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STUDY ON A QUADRATIC HADAMARD TYPE FRACTIONAL INTEGRAL EQUATION ON AN UNBOUNDED INTERVAL

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Abstract. In this paper, a quadratic Hadamard type fractional integral equations on an unbounded interval is studied. By applying a technique of measure of noncompactness and Schauder fixed point theorem, existence and uniform local attractivity of solutions are presented after overcoming some difficulty from the Hadamard type singular kernel. Moreover, three new solutions sets who tend to zero at infinity are constructed to obtain local stability of solutions. Finally, two examples are made to illustrate our theory results.

1. Introduction

Fractional integral and differential equations play increasingly important roles in the modeling of real world problems. Some problems in physics, mechanics and other fields can be described with the help of all kinds of fractional

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differential and integral equations. For more recent development on Riemann-Liouville, Caputo and Hadamard fractional calculus, the reader can refer to the monographs of Baleanu et al. [4], Kilbas et al. [15], Lakshmikantham et al. [17], Miller and Ross [21], Podlubny [22] and Tarasov [23] and the works on fractional integral and differential equations [1]–[3], [7]–[11], [13], [14], [16], [18], [19], [26]–[29].

Recently, Banás and O'Regan [6] study the existence and local attractivity of solutions of a quadratic Riemann–Liouville type fractional integral equation in the space of real functions defined, continuous and bounded on an unbounded interval. Further, Wang et al. [24], [25] study the existence, local attractivity and stability of solutions of Urysohn type and Erdélyi–Kober type fractional integral equations respectively.

It is worthwhile mentioning that up to now Hadamard type fractional integral equations in the space of real functions defined on a unbounded interval have not been studied. Motivated by [6], [24], [25], we will extend to study existence, uniform local attractivity and local stability of solutions for quadratic Hadamard type fractional integral equations of the form

$$x(t) = p(t) + f(t, x(t))({}_H D^{-\alpha}_{t_0,t} u(t, s, x(s))), \tag{1.1}$$

where $t \in [t_0, +\infty)$, $t_0 > 0$, $\alpha \in (0, 1)$, the symbol ${}_H D^{-\alpha}_{t_0,t} u$ denotes the Hadamard fractional integral on the continuous function u of the order α, which is defined by

$$ {}_H D^{-\alpha}_{t_0,t} u(t, s, x(t)) := \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \left(\ln \frac{t}{s}\right)^{\alpha-1} u(t, s, x(s)) \frac{ds}{s}, \tag{1.2}$$

the term $(\ln(t/s))^{\alpha-1}$ is so-called the Hadamard type singular kernel and $\Gamma(\,\cdot\,)$ is the Euler Gamma function.

Firstly, by applying the same technique of measure of noncompactness in the space $\mathrm{BC}([t_0, \infty))$ and Schauder fixed point theorem which appeared in [6], we obtain the existence and uniform local attractivity results of the solutions for the equation (1.1) after overcoming the main difficulty from the Hadamard type singular kernel $(\ln(t/s))^{\alpha-1}$, which is different from the Riemann–Liouville type singular kernel $(t - s)^{\alpha-1}$.

Secondly, local stability of the solutions of a special case the equation (1.1) with $t_0 = 1$ are studied by adopting some ideas in [11], [24], [25]. To achieve our aim, we try to construct three certain characters solutions sets: $X_{L,\gamma} = \{x : x \in \mathrm{BC}([1, \infty))$ and $|x(t)| \le L(\ln t)^{-\gamma}$ for $L > 0, t > 1\}$, $X_{1,\alpha} = \{x : x \in BC([1, \infty))$ and $|x(t)| \le (\ln t)^{-\alpha}$ for $t > 1\}$ and $X_{1,(1-(\alpha+\nu))} = \{x : x \in BC([1, \infty))$ and $|x(t)| \le (\ln t)^{-(1-(\alpha+\nu))}$ for $t > 1\}$. The constructing techniques considered here may be a stimulant for further investigations concerning local stability of solutions of other types nonlinear fractional integral equations.

2. Preliminaries

In this section we collect some definitions and results which will be needed later.

First we present some facts concerning measures of noncompactness (see Banás and Goebel [5]).

Let $(E, \|\cdot\|)$ be a Banach space. Denote by $B(x,r)$ the closed ball centered at x and with radius r. The symbol B_r stands for the ball $B(\theta, r)$ where θ is the zero element. If X is a subset of X we write $\overline{X}$, $\mathrm{Conv}X$ in order to denote the closure and convex closure of X, respectively. Moreover, we denote by $\mathfrak{M}_E$ the family of all nonempty and bounded subsets of E and by $\mathfrak{N}_E$ its subfamily consisting of all relatively compact sets.

We collect the following definition of the concept of a measure of noncompactness [5].

Definition 2.1. A mapping $\mu\colon \mathfrak{M}_E \to \mathbb{R}_+ = [0,\infty)$ is said to be a measure of noncompactness in E if it satisfies the following conditions:

(a) The family $\ker\mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $\ker\mu \subset \mathfrak{N}_E$.
(b) If $X \subset Y$ then $\mu(X) \le \mu(Y)$.
(c) $\mu(\overline{X}) = \mu(X)$.
(d) $\mu(\mathrm{Conv}\, X) = \mu(X)$.
(f) $\mu(\lambda X + (1-\lambda)Y) \le \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0,1]$.
(g) If (X_n) is a sequence of closed sets from $\mathfrak{M}_E$ such that $X_{n+1} \subset X_n$ (for $n = 1, 2, \ldots$) and if $\lim\limits_{n\to\infty} \mu(X_n) = 0$,then the intersection $X_\infty = \bigcap\limits_{n=1}^{\infty} X_n$ is nonempty.

The family $\ker\mu$ described in (a) is said to be the kernel of the measure of noncompactness μ. Let us observe that the intersection set X_∞ from (f) belongs to $\ker\mu$. In fact, since $\mu(X_\infty) \le \mu(X_n)$ for every n then we have that $\mu(X_\infty) = 0$. This simple observation will be essential later.

We introduce the space $\mathrm{BC}([t_0,\infty))$ which is consisting of all real functions defined, continuous and bounded on $[t_0,\infty)$. Endowed with the norm $\|x\| = \sup\{|x(t)| : t \ge t_0\}$, it is easy to see $(\mathrm{BC}([t_0,\infty)), \|\cdot\|)$ is a Banach space.

We will use a same measure of noncompactness in the space $\mathrm{BC}([t_0,\infty))$ [6]. We need the following preparations which are taken from [6].

Take a nonempty bounded subset $X \subset \mathrm{BC}([t_0,\infty))$ and a $T > 0$. For $x \in X$ and $\varepsilon \ge 0$, denote

$$\omega^T(x,\varepsilon) = \sup\{|x(t) - x(s)| : t, s \in [t_0, T],\ |t-s| \le \varepsilon\}.$$

Set $\omega^T(X,\varepsilon) = \sup\{\omega^T(x,\varepsilon) : x \in X\}$, $\omega_0^T(X) = \lim\limits_{\varepsilon\to 0} \omega^T(X,\varepsilon)$, $\omega_0(X) = \lim\limits_{T\to\infty} \omega_0^T(X)$. For $t \in [t_0,\infty)$, denote $X(t) = \{x(t) : x \in X\}$ and $\mathrm{diam}\, X(t) =$

$\sup\{|x(t)-y(t)| : x, y \in X\}$. Now, consider the function μ defined on the family $\mathfrak{M}_{\mathrm{BC}([t_0,\infty))}$ by the formula

$$\mu(X) = \omega_0(X) + \lim_{t\to\infty} \sup \operatorname{diam} X(t). \tag{2.1}$$

Then the function μ is a measure of noncompactness in the space $\mathrm{BC}([t_0,\infty))$ [6]. The kernel $\ker\mu$ of this measure consists of nonempty and bounded sets X such that functions from X are locally equicontinuous on $[t_0,\infty)$ and the thickness of the bundle formed by functions from X tends to zero at infinity. This property can help us to characterize solutions of the equation (1.1) and other kinds of integral equations.

