Topological **M**ethods in **N**onlinear **A**nalysis
Volume 42, No. 1, 2013, 91–104

EXISTENCE, UNIQUENESS AND STABILITY OF POSITIVE SOLUTIONS FOR A CLASS OF SEMILINEAR ELLIPTIC SYSTEMS

Renhao Cui — Ping Li — Junping Shi — Yunwen Wang

Abstract. We consider the stability of positive solutions to semilinear elliptic systems under a new general sublinear condition and its variants. Using the stability result and bifurcation theory, we prove the existence and uniqueness of positive solution and obtain the precise global bifurcation diagram of the system being a single monotone solution curve.

1. Introduction

We consider the positive solutions of a semilinear elliptic system:

$$\begin{cases} \Delta u + \lambda f(u,v) = 0, & x \in \Omega, \\ \Delta v + \lambda g(u,v) = 0, & x \in \Omega, \\ u(x) = v(x) = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\lambda > 0$ is a positive parameter, Ω is a bounded smooth domain in $\mathbb{R}^n$ for $n \geq 1$, and f and g are smooth real-valued functions defined on $\mathbb{R}^+ \times \mathbb{R}^+ =$

2010 *Mathematics Subject Classification.* 35J55, 35B32.

Key words and phrases. Semilinear elliptic systems, positive solution, stability, existence, uniqueness.

Partially supported by the National Natural Science Foundation of China (No. 11071051), Science Foundation of Heilongjiang Province (Grant A201009), Science Research Foundation of the Education Department of Heilongjiang Province (Grant No. 12521153) and Harbin Normal University advanced research Foundation (11xyg-02) and US-NSF grant DMS-1022648.

$[0,\infty)\times[0,\infty)$ satisfying $f_v(u,v)\geq 0$ and $g_u(u,v)\geq 0$ for $(u,v)\in\mathbb{R}^+\times\mathbb{R}^+$, which implies that the system is cooperative.

The existence, uniqueness and stability of positive solutions of sublinear semilinear elliptic systems have been recently studied in [2], [3], [26]. In [2], the stability of a positive solution was established under the condition

$$f(u,v) > f_v(u,v)v + g_v(u,v)u, \quad g(u,v) > g_u(u,v)u + f_u(u,v)v. \tag{1.2}$$

The sublinear condition (1.2) involves both f and g in the two inequalities, which is sometimes hard to achieve. In this article, we continue the effort in [2] to prove the stability of positive solution to (1.1) under some more reasonable sublinear conditions, and once again the stability implies the uniqueness of the positive solution. We also prove corresponding existence results using bifurcation and continuation theory.

For the scalar semilinear elliptic equation:

$$\begin{cases} \Delta u + \lambda f(u) = 0, & x\in\Omega,\\ u(x) = 0, & x\in\partial\Omega, \end{cases} \tag{1.3}$$

the exact multiplicity of positive solutions has been previously considered by many people, see for example, [14], [15], [18], [19]. In recent years, there have been some results on the existence and uniqueness of solution to the semilinear cyclic elliptic system:

$$\begin{cases} \Delta u + \lambda f(v) = 0, & x\in\Omega,\\ \Delta v + \lambda g(u) = 0, & x\in\Omega,\\ u(x) = v(x) = 0, & x\in\partial\Omega. \end{cases} \tag{1.4}$$

Dalmasso [7], [8] obtained the existence and uniqueness result for a more special sublinear system, and it was extended by Shi and Shivaji [26]. The uniqueness of positive solution for large λ was proved in Hai [9], [10], Hai and Shivaji [11]. If Ω is a finite ball or the whole space, then the positive solutions of systems are radially symmetric and decreasing in radial direction by the result of Troy [28], see also [1], [17]. Hence the system can be converted into a system of ODEs. Several authors have taken that approach for the existence of the solutions, see Serrin and Zou [21], [22]. Much success has been achieved for Lane–Emden systems. Using the scaling invariant, the uniqueness of the radial positive solution for the Lane–Emden system has been shown in Dalmasso [7], [8], Korman and Shi [16]. Cui, Wang and Shi [5], [6] considered cyclic systems with three equations, and the uniqueness and existence of positive solutions were obtained.

