Topological **M**ethods in **N**onlinear **A**nalysis
Volume 42, No. 2, 2013, 377–403

CONTROLLABILITY FOR SYSTEMS GOVERNED BY SECOND-ORDER DIFFERENTIAL INCLUSIONS WITH NONLOCAL CONDITIONS

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Abstract. We study a controllability problem for a system governed by a semilinear second-order differential inclusion involving control perturbations and nonlocal conditions in a Hilbert space. By using the fixed point theory for condensing multimaps, the (E_0, X_0)-controllability result for the mentioned problem is proved under the assumption that the corresponding linear system is (E_0, X_0)-controllable.

1. Introduction

We consider the following control problem

$$x''(t) - Ax(t) - Bu(t) \in F(t, x(t), u(t)), \quad t \in J := [0, T], \tag{1.1}$$

$$x(0) + g(x) = x_0, x'(0) + h(x) = x_1, \tag{1.2}$$

where the state function x takes values in a Hilbert space X, the control $u \in L^2(J; V)$, where V is a Hilbert space of controls. The linear operator A is the infinitesimal generator of a strongly continuous cosine function family $\{C(t)\}_{t\in\mathbb{R}}$,

2010 *Mathematics Subject Classification.* Primary 93B05, 34A60, 34G25, 47H04; Secondary 47H08, 47H10, 93C10, 93C25.

Key words and phrases. Controllability, control system, differential inclusion, nonlocal condition, condensing map, fixed point, measure of non-compactness, MNC-estimate.

The work of V. Obukhovskiĭ is partially supported by the Russian FBR grants 14-01-00468 and 12-01-00392. The work of T.D.Ke is partially supported by the Vietnam Ministry of Education and Training, Project No. B2012-17-12.

the control operator $B\colon V \to X$ is a bounded linear operator and the nonlinearity $F\colon J\times X\times V \multimap X$ is a multivalued map. The nonlocal functions $g, h\colon C(J;X)\to X$ and the initial data $(x_0, x_1) \in X^2$ are given.

The linear system corresponding to (1.1)–(1.2) is the following:

$$x''(t) = Ax(t) + Bu(t), \quad t \in J, \tag{1.3}$$

$$x(0) = x_0, x'(0) = x_1. \tag{1.4}$$

The mild solution $x \in C(J;X)$ of (1.3)–(1.4) with respect to a control u is represented by

$$x(t) = C(t)x_0 + S(t)x_1 + \int_0^t S(t-s)Bu(s)\,ds,$$

where $\{S(t)\}_{t\in\mathbb{R}}$ is the sine family associated to $\{C(t)\}_{t\in\mathbb{R}}$. As far as the nonlinear system (1.1)–(1.2) is concerned, a function $x \in C(J;X)$ is called a mild solution with respect to a control u if there exists a function $f \in L^1(J;X)$ such that $f(t) \in F(t, x(t), u(t))$ for almost every $t \in J$ and

$$x(t) = C(t)[x_0 - g(x)] + S(t)[x_1 - h(x)] + \int_0^t S(t-s)[Bu(s) + f(s)]\,ds.$$

For second-order differential equations in Banach spaces and cosine function theory, we refer the reader to [14], [26], [29].

The solvability for nonlinear second order differential equations with nonlocal conditions have been investigated by many authors. We refer the reader to the works in [2], [4], [17], [18], among others.

Set

$$W(x_0, x_1, u)(t) = C(t)x_0 + S(t)x_1 + \int_0^t S(t-s)Bu(s)\,ds,$$

and let $\Sigma(x_0, x_1, u)$ be the solution set of (1.1)–(1.2) with respect to a control u and initial data (x_0, x_1). It should be noted that there are several concepts of controllability for second-order differential system (see the survey [3]). We recall the definition of controllability along trajectory: the linear system (1.3)–(1.4) is said to be exactly controllable if for given $(x_0, x_1) \in X^2$, one has $W_T = X$, where

$$W_T := \{W(x_0, x_1, u)(T) : u \in L^2(J;V)\}.$$

Similarly, we say that the nonlinear system (1.1)–(1.2) is exactly controllable if for given initial data $(x_0, x_1) \in X^2$, we have $\Sigma_T = X$, where

$$\Sigma_T := \{y(T) : y \in \Sigma(x_0, x_1, u),\ u \in L^2(J;V)\}.$$

For the notions and facts of controllability for first-order differential systems, the reader is referred to, e.g. [3], [6], [8], [11], [13], [21].

Let us mention that there is an increasing interest in controllability of nonlinear second-order differential equations and inclusions in the recent research literature.

In [5], [10], the controllability results for nonlinear second-order integrodifferential systems were obtained under Lipschitz type conditions imposed on nonlinearity. The control problem involving neutral functional differential inclusions was studied in [22]. The reader can find some controllability results for nonlinear impulsive differential equations or impulsive neutral functional differential inclusions in [8], [23], [25]. Regarding control problems with nonlocal conditions, we refer the reader to some recent works, e.g. [4], [7], [16].

In order to deal with control problems for systems governed by nonlinear second-order differential systems in the mentioned works, the authors employed a crucial assumption that the linear controllability operator

$$\mathcal{B}_T u = \int_0^T S(T-s)Bu(s)\,ds$$

has a bounded inverse $\mathcal{B}_T^{-1}: X \to L^2(J;V)/\ker\mathcal{B}_T$. This requires that $\mathcal{B}_T$ is surjective and hence $W_T = X$.

It is known that, for the linear system (1.3)–(1.4), the reachable set W_T cannot coincide with X if, e.g. $S(\,\cdot\,)$ is compact and X is an infinite dimensional space (see [27], [28]). In this situation, W_T is a proper subspace of X. Hence the requirement that $\mathcal{B}_T$ is surjective cannot be applied, even to the standard wave equation (see the example in the last section for details).

By this limitation, the concept of exact controllability to a subspace is useful. Let X_0 be a closed subspace of X and $E_0 \subset X \times X$. Then the linear system is said to be exactly controllable to X_0 from E_0 (or (E_0, X_0)-controllable for short) if for any $(x_0, x_1) \in E_0, x_T \in X_0$, there exists a control $u \in L^2(J;V)$ such that $W(x_0, x_1, u)(T) = x_T$. Suppose that

$$\{C(T)x_0 + S(T)x_1 : (x_0, x_1) \in E_0\} \subset X_0.$$

Then the condition $R[\mathcal{B}_T] = X_0$ ensures (E_0, X_0)-controllability for (1.3)–(1.4), where $R[\mathcal{B}_T]$ is the range of $\mathcal{B}_T$. The aim of our work is to find suitable conditions imposed on the nonlinearity F and the nonlocal functions g, h in such a way that the nonlinear system (1.1)–(1.2) is (E_0, X_0)-controllable provided that the linear system (1.3)–(1.4) possesses this property.

Let us give a short description of the features in our work. In comparison with the above-mentioned works, our system allows to have the control perturbations, that is, the nonlinearity may contain the control function. In addition, instead of Lipschitz conditions required for nonlinearity and nonlocal functions, we assume some more general conditions being the regularity properties expressed in the terms of the Hausdorff measure of noncompactness (MNC) (see Remarks 3.1

and 3.2 for precise explanations). To prove the controllability results, we are applying the fixed point theory for condensing multivalued maps (see, e.g. [19]). More precisely, by constructing a suitable MNC in a product space and then using the MNC-estimates, we demonstrate the condensivity property for the solution multioperator, that allows to use the corresponding fixed point result. It is worth noting that at the present time this approach is widely used for the study of nonlinear differential equations, inclusions and control problems, see e.g. [19] and references therein and, among others, [9], [12], [20], [21].

The rest of our work is organized as follows. Section 2 gives some notions and facts related to measures of noncompactness, multivalued maps and the controllability of linear second-order differential systems. Section 3 is devoted to our main results: in particular, we prove the controllability assertion (Theorem 3.12) for the nonlinear system (1.1)–(1.2). The last section presents an application to the controllability problem for a system governed by the nonlinear wave equation. We demonstrate that the system is controllable to a dense subspace, that yields its approximate controllability.

2. Preliminaries

2.1. Measure of noncompactness and multivalued maps. Let us recall some basic facts from the theories of condensing and multivalued maps, which will be employed in this paper (for details, see, e.g. [1], [15], [19]).

Let $\mathcal{E}$ be a Banach space. We denote

$$\begin{aligned} C(\mathcal{E}) &= \{A \in \mathcal{P}(\mathcal{E}) : A \text{ is closed}\},\\ K(\mathcal{E}) &= \{A \in \mathcal{P}(\mathcal{E}) : A \text{ is compact}\},\\ Kv(\mathcal{E}) &= \{A \in K(\mathcal{E}) : A \text{ is convex}\}. \end{aligned}$$

We will use the following definition of the measure of noncompactness.

Definition 2.1. Let $(\mathcal{A},\geq)$ be a partially ordered set. A function $\beta\colon\mathcal{P}(\mathcal{E})\to\mathcal{A}$ is called *a measure of noncompactness* (MNC) in $\mathcal{E}$ if

$$\beta(\overline{\mathrm{co}}\,\Omega) = \beta(\Omega) \quad \text{for every } \Omega \in \mathcal{P}(\mathcal{E}),$$

where $\overline{\mathrm{co}}\,\Omega$ is the closure of the convex hull of Ω. An MNC β is called:

(a) *monotone*, if $\Omega_0, \Omega_1 \in \mathcal{P}(\mathcal{E})$, $\Omega_0 \subset \Omega_1$ implies $\beta(\Omega_0) \leq \beta(\Omega_1)$;
(b) *nonsingular*, if $\beta(\{a\} \cup \Omega) = \beta(\Omega)$ for any $a \in \mathcal{E}, \Omega \in \mathcal{P}(\mathcal{E})$;
(c) *invariant* with respect to union with compact set, if $\beta(K \cup \Omega) = \beta(\Omega)$ for every relatively compact set $K \subset \mathcal{E}$ and $\Omega \in \mathcal{P}(\mathcal{E})$;

If $\mathcal{A}$ is a cone in a normed space, we say that β is:

(d) *algebraically semi-additive*, if $\beta(\Omega_0 + \Omega_1) \leq \beta(\Omega_0) + \beta(\Omega_1)$ for any $\Omega_0, \Omega_1 \in \mathcal{P}(\mathcal{E})$;

(e) *regular*, if $\beta(\Omega) = 0$ is equivalent to the relative compactness of Ω.