In order to introduce some basic concepts, we choose a $\Omega \subset \mathrm{BC}([t_0,\infty))$, $\Omega \neq \emptyset$, define $Q\colon \Omega \to \mathrm{BC}([t_0,\infty))$ and consider the following operator equation:

$$x(t) = (Qx)(t), \quad t \geq t_0. \tag{2.2}$$

The following locally attractive concept for the solutions of the above operator equation (2.2) is introduced in [6].

Definition 2.2. A solution of the equation (2.2) is said to be locally attractive if there exists a ball $B(x_0, r)$ in the space $\mathrm{BC}([t_0,\infty))$ such that for arbitrary solutions $x, y \in B(x_0, r) \cap \Omega$ of the equation (2.2) satisfying

$$\lim_{t\to\infty} (x(t) - y(t)) = 0. \tag{2.3}$$

In the case when the limit (2.3) is uniform with respect to the set $B(x_0, r) \cap \Omega$, i.e. for each $\varepsilon > 0$ there exists $T > t_0$ such that

$$|x(t) - y(t)| \leq \varepsilon \tag{2.4}$$

for all $x, y \in B(x_0, r) \cap \Omega$ and for $t \geq T$, we will say that solutions of the equation (2.2) are uniformly locally attractive.

Definition 2.4. A solution of the equation (2.2) is said to be locally stable if $\lim_{t\to\infty} |x(t)| = 0$.

The following two inequalities are useful in the sequel. For more details, one can see Michalski [20].

Lemma 2.5. *For all $\beta > 0$ and $\vartheta > -1$,*

$$\int_0^t (t-s)^{\beta-1} s^{\vartheta}\, ds = \frac{\Gamma(\beta)\Gamma(\vartheta+1)}{\Gamma(\beta+\vartheta+1)}\, t^{\beta+\vartheta}.$$

Lemma 2.6. *For all $\lambda, v, w > 0$, then for any $t > 0$, we have*

$$\int_0^t (t-s)^{v-1} s^{\lambda-1} e^{-ws}\, ds \leq \max\{1, 2^{1-v}\}\Gamma(\lambda)\left(1 + \frac{\lambda}{v}\right) w^{-\lambda} t^{-(1-v)}.$$

3. Existence and uniform local attractivity of solutions

In this section, we will study the existence and uniform local attractivity of the solutions of the equation (1.1).

We introduce the following assumptions:

(H1) $p\colon [t_0,\infty) \to \mathbb{R}$ is continuous and bounded function on $[t_0,\infty)$.

(H2) $f\colon [t_0,\infty)\times\mathbb{R} \to \mathbb{R}$ is continuous and there exists a function $m\colon [t_0,\infty) \to \mathbb{R}_+$ being continuous on $[t_0,\infty)$ and such that

$$|f(t,x) - f(t,y)| \leq m(t)|x-y| \quad \text{for all } x,y \in \mathbb{R}.$$

(H3) $u\colon [t_0,\infty) \times [t_0,\infty) \times \mathbb{R} \to \mathbb{R}$ is continuous. Moreover, there exist a function $n\colon [t_0,\infty) \to \mathbb{R}_+$ being continuous on $[t_0,\infty)$ and a function $\Phi\colon \mathbb{R}_+ \to \mathbb{R}_+$ being continuous and nondecreasing on $\mathbb{R}_+$ with $\Phi(0) = 0$ and such that

$$|u(t,s,x) - u(t,s,y)| \leq n(t)\Phi(|x-y|) \quad \text{for all } x,y \in \mathbb{R}.$$

Define $u_1\colon [t_0,\infty) \to \mathbb{R}_+$ where $u_1 = \max\{|u(t,s,0)| : t_0 \leq s \leq t\}$. Clearly, u_1 is continuous on $[t_0,\infty)$.

We need the following additional conditions.

(H4) For some $0 < \alpha < 1$, the functions $a, b, c, d\colon [t_0,\infty) \to \mathbb{R}_+$ defined by

$$\begin{aligned} a(t) &= m(t)n(t)(\ln t - \ln t_0)^\alpha, & b(t) &= m(t)u_1(t)(\ln t - \ln t_0)^\alpha, \\ c(t) &= n(t)|f(t,0)|(\ln t - \ln t_0)^\alpha, & d(t) &= u_1(t)|f(t,0)|(\ln t - \ln t_0)^\alpha, \end{aligned}$$

are bounded on $[t_0,+\infty)$ and $a(\cdot)$, $c(\cdot)$ satisfying

$$\lim_{t\to\infty} a(t) = \lim_{t\to\infty} c(t) = 0.$$

For brevity, define

$$\begin{aligned} A &= \sup\{a(t) : t \in [t_0,\infty)\}, & B &= \sup\{b(t) : t \in [t_0,\infty)\}, \\ C &= \sup\{c(t) : t \in [t_0,\infty)\}, & D &= \sup\{d(t) : t \in [t_0,\infty)\}. \end{aligned}$$

(H5) There exists a number $r_0 > 0$ satisfying the following inequality

$$\|p\| + \frac{Ar\Phi(r) + Br + C\Phi(r) + D}{\Gamma(\alpha+1)} \leq r.$$

And, the inequality $A\Phi(r_0) + B < \Gamma(\alpha+1)$ also holds.

In order to use the technique of fixed point theorem, we introduce the operator $V\colon \mathrm{BC}([t_0,\infty)) \to \mathrm{BC}([t_0,\infty))$ defined by

$$(Vx)(t) = p(t) + (Fx)(t)(Ux)(t),$$

where $F,U\colon \mathrm{BC}([t_0,\infty))\to \mathrm{BC}([t_0,\infty))$ defined by

$$(Fx)(t)=f(t,x(t)),\qquad (Ux)(t)=\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\frac{u(t,s,x(s))}{(\ln t-\ln s)^{1-\alpha}}\,\frac{ds}{s}.$$

LEMMA 3.1. *Let the assumptions* (H1)–(H5) *be satisfied. Then we have*:

(a) *the operator* $V\colon B_{r_0}\subset \mathrm{BC}([t_0,\infty))\to B_{r_0}$, *where* r_0 *satisfying the assumption* (H5),

(b) *the fixed points of the operator* V *is just the solutions of the equation* (1.1).

PROOF. (a) We firstly verify that V is continuous operator.

To achieve our aim, we only need to verify that F, U are continuous operators. In fact, for any function $x\in \mathrm{BC}([t_0,\infty))$, it is clear that the function Fx is continuous on $[t_0,\infty)$. We only need to show that the function Ux is continuous on $[t_0,\infty)$.