The approach in this article includes several ingredients. We recall the maximum principle and prove the main stability result in Section 2. In Sections 3 and 4, we use the stability result and bifurcation theory to prove the existence and uniqueness of positive solution for two types of semilinear system. We also obtain the precise global bifurcation diagrams of the system and the bifurcation diagram is a single monotone solution curve in all cases. We use $W^{2,p}(\Omega)$ and $W^{2,p}_{\rm loc}(\Omega)$ for the standard Sobolev space, $C(\overline{\Omega})$ for the space of continuous functions defined on $\overline{\Omega}$, and $C_0(\overline{\Omega}) = \{u \in C(\overline{\Omega}) : u(x) = 0, x \in \partial\Omega\}$. We use $N(L)$ and $R(L)$ to denote the null space and the range space of linear operator L.

2. Stability and linearized equations

Let (u, v) be a positive solution of (1.1). The stability of (u, v) is determined by the eigenvalue equation:

$$\begin{cases} \Delta\xi + \lambda f_u(u,v)\xi + \lambda f_v(u,v)\eta = -\mu\xi, & x \in \Omega, \\ \Delta\eta + \lambda g_u(u,v)\xi + \lambda g_v(u,v)\eta = -\mu\eta, & x \in \Omega, \\ \xi(x) = \eta(x) = 0, & x \in \partial\Omega, \end{cases} \tag{2.1}$$

which can be written as

$$L\mathbf{u} = H\mathbf{u} + \mu\mathbf{u}, \tag{2.2}$$

where

$$\mathbf{u} = \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \qquad L\mathbf{u} = \begin{pmatrix} -\Delta\xi \\ -\Delta\eta \end{pmatrix} \quad \text{and} \quad H = \begin{pmatrix} f_u(u,v) & f_v(u,v) \\ g_u(u,v) & g_v(u,v) \end{pmatrix}. \tag{2.3}$$

For linear elliptic systems of cooperative type, the maximum principle holds and here we recall some known results:

LEMMA 2.1. *Let $X = [W^{2,p}_{\rm loc}(\Omega) \cap C_0(\overline{\Omega})]^2$, and let $Y = [L^p(\Omega)]^2$ for $p > n$. Suppose that L and H are given as in* (2.3)*, the partial derivatives of f and g are continuous on $\mathbb{R}^+ \times \mathbb{R}^+$, and $f_v(u,v) \geq 0$, $g_u(u,v) \geq 0$ for $(u,v) \in \mathbb{R}^+ \times \mathbb{R}^+$. Then*:

- (a) $\mu_1 = \inf\{\mu \in \mathrm{spt}(L-H)\}$ *is a real eigenvalue of $L-H$, where* $\mathrm{spt}(L-H)$ *is the spectrum of $L - H$.*
- (b) *For $\mu = \mu_1$, there exists a unique eigenfunction $\mathbf{u}_1 \in [W^{2,n}_{\rm loc}(\Omega) \cap C_0(\overline{\Omega})]^2$ of $L - H$ (up a constant multiple), and $\mathbf{u}_1 > 0$ in Ω.*
- (c) *For $\mu < \mu_1$, the equation $L\mathbf{u} = H\mathbf{u} + \mu\mathbf{u} + \mathbf{f}$ has a unique solution $\mathbf{u} \in X$ for any $\mathbf{f} \in Y$, and $\mathbf{u} > 0$ as long as $\mathbf{f} \geq (\not\equiv) 0$.*
- (d) (Maximum principle) *For $\mu \leq \mu_1$, suppose that there exists $\mathbf{u} \in [W^{2,p}_{\rm loc}(\Omega) \cap C(\overline{\Omega})]^2$, satisfies $L\mathbf{u} \geq H\mathbf{u} + \mu\mathbf{u}$ in Ω, $\mathbf{u} \geq 0$ on $\partial\Omega$, then $\mathbf{u} \geq 0$ in Ω.*
- (e) *If there exists $\mathbf{u} \in [W^{2,p}(\Omega) \cap C(\overline{\Omega})]^2$, satisfies $L\mathbf{u} \geq H\mathbf{u}$ and $\mathbf{u} \geq 0$ in Ω, and either $\mathbf{u} \not\equiv 0$ on $\partial\Omega$ or $L\mathbf{u} \not\equiv H\mathbf{u}$ in Ω, then $\mu_1 > 0$.*

