_u F(x, u) + h(x), \quad u(0) = u(T), \quad u'(0) = u'(T).$$

Here it is assumed that

$(H_\Phi 2)$ ϕ a homeomorphism from $B_a \subset \mathbb{R}^n$ onto $\mathbb{R}^n$ such that $\phi(0) = 0$, $\phi = \nabla \Phi$ with $\Phi \colon \overline{B_a} \to (-\infty, 0]$ of class C^1 on B_a, continuous and strictly convex on $\overline{B_a}$.

(H_F) $F(\,\cdot\,, u)$ is measurable on $[0, T]$ for every $u \in \mathbb{R}^n$, $F(x, \cdot)$ is continually differentiable on $\mathbb{R}^n$ for almost every $x \in [0, T]$, and $\nabla_u F$ satisfies the L^1-Carathéodory conditions (e.g. [20]).

THEOREM 1.2 ([7]). *Under* $(H_\Phi 2)$ *and* (H_F) *and if F is also periodic in each component of u and $\bar{h} = 0$ the system (1.6) has at least one periodic solution.*

For the classical pendulum equations and systems while there was the original paper [12] there was a surge of interests in the existence and more in multiplicity of periodic solutions in the 1980's ([8], [9], [11], [13], [14], [19], [22], [26] and references therein). In particular for the classical pendulum systems the existence of $n + 1$ periodic solutions was given in [9], [14], [22] independently. With this background Brezis and Mawhin raised an open problem on the multiplicity of periodic solutions for the relativistic pendulum type systems (1.6). More precisely, the question is whether there exist $n + 1$ periodic solutions for (1.6) in the setting of Theorem 1.2. Several other open problems were also raised in [7] concerning existence of periodic solutions of the relativistic systems.

This paper is motivated by the results in [6], [7] and devoted to the existence of multiplicity results of periodic solutions. We give a positive answer to the above question. We use a different method and our method applies to more general situations than those treated in the above mentioned papers. Our approach allows us to give an answer to another open problem in [7] for anti-coercive potentials. See Remarks 2.2 and 2.3. The main idea is to introduce Hamiltonian coordinates to study the first order version of the systems of equations for which a more standard variational formulation exists and for which we can use an abstract theorem in [13] by Liu on multiplicity of critical points for

functionals defined on product spaces of a linear space and a manifold. We make the following assumptions.

(H1) $\phi \colon B_a \to \mathbb{R}^n$ is a homeomorphism with $\psi := \phi^{-1} \colon \mathbb{R}^n \to B_a$ and there is a C^1 function $\Psi \colon \mathbb{R}^n \to \mathbb{R}$ such that $\nabla \Psi = \psi$ and Ψ is bounded below.

EXAMPLE 1.3.

$$\phi(x) = \frac{x}{\sqrt{1 - |x|^2}}, \quad \psi(x) = \frac{x}{\sqrt{1 + |x|^2}} \quad \text{and} \quad \Psi(x) = \sqrt{1 + |x|^2}.$$

For conditions on potentials we shall consider a more general case and prove a more general theorem which contains both Theorems 1.1 and 1.2 with slightly stronger smoothness condition. Our result answers a couple of open problems in [7] (see Remarks 2.2 and 2.3 below).

(H2) $F \in C^1(S, \mathbb{R}^n)$. For an integer $0 \le k \le n$, F is T-periodic in x, 2π-periodic in $u_1, \dots, u_k$ and $\nabla_u F$ is bounded. $h \in (L_T^2(\mathbb{R}))^n$ is 2π-periodic in x and $\int_0^T h_i(x)\,dx = 0$ for $i = 1, \dots, k$. Writing $u = (v, w)$ with $v \in \mathbb{R}^k$ and $w \in \mathbb{R}^{n-k}$. Assume $\int_0^T (F(x, u) + h(x)u)\,dx \to -\infty$ as $|w| \to \infty$ uniformly in $v \in \mathbb{R}$.

THEOREM 1.4. *Assume* (H1) *and* (H2). *Then the system* (1.6) *has as least* $k + 1$ *classical T-periodic solutions.*

2. The proof and further remarks

For the proof of our main theorem we will transform the second order system to a first order system. For the first order system we follow closely the methods developed in [10] which was also used in [9], [13] in late 1980's.

First we use the idea of relativistic kinetic momentum and set $v = \phi(u')$. Due to the condition (H1) we have $u' = \psi(v)$. Now the original system is equivalent to the following first order Hamiltonian system

$$u' = \psi(v), \quad v' = \nabla_u F(x, u) + h(x), \quad u(0) = u(T), \quad v(0) = v(T).$$

It is well known that under (H_F), (H1), weak solutions of (1.6) are classical. Thus we just need to consider the weak solutions. Without loss of generality we assume $T = 2\pi$. The Euler–Lagrange functional associated with the above system is

$$I(u, v) = \int_0^T u'v\,dx - \int_0^T \Psi(v)\,dx + \int_0^T F(x, u)\,dx + \int_0^T h(x)u\,dx$$

The functional I is defined on the product space $H^{1/2}(S, \mathbb{R}^n) \times H^{1/2}(S, \mathbb{R}^n)$ with $S := \mathbb{R}/\{2\pi\mathbb{Z}\}$.