For an arbitrary $x\in \mathrm{BC}([t_0,\infty))$ and fix $T>t_0$ and $\varepsilon>0$. Without loss of generality we can assume that $t_0\le t_1<t_2\le T$ with $|t_2-t_1|\le\varepsilon$. After some standard computation, we obtain

$$\begin{aligned}
|(Ux)(t_2)-(Ux)(t_1)| &= \frac{1}{\Gamma(\alpha)}\bigg|\int_{t_0}^{t_1}\frac{u(t_2,s,x(s))}{(\ln t_2-\ln s)^{1-\alpha}}\,\frac{ds}{s}\\
&\quad+\int_{t_1}^{t_2}\frac{u(t_2,s,x(s))}{(\ln t_2-\ln s)^{1-\alpha}}\,\frac{ds}{s}-\int_{t_0}^{t_1}\frac{u(t_1,s,x(s))}{(\ln t_1-\ln s)^{1-\alpha}}\,\frac{ds}{s}\bigg|\\
&\le \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_1}\bigg|\frac{u(t_2,s,x(s))}{(\ln t_2-\ln s)^{1-\alpha}}-\frac{u(t_1,s,x(s))}{(\ln t_2-\ln s)^{1-\alpha}}\bigg|\,\frac{ds}{s}\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_1}\bigg|\frac{u(t_1,s,x(s))}{(\ln t_2-\ln s)^{1-\alpha}}-\frac{u(t_1,s,x(s))}{(\ln t_1-\ln s)^{1-\alpha}}\bigg|\,\frac{ds}{s}\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\frac{|u(t_2,s,x(s))|}{(\ln t_2-\ln s)^{1-\alpha}}\,\frac{ds}{s}\\
&= \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_1}\frac{|u(t_2,s,x(s))-u(t_1,s,x(s))|}{(\ln t_2-\ln s)^{1-\alpha}}\,\frac{ds}{s}\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_1}|u(t_1,s,x(s))|\bigg[\frac{1}{(\ln t_1-\ln s)^{1-\alpha}}-\frac{1}{(\ln t_2-\ln s)^{1-\alpha}}\bigg]\frac{ds}{s}\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\frac{|u(t_2,s,x(s))|}{(\ln t_2-\ln s)^{1-\alpha}}\,\frac{ds}{s}\\
&\le \frac{\omega_1^T(u,\varepsilon;\|x\|)}{\Gamma(\alpha)}\int_{t_0}^{t_1}\frac{1}{(\ln t_2-\ln s)^{1-\alpha}}\,\frac{ds}{s}\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_1}[|u(t_1,s,x(s))-u(t_1,s,0)|+|u(t_1,s,0)|]\\
&\quad\times\bigg[\frac{1}{(\ln t_1-\ln s)^{1-\alpha}}-\frac{1}{(\ln t_2-\ln s)^{1-\alpha}}\bigg]\frac{ds}{s}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\frac{|u(t_2,s,x(s))-u(t_2,s,0)|+|u(t_2,s,0)|}{(\ln t_2-\ln s)^{1-\alpha}}\frac{ds}{s}\\
\le{}&\frac{\omega_1^T(u,\varepsilon;\|x\|)}{\Gamma(\alpha)}\frac{(\ln t_2-\ln t_0)^\alpha-(\ln t_2-\ln t_1)^\alpha}{\alpha}\\
&+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_1}[n(t_1)\Phi(|x(s)|)+u_1(t_1)]\\
&\times\left[\frac{1}{(\ln t_1-\ln s)^{1-\alpha}}-\frac{1}{(\ln t_2-\ln s)^{1-\alpha}}\right]\frac{ds}{s}\\
&+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\frac{n(t_2)\Phi(|x(s)|)+u_1(t_2)}{(\ln t_2-\ln s)^{1-\alpha}}\frac{ds}{s}\\
\le{}&\frac{\omega_1^T(u,\varepsilon;\|x\|)}{\Gamma(\alpha+1)}(\ln t_1-\ln t_0)^\alpha\\
&+\frac{n(t_1)\Phi(\|x\|)+u_1(t_1)}{\Gamma(\alpha+1)}[(\ln t_2-\ln t_1)^\alpha+(\ln t_1-\ln t_2)^\alpha]\\
&+\frac{n(t_2)\Phi(\|x\|)+u_1(t_2)}{\Gamma(\alpha+1)}(\ln t_2-\ln t_1)^\alpha\\
\le{}&\frac{1}{\Gamma(\alpha+1)}\{(\ln t_1-\ln t_0)^\alpha\omega_1^T(u,\varepsilon;\|x\|)\\
&+(\ln t_2-\ln t_1)^\alpha[n(t_1)\Phi(\|x\|)+u_1(t_1)]\\
&+(\ln t_2-\ln t_1)^\alpha[n(t_2)\Phi(\|x\|)+u_1(t_2)]\},
\end{aligned}
$$

where $\omega_1^T(u,\varepsilon;\|x\|)=\sup\{|u(t_2,s,y)-u(t_1,s,y)| : s,t_1,t_2\in[t_0,T],\ s\le t_1,\ s\le t_2,\ |t_2-t_1|\le\varepsilon,\ |y|\le\|x\|\}$.

Obviously, we have that $\omega_1^T(u,\varepsilon;\|x\|)\to 0$ as $\varepsilon\to 0$ due to the uniform continuity of the function $u(t,s,y)$ on the set $[t_0,T]\times[t_0,T]\times[-\|x\|,\|x\|]$.

Denote $\overline{n}(T)=\max\{n(t):[t_0,T]\}$, $\overline{u_1}(T)=\max\{u_1(t):[t_0,T]\}$. Then, we have

$$
\begin{aligned}
\omega^T(Ux,\varepsilon)\le\frac{1}{\Gamma(\alpha+1)}\{&(\ln T-\ln t_0)^\alpha\omega_1^T(u,\varepsilon;\|x\|)\\
&+2\ln(1+\varepsilon)^\alpha[\overline{n}(T)\Phi(\|x\|)+\overline{u_1}(T)]\},
\end{aligned}
$$

which implies the function Ux is continuous on the interval $[t_0,T]$ for any $T>t_0$. Further, we can obtain the continuity of Ux on $[t_0,\infty)$. As a result, the continuity of Vx on $[t_0,\infty)$ can be derived.

Secondly, we show that the function Vx is bounded on $[t_0,\infty)$.

For an arbitrary $x\in \mathrm{BC}([t_0,\infty))$ and a fixed $t\in[t_0,+\infty)$ we have

$$
\begin{aligned}
|(Vx)(t)|\le|p(t)|+&\frac{1}{\Gamma(\alpha)}[|f(t,x(t))-f(t,0)|+f(t,0)]\\
&\times\int_{t_0}^{t}\frac{|u(t,s,x(s))-u(t,s,0)|+|u(t,s,0)|}{(\ln t-\ln s)^{1-\alpha}}\frac{ds}{s}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \|p\| + \frac{m(t)|x(t)| + |f(t,0)|}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{n(t)\Phi(|x(s)|) + u_1(t)}{(\ln t - \ln s)^{1-\alpha}} \frac{ds}{s} \\
&\leq \|p\| + \frac{m(t)\|x\| + |f(t,0)|}{\Gamma(\alpha)} [n(t)\Phi(\|x\|) + u_1(t)] \int_{t_0}^{t} \frac{1}{(\ln t - \ln s)^{1-\alpha}} \frac{ds}{s} \\
&\leq \|p\| + \frac{1}{\Gamma(\alpha+1)} [m(t)n(t)(\ln t - \ln t_0)^{\alpha} \|x\|\Phi(\|x\|) \\
&\quad + m(t)u_1(u)(\ln t - \ln t_0)^{\alpha}\|x\| \\
&\quad + n(t)|f(t,0)|(\ln t - \ln t_0)^{\alpha}\Phi(\|x\|) + |f(t,0)|u_1(u)(\ln t - \ln t_0)^{\alpha}] \\
&= \|p\| + \frac{1}{\Gamma(\alpha+1)} [a(t)\|x\|\Phi(\|x\|) + b(t)\|x\| + c(t)\Phi(\|x\|) + d(t)].
\end{aligned}
$$

By assumption (H4), we can show the function Vx is bounded on $[t_0,\infty)$. Combining with the continuity of Vx on $[t_0,\infty)$, we can obtian $Vx \in \mathrm{BC}([t_0,\infty))$. Moreover, we can obtain the following inequality

$$
\|Vx\| \leq \|p\| + \frac{1}{\Gamma(\alpha+1)} [A\|x\|\Phi(\|x\|) + B\|x\| + C\Phi(\|x\|) + D].
$$

Combining the above estimate with assumption (H5) we deduce that there exists $r_0 > 0$ such that the operator $V : B_{r_0} \to B_{r_0}$.