For the result and proof of Lemma 2.1, see Sweers [26], Proposition 3.1 and Theorem 1.1. Moreover, from a standard compactness argument, the eigenvalues $\{\mu_i\}$ of $L - H$ are countably many, and $Re(\mu_i - \mu_1) \to \infty$ as $i \to \infty$. We notice that μ_i is not necessarily real-valued for $i \geq 2$. We call a solution (u,v) is *stable* if $\mu_1 > 0$, and otherwise it is *unstable* ($\mu_1 \leq 0$).

For our purpose in this section, we also need to consider the adjoint operator of $L - H$. Let the transpose matrix of the Jacobian be

$$H^{\mathrm{T}} = \begin{pmatrix} f_u(u,v) & g_u(u,v) \\ f_v(u,v) & g_v(u,v) \end{pmatrix}. \tag{2.4}$$

Then evidently the results in Lemma 2.1 also hold for the eigenvalue problem

$$L\mathbf{u}^* = H^{\mathrm{T}}\mathbf{u}^* + \mu\mathbf{u}^*, \tag{2.5}$$

which is

$$\begin{cases} \Delta\xi^* + \lambda f_u(u,v)\xi^* + \lambda g_u(u,v)\eta^* = -\mu\xi^*, & x \in \Omega, \\ \Delta\eta^* + \lambda f_v(u,v)\xi^* + \lambda g_v(u,v)\eta^* = -\mu\eta^*, & x \in \Omega, \\ \xi^*(x) = \eta^*(x) = 0, & x \in \partial\Omega. \end{cases} \tag{2.6}$$

It is easy to verify that $L - H^T$ is the adjoint operator of $L - H$, while both are considered as operators defined on subspaces of $[L^2(\Omega)]^2$. By using the well-known functional analytic techniques (see [12], [26]), one can show that:

LEMMA 2.2. *Let X, Y, L, H and f, g be same as in Lemma* 2.1. *Then the principal eigenvalue μ_1 of $L - H$ is also a real eigenvalue of $L - H^T$, $\mu_1 = \inf\{\mu \in \mathrm{spt}(L - H^T)\}$, and for $\mu = \mu_1$, there exists a unique eigenfunction $\mathbf{u}_1^* \in [W^{2,p}_{\mathrm{loc}}(\Omega) \cap C_0(\overline{\Omega})]^2$ of $L - H^T$ (up a constant multiple), and $\mathbf{u}_1^* > 0$ in Ω.*

Now we are ready to establish the main stability result:

THEOREM 2.3. *Suppose that (u,v) is a positive solution of* (1.1), *and f and g are smooth real-valued functions defined on $\mathbb{R}^+ \times \mathbb{R}^+$ satisfying $f_v(u,v) \geq 0$ and $g_u(u,v) \geq 0$ for $(u,v) \in \mathbb{R}^+ \times \mathbb{R}^+$. Then (u,v) is stable if (f,g) satisfies one of the following conditions: for any $(u,v) \in \mathbb{R}^+ \times \mathbb{R}^+$,*

(A_1) $f(u,v) > f_u(u,v)u + f_v(u,v)v$, $\quad g(u,v) > g_u(u,v)u + g_v(u,v)v$; *or*
(A_2) $f(u,v) > f_u(u,v)u + g_u(u,v)v$, $\quad g(u,v) > g_v(u,v)v + f_v(u,v)u$; *or*
(A_3) $f(u,v) > f_v(u,v)v + g_v(u,v)u$, $\quad g(u,v) > g_u(u,v)u + f_u(u,v)v$; *or*
(A_4) $f(u,v) > g_v(u,v)u + g_u(u,v)v$, $\quad g(u,v) > f_v(u,v)u + f_u(u,v)v$.