An important example of MNC is *the Hausdorff* MNC, which satisfies all above properties:

$$\chi(\Omega) = \inf\{\varepsilon : \Omega \text{ has a finite } \varepsilon\text{-net}\}.$$

Based on the Hausdorff MNC χ in $\mathcal{E}$, one can define *the sequential* MNC χ_0 as follows:

$$\chi_0(\Omega) = \sup\{\chi(D) : D \in \Delta(\Omega)\}, \tag{2.1}$$

where $\Delta(\Omega)$ is the collection of all at-most-countable subsets of Ω (see [1]). We know that

$$\frac{1}{2}\chi(\Omega) \le \chi_0(\Omega) \le \chi(\Omega), \tag{2.2}$$

for all bounded set $\Omega \subset \mathcal{E}$. Then the following property is evident:

Proposition 2.2. *Let χ be the Hausdorff* MNC *in $\mathcal{E}$ and $\Omega \subset \mathcal{E}$ be a bounded set. Then for every $\varepsilon > 0$, there exists a sequence $\{x_n\} \subset \Omega$ such that*

$$\chi(\Omega) \le 2\chi(\{x_n\}) + \varepsilon.$$

Let us remind that X and V are Hilbert spaces of states and controls, respectively. We will denote by χ_X and χ_V the Hausdorff MNCs in these spaces. For $J = [0, T]$, let χ_{CX} and χ_{CV} be the Hausdorff MNCs in the spaces of continuous functions $C(J; X)$ and $C(J; V)$, respectively. We recall the following facts (see e.g. [1], [19]), which will be used in our paper: for each bounded set $D \subset C(J; X)$, one has

- $\chi_X(D(t)) \le \chi_{CX}(D)$, for all $t \in J$, where $D(t) := \{x(t) : x \in D\}$.
- If D is an equicontinuous set then

$$\chi_{CX}(D) = \sup_{t \in J} \chi_X(D(t)).$$

In this paper, we use the MNC κ_C in the product space $C(J; X) \times C(J; V)$, which is defined as follows: let π_1 and π_2 be the canonical projections to $C(J; X)$ and $C(J; V)$, respectively, then

$$\kappa_C(\mathcal{A}) = \chi_{CX}(\pi_1(\mathcal{A})) + \chi_{CV}(\pi_2(\mathcal{A})), \tag{2.3}$$

for all bounded set $\mathcal{A} \subset C(J; X) \times C(J; V)$. It is easy to check that κ_C has all properties described in Definition 2.1, including the regularity.

Let us now recall the notion of MNC-norm (see [1], [19]), which will be used in the sequel. Assume that $\mathcal{E}_1$, $\mathcal{E}_2$ are Banach spaces and $\mathcal{T} : \mathcal{E}_1 \to \mathcal{E}_2$ is a bounded linear operator. Let β_1 and β_2 be MNCs in $\mathcal{E}_1$ and $\mathcal{E}_2$, respectively. Define

$$\|\mathcal{T}\|_{\beta_1,\beta_2} = \inf\{k : \beta_2(\mathcal{T}(\Omega)) \le k\beta_1(\Omega) \text{ for all bounded sets } \Omega\}.$$

It is known that, for a linear bounded operator $\mathcal{T}$, the number $\|\mathcal{T}\|_{\beta_1,\beta_2}$ is finite and it is called (β_1,β_2)-norm of $\mathcal{T}$. In particular,

$$\beta_2(\mathcal{T}(\Omega)) \leq \|\mathcal{T}\|_{\beta_1,\beta_2}\beta_1(\Omega). \tag{2.4}$$

Let Y be a metric space.

Definition 2.3. A multivalued map (multimap) $\mathcal{F}\colon Y \to \mathcal{P}(\mathcal{E})$ is said to be:

(a) *upper semi-continuous* (u.s.c) if $\mathcal{F}^{-1}(V) = \{y \in Y : \mathcal{F}(y) \subset V\}$ is an open subset of Y for every open set $V \subset \mathcal{E}$;
(b) *closed* if its graph $\Gamma_{\mathcal{F}} = \{(y,z) : z \in \mathcal{F}(y)\}$ is a closed subset of $Y \times \mathcal{E}$;
(c) *compact* if its range $\mathcal{F}(Y)$ is relatively compact in $\mathcal{E}$;
(d) *quasicompact* if its restriction to any compact subset $A \subset Y$ is compact.

Definition 2.4. A multimap $\mathcal{F}\colon D(\mathcal{F}) \subseteq \mathcal{E} \to K(\mathcal{E})$ is said to be *condensing with respect to an* MNC β (β-condensing) if for every bounded set $\Omega \subset D(\mathcal{F})$ that is not relatively compact, we have $\beta(\mathcal{F}(\Omega)) \ngeq \beta(\Omega)$.

The application of the topological degree theory for condensing multimaps (see e.g. [19]) yields the following fixed point result.

Theorem 2.5 ([19, Corollary 3.3.1]). *Let $\mathcal{M}$ be a bounded convex closed subset of $\mathcal{E}$ and $\mathcal{F}\colon \mathcal{M} \to Kv(\mathcal{M})$ an u.s.c. β-condensing multimap, where β be a monotone nonsingular* MNC *in $\mathcal{E}$. Then the fixed point set* $\operatorname{fix}\mathcal{F} := \{x : x \in \mathcal{F}(x)\}$ *is nonempty and compact.*

Definition 2.6. Let $G\colon J \to K(\mathcal{E})$ be a multivalued function (multifunction). Then G is said to be

(a) *integrable*, if it admits a Bochner integrable selection. That is there exists $g\colon J \to \mathcal{E}$, $g(t) \in G(t)$ for almost every $t \in [0,d]$ such that
$$\int_0^T \|g(s)\|_{\mathcal{E}}\, ds < \infty;$$
(b) *integrably bounded*, if there exists a function $\xi \in L^1(J)$ such that
$$\|G(t)\| := \sup\{\|g\|_{\mathcal{E}} : g \in G(t)\} \leq \xi(t) \text{ for a.e. } t \in J.$$

The set of all integrable selections of G will be denoted by S^1_G.

The multifunction G is called *measurable* if $G^{-1}(V)$ measurable (with respect to the Lebesgue measure on J) for every open subset V of $\mathcal{E}$. We say that G is *strongly measurable* if there exists a sequence $G_n\colon J \to K(\mathcal{E})$, $n = 1,2,\ldots$ of step multifunctions such that

$$\lim_{n\to\infty} \mathcal{H}(G_n(t),G(t)) = 0 \quad \text{for a.e. } t \in J,$$

where $\mathcal{H}$ is the Hausdorff metric on $K(\mathcal{E})$.

It is known that, when $\mathcal{E}$ is separable, the notions of measurable and strongly measurable multifunctions are equivalent, and they are also equivalent to the assertion that the function $t \mapsto \operatorname{dist}(x, G(t))$ is measurable for each $x \in \mathcal{E}$. Furthermore, if G is measurable and integrably bounded, then it is integrable. In this case, we have a multifunction $t \in J \mapsto \int_0^t G(s)\,ds$ defined by

$$\int_0^t G(s)\,ds := \left\{ \int_0^t g(s)\,ds : g \in S_G^1 \right\}.$$

We have the following χ-estimate for the multivalued integral in the case when $\mathcal{E}$ is separable.

Proposition 2.7 ([19, Theorem 4.2.3]). *Let $\mathcal{E}$ be a separable Banach space and $G\colon J \to \mathcal{P}(\mathcal{E})$ an integrable, integrably bounded multifunction such that*

$$\chi(G(t)) \le q(t)$$

for almost every $t \in J$, where $q \in L^1(J)$. Then

$$\chi\biggl(\int_0^t G(s)\,ds \biggr) \le \int_0^t q(s)\,ds$$

for all $t \in J$. In particular, if $G\colon J \to K(\mathcal{E})$ is measurable and integrably bounded then the function $\chi(G(\cdot))$ is integrable and

$$\chi\biggl(\int_0^t G(s)\,ds \biggr) \le \int_0^t \chi(G(s))\,ds$$

for all $t \in J$.

Consider an abstract operator $L\colon L^1(J;\mathcal{E}) \to C(J;\mathcal{E})$ satisfying the following conditions:

(L1) there exists a constant $C > 0$ such that

$$\|L(f)(t) - L(g)(t)\|_{\mathcal{E}} \le C \int_0^t \|f(s) - g(s)\|_{\mathcal{E}}\,ds,$$

for all $f, g \in L^1(J;\mathcal{E})$, $t \in J$;

(L2) for each compact set $K \subset \mathcal{E}$ and sequence $\{f_n\} \subset L^1(J;\mathcal{E})$ such that $\{f_n(t)\} \subset K$ for almost every $t \in J$, the weak convergence $f_n \rightharpoonup f_0$ implies $L(f_n) \to L(f_0)$ strongly in $C(J;\mathcal{E})$.

As mentioned in [19, Remark 4.2.3], the integral operator

$$\mathcal{G}_I(f)(t) = \int_0^t f(s)\,ds, \tag{2.5}$$

presents an example of operator which satisfies (L1)–(L2).

One has the following assertion, which is a basic MNC-estimate for our purpose.

Proposition 2.8 ([19]). *Let L satisfy* (L1)–(L2) *and* $\{\xi_n\} \subset L^1(J;\mathcal{E})$ *be integrably bounded, that is*

$$\|\xi_n(t)\|_{\mathcal{E}} \leq \nu(t), \quad \text{for a.e. } t \in J,$$

where $\nu \in L^1(J)$. *Assume that there exists* $q \in L^1(J)$ *such that*

$$\chi(\{\xi_n(t)\}) \leq q(t), \quad \text{for a.e. } t \in J.$$

Then

$$\chi(\{L(\xi_n)(t)\}) \leq 2C \int_0^t q(s)\, ds$$

for any $t \in J$, *where* C *comes from assumption* (L1).