(b) It comes from the fact $V\colon B_{r_0} \to B_{r_0}$ we have the second assertion immediately. □

Now, we are ready to state and prove one of our main results in this paper.

Theorem 3.2. *Let the assumptions* (H1)–(H5) *be satisfied. Then we have*:

(a) *the equation* (1.1) *has at least one solution* $x \in \mathrm{BC}([t_0,\infty))$,
(b) *the solutions of the equation* (1.1) *are uniformly locally attractive.*

Proof. (a) Take $X \in B_{r_0}$ and $X \neq \emptyset$, where B_{r_0} is just described the ball in Lemma 3.1. Then, for $x, y \in X$ and for an arbitrarily fixed $t \in [t_0,\infty)$, using our assumptions (H2)–(H4) we obtain

$$
\begin{aligned}
|(Vx)(t) &- (Vy)(t)| \\
&= \bigg| \frac{f(t,x(t))}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{u(t,s,x(s))}{(\ln t - \ln s)^{1-\alpha}} \frac{ds}{s} - \frac{f(t,y(t))}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{u(t,s,y(s))}{(\ln t - \ln s)^{1-\alpha}} \frac{ds}{s} \bigg| \\
&\leq \frac{1}{\Gamma(\alpha)} |f(t,x(t)) - f(t,y(t))| \int_{t_0}^{t} \frac{|u(t,s,x(s))|}{(\ln t - \ln s)^{1-\alpha}} \frac{ds}{s} \\
&\quad + \frac{|f(t,y(t))|}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{|u(t,s,x(s)) - u(t,s,y(s))|}{(\ln t - \ln s)^{1-\alpha}} \frac{ds}{s} \\
&\leq \frac{1}{\Gamma(\alpha)} m(t)|x(t) - y(t)| \int_{t_0}^{t} \frac{|u(t,s,x(s)) - u(t,s,0)| + |u(t,s,0)|}{(\ln t - \ln s)^{1-\alpha}} \frac{ds}{s} \\
&\quad + \frac{1}{\Gamma(\alpha)} [|f(t,y(t)) - f(t,0)| + |f(t,0)|] \int_{t_0}^{t} \frac{n(t)\Phi(|x(s) - y(s)|)}{(\ln t - \ln s)^{1-\alpha}} \frac{ds}{s}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{m(t)|x(t)-y(t)|}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{n(t)\Phi(|x(s)|)+u_1(t)}{(\ln t-\ln s)^{1-\alpha}} \frac{ds}{s} \\
&\quad + \frac{[m(t)|y(t)|+|f(t,0)|]n(t)}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{\Phi(|x(s)-y(s)|)}{(\ln t-\ln s)^{1-\alpha}} \frac{ds}{s} \\
&\leq \frac{m(t)n(t)(|x(t)|+|y(t)|)}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{\Phi(|x(s)|)}{(\ln t-\ln s)^{1-\alpha}} \frac{ds}{s} \\
&\quad + \frac{m(t)u_1(t)}{\Gamma(\alpha)} |x(t)-y(t)| \int_{t_0}^{t} \frac{1}{(\ln t-\ln s)^{1-\alpha}} \frac{ds}{s} \\
&\quad + \frac{m(t)n(t)|y(t)|}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{\Phi(|x(s)|+|y(s)|)}{(\ln t-\ln s)^{1-\alpha}} \frac{ds}{s} \\
&\quad + \frac{n(t)|f(t,0)|}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{\Phi(|x(s)|+|y(s)|)}{(\ln t-\ln s)^{1-\alpha}} \frac{ds}{s} \\
&\leq \frac{2m(t)n(t)r_0\Phi(r_0)}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{1}{(\ln t-\ln s)^{1-\alpha}} \frac{ds}{s} \\
&\quad + \frac{m(t)u_1(t)}{\Gamma(\alpha)} \operatorname{diam} X(t) \int_{t_0}^{t} \frac{1}{(\ln t-\ln s)^{1-\alpha}} \frac{ds}{s} \\
&\quad + \frac{m(t)n(t)r_0\Phi(2r_0)}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{1}{(\ln t-\ln s)^{1-\alpha}} \frac{ds}{s} \\
&\quad + \frac{n(t)|f(t,0)|\Phi(2r_0)}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{1}{(\ln t-\ln s)^{1-\alpha}} \frac{ds}{s} \\
&= \frac{2a(t)}{\Gamma(\alpha+1)} r_0\Phi(r_0) + \frac{a(t)}{\Gamma(\alpha+1)} r_0\,\Phi(2r_0) \\
&\quad + \frac{c(t)}{\Gamma(\alpha+1)} \Phi(2r_0) + \frac{b(t)}{\Gamma(\alpha+1)} \operatorname{diam} X(t).
\end{aligned}
$$

From the above estimate we derive the following inequality:

$$
\operatorname{diam}(Vx)(t) \leq \frac{2a(t)r_0\Phi(r_0)}{\Gamma(\alpha+1)} + \frac{a(t)r_0\Phi(2r_0)}{\Gamma(\alpha+1)} + \frac{c(t)\Phi(2r_0)}{\Gamma(\alpha+1)} + \frac{b(t)\operatorname{diam} X(t)}{\Gamma(\alpha+1)}.
$$

Hence, by assumption (H4) we get

$$
\lim_{t\to\infty} \sup \operatorname{diam}(Vx)(t) \leq k \lim_{t\to\infty} \sup \operatorname{diam} X(t), \tag{3.1}
$$

where $k = (A\Phi(r_0)+B)/(\Gamma(\alpha+1)) < 1$ due to assumption (H5).

Further, take $T > t_0$ and $\varepsilon > 0$. For an arbitrary $x \in X$ and $t_1, t_2 \in [t_0, T]$ with $|t_2 - t_1| \leq \varepsilon$. Without loss of generality we assume that $t_1 < t_2$. Then, using our assumptions and the previously obtained estimate we get