The quadratic part defines a linear operator A and the domain of A is $W^{1,2}(S, \mathbb{R}^{2n})$. The spectrum of the operator A is $\sigma(A) = \mathbb{Z}$ with each eigenvalue

being of multiplicity $2n$. The eigenspace of A corresponding to the eigenvalue $k \in \mathbb{Z}$ is

$$E_k = \exp{(ktJ)}\mathbb{R}^{2n} = ((\cos kt)I + (\sin kt)J)\mathbb{R}^{2n}.$$

In particular $\ker A = E_0 = \mathbb{R}^{2n}$. Define

$$E = \left\{ u \in H^{1/2}(S, \mathbb{R}^n) \,\middle|\, \int_0^T h(x)dx = 0 \right\}.$$

Then we have $H^{1/2}(S, \mathbb{R}^n) = E \oplus \mathbb{R}^n$. By the conditions we have I is translation-invariant in $u_1, \dots, u_k$ with integer multiple of 2π. Then I can be regarded as defined on $X = (E \oplus E \oplus \mathbb{R}^n \oplus \mathbb{R}^{n-k}) \times \mathbb{R}^k / \{2\pi\mathbb{Z}\} = (E \oplus E \oplus \mathbb{R}^{2n-k}) \times \mathbb{T}^k$.

The quadratic form $\int_0^T u'v\, dx$ has kernel $\mathbb{R}^n \oplus \mathbb{R}^n$ and is non-degenerate on $E \oplus E$. We verify that we can apply the following abstract theorem from [13] to this functional I to establish the existence of $k + 1$ critical points, which give $k + 1$ T-periodic solutions up to translations.

THEOREM 2.1 ([13]). *Let H be a Hilbert space and A be a bounded self-adjoint operator on H which splits the apace H into $H_+ = H_- + H_0$ according to its spectral decomposition. Denote by $P_\pm$ and P_0 the orthogonal projections onto positive, negative spectrum space $H_\pm$ and the kernel of A, H_0, respectively. Assume that*

(A1) *The restriction $A|H_\pm$ is invertible, i.e. $A|H_\pm$ has a bounded inverse on $H_\pm$.*

(A2) *The space H_0 is finite-dimensional.*

(A3) *$G: H \times V \to \mathbb{R}$ is a C^1-function, where V is a finite-dimensional compact C^2-manifold. Suppose that G has a bounded, compact gradient dG and $G(P_0x, y) \to -\infty$ (or $+\infty$), uniformly in y as $|P_0x| \to \infty$.*

Then the functional $f: H \times V \to \mathbb{R}$ defined by

$$f(x, y) = \frac{1}{2}(Ax, x) + G(x, y)$$

has at least cuplength $V + 1$ critical points.

With the spectral decomposition of the operator A we may identify $A_0 = \mathbb{R}^{2n-k}$ and $V = \mathbb{T}^k$. The fact that cuplength $\mathbb{T}^k = k$ gives the result once we verify the other conditions.

We need to verify the Landersman–Lazer type condition. By (H2) it suffices to show $\Psi(v) \to \infty$ as $|v| \to \infty$. We may assume $\psi(0) = 0$. Consider the flow generated by the negative pseudo-gradient vector field $V(\eta)$ of Ψ

$$\frac{d\eta}{dt} = -V(\eta), \qquad \eta(0) = x.$$

Here V is a pseudo-gradient vector field of Ψ such that

$$\langle V(x), \psi(x) \rangle \geq ||\psi||, \qquad ||V(x)|| \leq 2.$$

Since ψ is a homeomorphism from $\mathbb{R}^n$ onto B_a, there is $\delta > 0$ such that, for all $v \in \mathbb{R}^n$ with $|v| \geq 1$,

$$|\nabla \Psi(v)| = |\psi(v)| \geq \delta.$$

First, for each v with $|v| = 1$, consider the flow line $\eta(t, v)$. Then, as Ψ is non-increasing for t and Ψ is bounded from below, there is $t_0 \leq 0$ such that $\eta(t, v)$ is outside B_1 for $t \leq t_0$ and

$$\Psi(\eta(t)) \geq -\delta t + \Psi(\eta(t_0, v)) \to \infty \quad \text{and} \quad |\eta(t)| \to \infty \quad \text{as } t \to -\infty.$$

Now, if there is a $C > 0$ and $w_n \in \mathbb{R}^n$ such that $|w_n| \to \infty$ and $\Psi(w_n) \leq C$, there are $t_n > 0$ such that $v_n = \eta(t_n, w_n) \in \partial B_1$ (in fact $\eta(t, w) \to 0$ as $t \to +\infty$ due to Ψ being bounded below). Then $w_n = \eta(-t_n, v_n)$.