Using Proposition 2.8, we have the following result:

Proposition 2.9. *Let* $\Omega \subset L^1(J;\mathcal{E})$ *be a bounded set such that*

(a) *for all* $\xi \in \Omega$, $\|\xi(t)\|_{\mathcal{E}} \leq \nu(t)$ *for almost every* $t \in J$,

(b) $\chi(\Omega(t)) \leq q(t)$ *for almost every* $t \in J$,

where $\Omega(t) = \{\xi(t) : \xi \in \Omega\}$, ν *and* q *are functions in* $L^1(J)$. *Let* L *satisfy* (L1)–(L2). *Then*

$$\chi(L(\Omega)(t)) \leq 4C \int_0^t q(s)\, ds.$$

Proof. Using Proposition 2.2, for any $\varepsilon > 0$, there exists a sequence $\{\xi_n\} \subset \Omega$ such that

$$\chi(L(\Omega)(t)) \leq 2\chi(\{L(\xi_n)(t)\}) + \varepsilon, \tag{2.6}$$

for any $t \in J$. Since Ω is integrably bounded, so is $\{\xi_n\}$. Furthermore,

$$\chi(\{\xi_n(t)\}) \leq \chi(\Omega(t)) \leq q(t), \quad \text{for a.e. } t \in J.$$

Applying Proposition 2.8, one has

$$\chi(\{L(\xi_n)(t)\}) \leq 2C \int_0^t q(s)\, ds, \quad t \in J.$$

Putting this into (2.6), we have

$$\chi(L(\Omega)(t)) \leq 4C \int_0^t q(s)\, ds + \varepsilon, \quad t \in J.$$

Since ε is arbitrary, we get the conclusion as desired. □

In what follows, we employ the notion of semicompact sequence:

Definition 2.10. A sequence $\{\xi_n\} \subset L^1(J;\mathcal{E})$ is called semicompact if it is integrably bounded and the set $\{\xi_n(t)\}$ is relatively compact in $\mathcal{E}$ for almost every $t \in J$.

Following [19, Theorems 4.2.1 and 5.1.1], we have

Proposition 2.11. *If* $\{\xi_n\} \subset L^1(J;\mathcal{E})$ *is a semicompact sequence, then* $\{\xi_n\}$ *is weakly compact in* $L^1(J;\mathcal{E})$ *and* $\{L(\xi_n)\}$ *is relatively compact in* $C(J;\mathcal{E})$. *Moreover, if* $\xi_n \rightharpoonup \xi_0$ *then* $L(\xi_n) \to L(\xi_0)$.

2.2. Cosine function and controllability of a second-order linear system. Let us recall that a family $\{C(t)\}_{t\in\mathbb{R}}$ of bounded linear operators in X is called a strongly continuous cosine family if

(a) $C(0) = I$;
(b) $C(t+s) + C(t-s) = 2C(t)C(s)$, for all $t, s \in \mathbb{R}$;
(c) for each $x \in X$, the map $t \mapsto C(t)x$ is strongly continuous.

The sine family $\{S(t)\}_{t\in\mathbb{R}}$, associated to a given strongly continuous cosine family $\{C(t)\}_{t\in\mathbb{R}}$, is defined by

$$S(t)x = \int_0^t C(s)x\,ds, \quad x \in X,\ t \in \mathbb{R}.$$

The operator $A\colon D(A) \subset X \to X$ is said to be the infinitesimal generator of a cosine family $\{C(t)\}_{t\in\mathbb{R}}$ if and only if

$$Ax = \frac{d^2}{dt^2}C(t)x\bigg|_{t=0}.$$

We have the following proposition.

Proposition 2.12 ([26]). *Let* $\{C(t)\}_{t\in\mathbb{R}}$ *be a strongly continuous cosine family in* X. *Then there exist constants* $M \geq 1$ *and* $\omega \geq 0$ *such that*

(a) $\|C(t)\| \leq Me^{\omega|t|}$ *for all* $t \in \mathbb{R}$;
(b) $\|S(t_2) - S(t_1)\| \leq M\int_{t_1}^{t_2} e^{\omega|s|}\,ds$ *for all* $t_1, t_2 \in \mathbb{R}, t_1 < t_2$.

For more details on the cosine function theory, the reader is referred to [14] and [26].

Let X_0, E_0 be the spaces mentioned in Section 1. It is known that the linear system (1.3)–(1.4) is (E_0, X_0)-controllable if and only if $R[\mathcal{B}_T] = X_0$. The following result is useful for our purpose:

Lemma 2.13 ([11, Corollary 3.5]). *Let* $\mathcal{V}$, $\mathcal{W}$, $\mathcal{Z}$ *be reflexive Banach spaces and* $\mathcal{G}_0 \in L(\mathcal{V};\mathcal{Z})$, $\mathcal{G}_1 \in L(\mathcal{W};\mathcal{Z})$. *Then the following statements are equivalent*:

(a) $R[\mathcal{G}_0] \subset R[\mathcal{G}_1]$,
(b) *there exists* $\gamma > 0$ *such that* $\sqrt{\gamma}\|\mathcal{G}_0^* z^*\|_{\mathcal{V}^*} \leq \|\mathcal{G}_1^* z^*\|_{\mathcal{W}^*}$, *for all* $z^* \in \mathcal{Z}^*$.

By using this lemma with $\mathcal{V} = X_0$, $\mathcal{W} = L^2(J;V)$, $\mathcal{Z} = X$, $\mathcal{G}_0$ being the injection from X_0 into X and $\mathcal{G}_1 = \mathcal{B}_T$, the controllability condition is equivalent to the inequality

$$\|\mathcal{B}_T^* z\|_{L^2(J;V)} \geq \sqrt{\gamma}\|z\|_{X_0^*}, \quad \gamma > 0, \tag{2.7}$$

for all $z \in X_0^*$. Here $\mathcal{B}_T^* : X \to L^2(J;V)$ is the adjoint operator of $\mathcal{B}_T$. The last inequality implies that $(\mathcal{B}_T\mathcal{B}_T^* z, z)_X \geq \gamma \|z\|^2_{X_0^*}$, for all $z \in X$. Moreover, the arguments in the proof of [11, Theorem 3.7] show that $\mathcal{B}_T^* = B^*S^*(T - \,\cdot\,)$, and then the operator $\Gamma_0^T\colon X \to X_0$ defined by

$$\Gamma_0^T(z) = \mathcal{B}_T\mathcal{B}_T^* z = \int_0^T S(T-s)BB^*S^*(T-s)z\,ds, \quad z \in X \tag{2.8}$$

is invertible and

$$\|(\Gamma_0^T)^{-1}\| \leq \frac{1}{\gamma}. \tag{2.9}$$

By the hypothesis that the linear system is (E_0, X_0)-controllable, for given $x_T \in X_0$, the feedback control is defined by

$$u(t) = B^*S^*(T-t)(\Gamma_0^T)^{-1}[x_T - C(T)x_0 - S(T)x_1].$$

3. Controllability result

We start this section by giving our assumptions on the control problem (1.1)–(1.2). The following suggestion will be used in this section:

- (SA) The linear system (1.3)–(1.4) is (E_0, X_0)-controllable. Furthermore,
  - (a) $\{C(t)x_0 + S(t)x_1 : (x_0, x_1) \in E_0\} \subset X_0$,
  - (b) $\int_0^T S(T-s)f(s)\,ds \in X_0$ for all $f \in L^1(J;X)$.

In addition, let us impose some regularity conditions on F, g and h. Concerning the multivalued nonlinearity F, we assume that:

- (F1) $F\colon J \times X \times V \to Kv(X)$ is such that $F(\,\cdot\,, x(\,\cdot\,), v(\,\cdot\,))$ admits a strongly measurable selection for each $(x, v) \in C(J;X) \times L^2(J;V)$;
- (F2) For almost every $t \in J$, $F(t, \,\cdot\,, \,\cdot\,)\colon X \times V \to Kv(X)$ is u.s.c.;
- (F3) There exists a continuous nondecreasing function $\Psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that

  $$\|F(t,\eta,\zeta)\| := \sup\{\|z\|_X : z \in F(t,\eta,\zeta)\} \leq \mu(t)\Psi(\|\eta\|_X + \|\zeta\|_V),$$

  for almost every $t \in J, (\eta,\zeta) \in X \times V$, where $\mu \in L^1(J)$;
- (F4) There are functions $k, q \in L^1(J)$ such that

  $$\chi_X(F(t,\Omega,Q)) \leq k(t)\chi_X(\Omega) + q(t)\chi_V(Q), \quad \text{for a.e. } t \in J,$$

  for all bounded subsets $\Omega \subset X$ and $Q \subset V$.

Remark 3.1. We now give a comment on hypothesis (F4). If X is a finite dimension space, one can drop (F4) since it can be deduced from (F2)–(F3). That is $F(t, \,\cdot\,, \,\cdot\,)$ maps bounded set in $X \times V$ into bounded set in X, and therefore $\chi_X(F(t,\Omega,Q)) = 0$ for all bounded sets $\Omega \subset X$ and $Q \subset V$.

Now we show that if $F(t,\cdot,\cdot)$ is a Lipschitz multifunction with respect to the Hausdorff metric $\mathcal{H}$ on $K(X)$, that is for all $x,y \in X$, $\xi,\eta \in V$:

$$\mathcal{H}(F(t,x,\xi),F(t,y,\eta)) \le k(t)\|x-y\|_X + q(t)\|\xi-\eta\|_V, \tag{3.1}$$

then (F4) is satisfied. Indeed, by definition of the Hausdorff MNC, for given $\varepsilon > 0$, one can choose $\{y_1,\dots,y_m\} \subset X$ and $\{\eta_1,\dots,\eta_p\} \subset V$ such that

$$\Omega \subset \bigcup_{i=1}^{m} B(y_i,\chi_X(\Omega)+\varepsilon), \qquad Q \subset \bigcup_{k=1}^{p} B(\eta_k,\chi_V(Q)+\varepsilon).$$

Now for any $z \in F(t,\Omega,Q)$, there exists $(x,\xi) \in \Omega \times Q$ such that $z \in F(t,x,\xi)$. Taking y_i and η_k such that

$$\|x-y_i\|_X \le \chi_X(\Omega)+\varepsilon, \qquad \|\xi-\eta_k\|_V \le \chi_V(Q)+\varepsilon,$$

we obtain

$$\|z-z_{ik}\|_X \le k(t)\|x-y_i\|_X + q(t)\|\xi-\eta_k\|_V \le k(t)(\chi_X(\Omega)+\varepsilon)+q(t)(\chi_V(Q)+\varepsilon),$$

due to (3.1), here $z_{ik} \in F(t,y_i,\eta_k)$. Thus

$$F(t,\Omega,Q) \subset \bigcup_{\substack{i=1,\dots,m\\ k=1,\dots,p}} B(z_{ik},k(t)(\chi_X(\Omega)+\varepsilon)+q(t)(\chi_V(Q)+\varepsilon)).$$

The last inequality implies (F4).