$$
\begin{aligned}
|(Vx)(t_2)-(Vx)(t_1)| &\leq |p(t_2)-p(t_1)| + |(Fx)(t_2)(Ux)(t_2)-(Fx)(t_1)(Ux)(t_2)| \\
&\quad + |(Fx)(t_1)(Ux)(t_2)-(Fx)(t_1)(Ux)(t_1)| \\
&\leq \omega^T(p,\varepsilon) + |f(t_2,x(t_2))-f(t_1,x(t_1))| \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_2} \frac{|u(t_2,s,x(s))|}{(\ln t_2-\ln s)^{1-\alpha}} \frac{ds}{s}
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{|f(t_1,x(t_1))|}{\Gamma(\alpha+1)}\{(\ln T-\ln t_0)^\alpha\omega_1^T(u,\varepsilon;r_0)\\
&+ 2\ln(1+\varepsilon)^\alpha[\overline{n}(T)\Phi(r_0)+\overline{u_1}(T)]\}\\
\le{}& \omega^T(p,\varepsilon)+\frac{|f(t_2,x(t_2))-f(t_2,x(t_1))|+|f(t_2,x(t_1))-f(t_1,x(t_1))|}{\Gamma(\alpha)}\\
&\times\int_{t_0}^{t_2}\frac{|u(t_2,s,x(s))-u(t_2,s,0)|+|u(t_2,s,0)|}{(\ln t_2-\ln s)^{1-\alpha}}\frac{ds}{s}\\
&+\frac{|f(t_1,x(t_1))-f(t_1,0))|+|f(t_1,0))|}{\Gamma(\alpha+1)}\\
&\times\{(\ln T-\ln t_0)^\alpha\omega_1^T(u,\varepsilon;r_0)+2\ln(1+\varepsilon)^\alpha[\overline{n}(T)\Phi(r_0)+\overline{u_1}(T)]\}\\
\le{}& \omega^T(p,\varepsilon)+\frac{m(t_2)|x(t_2)-x(t_1)|+\omega_1^T(f,\varepsilon)}{\Gamma(\alpha)}\int_{t_0}^{t_2}\frac{n(t_2)\Phi(|x(s)|)+u_1(t_2)}{(\ln t_2-\ln s)^{1-\alpha}}\frac{ds}{s}\\
&+\frac{m(t_1)|x(t_1)|+|f(t_1,0))|}{\Gamma(\alpha+1)}\\
&\times\{(\ln T-\ln t_0)^\alpha\omega_1^T(u,\varepsilon;r_0)+2\ln(1+\varepsilon)^\alpha[\overline{n}(T)\Phi(r_0)+\overline{u_1}(T)]\}\\
\le{}& \omega^T(p,\varepsilon)+\frac{[m(t_2)\omega^T(x,\varepsilon)+\omega_1^T(f,\varepsilon)](\ln t_2-\ln t_0)^\alpha[n(t_2)\Phi(r_0)+u_1(t_2)]}{\Gamma(\alpha+1)}\\
&+\frac{\overline{m}(T)r_0+\overline{f}(T)}{\Gamma(\alpha+1)}\{(\ln T-\ln t_0)^\alpha\omega_1^T(u,\varepsilon;r_0)\\
&+2\ln(1+\varepsilon)^\alpha[\overline{n}(T)\Phi(r_0)+\overline{u_1}(T)]\}\\
\le{}& \omega^T(p,\varepsilon)+\frac{1}{\Gamma(\alpha+1)}[m(t_2)n(t_2)(\ln t_2-\ln t_0)^\alpha\Phi(r_0)\\
&+m(t_2)u_1(t_2)(\ln t_2-\ln t_0)^\alpha]\omega^T(x,\varepsilon)\\
&+\frac{\omega_1^T(f,\varepsilon)\,(\ln T-\ln t_0)^\alpha[\overline{n}(T)\Phi(r_0)+\overline{u_1}(T)]}{\Gamma(\alpha+1)}\\
&+\frac{\overline{m}(T)r_0+\overline{f}(T)}{\Gamma(\alpha+1)}\{(\ln T-\ln t_0)^\alpha\omega_1^T(u,\varepsilon;r_0)\\
&+2\ln(1+\varepsilon)^\alpha[\overline{n}(T)\Phi(r_0)+\overline{u_1}(T)]\}\\
\le{}& \omega^T(p,\varepsilon)+\frac{A\Phi(r_0)+B}{\Gamma(\alpha+1)}\omega^T(x,\varepsilon)\\
&+\frac{\omega_1^T(f,\varepsilon)\,(\ln T-\ln t_0)^\alpha[\overline{n}(T)\Phi(r_0)+\overline{u_1}(T)]}{\Gamma(\alpha+1)}\\
&+\frac{\overline{m}(T)r_0+\overline{f}(T)}{\Gamma(\alpha+1)}\{(\ln T-\ln t_0)^\alpha\omega_1^T(u,\varepsilon;r_0)\\
&+2\ln(1+\varepsilon)^\alpha[\overline{n}(T)\Phi(r_0)+\overline{u_1}(T)]\},
\end{aligned}
$$

where

$$
\begin{aligned}
\omega_1^T(f,\varepsilon)&=\sup\{|f(t_2,x)-f(t_1,x)|:t_1,t_2\in[t_0,T],|t_2-t_1|\le\varepsilon,x\in[-r_0,r_0]\},\\
\overline{m}(T)&=\max\{m(t):t\in[t_0,T]\},\qquad \overline{f}(T)=\max\{|f(t,0)|:t\in[t_0,T]\}.
\end{aligned}
$$

Now, noting the uniform continuity of the function $u = u(t, s, x)$ on $[t_0, T] \times [t_0, T] \times [-r_0, r_0]$ and the uniform continuity of the function $f = f(t, x)$ on $[t_0, T] \times [-r_0, r_0]$, from the estimate above we obtain

$$\omega_0^T(Vx) \le k\omega_0^T(X).$$

Further, we obtain

$$\omega_0(Vx) \le k\omega_0(X). \tag{3.2}$$

Combining (3.1) and (3.2) and noting the definition of the measure of noncompactness μ given by the formula (2.1), we get

$$\mu(Vx) \le k\mu(X). \tag{3.3}$$

Now, put $B_{r_0}^1 = \operatorname{Conv} V(B_{r_0})$, $B_{r_0}^2 = \operatorname{Conv} V(B_r^1)$ and so on. Clearly, from Lemma 3.1 we have $B_{r_0}^1 \subset B_{r_0}$. Further, $B_{r_0}^{n+1} \subset B_{r_0}^n$ for $n = 1, 2, \dots$ The sets $B_{r_0}^n$ are closed, convex and nonempty. Moreover, we get $\mu(B_{r_0}^n) \le k^n \mu(B_{r_0})$ for any $n = 1, 2, \dots$ due to (3.3). Combining the fact $\mu(B_{r_0}) = 4r_0$ with the inequality $\mu(B_{r_0}^n) \le k^n \mu(B_{r_0})$, $0 < k < 1$ we obtain $\lim_{n\to\infty} \mu(B_{r_0}^n) = 0$. From Definition 2.1 we can derive the set $Y = \bigcap_{n=1}^{\infty} B_{r_0}^n$ is nonempty, bounded, closed and convex. And, the set Y is a member of the kernel $\ker \mu$ of the measure of noncompactness μ. Particularly,

$$\lim_{t\to\infty} \sup \operatorname{diam} Y(t) = \lim_{t\to\infty} \operatorname{diam} Y(t) = 0. \tag{3.4}$$

Thus, the operator $V\colon Y \to Y$.

Next, we prove that V is continuous on the set Y. To show this fact, fix $\varepsilon > 0$ and take $x, y \in Y$ such that $\|x - y\| \le \varepsilon$. Follows (3.4) and the fact that $V(Y) \subset Y$, there exists $T > t_0$ such that for an arbitrary $t \ge T$,

$$|(Vx)(t) - (Vy)(t)| \le \varepsilon. \tag{3.5}$$

For $t \in [t_0, T]$. Applying our assumptions, after some standard computation, we obtain

$$\begin{aligned}
|(Vx)(t) - (Vy)(t)| &\le \frac{m(t)|x(t) - y(t)|}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{n(t)\Phi(|x(s)|) + u_1(t)}{(\ln t - \ln s)^{1-\alpha}} \frac{ds}{s} \\
&\quad + \frac{[m(t)|y(t)| + |f(t,0)|]n(t)}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{\Phi(|x(s) - y(s)|)}{(\ln t - \ln s)^{1-\alpha}} \frac{ds}{s} \\
&\le \frac{[m(t)n(t)\Phi(r_0) + m(t)u_1(t)]\varepsilon}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{1}{(\ln t - \ln s)^{1-\alpha}} \frac{ds}{s} \\
&\quad + \frac{[m(t)n(t)r_0 + |f(t,0)|n(t)]\Phi(\varepsilon)}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{1}{(\ln t - \ln s)^{1-\alpha}} \frac{ds}{s} \\
&= \frac{a(t)\Phi(r_0) + b(t)}{\Gamma(\alpha+1)}\varepsilon + \frac{a(t)r_0 + c(t)}{\Gamma(\alpha+1)}\Phi(\varepsilon) \le \frac{A\Phi(r_0) + B}{\Gamma(\alpha+1)}\varepsilon + \frac{Ar_0 + C}{\Gamma(\alpha+1)}\Phi(\varepsilon).
\end{aligned}$$

Then, it follows the above inequality, (3.5) and assumption [H4], we known that the operator V is continuous on the set Y.