PROOF. The result under (A_3) has been proved in [3], hence here we prove the stability result when (f,g) satisfies one of (A_1), (A_2) and (A_4). Let (u,v) be a positive solution of (1.1), and let (μ_1, ξ, η) and (μ_1, ξ^*, η^*) be the corresponding principal eigen-pair of (2.1) and (2.6) respectively, such that ξ, η, ξ^*, $\eta^* > 0$ in Ω.

First we assume that (f,g) satisfies (A_1). Multiplying the equation of u in (1.1) by ξ^*, the equation of ξ^* in (2.6) by u, integrating over Ω and subtracting, we obtain that

$$\lambda\int_\Omega f\xi^*\,dx = \lambda\int_\Omega (f_u\xi^* + g_u\eta^*)u\,dx + \mu_1\int_\Omega u\xi^*\,dx. \tag{2.7}$$

Similarly from the equation of v and η^*, we find

$$\lambda \int_\Omega g\eta^*\, dx = \lambda \int_\Omega (f_v\xi^* + g_v\eta^*)v\, dx + \mu_1 \int_\Omega v\eta^*\, dx. \tag{2.8}$$

Adding (2.7) and (2.8), we get

$$\mu_1 \int_\Omega (u\xi^* + v\eta^*)\, dx = \lambda \int_\Omega [f - f_u u - f_v v]\eta^*\, dx + \lambda \int_\Omega [g - g_u u - g_v v]\xi^*\, dx. \tag{2.9}$$

Hence $\mu_1 > 0$ if (A_1) is satisfied.

Secondly we assume that (f, g) satisfies (A_2). Similar to the proof above, multiplying the equation of u in (1.1) by ξ, multiplying the equation of ξ in (2.1) by u, integrating over Ω and subtracting, and also doing the same operations for the equations of v and η, we can get

$$\mu_1 \int_\Omega (u\xi + v\eta)\, dx = \lambda \int_\Omega [f - f_u u - g_u v]\xi\, dx + \lambda \int_\Omega [g - f_v u - g_v v]\eta\, dx, \tag{2.10}$$

which implies $\mu_1 > 0$ if (A_2) is satisfied.

Finally we assume that (f, g) satisfies (A_4). We repeat the above calculation for the equations of u and η^*, and the equations of v and ξ^*, then we obtain

$$\mu_1 \int_\Omega (u\eta + v\xi)\, dx = \lambda \int_\Omega [f - g_u v - g_v u]\eta\, dx + \lambda \int_\Omega [g - f_u v - g_v u]\xi\, dx. \tag{2.11}$$

Therefore $\mu_1 > 0$ if (A_4) is satisfied. □

On the other hand, the same proof also implies the following instability result under the opposite condition of (A_i) for $i = 1, 2, 3$ and 4:

Theorem 2.4. *Suppose that (u, v) is a positive solution of* (1.1), *and f and g are smooth real-valued functions defined on $\mathbb{R}^+ \times \mathbb{R}^+$ satisfying $f_v(u, v) \ge 0$ and $g_u(u, v) \ge 0$ for $(u, v) \in \mathbb{R}^+ \times \mathbb{R}^+$. Then (u, v) is unstable if (f, g) satisfies one of the following conditions*:

- (A_1') $f(u,v) < f_u(u,v)u + f_v(u,v)v$, $\quad g(u,v) < g_u(u,v)u + g_v(u,v)v$; *or*
- (A_2') $f(u,v) < f_u(u,v)u + g_u(u,v)v$, $\quad g(u,v) < g_v(u,v)v + f_v(u,v)u$; *or*
- (A_3') $f(u,v) < f_v(u,v)v + g_v(u,v)u$, $\quad g(u,v) < g_u(u,v)u + f_u(u,v)v$; *or*
- (A_4') $f(u,v) < g_v(u,v)u + g_u(u,v)v$, $\quad g(u,v) < f_v(u,v)u + f_u(u,v)v$,

for any $(u, v) \in \mathbb{R}^+ \times \mathbb{R}^+$,

Remark 2.5. (a) Theorems 2.3 and 2.4 are generalizations of corresponding results for the positive solutions of scalar equation:

$$\Delta u + \lambda h(u) = 0, \quad x \in \Omega, \quad u(x) = 0, \quad x \in \partial\Omega.$$

In [19], the function $h(u)$ is called a sublinear function if $h(u) > uh'(u)$, and it is superlinear if $h(u) < uh'(u)$. It was proved in Proposition 3.14 of [19] that a positive solution u is stable if h is sublinear, and u is unstable if h is superlinear.