Assume $v_n \to v_0$. There is $t_0 < 0$ such that $\Psi(\eta(-t_0, v_0)) \geq C + 1$. Using the continuity of the flow in initial values we have

$$\Psi(\eta(-t_0, v_n)) \to \Psi(\eta(-t_0, v_0)) \geq C + 1 \quad \text{and} \quad \Psi(\eta(-t_n, v_n)) \geq \Psi(\eta(-t_0, v_n))$$

a contradiction for n large.

Applying Theorem 2.1 we complete the proof of the main theorem, establishing the existence of at least $k + 1$ critical points of I.

REMARK 2.2. When $k = n$ we get $n + 1$ periodic solutions for the relativistic systems, answering an open problem in Remark 9.3 of [7].

REMARK 2.3. When $k = 0$ our result answers an open problem raised in Remark 7.4 of [7] giving the existence of one T-periodic solution. The classical case was established in [20, Theorem 4.8].

REMARK 2.4. When $n = 1$ and $k = 1$ we obtain two periodic solutions for the relativistic pendulum equation. In this case (H1) can be weakened as the following (H_ϕ) which does not need Ψ be defined at $x = \pm a$ and the convexity of Φ as in (H_Φ1).

(H_ϕ) $\phi: (-a, a) \to \mathbb{R}$ is a homeomorphism.

Thus we have

THEOREM 2.5. *Assume* (H_ϕ). *For any $T > 0$, $A \in \mathbb{R}$ and $h \in L_T^2(\mathbb{R})$ such that $\overline{h} = 0$, the relativistic pendulum equation (1.1) has at least two classical T-periodic solutions.*

For a proof of this we just need to check (H1). In fact, let $\psi = \phi^{-1}$ then $\psi: \mathbb{R} \to (-a, a)$ and $\psi(t) \to \pm a$ (if ϕ is increasing) as $t \to \pm\infty$. Let $\Psi(t) = \int_0^t \psi(s)\,ds$. Then we have $\Psi(t) \to +\infty$ as $t \to \infty$. Thus (H1) is satisfied.

REMARK 2.6. Finally we compare (H_Φ2) in [7] and our condition (H1). As discussed in [7] under condition (H_Φ2) the Legendre–Fenchel transform of Φ is

well defined and is also strictly convex and also of class C^1 (e.g. [20]). Furthermore, $\phi^{-1} = \nabla\Phi^*$. As Φ is bounded from above and Φ^* is bounded from below. Thus $(H_\Phi 2)$ implies (H1). We do not need Φ to be defined on the close ball $\overline{B}_a$.

3. Further results and remarks

In [6], [7] the convexity of Φ (and thus Ψ) was assumed. If this is the case, with our first order systems approach some further results follow immediately from some classical work as in [20]. These results do not require bounded derivative of F. We mention a couple of result here.

(A1) There is a $l \in L^4$ such that for all x, u, $F(x, u) \geq (l, u)$.

(A2) There is $\alpha \in (0, 2\pi/T)$ and $\gamma \in L^2$ such that $F(x, u) \leq \alpha/2|u|^2 + \gamma(x)$.

(A3) $\int_0^T F(x, u)\, dx \to +\infty$ as $|u| \to \infty$ for $u \in \mathbb{R}^n$.

(a) Assume (H1) with Ψ being convex. Assume (A1)–(A3) and assume $-F(x, u) \in C^1$ is convex in u. Then (1.6) has at least one T-periodic solution. This follows from Theorem 3.5 in [20] because $\Psi(v) - F(x, u)$ is convex in (u, v) now.

Under stronger conditions one can get subharmonic solutions with large minimal periods.

(A4) Uniformly in x, $F(x, u)/u^2 \to 0$ and $F(x, u) \to -\infty$ as $|u| \to \infty$.

(b) Assume (H1) with Ψ being convex. Assume (A4) and $-F(x, u) \in C^1$ is convex in u. Then for each integer $k \geq 1$ (1.6) has a kT-periodic solution u_k with minimal period T_k such that $T_k \to \infty$ as $k \to \infty$. When $F = F(u)$ for each large k the minimal period of u_k is kT. This follows from Theorem 3.2 in [20]. We refer [20] for more source of references therein.

While finalizing the note we wrote to Professor Jean Mawhin to check on related references. Professor Mawhin kindly informed of us, among several other references, his recent paper [17]. In [17] he had used the Hamiltonian approach and obtained the multiplicity result of $n+1$ periodic solutions for the relativistic pendulum type systems (1.6). He also had generalizations to Neumann problems and difference systems and we refer [16], [18] for his nice surveys on recent progress and references therein. The treatment in [17] is somewhat different from ours, based on mainly a result of Szulkin [23] for indefinite functionals. We also learnt of [4] which gives the existence of two periodic solutions for the scalar relativistic pendulum type equations, though the method there, based on a variant of mountain pass theorem (e.g. [19], [21]), may not be suitable for producing more solutions.

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