By means of Schauder fixed point theorem, the operator V has at least one fixed point $x \in Y$. Using Lemma 3.1., this fixed point x must be a solution of the equation (1.1).

(b) It follows the fact $Y \in \ker\mu$ and the characterization of sets belonging to $\ker\mu$, all solutions of the equation (1.1) must be uniformly locally attractive. □

4. Local stability of solutions

In this section, we will study the local stability of solutions for the equation (1.1) with $t_0 = 1$, i.e. the following quadratic Hadamard type fractional integral equations

$$x(t) = p(t) + f(t, x(t))({}_H D_{1,t}^{-\alpha} u(t, s, x(s))), \quad t \in [1, +\infty), \ \alpha \in (0, 1), \tag{4.1}$$

where

$${}_H D_{1,t}^{-\alpha} u(t, s, x(t)) := \frac{1}{\Gamma(\alpha)} \int_1^t (\ln \frac{t}{s})^{\alpha-1} u(t, s, x(s)) \frac{ds}{s}.$$

We begin to introduce a stronger assumption:

(H6) There are two constants $L, \gamma > 0$ such that

$$\bigg| p(t) + \frac{f(t, x(t))}{\Gamma(\alpha)} \int_1^t \frac{u(t, s, x(s))}{(\ln t - \ln s)^{1-\alpha}} \frac{ds}{s} \bigg| \leq L(\ln t)^{-\gamma}.$$

Theorem 4.1. *Let the assumptions* ((H1)–(H3) *with* $t_0 = 1$ *and* (H6) *be satisfied. The equation* (4.1) *has at least one solution which tends to zero as* $t \to +\infty$*. In other word, the equation* (4.1) *has a solution which is locally stable.*

Proof. Define the set

$$X_{L,\gamma} = \{x : x \in \mathrm{BC}([1, \infty)) \text{ and } |x(t)| \leq L(\ln t)^{-\gamma} \text{ for } L > 0, \ t > 1\}.$$

It is easy to know that $X_{L,\gamma}$ is a closed, bounded and convex set.

We define a operator V as follows

$$(Vx)(t) = p(t) + \frac{f(t, x(t))}{\Gamma(\alpha)} \int_1^t \frac{u(t, s, x(s))}{(\ln t - \ln s)^{1-\alpha}} \frac{ds}{s}, \tag{4.2}$$

for $t \in [1, +\infty)$, $\alpha \in (0, 1)$.

We shall prove that the operator V has a fixed point in $X_{L,\gamma}$.

We firstly show that V transforms the set $X_{L,\gamma}$ into itself. In fact, for $t > 1$, applying assumption (H6) to the operator V, we have $|(Vx)(t)| \leq L(\ln t)^{-\gamma}$, then $V(X_{L,\gamma}) \subset X_{L,\gamma}$.

Next, we show that V is continuous. For any $x_n, x \in X_{L,\gamma}$, $n = 1, 2, \ldots$, and $\lim\limits_{n\to\infty} |x_n - x| = 0$, we will show $Vx_n \to Vx$ in $X_{L,\gamma}$. Let $\varepsilon > 0$ be given, there exists a $T > 1$ such that $t \geq T$ implies that $L(\ln t)^{-\gamma} < \varepsilon/2$. For $1 < t \leq T$, we have

$$
\begin{aligned}
&|(Vx_n)(t) - (Vx)(t)| \\
&\quad \leq \frac{m(t)|x_n(t) - x(t)|}{\Gamma(\alpha)} \int_1^t \frac{n(t)\Phi(|x_n(s)|) + u_1(t)}{(\ln t - \ln s)^{1-\alpha}} \frac{ds}{s} \\
&\qquad + \frac{m(t)|x(t)| + |f(t,0)|}{\Gamma(\alpha)} \int_1^t \frac{|u(t,s,x_n(s)) - u(t,s,x(s))|}{(\ln t - \ln s)^{1-\alpha}} \frac{ds}{s} \\
&\quad \leq \frac{1}{\Gamma(\alpha+1)} \sup_{t\in[1,T]} m(t)|x_n(t) - x(t)| \Big(\sup_{t\in[1,T]} n(t)\Phi(|x_n(t)|) + u_1(t) \Big) (\ln T)^{\alpha} \\
&\qquad + \frac{1}{\Gamma(\alpha+1)} \sup_{t\in[1,T]} (m(t)\Psi + |f(t,0)|)(\ln T)^{\alpha} n(t)\Phi(|x_n(t) - x(t)|),
\end{aligned}
$$

where $\Psi = \max\{|x(t)| : 1 < t \leq T\}$. Therefore $|(Vx_n)(t) - (Vx)(t)| \to 0$ as $n \to \infty$.

For $t \geq T$, we have $|(Vx_n)(t) - (Vx)(t)| \leq 2L(\ln t)^{-\gamma} \leq \varepsilon$. Then, for $t > 1$, it is clear that $|(Vx_n)(t) - (Vx)(t)| \to 0$ as $n \to \infty$. Therefore, V is continuous.

Lastly, we prove that $V(X_{L,\gamma})$ is equicontinuous. Let $\varepsilon > 0$ be given, there is a $T > 1$ such that $L(\ln t)^{-\gamma} < \varepsilon/2$ for $t > T$.