Concerning g and h, we suppose that:

(GH1) $g,h\colon C(J;X) \to X$ are continuous and for $x \in C(J;X)$, $(g(x),h(x)) \in E_0$;

(GH2) There exist numbers $C_g,C_h \ge 0$ and nondecreasing functions $\Psi_g,\Psi_h\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|g(x)\|_X \le C_g\Psi_g(\|x\|_C), \qquad \|h(x)\|_X \le C_h\Psi_h(\|x\|_C),$$

where $\|x\|_C = \|x\|_{C(J;X)}$;

(GH3) We have

$$\chi_{CX}(C(\cdot)g(D)) \le m_g\chi_{CX}(D), \qquad \chi_{CX}(S(\cdot)h(D)) \le m_h\chi_{CX}(D),$$

for all bounded subsets $D \subset C(J;X)$, where m_g, m_h are nonnegative constants.

Remark 3.2. (a) If g and h are Lipschitz continuous, then (GH3) is true. Indeed, we can show this fact, e.g. for g. Let

$$\|g(x)-g(y)\|_X \le l_g\|x-y\|_C, \quad l_g \ge 0, \quad \text{for all } x,y \in C(J;X).$$

Then

$$\sup_{t\in J}\|C(t)g(x)-C(t)g(y)\|_X \le l_g \sup_{t\in J} C(t)\|x-y\|_C.$$

It implies

$$\|C(\,\cdot\,)g(x) - C(\,\cdot\,)g(y)\|_C \le m_g\|x-y\|_C,$$

where $m_g := l_g \sup_{t\in J} C(t)$. This leads to the first inequality in (GH3). In addition, it is easy to check that g and h satisfy (GH2), due to the fact that

$$\|g(x)\|_X \le l_g\|x\|_C + \|g(0)\|_X,$$

and a similar estimate for h.

(b) If g and h are completely continuous, i.e. they send any bounded set in $C(J;X)$ to a relatively compact set in X, then (GH3) is fulfilled with $m_g = m_h = 0$. Indeed, let $D \subset C(J;X)$ be a bounded set. Then $g(D)$ is a relatively compact set in X. It follows that $C(\,\cdot\,)g(D)$ is equicontinuous and then

$$\chi_{CX}(C(\,\cdot\,)g(D)) = \sup_{t\in J}\chi_X(C(t)g(D)) = 0,$$

due to the fact that $C(t)g(D)$ is relatively compact for each $t \in J$. The same arguments show that

$$\chi_{CX}(S(\,\cdot\,)g(D)) = \sup_{t\in J}\chi_X(S(t)g(D)) = 0.$$

For each $(x,u) \in C(J;X)\times L^2(J;V)$, set

$$S^1_F(x,u) = \{f \in L^1(J;X) : f(t) \in F(t,x(t),u(t)),\ t\in J\}.$$

Definition 3.3. A function $x \in C(J;X)$ is said to be a mild solution of the nonlinear system (1.1)–(1.2) if there exists $f \in S^1_F(x,u)$ such that

$$x(t) = C(t)[x_0 - g(x)] + S(t)[x_1 - h(x)] + \int_0^t S(t-s)[Bu(s) + f(s)]\,ds.$$

For the sake of convenience, to obtain the controllability of (1.1)–(1.2), we will divide our arguments into steps. As the first step, we define a solution mutioperator, whose fixed points are the solutions of the control problem (1.1)–(1.2).

Define the evaluation operator $\mathcal{Q}\colon C(J;X) \to X$ by $\mathcal{Q}y = y(T)$ and the integral operator $\mathcal{L}$ as follows:

$$\mathcal{L}\ :\ L^1(J;X) \to C(J;X), \tag{3.2}$$

$$\mathcal{L}(f)(t) = \int_0^t S(t-s)f(s)\,ds. \tag{3.3}$$

In addition, define the operator $\mathcal{G}$ on $C(J;X)$:

$$\mathcal{G}(x)(t) = C(t)[x_0 - g(x)] + S(t)[x_1 - h(x)]. \tag{3.4}$$

We are in a position to construct the multioperator

$$\mathcal{F}\colon C(J;X)\times L^2(J;V)\to \mathcal{P}(C(J;X)\times C(J;V)), \tag{3.5}$$

$$\mathcal{F}(x,u)=\{(y(x,u,f),z(x,u,f)) : f\in S^1_F(x,u)\}, \tag{3.6}$$

where

$$z(x,u,f) = B^*S^*(T-\,\cdot\,)(\Gamma_0^T)^{-1}[x_T-\mathcal{Q}\mathcal{G}(x)-\mathcal{Q}\mathcal{L}(f)], \tag{3.7}$$

$$y(x,u,f) = \mathcal{G}(x)+\mathcal{L}Bz(x,u,f)+\mathcal{L}(f). \tag{3.8}$$

Here the operator Γ_0^T is defined by (2.8) and $x_T\in X$ is given.

Notice that the multioperator $\mathcal{F}$ is well-defined by virtue of assumption (SA). Let us mention also that the projections of $\mathcal{F}$ on $C(J;X)$ and $C(J;V)$, respectively, can be written as

$$\begin{aligned}\pi_1\mathcal{F}(x,u) &= \{y(x,u,f) : f\in S^1_F(x,u)\},\\ \pi_2\mathcal{F}(x,u) &= \{z(x,u,f) : f\in S^1_F(x,u)\}.\end{aligned}$$

Moreover, if (x^*,u^*) is a fixed point of $\mathcal{F}$, then there exists $f\in S^1_F(x^*,u^*)$ such that

$$x^* = \mathcal{G}(x^*)+\mathcal{L}(Bu^*+f), \tag{3.9}$$

$$u^* = B^*S^*(T-\,\cdot\,)(\Gamma_0^T)^{-1}[x_T-\mathcal{Q}\mathcal{G}(x^*)-\mathcal{Q}\mathcal{L}(f)]. \tag{3.10}$$

Therefore, it is easy to check that the control u^* given above steers the initial profile (x_0,x_1) to $x_T=x^*(T)$.

Since we are searching for a fixed point of $\mathcal{F}$ satisfying (3.9)–(3.10), the multioperator $\mathcal{F}$ can be restricted to $C(J;X)\times C(J;V)$. We call $\mathcal{F}$ *the solution multioperator*.

As the second step, we will study some properties of the solution multioperator $\mathcal{F}$. The following results will be useful in the sequel:

Proposition 3.4. *The operator $\mathcal{L}$ defined by* (3.2)–(3.3) *satisfies* (L1)–(L2) *with constant $C=M_0:=\sup_{t\in J}\|S(t)\|$. Furthermore, it maps any bounded set in $L^1(J;X)$ into an equicontinuous one in $C(J;X)$.*

Proof. Taking the arguments from [19, Lemma 4.2.1], we see that $\mathcal{L}$ fulfills (L1)–(L2). On the other hand, if $Q\subset L^1(J;X)$ is a bounded set, then for all $f\in Q$ and $t_1,t_2\in J$, $t_2>t_1$, we have

$$\begin{aligned}\|\mathcal{L}(f)(t_2)-\mathcal{L}(f)(t_1)\|_X &= \bigg\|\int_0^{t_2}S(t_2-s)f(s)\,ds-\int_0^{t_1}S(t_1-s)f(s)\,ds\bigg\|_X\\ &\le \int_0^{t_1}\|S(t_2-s)-S(t_1-s)\|\|f(s)\|_X\,ds+\int_{t_1}^{t_2}\|S(t_2-s)\|\|f(s)\|_X\,ds.\end{aligned}$$

Using Proposition 2.12, one obtains that

$$\|S(t_2 - s) - S(t_1 - s)\| \le M \int_{t_1-s}^{t_2-s} e^{\omega\zeta}\, d\zeta \le M(t_2 - t_1)e^{\omega T}.$$

Making use of this estimate, we arrive at

$$\|\mathcal{L}(f)(t_2) - \mathcal{L}(f)(t_1)\|_X \le M(t_2 - t_1)e^{\omega T} \int_0^{t_1} \|f(s)\|_X\, ds + M_0 \int_{t_1}^{t_2} \|f(s)\|_X\, ds.$$

The last inequality ensures the second conclusion of Proposition 3.4. □

PROPOSITION 3.5. *Let $\mathcal{A}$ be a bounded set in $C(J;X) \times C(J;V)$. Then the set $\pi_2\mathcal{F}(\mathcal{A})$ is equicontinuous in $C(J;V)$.*

PROOF. Let $D = \pi_1(\mathcal{A})$. Then D is a bounded set in $C(J;X)$. For any $v \in \pi_2\mathcal{F}(\mathcal{A})$, there exists an $(x,u) \in \mathcal{A}$ and $f \in S_F^1(x,u)$ such that

$$v(t) = B^*S^*(T-t)(\Gamma_0^T)^{-1}[x_T - \mathcal{Q}\mathcal{G}(x) - \mathcal{Q}\mathcal{L}(f)].$$

Then

$$\begin{aligned} \|v(t_2) - v(t_1)\|_V &\le \|B^*\| \cdot \|S^*(T-t_2) - S^*(T-t_1)\| \\ &\quad \cdot \|(\Gamma_0^T)^{-1}\|(\|x_T\|_X + \|\mathcal{Q}\mathcal{G}(x)\|_X + \|\mathcal{Q}\mathcal{L}(f)\|_X) \\ &\le \frac{1}{\gamma}\|B\| \cdot \|S(T-t_2) - S(T-t_1)\| \\ &\quad \cdot (\|x_T\|_X + \|\mathcal{Q}\mathcal{G}(x)\|_X + \|\mathcal{Q}\mathcal{L}(f)\|_X) \\ &\le \frac{1}{\gamma}\|B\|M|t_2 - t_1|e^{\omega T}(\|x_T\|_X + \|\mathcal{Q}\mathcal{G}(x)\|_X + \|\mathcal{Q}\mathcal{L}(f)\|_X). \end{aligned} \tag{3.11}$$