The conditions (A_i) (or (A'_i)) for $1 \le i \le 4$ are the generalization of sublinearity (or superlinearity) to two-variable vector fields.

(b) The conditions (A_i) for $1 \le i \le 4$ can be written in a vector form $F(\mathbf{u}) > J_i(\mathbf{u})\mathbf{u}$, where $\mathbf{u} = (u,v)^T$, and

$$J_1 = \begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix}, \quad J_2 = \begin{pmatrix} f_u & g_u \\ f_v & g_v \end{pmatrix}, \quad J_3 = \begin{pmatrix} g_v & f_v \\ g_u & f_u \end{pmatrix}, \quad J_4 = \begin{pmatrix} g_v & g_u \\ f_v & f_u \end{pmatrix}.$$

Notice that J_1 is the original Jacobian matrix of the vector field $(f(\mathbf{u}), g(\mathbf{u}))$, and J_i ($2 \le i \le 4$) are reflections of J_1 with respect to the two diagonal lines. The condition with original Jacobian is clearly more natural as the conditions for f and g are separate. Hence the sublinearity can be defined for a single two-variable function $f(u,v)$ or $g(u,v)$. Other conditions are defined for the whole vector field (f,g).

(c) We can weaken the strict inequalities in (A_i) or (A'_i) to $\ge$ or $\le$ respectively, but assume that the strict inequalities hold at least for (u,v) in a set of positive measure. On the other hand, if we have $f \equiv f_u u + f_v v$ and $g \equiv g_u u + g_v v$, then f and g are necessarily linear functions of u and v, and the corresponding positive solution (u,v) is neutrally stable with $\mu_1 = 0$. Indeed (u,v) is the principal eigenfunction for linear f and g.

(d) If a solution (u,v) is stable, then it is necessarily a non-degenerate solution. That is, any eigenvalue μ_i of (2.1) has positive real part. But when a solution is proved to be unstable, it can be a degenerate one with zero or pure imaginary eigenvalues.

3. Application: positive nonlinearities

In this section, we consider the uniqueness and existence of positive solutions for the following problem:

$$\begin{cases} \Delta u + \lambda(f_1(v) + f_2(u)) = 0, & x \in \Omega, \\ \Delta v + \lambda(g_1(u) + g_2(v)) = 0, & x \in \Omega, \\ u(x) = v(x) = 0, & x \in \partial\Omega. \end{cases} \tag{3.1}$$

Suppose that each of the functions f_1, f_2, g_1 and g_2 is a smooth real-valued function defined on $\mathbb{R}^+$ and satisfies (denote f_1, f_2, g_1 or g_2 by h):

- (B1) $h(0) \ge 0$;
- (B2) $h'(x) \ge 0$, $(h(x)/x)' \le 0$, for all $x \ge 0$, and $(h(x)/x)' \not\equiv 0$ for any open interval $(a,b) \subset \mathbb{R}^+$.

Here let (λ_1, φ_1) be the principal eigen-pair of

$$-\Delta\varphi = \lambda\varphi, \quad x \in \Omega, \quad \varphi(x) = 0, \quad x \in \partial\Omega, \tag{3.2}$$

such that $\varphi_1(x) > 0$ in Ω and $\|\varphi_1\|_\infty = 1$. Then we have the following result about this sublinear problem:

THEOREM 3.1. *Assume that each of* f_1, f_2, g_1 *and* g_2 *satisfies* (B1), (B2) *and*

(B3) $\lim_{x\to\infty} \frac{h(x)}{x} = 0.$

(a) *If at least one of* $f_i(0)$ *and* $g_i(0)$ $(i = 1,2)$ *is positive, then* (3.1) *has a unique positive solution* (u_λ, v_λ) *for all* $\lambda > 0$;
(b) *If* $h(0) = 0$, *and* $h'(0) \geq 0$, *then for some* $\lambda_* > 0$, (3.1) *has no positive solution when* $\lambda \leq \lambda_*$, *and* (3.1) *has a unique positive solution* $(u(\lambda), v(\lambda))$ *for* $\lambda > \lambda_*$.