Let $t_1, t_2 > 1$ and $t_2 > t_1$. For $t_1, t_2 \in [1, T]$, then we have

$$
\begin{aligned}
&|(Vx)(t_2) - (Vx)(t_1)| \leq |p(t_2) - p(t_1)| \\
&\qquad + \left| \frac{f(t_2, x(t_2))}{\Gamma(\alpha)} \int_1^{t_2} \frac{u(t_2,s,x(s))}{(\ln t_2 - \ln s)^{1-\alpha}} \frac{ds}{s} - \frac{f(t_1, x(t_1))}{\Gamma(\alpha)} \int_1^{t_1} \frac{u(t_1,s,x(s))}{(\ln t_1 - \ln s)^{1-\alpha}} \frac{ds}{s} \right| \\
&\leq \omega^T(p,\varepsilon) + \frac{|f(t_2,x(t_2)) - f(t_1,x(t_1))|}{\Gamma(\alpha)} \left| \int_1^{t_2} \frac{u(t_2,s,x(s))}{(\ln t_2 - \ln s)^{1-\alpha}} \frac{ds}{s} \right| \\
&\qquad + \frac{|f(t_1,x(t_1))|}{\Gamma(\alpha)} \left| \int_1^{t_2} \frac{u(t_2,s,x(s))}{(\ln t_2 - \ln s)^{1-\alpha}} \frac{ds}{s} - \int_1^{t_1} \frac{u(t_1,s,x(s))}{(\ln t_1 - \ln s)^{1-\alpha}} \frac{ds}{s} \right| \\
&\leq \omega^T(p,\varepsilon) + \frac{|f(t_2,x(t_2)) - f(t_2,x(t_1))| + |f(t_2,x(t_1)) - f(t_1,x(t_1))|}{\Gamma(\alpha)} \\
&\qquad \times \left| \int_1^{t_2} \frac{u(t_2,s,x(s))}{(\ln t_2 - \ln s)^{1-\alpha}} \frac{ds}{s} \right| \\
&\qquad + \frac{|f(t_1,x(t_1)) - f(t_1,0)| + |f(t_1,0)|}{\Gamma(\alpha)} \int_1^{t_2} \frac{|u(t_2,s,x(s)) - u(t_1,s,x(s))|}{(\ln t_2 - \ln s)^{1-\alpha}} \frac{ds}{s} \\
&\qquad + \frac{|f(t_1,x(t_1)) - f(t_1,0)| + |f(t_1,0)|}{\Gamma(\alpha)} \\
&\qquad \times \left| \int_1^{t_1} u(t_1,s,x(s)) \left[\frac{1}{(\ln t_2 - \ln s)^{1-\alpha}} - \frac{1}{(\ln t_1 - \ln s)^{1-\alpha}} \right] \frac{ds}{s} \right| \\
&\qquad + \frac{|f(t_1,x(t_1)) - f(t_1,0)| + |f(t_1,0)|}{\Gamma(\alpha)} \left| \int_{t_1}^{t_2} \frac{u(t_1,s,x(s))}{(\ln t_2 - \ln s)^{1-\alpha}} \frac{ds}{s} \right|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \omega^T(p,\varepsilon) + \frac{m(t_2)|x(t_2)-x(t_1)| + \omega^T(f,\varepsilon)}{\Gamma(\alpha)} \\
&\quad \times \int_1^{t_2} \frac{|u(t_2,s,x(s)) - u(t_2,s,0)| + |u(t_2,s,0)|}{(\ln t_2 - \ln s)^{1-\alpha}} \frac{ds}{s} \\
&\quad + \frac{m(t_1)|x(t_1)| + |f(t_1,0)|}{\Gamma(\alpha)} \int_1^{t_2} \frac{|u(t_2,s,x(s)) - u(t_1,s,x(s))|}{(\ln t_2 - \ln s)^{1-\alpha}} \frac{ds}{s} \\
&\quad + \frac{m(t_1)|x(t_1)| + |f(t_1,0)|}{\Gamma(\alpha)} \int_1^{t_1} (|u(t_1,s,x(s)) - u(t_1,s,0)| + |u(t_1,s,0)|) \\
&\quad \times \left[\frac{1}{(\ln t_1 - \ln s)^{1-\alpha}} - \frac{1}{(\ln t_2 - \ln s)^{1-\alpha}}\right] \frac{ds}{s} \\
&\quad + \frac{m(t_1)|x(t_1)| + |f(t_1,0)|}{\Gamma(\alpha)} \int_{t_1}^{t_2} \frac{|u(t_1,s,x(s)) - u(t_1,s,0)| + |u(t_1,s,0)|}{(\ln t_2 - \ln s)^{1-\alpha}} \frac{ds}{s} \\
&\leq \omega^T(p,\varepsilon) + \frac{\sup\limits_{t_2 \leq T} m(t_2)\omega^T(x,\varepsilon) + \omega^T(f,\varepsilon)}{\Gamma(\alpha)} \int_1^{t_2} \frac{n(t_2)\Phi(|x(s)|) + u_1(t_2)}{(\ln t_2 - \ln s)^{1-\alpha}} \frac{ds}{s} \\
&\quad + \frac{\sup_{t_1 \leq T}(m(t_1)\Psi + |f(t_1,0)|)}{\Gamma(\alpha)} \int_1^{t_2} \frac{|u(t_2,s,x(s)) - u(t_1,s,x(s))|}{(\ln t_2 - \ln s)^{1-\alpha}} \frac{ds}{s} \\
&\quad + \frac{\sup\limits_{t_1 \leq T}(m(t_1)\Psi + |f(t_1,0)|)}{\Gamma(\alpha)} \\
&\quad \times \int_1^{t_2} (n(t_1)\Phi(|x(s)|) + u_1(t_1)) \left[\frac{1}{(\ln t_1 - \ln s)^{1-\alpha}} - \frac{1}{(\ln t_2 - \ln s)^{1-\alpha}}\right] \frac{ds}{s} \\
&\quad + \frac{\sup\limits_{t_1 \leq T}(m(t_1)\Psi + |f(t_1,0)|)}{\Gamma(\alpha)} \int_{t_1}^{t_2} \frac{n(t_1)\Phi(|x(s)|) + u_1(t_1)}{(\ln t_2 - \ln s)^{1-\alpha}} \frac{ds}{s} \\
&\leq \omega^T(p,\varepsilon) + \frac{\Big(\sup\limits_{t_2 \leq T} m(t_2)\omega^T(x,\varepsilon) + \omega^T(f,\varepsilon))(n(t_2)\Phi(\Psi) + u_1(t_2)\Big)(\ln T)^\alpha}{\Gamma(\alpha+1)} \\
&\quad + \frac{\sup\limits_{t_1 \leq T}(m(t_1)\Psi + |f(t_1,0)|)}{\Gamma(\alpha)} \int_1^{t_2} \frac{|u(t_2,s,x(s)) - u(t_1,s,x(s))|}{(\ln t_2 - \ln s)^{1-\alpha}} \frac{ds}{s} \\
&\quad + \frac{\sup\limits_{t_1 \leq T}(m(t_1)\Psi + |f(t_1,0)|)(n(t_1)\Phi(\Psi) + u_1(t_1))}{\Gamma(\alpha+1)} \\
&\quad \times [(\ln t_1)^\alpha - (\ln t_2)^\alpha + (\ln t_2 - \ln t_1)^\alpha] \\
&\quad + \frac{\sup\limits_{t_1 \leq T}(m(t_1)\Psi + |f(t_1,0)|)(n(t_1)\Phi(\Psi) + u_1(t_1))}{\Gamma(\alpha+1)} (\ln t_2 - \ln t_1)^\alpha.
\end{aligned}
$$

Therefore,

$$|(Vx)(t_2) - (Vx)(t_1)| \to 0 \quad \text{as } t_1 \to t_2.$$

If $t_1, t_2 > T$, then we have

$$|(Vx)(t_2) - (Vx)(t_1)| \leq 2L(\ln t_2)^{-\gamma} \leq \varepsilon.$$

If $1 < t_1 < T < t_2$, note that $t_2 \to t_1$ implies that $t_2 \to T$ and $T \to t_1$, according to the above discussion we have

$$|(Vx)(t_2) - (Vx)(t_1)| \le |(Vx)(t_2) - (Vx)(T)| + |(Vx)(T) - (Vx)(t_1)| \to 0$$

as $t_2 \to t_1$. From the above we have $|(Vx)(t_2) - (Vx)(t_1)| \to 0$ as $t_1 \to t_2$ for $t_1, t_2 > 1$. Then $V(X_{L,\gamma})$ is equicontinuous. Meanwhile, $V(X_{L,\gamma})$ is relatively compact because that $V(X_{L,\gamma}) \subset X_{L,\gamma}$ is uniformly bounded. Thus V is completely continuous on $X_{L,\gamma}$. By Schauder fixed point theorem, we deduce that V has a fixed point x in $X_{L,\gamma}$.

It is easy to see that the fixed point x is just the solution of the equation (4.1) which tends to zero as $t \to \infty$. Thus, the solution of the equation (4.1) is locally stable. □

Next, we give another sufficient conditions to guarantee the local stability of solutions of the equation (4.1).

We replace the assumptions (H6) by the following easy checked condition:

(H6′) Suppose that $|p(t)| \le (\ln t)^{-\alpha}/2$ and there are two constants $L_f, L_u > 0$ such that

$$|f(t,x)| \le L_f|x| \quad \text{and} \quad |u(t,s,x)| \le L_u|x|,$$

where $L_f L_u = 1/2\Gamma(1-\alpha)$.

THEOREM 4.2. *Suppose that assumptions* (H1)–(H3) *and* (H6′) *hold. Then the equation* (4.1) *has at least one solution which is locally stable.*

PROOF. Define the set

$$X_{1,\alpha} = \{x : x \in \mathrm{BC}([1,\infty)) \text{ and } |x(t)| \le (\ln t)^{-\alpha} \text{ for } t > 1\}.$$

It is easy to know that $X_{1,\alpha}$ is a closed, bounded and convex set.