Here we have used (2.9) and Proposition 2.12. Applying (GH2), we have

$$\begin{aligned} \|\mathcal{Q}\mathcal{G}(x)\|_X &= \|C(T)[x_0 - g(x)] + S(T)[x_1 - h(x)]\|_X \\ &\le \|C(T)\|(\|x_0\|_X + C_g\Psi_g(\|x\|_C)) + \|S(T)\|(\|x_1\|_X + C_h\Psi_h(\|x\|_C)). \end{aligned}$$

Thus one can find $M_1 > 0$ such that

$$\|\mathcal{Q}\mathcal{G}(x)\|_X \le M_1, \quad \text{for all } x \in D. \tag{3.12}$$

In addition, by (F3), we see that

$$\|\mathcal{Q}\mathcal{L}(f)\|_X = \left\| \int_0^T S(T-s)f(s)\, ds \right\|_X \le M_0\Psi(\|x\|_C + \|u\|_C) \int_0^T \mu(s)\, ds$$

for any $f \in S_F^1(x,u)$. Since $\mathcal{A}$ is bounded, there is a number $M_2 > 0$ such that

$$\|\mathcal{Q}\mathcal{L}(f)\|_X \le M_2, \tag{3.13}$$

for all $(x,u) \in \mathcal{A}$. Hence putting (3.12) and (3.13) into inequality (3.11), we obtain that $\pi_2\mathcal{F}(\mathcal{A})$ is equicontinuous in $C(J;V)$. □

Lemma 3.6 ([19, Lemma 5.1.1]). *Let F satisfy* (F1)–(F2) *and* $\{(x_n, u_n)\} \subset C(J;X) \times C(J;V)$ *be a sequence converging to* $(x^*, u^*) \in C(J;X) \times C(J;V)$. *Suppose that the sequence* $\{\phi_n\}$ *such that* $\phi_n \in S^1_F(x_n, u_n)$ *weakly converges to* ϕ^* *in* $L^1(J;X)$, *then* $\phi^* \in S^1_F(x^*, u^*)$.

Lemma 3.7 ([19, Theorem 1.1.12]). *Let X and Y be metric spaces and* $G\colon X \to \mathcal{P}(Y)$ *a closed quasi-compact multimap with compact values. Then G is u.s.c.*

We can describe now the first property for the solution multioperator.

Lemma 3.8. *Let* (F1)–(F3) *and* (GH1)–(GH2) *hold. Then the multioperator* $\mathcal{F}$ *given by* (3.5)–(3.8) *is a quasi-compact multimap.*

Proof. Let $K \subset C(J;X) \times C(J;V)$ be a compact set and $D = \pi_1(K)$, $\mathcal{C} = \pi_2(K)$. It follows from (3.7) that

$$\begin{aligned}\chi_V(\pi_2\mathcal{F}(K)(t)) &= \chi_V(B^*S^*(T-t)(\Gamma_0^T)^{-1}[x_T - \mathcal{Q}\mathcal{G}(D) - \mathcal{Q}\mathcal{L}S^1_F(K)])\\ &\le \|B^*S(T-t)(\Gamma_0^T)^{-1}\|_{\chi_X,\chi_V}\chi_X(x_T - \mathcal{Q}\mathcal{G}(D) - \mathcal{Q}\mathcal{L}S^1_F(K))\\ &\le \|B^*S(T-t)(\Gamma_0^T)^{-1}\|_{\chi_X,\chi_V}[\chi_X(\mathcal{Q}\mathcal{G}(D)) + \chi_X(\mathcal{Q}\mathcal{L}S^1_F(K))].\end{aligned}$$

Here we used the MNC-norm estimate (2.4) and the fact that x_T is singleton. We have

$$\begin{aligned}\chi_X(\mathcal{Q}\mathcal{G}(D)) &= \chi_X(C(T)[x_0 - g(D)] + S(T)[x_1 - h(D)])\\ &\le \|C(T)\|\chi_X(g(D)) + \|S(T)\|\chi_X(h(D)) = 0,\end{aligned}$$

due to the fact that $g, h : C(J;X) \to X$ are continuous and $D \subset C(J;X)$ is compact, which leads to the compactness of $g(D)$ and $h(D)$. On the other hand,

$$\chi_X(\{f(s) : f \in S^1_F(K)\}) \le \chi_X(F(s, D(s), \mathcal{C}(s))) = 0 \quad \text{for each } s \in J, \tag{3.14}$$

thanks to the fact that $F(s,\,\cdot\,,\,\cdot\,)$ is u.s.c and $D(s) \subset X$, $\mathcal{C}(s) \subset V$ are compact. Thus, the use of Proposition 2.9 yields

$$\chi_X(\mathcal{Q}\mathcal{L}S^1_F(K)) = \chi_X\left(\left\{\int_0^T S(T-s)f(s)\,ds : f \in S^1_F(K)\right\}\right) = 0. \tag{3.15}$$

Hence

$$\chi_V(\pi_2\mathcal{F}(K)(t)) = 0 \quad \text{for each } t \in J. \tag{3.16}$$

Since K is a bounded set, by virtue of Proposition 3.5, $\pi_2\mathcal{F}(K)$ is equicontinuous in $C(J;V)$. Taking into account (3.16) and using the Arzela-Ascoli theorem, one concludes that $\pi_2\mathcal{F}(K)$ is a relatively compact set in $C(J;V)$.

Regarding $\pi_1\mathcal{F}(K)$, one has

$$\begin{aligned}\chi_X(\pi_1\mathcal{F}(K)(t)) &= \chi_X(\mathcal{G}(D)(t) + \mathcal{L}[B\pi_2\mathcal{F}(K) + S^1_F(K)])\\ &\le \chi_X(\mathcal{G}(D)(t)) + \chi_X(\mathcal{L}B\pi_2\mathcal{F}(K)(t)) + \chi_X(\mathcal{L}S^1_F(K)(t)).\end{aligned} \tag{3.17}$$

By the continuity of $C(t)$, $S(t)$, g, h and the compactness of D, one has

$$(3.18)\qquad \begin{aligned}\chi_X(\mathcal{G}(D)(t)) &= \chi_X(C(t)[x_0 - g(D)] + S(t)[x_1 - h(D)])\\ &\le \chi_X(C(t)g(D)) + \chi_X(S(t)h(D)) = 0,\end{aligned}$$

for $t \in J$. Since $\pi_2\mathcal{F}(K)$ is relatively compact and B is linear bounded, we have

$$(3.19)\qquad \chi_X(\mathcal{L}B\pi_2\mathcal{F}(K)(t)) = 0, \quad t \in J.$$

Moreover, by using the arguments as in (3.14)–(3.15), we get

$$\chi_X(\mathcal{L}S^1_F(K)(t)) = 0, \quad t \in J.$$

This fact together with (3.18)–(3.19) implies

$$\chi_X(\pi_1\mathcal{F}(K)(t)) = 0, \quad t \in J.$$

In order to show that $\pi_1\mathcal{F}(K)$ is a compact set in $C(J;X)$, it remains to prove that $\pi_1\mathcal{F}(K)$ is equicontinuous. In view of the compactness of $g(D)$ and $h(D)$, we deduce that $\mathcal{G}(D) = C(\,\cdot\,)[x_0 - g(D)] + S(\,\cdot\,)[x_1 - h(D)]$ is equicontinuous. In addition, $B\pi_2\mathcal{F}(K)$ and $S^1_F(K)$ are bounded sets in $L^1(J;X)$, then, by applying Proposition 3.4 we see that $\mathcal{L}B\pi_2\mathcal{F}(K)$ and $\mathcal{L}S^1_F(K)$ are equicontinuous as well. Therefore $\pi_1\mathcal{F}(K)$ is an equicontinuous set in $C(J;X)$. □

Lemma 3.9. *Let the hypotheses of Lemma* 3.8 *hold. Then the solution multioperator $\mathcal{F}$ is u.s.c.*

Proof. Using Lemmas 3.7 and 3.8, it suffices to prove that $\mathcal{F}$ is a closed multimap. Let $\{(x_n, u_n)\} \subset C(J;X) \times C(J;V)$ converge to (x^*, u^*) and $(y_n, z_n) \in \mathcal{F}(x_n, u_n)$. Suppose that (y_n, z_n) converges to (y^*, z^*). We will show that $(y^*, z^*) \in \mathcal{F}(x^*, u^*)$. Indeed, by the definition of $\mathcal{F}$, for each $n \ge 1$, there exists $f_n \in S^1_F(x_n, u_n)$ such that

$$(3.20)\qquad y_n = \mathcal{G}(x_n) + \mathcal{L}(Bu_n + f_n),$$

$$(3.21)\qquad z_n = B^*S^*(T - \,\cdot\,)(\Gamma_0^T)^{-1}[x_T - \mathcal{Q}\mathcal{G}(x_n) - \mathcal{Q}\mathcal{L}f_n].$$

Let $K = \{(x_n, u_n)\}$. Then K is a compact set in $C(J;X) \times C(J;V)$. Noting that $f_n(t) \in F(t, K(t))$, we see that $\{f_n(t)\} \subset X$ is compact. In addition, it follows from (F3) that $\{f_n\}$ is integrably bounded, thanks to the fact that K is a bounded set. Thus $\{f_n\}$ is a semicompact sequence. By Proposition 2.11, $\{f_n\}$ converges weakly to a function f^* in $L^1(J;X)$ and then $\mathcal{L}f_n$ converges to $\mathcal{L}f^*$ strongly in $C(J;X)$. Now one can pass to the limits in equalities (3.20)–(3.21) to get that

$$(3.22)\qquad y^* = \mathcal{G}(x^*) + \mathcal{L}(Bu^* + f^*),$$

$$(3.23)\qquad z^* = B^*S^*(T - \,\cdot\,)(\Gamma_0^T)^{-1}[x_T - \mathcal{Q}\mathcal{G}(x^*) - \mathcal{Q}\mathcal{L}f^*].$$

Finally, by using Lemma 3.6, we have $f^* \in S^1_F(x^*, u^*)$ and then (3.22)–(3.23) guarantees that $(y^*, z^*) \in \mathcal{F}(x^*, u^*)$. □

For the sake of simplicity, we set

$$N_0 = \sup_{t\in J} \|C(t)\|, \tag{3.24}$$

$$M^* = \sup_{t\in J} \|B^* S^*(T-t)(\Gamma_0^T)^{-1}\|_{\chi_X,\chi_V}, \tag{3.25}$$

$$k_0 = m_g + m_h + 2M_0 \int_0^T k(s)\,ds, \tag{3.26}$$

$$q_0 = 2M_0 \int_0^T q(s)\,ds. \tag{3.27}$$

We are in a position to present the following key statement.