We also introduce a operator V defined by (4.2) in Theorem 4.1. We only need to show that V transforms the set $X_{1,\alpha}$ into itself. In fact, for $t > 1$, applying assumption (H6′) and Lemma 2.1 to the operator V, we have

$$\begin{aligned}
|(Vx)(t)| &\le |p(t)| + \frac{|f(t,x(t))|}{\Gamma(\alpha)} \int_1^t \frac{|u(t,s,x(s))|}{(\ln t - \ln s)^{1-\alpha}} \frac{ds}{s} \\
&\le \frac{1}{2}(\ln t)^{-\alpha} + \frac{L_f L_u (\ln t)^{-\alpha}}{\Gamma(\alpha)} \int_1^t (\ln t - \ln s)^{\alpha-1} (\ln s)^{-\alpha} \frac{ds}{s} \\
&\le \frac{1}{2}(\ln t)^{-\alpha} + \frac{L_f L_u (\ln t)^{-\alpha}}{\Gamma(\alpha)} (\Gamma(\alpha)\Gamma(1-\alpha)) \\
&\le \frac{1}{2}(\ln t)^{-\alpha} + \frac{1}{2}(\ln t)^{-\alpha} = (\ln t)^{-\alpha}.
\end{aligned}$$

Then $V(X_{1,\alpha}) \subset X_{1,\alpha}$. The next process is similar to the proof of Theorem 4.1, one can complete it easily. □

Further, we replace the assumption (H6′) by the following condition:

(H6″) Suppose that $|p(t)| \leq (\ln t)^{-(1-(\alpha+\nu))}/2$ and there exist $M_f, \nu > 0$ such that

$$|f(t,x)| \leq M_f \quad \text{and} \quad |u(t,s,x)| \leq (\ln t - \ln s)^{\nu} e^{-\alpha s}|x|$$

where $M_f \leq \Gamma(\alpha)/(4\max\{1, 2^{1-\alpha-\nu}\}\Gamma(\alpha+\nu)\alpha^{-\alpha-\nu})$ and $0<\alpha+\nu<1$.

THEOREM 4.3. *Assume that conditions* (H1)–(H3) *and* (H6″) *hold, then the equation* (4.1) *has at least one solution which is locally stable.*

PROOF. Define the set

$$X_{1,(1-(\alpha+\nu))} = \{x : x \in \mathrm{BC}([1,\infty)) \text{ and } |x(t)| \leq (\ln t)^{-(1-(\alpha+\nu))} \text{ for } t > 1\}.$$

It is easy to know that $X_{1,(1-(\alpha+\nu))}$ is a closed, bounded and convex set. We also introduce a operator V defined by (4.2) in Theorem 4.1. We only need to show that V transforms the set $X_{1,(1-(\alpha+\nu))}$ into itself. In fact, for $t > 1$, applying assumption (H6″) and Lemma 2.6 to the operator V, we have

$$\begin{aligned}
|(Vx)(t)| &\leq \frac{1}{2}(\ln t)^{-(1-(\alpha+\nu))} + \frac{M_f}{\Gamma(\alpha)}\int_1^t (\ln t - \ln s)^{\alpha+\nu-1}(\ln s)^{\alpha+\nu-1}e^{-\alpha s}\frac{ds}{s} \\
&\leq \frac{1}{2}(\ln t)^{-(1-(\alpha+\nu))} \\
&\quad + \frac{M_f}{\Gamma(\alpha)}\max\{1, 2^{1-\alpha-\nu}\}\Gamma(\alpha+\nu)(1+1)\alpha^{-\alpha-\nu}(\ln t)^{-(1-\alpha-\nu)} \\
&\leq \frac{1}{2}(\ln t)^{-(1-(\alpha+\nu))} + \frac{1}{2}(\ln t)^{-(1-(\alpha+\nu))} = (\ln t)^{-(1-(\alpha+\nu))}.
\end{aligned}$$

Then $V(X_{1,(1-(\alpha+\nu))}) \subset X_{1,(1-(\alpha+\nu))}$. The next process is similar to the proof of Theorem 4.1, one can complete it easily. $\square$

5. Examples

In this section we make two examples illustrating the main results contained in Theorem 3.2 and Theorem 4.1.

EXAMPLE 5.1. Consider a quadratic Hadamard fractional integral equation

$$\begin{aligned}
x(t) &= te^{-3t^2} \\
&\quad + \frac{(\ln t)^{-1/2} + (\ln t)^{-1/2}x(t)}{\Gamma(1/2)}\int_1^t \frac{e^{-3t-s}\sqrt[3]{x^2(t)} + 1/(10t^{8/3}+1)}{(\ln t - \ln s)^{1/2}}\frac{ds}{s},
\end{aligned} \tag{5.1}$$

for $t \in [1,\infty)$. Observe that the above equation is a special case of the equation (1.1). Put $\alpha = 1/2$, $p(t) = te^{-3t^2}$, $f(t,x) = (\ln t)^{-1/2} + (\ln t)^{-1/2}x(t)$, $|f(t,0)| = (\ln t)^{-1/2}$, $u(t,s,x) = e^{-3t-s}\sqrt[3]{x^2(t)} + 1/(10t^{8/3}+1)$, $m(t) = (\ln t)^{-1/2}$, $n(t) =$

e^{-3t}, $\Phi(r) = r^{2/3}$ and $u(t,s,0) = u_1(t) = 1/(10t^{8/3}+1)$. The functions a, b, c, d take the form

$$a(t) = c(t) = e^{-3t}, \qquad b(t) = d(t) = \frac{1}{10t^{8/3}+1}.$$

It is easily seen that $a(t) \to 0$ as $t \to \infty$ and $A = e^{-3}$. Further we have that the function $b(t)$ is bounded on $[1,\infty)$ and $B = 1/11$. It is also easy to check that $c(t) \to 0$ as $t \to \infty$. Moreover, we have that $C = e^{-3}$. Also we see that $d(t) \to 0$ as $t \to \infty$ and $D = 1/11$. Further,

$$L(r) = \frac{\sqrt{\pi}}{2}e^{-3} + e^{-3}r^{5/3} + \frac{1}{11}r + e^{-3}r^{2/3} + \frac{1}{11}$$

has a solution $r_0 = 1$ and $e^{-3} + 1/11 < \sqrt{\pi}/2$. Then all assumptions given in Theorem 3.2 are satisfied, our results can be applied to the equation (5.1).

EXAMPLE 5.2. Consider another quadratic Hadamard type fractional integral equations

$$x(t) = \frac{2\cos(x(t))}{5\Gamma(\frac{1}{2})}\int_1^t \frac{(\ln t)^{-3/4}\sin(x(s)-s)}{(\ln t - \ln s)^{1/2}}\,\frac{ds}{s}, \quad t \in [1,\infty). \tag{5.2}$$

Observe that the above equation is a special case of the equation (4.1). Indeed, if we put $\alpha = 1/2$ and $p(t) = 0$, $f(t,x) = (2/5)\cos(x(t))$, $u(t,s,x) = (\ln t)^{-3/4}\sin(x(s)-s)$. Moreover,

$$\begin{aligned} \left| p(t) + \frac{f(t,x(t))}{\Gamma(\alpha)}\int_1^t \frac{u(t,s,x(s))}{(\ln t - \ln s)^{1-\alpha}}\,\frac{ds}{s}\right| & \\ \le \frac{2}{5\Gamma(1/2)}\int_1^t (\ln t - \ln s)^{-1/2}(\ln s)^{-3/4}\frac{ds}{s} & \\ \le \frac{2\Gamma(1/4)}{5\Gamma(3/4)}(\ln t)^{-3/4} \le 2(\ln t)^{-3/4}. & \end{aligned}$$

Then all assumptions given in Theorem 4.1 are satisfied, our results can be applied to the equation (5.2).

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