THEOREM 3.10. *Let the nonlinearity F satisfy* (F1)–(F2) *and* (F4)*, g and h obey* (GH1), (GH3)*. Then the solution operator $\mathcal{F}$ is κ_C-condensing, provided*

$$\ell := \max\{2k_0(1 + 2M_0 M^* T\|B\|_{\chi_X,\chi_V}), \quad 2q_0(1 + 2M_0 M^* T\|B\|_{\chi_X,\chi_V})\} < 1, \tag{3.28}$$

where the MNC *κ_C is defined in* (2.3).

PROOF. Let $\mathcal{A}$ be a bounded set in $C(J;X) \times C(J;V)$ such that

$$\kappa_C(\mathcal{F}(\mathcal{A})) \ge \kappa_C(\mathcal{A}). \tag{3.29}$$

We will demonstrate that $\mathcal{A}$ is relatively compact. By the definitions of κ_C in (2.3) and $\mathcal{F}$ in (3.5)–(3.8), we have

$$\kappa_C(\mathcal{F}(\mathcal{A})) = \chi_{CX}(\pi_1\mathcal{F}(\mathcal{A})) + \chi_{CV}(\pi_2\mathcal{F}(\mathcal{A})). \tag{3.30}$$

We first give some estimates for $\chi_{CV}(\pi_2\mathcal{F}(\mathcal{A}))$. In view of Proposition 3.5, $\pi_2\mathcal{F}(\mathcal{A})$ is equicontinuous. Let $D = \pi_1(\mathcal{A})$, $\mathcal{C} = \pi_2(\mathcal{A})$. Then

$$\begin{aligned} \chi_{CV}(\pi_2\mathcal{F}(\mathcal{A})) &= \sup_{t\in J} \chi_V(\pi_2\mathcal{F}(\mathcal{A})(t)) \\ &\le \sup_{t\in J} \|B^* S^*(T-t)(\Gamma_0^T)^{-1}\|_{\chi_X,\chi_V} \chi_X(x_T - \mathcal{Q}\mathcal{G}(D) - \mathcal{Q}\mathcal{L}S^1_F(\mathcal{A})) \\ &\le \sup_{t\in J} \|B^* S^*(T-t)(\Gamma_0^T)^{-1}\|_{\chi_X,\chi_V} [\chi_X(\mathcal{Q}\mathcal{G}(D)) + \chi_X(\mathcal{Q}\mathcal{L}S^1_F(\mathcal{A}))]. \end{aligned} \tag{3.31}$$

Dealing with $\chi_X(\mathcal{Q}\mathcal{G}(D))$, one gets

$$\begin{aligned} \chi_X(\mathcal{Q}\mathcal{G}(D)) &= \chi_X(C(T)[x_0 - g(D)] + S(T)[x_1 - h(D)]) \\ &\le \chi_X(C(T)g(D)) + \chi_X(S(T)h(D)) \\ &\le \chi_{CX}(C(\cdot)g(D)) + \chi_{CX}(S(\cdot)h(D)) \\ &\le (m_g + m_h)\chi_{CX}(D), \end{aligned} \tag{3.32}$$

according to (GH3). On the other hand, by (F4) we have

$$\begin{aligned}\chi_X(F(s,D(s),\mathcal{C}(s))) &\le k(s)\chi_X(D(s)) + q(s)\chi_V(\mathcal{C}(s))\\ &\le k(s)\chi_{CX}(D) + q(s)\chi_{CV}(\mathcal{C}), \quad s\in J.\end{aligned}$$

Therefore,

$$\begin{aligned}(3.33)\quad &\chi_X(\mathcal{Q}\mathcal{L}S^1_F(\mathcal{A}))\\ &\le \chi_X\bigg(\int_0^T S(T-s)f(s)\,ds : f\in L^1(J;X),\ f(s)\in F(s,D(s),\ \mathcal{C}(s))\bigg)\\ &\le 4M_0\bigg(\chi_{CX}(D)\int_0^T k(s)\,ds + \chi_{CX}(\mathcal{C})\int_0^T q(s)\,ds\bigg),\end{aligned}$$

due to Proposition 2.9. Putting the last inequality and (3.32) into (3.31), we get

$$(3.34)\qquad \chi_{CV}(\pi_2\mathcal{F}(\mathcal{A})) \le M^*(k_0\chi_{CX}(D) + q_0\chi_{CV}(\mathcal{C})).$$

Now we implement estimates for $\chi_{CX}(\pi_1\mathcal{F}(\mathcal{A}))$. One can write

$$\begin{aligned}(3.35)\quad \chi_{CX}(\pi_1\mathcal{F}(\mathcal{A})) &= \chi_{CX}(\mathcal{G}(D) + \mathcal{L}[B\pi_2\mathcal{F}(\mathcal{A}) + S^1_F(\mathcal{A})])\\ &\le \chi_{CX}(\mathcal{G}(D)) + \chi_{CX}(\mathcal{L}B\pi_2\mathcal{F}(\mathcal{A}) + \chi_{CX}(\mathcal{L}S^1_F(\mathcal{A})).\end{aligned}$$

Taking similar estimates as in (3.32), we have

$$(3.36)\qquad \chi_{CX}(\mathcal{G}(D)) \le (m_g + m_h)\chi_{CX}(D).$$

Moreover, by virtue of the boundedness of $B\pi_2\mathcal{F}(\mathcal{A})$ and Proposition 3.4, the set $\mathcal{L}B\pi_2\mathcal{F}(\mathcal{A})$ is equicontinuous. Then

$$\chi_{CX}(\mathcal{L}B\pi_2\mathcal{F}(\mathcal{A})) = \sup_{t\in J}\chi_X(\mathcal{L}B\pi_2\mathcal{F}(\mathcal{A})(t)).$$

In order to get estimates for the last term, we observe that

$$\begin{aligned}\chi_X(B\pi_2\mathcal{F}(\mathcal{A})(t)) &\le \|B\|_{\chi_X,\chi_V}\chi_V(\pi_2\mathcal{F}(\mathcal{A})(t)) \le \|B\|_{\chi_X,\chi_V}\chi_{CV}(\pi_2\mathcal{F}(\mathcal{A}))\\ &\le \|B\|_{\chi_X,\chi_V}M^*(k_0\chi_{CX}(D) + q_0\chi_{CV}(\mathcal{C})),\end{aligned}$$

thanks to (3.34). Accordingly, by Proposition 2.9 we have

$$(3.37)\qquad \chi_{CX}(\mathcal{L}B\pi_2\mathcal{F}(\mathcal{A})) \le 4M_0T\|B\|_{\chi_X,\chi_V}M^*(k_0\chi_{CX}(D) + q_0\chi_{CV}(\mathcal{C})).$$

Since the set $S^1_F(\mathcal{A})$ is integrably bounded, $\mathcal{L}S^1_F(\mathcal{A})$ is an equicontinuous set and the last term in (3.35) is proceeded similarly to (3.33):

$$\begin{aligned}(3.38)\quad &\chi_{CX}(\mathcal{L}S^1_F(\mathcal{A})) = \sup_{t\in J}\chi_X(\mathcal{L}S^1_F(\mathcal{A})(t))\\ &\le \sup_{t\in J}\chi_X\bigg(\int_0^t S(t-s)f(s)\,ds : f\in L^1(J;X),\ f(s)\in F(s,D(s),\mathcal{C}(s))\bigg)\\ &\le 4M_0\bigg(\chi_{CX}(D)\int_0^T k(s)\,ds + \chi_{CV}(\mathcal{C})\int_0^T q(s)\,ds\bigg).\end{aligned}$$

The combination of (3.35)–(3.38) gives

$$\begin{aligned}\chi_{CX}(\pi_1\mathcal{F}(\mathcal{A})) \le{}& \bigg(m_g + m_h + 4M_0M^*k_0T\|B\|_{\chi_X,\chi_V} + 4M_0\int_0^T k(s)\,ds\bigg)\chi_{CX}(D)\\ &+ \bigg(4M_0M^*q_0T\|B\|_{\chi_X,\chi_V} + 4M_0\int_0^T q(s)\,ds\bigg)\chi_{CV}(\mathcal{C})\\ ={}& (k_0 + 4M_0M^*k_0T\|B\|_{\chi_X,\chi_V})\chi_{CX}(D)\\ &+ (q_0 + 4M_0M^*q_0T\|B\|_{\chi_X,\chi_V})\chi_{CV}(\mathcal{C}).\end{aligned}$$

The last inequality together with (3.34) implies

$$\begin{aligned}\kappa_C(\mathcal{F}(\mathcal{A})) &= \chi_{CX}(\pi_1\mathcal{F}(\mathcal{A})) + \chi_{CV}(\pi_2\mathcal{F}(\mathcal{A}))\\ &\le 2k_0(1 + 2M_0M^*T\|B\|_{\chi_X,\chi_V})\chi_{CX}(D)\\ &\quad + 2q_0(1 + 2M_0M^*T\|B\|_{\chi_X,\chi_V})\chi_{CV}(\mathcal{C}).\end{aligned}$$

It means that $\kappa_C(\mathcal{F}(\mathcal{A})) \le \ell(\chi_{CX}(D)+\chi_{CV}(\mathcal{C})) = \ell\kappa_C(\mathcal{A})$. Taking into account (3.29) and the fact that $\ell < 1$, we obtain $\kappa_C(\mathcal{A}) = 0$ and then $\mathcal{A}$ is a relatively compact set, due to the regularity of κ_C. □

Remark 3.11. If $S(t)$ is compact for $t \in J$ and X is separable, one can drop hypothesis (F4). In fact, we use (F4) for estimates (3.33) and (3.38). Now with the assumptions of this remark, one gets estimate (3.33) directly due to Proposition 2.7:

$$\begin{aligned}&\chi_X(\mathcal{QL}S^1_F(\mathcal{A}))\\ &\quad= \chi_X\bigg(\int_0^T S(T-s)f(s)\,ds : f \in L^1(J;X),\ f(s) \in F(s, D(s),\ \mathcal{C}(s))\bigg)\\ &\quad\le \int_0^T \chi_X(S(T-s)f(s) : f \in L^1(J;X),\ f(s) \in F(s, D(s),\ \mathcal{C}(s))\,ds = 0\end{aligned}$$

due to the compactness of $S(T-s)$. Estimate (3.38) can be obtained by the same manner. Thus, in this case $q_0 = 0$, $k_0 = m_g + m_h$ in (3.28) and then condition (3.28) is reduced to

$$2k_0(1 + 2M_0M^*T\|B\|_{\chi_X,\chi_V}) < 1.$$

It should be noted that, the situation mentioned in this remark appears in numerous models of control problems involving PDEs, including parabolic equations. It follows that the exact controllability for these models cannot be obtained.

We are in the last step to prove the main theorem of this section. It is the controllability result, that is, we will demonstrate the (E_0, X_0)-controllability of the system (1.1)–(1.2).

Theorem 3.12. *Let* (SA), (F1)–(F4) *and* (GH1)–(GH3) *hold. Suppose that* (3.28) *and the following inequality*

$$\lim_{n\to\infty}\frac{1}{n}\bigg(N_0C_g\Psi_g(n)+M_0C_h\Psi_h(n)+M_0\Psi(n)\int_0^T\mu(s)\,ds\bigg)\\ <\frac{1}{1+M^*(1+M_0T\|B\|)}$$

hold. Then the nonlinear system (1.1)–(1.2) *is* (E_0,X_0)*-controllable.*

Proof. We will apply Theorem 2.5 to show that the solution multioperator $\mathcal{F}$ has a fixed point. We have proved in Lemma 3.9 and Theorem 3.10 that $\mathcal{F}$ is u.s.c. and κ_C-condensing. It remains to show that there exists $R>0$ such that $\mathcal{F}(B_R)\subset B_R$, where

$$B_R=\{(x,u)\in C(J;X)\times C(J;V):\|x\|_C+\|u\|_C\le R\}.$$

Assume to the contrary that for each $n\in\mathbb{N}$, there exists $(x_n,u_n)\in C(J;X)\times C(J;V)$ such that

$$\|x_n\|_C+\|u_n\|_C\le n, \tag{3.39}$$

$$\|y_n\|_C+\|z_n\|_C> n, \tag{3.40}$$

for some $(y_n,z_n)\in\mathcal{F}(x_n,u_n)$. Let $f_n\in S^1_F(x_n,u_n)$ be such that

$$\begin{aligned} y_n&=\mathcal{G}(x_n)+\mathcal{L}(Bz_n+f_n),\\ z_n&=B^*S^*(T-\,\cdot\,)(\Gamma_0^T)^{-1}[x_T-\mathcal{Q}\mathcal{G}(x_n)-\mathcal{Q}\mathcal{L}(f_n)].\end{aligned}$$

Estimating z_n, one has

$$\begin{aligned}\|z_n\|_C\le{}& M^*(\|x_T\|_X+\|C(T)[x_0-g(x_n)]\|_X)\\ &+M^*\bigg(\|S(T)[x_1-h(x_n)]\|_X+\bigg\|\int_0^T S(T-s)f_n(s)\,ds\bigg\|_X\bigg)\\ \le{}& M^*(\|x_T\|_X+\|C(T)\|[\|x_0\|+C_g\Psi_g(\|x_n\|_C)])\\ &+M^*(\|S(T)\|[\|x_1\|+C_h\Psi_h(\|x_n\|_C)])\\ &+M^*M_0\int_0^T\mu(s)\Psi(\|x_n(s)\|_X+\|u_n(s)\|_V)\,ds\\ \le{}& C^*+M^*(\|C(T)\|C_g\Psi_g(\|x_n\|_C)+\|S(T)\|C_h\Psi_h(\|x_n\|_C))\\ &+M^*M_0\Psi(\|x_n\|_C+\|u_n\|_C)\int_0^T\mu(s)\,ds,\end{aligned}\tag{3.41}$$

where $C^*=M^*(\|x_T\|_X+\|C(T)\|\|x_0\|+\|S(T)\|\|x_1\|_X)$. In the foregoing estimates we have used (GH2) and (F3).

Regarding y_n, we have

$$\begin{aligned}\|y_n(t)\|_X &\le \|C(t)[x_0 - g(x_n)] + S(t)[x_1 - h(x_n)]\|_X \\ &\quad + \bigg\| \int_0^t S(t-s)Bz_n(s)\,ds \bigg\|_X + \bigg\| \int_0^t S(t-s)f_n(s)\,ds \bigg\|_X \\ &\le N_0(\|x_0\|_X + C_g\Psi_g(\|x_n\|_C)) + M_0(\|x_1\|_X + C_h\Psi_h(\|x_n\|_C)) \\ &\quad + M_0T\|B\|\|z_n\|_C + M_0\int_0^t \mu(s)\Psi(\|x_n(s)\|_X + \|u_n(s)\|_V)\,ds\end{aligned}$$

where $N_0 = \sup_{t\in J} \|C(t)\|$, again due to (GH2) and (F3). Hence

$$\begin{aligned}\|y_n\|_C &\le C_0^* + N_0C_g\Psi_g(\|x_n\|_C) + M_0C_h\Psi_h(\|x_n\|_C) \\ &\quad + M_0T\|B\|\|z_n\|_C + M_0\Psi(\|x_n\|_C + \|u_n\|_C)\int_0^T \mu(s)ds,\end{aligned} \tag{3.42}$$

where $C_0^* = N_0\|x_0\|_X + M_0\|x_1\|_X$.

It is convenient to denote

$$\Lambda(r,\zeta) = N_0C_g\Psi_g(r) + M_0C_h\Psi_h(r) + M_0\Psi(r+\zeta)\int_0^T \mu(s)\,ds.$$

Then it follows from (3.41) and (3.42) that

$$\begin{aligned}\|z_n\|_C &\le C^* + M^*\Lambda(\|x_n\|_C, \|u_n\|_C), \\ \|y_n\|_C &\le C_0^* + M_0T\|B\|\|z_n\|_C + \Lambda(\|x_n\|_C, \|u_n\|_C).\end{aligned}$$

Therefore

$$\begin{aligned}\|y_n\|_C + \|z_n\|_C &\le C_0^* + (1 + M_0T\|B\|)\|z_n\|_C + \Lambda(\|x_n\|_C, \|u_n\|_C) \\ &\le C_0^* + (1 + M_0T\|B\|)C^* \\ &\quad + [1 + M^*(1 + M_0T\|B\|)]\Lambda(\|x_n\|_C, \|u_n\|_C).\end{aligned} \tag{3.43}$$

Taking into account (3.39)–(3.40), one deduces from (3.43) that

$$\begin{aligned}1 < \frac{1}{n}(\|y_n\|_C + \|z_n\|_C) &\le \frac{1}{n}(C_0^* + (1 + M_0T\|B\|)C^*) \\ &\quad + [1 + M^*(1 + M_0T\|B\|)]\,\frac{1}{n}\Lambda(\|x_n\|_C, \|u_n\|_C).\end{aligned}$$

Then

$$\begin{aligned}\frac{1}{1 + M^*(1 + M_0T\|B\|)} &\le \lim_{n\to\infty} \frac{1}{n}\Lambda(\|x_n\|_C, \|u_n\|_C) \\ &\le \lim_{n\to\infty} \frac{1}{n}\bigg(N_0C_g\Psi_g(n) + M_0C_h\Psi_h(n) + M_0\Psi(n)\int_0^T \mu(s)\,ds\bigg),\end{aligned}$$

which is the contradiction. □

4. Application

Consider the following control system

$$\frac{\partial^2 x(t,\theta)}{\partial t^2} = \frac{\partial^2 x(t,\theta)}{\partial \theta^2} + u(t,\theta) + f(t, x(t,\theta), u(t,\theta)), \quad t \in [0,T], \theta \in [0,\pi], \tag{4.1}$$

$$x(t,0) = x(t,\pi) = 0, \tag{4.2}$$

$$x(0,\theta) = x_0(\theta) - \sum_{k=1}^{m} \int_0^\theta g_k(\eta) x(t_k, \eta)\, d\eta, \quad t_k \in [0,T],\ k = 1, \ldots, m, \tag{4.3}$$

$$\frac{\partial}{\partial t} x(0,\theta) = x_1(\theta) - \sum_{k=1}^{m} \int_0^{t_k} \int_0^\pi h_k(\theta,\eta) x(s,\eta)\, d\eta\, ds, \tag{4.4}$$

where control $u \in L^2(0,T; L^2(0,\pi))$.

Let $X = L^2(0,\pi)$. Define $A: X \to X$ by $Ay = y''$ with the domain

$$D(A) = H^2(0,\pi) \cap H_0^1(0,\pi).$$

It is well known that A is the infinitesimal generator of a strongly continuous cosine family $\{C(t)\}_{t\in\mathbb{R}}$ on X. More precisely,

$$(C(t)y)(\theta) = \sum_{n=1}^{\infty} \frac{2}{\pi} \bigg(\int_0^\pi y(\eta) \sin n\eta\, d\eta \bigg) \cos nt \sin n\theta.$$

Here $\{\phi_n(\theta) = \sqrt{2/\pi} \sin n\theta : n = 1, 2, \ldots\}$ is the orthonormal basis of $L^2(0,\pi)$ and its elements are the eigenfunctions corresponding to the eigenvalues $\{\lambda_n = n^2 : n = 1, 2, \ldots\}$ of $-A$. The norm in $L^2(0,\pi)$ is defined as:

$$\|y\|^2 = \sum_{n=1}^{\infty} \frac{2}{\pi} \bigg(\int_0^\pi y(\theta) \sin n\theta\, d\theta \bigg)^2.$$

In addition, the associated sine family $\{S(t)\}_{t\in\mathbb{R}}$ is given by

$$(S(t)y)(\theta) = \sum_{n=1}^{\infty} \frac{2}{n\pi} \bigg(\int_0^\pi y(\eta) \sin n\eta\, d\eta \bigg) \sin nt \sin n\theta. \tag{4.5}$$

Noting that $-A$ is positive definite and self-adjoint, one can define the fractional operator $(-A)^\alpha$, $\alpha \in \mathbb{R}$ as follows:

$$(-A)^\alpha y(\theta) = \sum_{n=1}^{\infty} \lambda_n^\alpha \langle y, \phi_n \rangle_{L^2(0,\pi)} \phi_n = \sum_{n=1}^{\infty} n^{2\alpha} \frac{2}{\pi} \bigg(\int_0^\pi y(\eta) \sin n\eta\, d\eta \bigg) \sin n\theta.$$

This implies

$$\|(-A)^\alpha y\|^2 = \sum_{n=1}^{\infty} \frac{2n^{4\alpha}}{\pi} \bigg(\int_0^\pi y(\eta) \sin n\eta\, d\eta \bigg)^2.$$

On the other hand

$$\|y\|^2_{H^1_0(0,\pi)} = \langle -Ay, y\rangle_{L^2(0,\pi)} = \langle (-A)^{1/2}y, (-A)^{1/2}y\rangle_{L^2(0,\pi)} = \|(-A)^{1/2}y\|^2.$$

Then

$$\|y\|^2_{H^1_0(0,\pi)} = \sum_{n=1}^{\infty} \frac{2n^2}{\pi}\bigg(\int_0^{\pi} y(\eta)\sin n\eta\, d\eta\bigg)^2$$

and, moreover, we have $D((-A)^{1/2}) = H^1_0(0,\pi)$. Let H^{-1} be the dual space of $H^1_0(0,\pi)$. Then one can easily see that $H^{-1} = D((-A)^{-1/2})$ and the norm in H^{-1} is given by

$$\|y\|^2_{H^{-1}} = \sum_{n=1}^{\infty} \frac{2}{n^2\pi}\bigg(\int_0^{\pi} y(\eta)\sin n\eta\, d\eta\bigg)^2.$$

Notice that $S(t)$ is compact. Indeed, since the embedding $H^1_0(0,\pi) \subset L^2(0,\pi)$ is compact, it suffices to show that $S(t)D$ is bounded in $H^1_0(0,\pi)$ provided that D is bounded in $L^2(0,\pi)$. We deduce from (4.5) that

$$\|S(t)y\|^2_{H^1_0(0,\pi)} = \sum_{n=1}^{\infty} \frac{2}{\pi}\bigg(\int_0^{\pi} y(\eta)\sin n\eta\, d\eta\bigg)^2 \sin^2 nt \le \|y\|^2_{L^2(0,\pi)}.$$

Thus, we obtain that $S(t)D$ is bounded in $H^1_0(0,\pi)$.

Let $X_0 = H^1_0(0,\pi)$ and $E_0 = H^1_0(0,\pi)\times L^2(0,\pi)$. Then we can verify the (E_0, X_0)-controllability for the linear system

$$\begin{aligned}
\frac{\partial^2 x(t,\theta)}{\partial t^2} &= \frac{\partial^2 x(t,\theta)}{\partial \theta^2} + u(t,\theta), \quad t\in[0,T],\ \theta\in[0,\pi],\\
x(t,0) &= x(t,\pi) = 0,\\
x(0,\theta) &= x_0(\theta), \quad \frac{\partial}{\partial t}x(0,\theta) = x_1(\theta).
\end{aligned}$$

Indeed, we have

$$\begin{aligned}
\|B^*S^*(T-\,\cdot\,)y\|^2_{L^2(J;X)} = \|S^*(T-\cdot)y\|^2_{L^2(J;X)} &= \int_0^T \|S(T-s)y\|^2_X\, ds\\
&= \sum_{n=1}^{\infty}\frac{2}{n^2\pi}\bigg(\int_0^{\pi} y(\eta)\sin n\eta\, d\eta\bigg)^2\int_0^T \sin^2 n(T-s)\, ds\\
&= \sum_{n=1}^{\infty}\frac{(T-(1/2n)\sin 2nT)}{n^2\pi}\bigg(\int_0^{\pi} y(\eta)\sin n\eta\, d\eta\bigg)^2
\end{aligned}$$

due to the presentation of $S(t)$ in (4.5). Therefore, if $T > 1/2$, one can find a number $\gamma > 0$ such that

$$\|B^*S^*(T-s)y\|^2_{L^2(J;X)} \ge \gamma\|y\|^2_{H^{-1}}.$$

Now taking into account (4.5) again, one observes that $S(t)y \in H_0^1(0,\pi)$ for all $y \in L^2(0,\pi)$. Additionally, if $y \in H_0^1(0,\pi)$ then $C(t)y \in H_0^1(0,\pi)$ as well. Hence

$$\{C(t)x_0 + S(t)x_1 : (x_0,x_1) \in H_0^1(0,\pi) \times L^2(0,\pi)\} \subset H_0^1(0,\pi)$$

and condition (SA)(a) is satisfied. Now for $f \in L^1(0,T;L^2(0,\pi))$, we have

$$S(T-s)f(s,\,\cdot\,) \in H_0^1(0,\pi) \quad \text{for a.e. } s \in [0,T].$$

Thus $\int_0^T S(T-s)f(s,\,\cdot\,)\,ds \in H_0^1(0,\pi)$ and condition (SA)(b) is verified.

As far as the nonlinear system (4.1)–(4.4) is concerned, we assume that

- (N1) The nonlinearity $f\colon [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous. Furthermore, there exists a function $\mu \in L^1(0,T)$ such that $|f(t,\xi,\eta)| \le \mu(t)(|\xi| + |\eta|)$ for all $\xi,\eta \in \mathbb{R}$;
- (N2) For each $k = 1,\dots,m$, $g_k \in L^2(0,\pi)$ and $h_k \in L^2([0,\pi]^2)$.

It is easy to check that f satisfies (F1)–(F3) due to (N1). Since $S(t)$ is compact and X is separable, one can drop (F4) as it was mentioned in Remark 3.11.

Now setting

$$\begin{aligned} g(x)(\theta) &= \sum_{k=1}^m \int_0^\theta g_k(\eta)x(t_k,\eta)\,d\eta, \\ h(x)(\theta) &= \sum_{k=1}^m \int_0^{t_k}\int_0^\pi h_k(\theta,\eta)x(s,\eta)\,d\eta\,ds, \end{aligned}$$

we see that $g,h\colon C([0,T];L^2(0,\pi)) \to L^2(0,\pi)$ are Lipschitz functions. Indeed,

$$\begin{aligned} |g(x)(\theta) - g(y)(\theta)| &\le \sum_{k=1}^m \int_0^\pi |g_k(\theta)||x(t_k,\eta) - y(t_k,\eta)|\,d\eta \\ \le \sum_{k=1}^m \|g_k\|_{L^2(0,\pi)} &\|x(t_k,\,\cdot\,) - y(t_k,\,\cdot\,)\|_{L^2(0,\pi)} \le \bigg(\sum_{k=1}^m \|g_k\|_{L^2(0,\pi)}\bigg)\|x-y\|_C. \end{aligned}$$

Thus

$$\|g(x) - g(y)\|_{L^2(0,\pi)} \le \bigg(\sqrt{\pi}\sum_{k=1}^m \|g_k\|_{L^2(0,\pi)}\bigg)\|x-y\|_C. \tag{4.6}$$

For the nonlocal function h, we have

$$\begin{aligned} |h(x)(\theta) - h(y)(\theta)| &\le \sum_{k=1}^m \int_0^{t_k}\int_0^\pi |h_k(\theta,\eta)||x(s,\eta) - y(s,\eta)|\,d\eta\,ds \\ &\le \sum_{k=1}^m \int_0^{t_k} \|h_k(\theta,\,\cdot\,)\|_{L^2(0,\pi)}\|x(s,\,\cdot\,) - y(s,\,\cdot\,)\|_{L^2(0,\pi)}\,ds \\ &\le \bigg(T\sum_{k=1}^m \|h_k(\theta,\,\cdot\,)\|_{L^2(0,\pi)}\bigg)\|x-y\|_C. \end{aligned}$$

Then

$$\|h(x)-h(y)\|_{L^2(0,\pi)} \le \bigg(\sqrt{2}\,T\sum_{k=1}^{m}\|h_k\|_{L^2([0,\pi]^2)}\bigg)\|x-y\|_C. \tag{4.7}$$

Hence, by Remark 3.2, g and h satisfy (GH2)–(GH3). It is obvious that $g(x) \in H_0^1(0,\pi)$ for all $x \in L^2(0,\pi)$, thanks to the definition of g. Then $(g(x),h(x)) \in E_0 = H_0^1(0,\pi)\times L^2(0,\pi)$ for all $x \in L^2(0,\pi)$ and (GH1) is verified.

Let

$$C_g = \sqrt{\pi}\sum_{k=1}^{m}\|g_k\|_{L^2(0,\pi)}, \qquad C_h = \sqrt{2}\,T\sum_{k=1}^{m}\|h_k\|_{L^2([0,\pi]^2)},$$
$$M^* = \sup_{t\in[0,T]}\|S^*(T-t)(\Gamma_0^T)^{-1}\|_{\chi X},$$
$$m_g = N_0C_g, \quad m_h = M_0C_h, \quad k_0 = m_g + m_h.$$

We have the following controllability result for (4.1)–(4.4).

Theorem 4.1. *Assume* (N1)–(N2). *Let the following inequalities hold:*

$$2k_0(1+2M_0M^*T) < 1,$$
$$N_0C_g + M_0C_h + M_0\int_0^T \mu(s)\,ds < \frac{1}{1+M^*(1+M_0T)}.$$

Then the nonlinear system (4.1)–(4.4) *is* $(H_0^1\times L^2, H_0^1)$*-controllable.*

Acknowledgments. The authors are grateful to the referees for their valuable comments and suggestions that helped to improve the manuscript.

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Manuscript received March 23, 2012

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