Moreover, $\{(\lambda, u(\lambda), v(\lambda)) : \lambda > \lambda_*\}$ *(in the first case, we assume* $\lambda_* = 0$*) is a smooth curve so that* $u(\lambda)$, $v(\lambda)$ *are strictly increasing in* λ, *and* $(u(\lambda), v(\lambda)) \to (0,0)$ *as* $\lambda \to \lambda_*^+$.

PROOF. Our proof follows that of Theorem 6.1 in [26]. First we extend f_i, g_i to be defined on $\mathbb{R}$ for u, $v < 0$ properly so they are continuous differentiable on $\mathbb{R}$. From the assumptions, $f(u,v) = f_1(v) + f_2(u)$, $g(u,v) = g_1(u) + g_2(v)$ satisfy (A_1). Hence from Theorem 2.3, any positive solution of (3.1) is stable.

We define

$$F(\lambda, u, v) = \begin{pmatrix} \Delta u + \lambda[f_1(v) + f_2(u)] \\ \Delta v + \lambda[g_1(u) + g_2(v)] \end{pmatrix}, \tag{3.3}$$

where $\lambda \in \mathbb{R}$ and $u, v \in C_0^{2,\alpha}(\overline{\Omega})$. Here f, g are at least C^1, then $F: \mathbb{R} \times X \to Y$ is continuously differentiable, where $X = [C_0^{2,\alpha}(\overline{\Omega})]^2$ and $Y = [C^\alpha(\overline{\Omega})]^2$. For weak solutions (u,v), one can also consider $X = [W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)]^2$ and $Y = [L^p(\Omega)]^2$ where $p > 1$ is properly chosen.

Apparently $(\lambda, u, v) = (0,0,0)$ is a solution of (3.1). We apply the implicit function theorem at $(\lambda, u, v) = (0,0,0)$. The Fréchet derivative of F is given by

$$F_{(u,v)}(\lambda, u, v) \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} \Delta\phi + \lambda[f_1'(v)\psi + f_2'(u)\phi] \\ \Delta\psi + \lambda[g_1'(u)\phi + g_2'(v)\psi] \end{pmatrix}. \tag{3.4}$$

Then $F_{(u,v)}(0,0,0)(\phi,\psi)^{\mathrm{T}} = (\Delta\phi, \Delta\psi)^{\mathrm{T}}$ is an isomorphism from X to Y, and the implicit function theorem implies that $F(\lambda, u, v) = 0$ has a unique solution $(\lambda, u(\lambda), v(\lambda))$ for $\lambda \in (0, \delta)$ for some small $\delta > 0$, and $(u'(0), v'(0))$ is the unique solution of

$$\begin{cases} \Delta\phi + \lambda(f_1(0) + f_2(0)) = 0, & x \in \Omega, \\ \Delta\psi + \lambda(g_1(0) + g_2(0)) = 0, & x \in \Omega, \\ \phi(x) = \psi(x) = 0, & x \in \partial\Omega. \end{cases} \tag{3.5}$$

Then $(u'(0), v'(0)) = ((f_1(0) + f_2(0))e, (g_1(0) + g_2(0))e)$ where e is the unique positive solution of

$$\Delta e + 1 = 0, \quad x \in \Omega, \quad e(x) = 0, \quad x \in \partial\Omega. \tag{3.6}$$

If $f_1(0)+f_2(0)>0$ and $g_1(0)+g_2(0)>0$, then (u_λ,v_λ) is positive for $\lambda\in(0,\delta)$. If $f_1(0)+f_2(0)=0$ and $g_1(0)+g_2(0)>0$, then $v(\lambda)>0$ for $\lambda\in(0,\delta)$. But $\Delta u(\lambda)=-\lambda[f_1(v(\lambda))+f_2(u(\lambda))]$ and f_i is positive, hence $u(\lambda)>0$ as well. Similar conclusion holds when $f_1(0)+f_2(0)>0$ and $g_1(0)+g_2(0)=0$. Therefore (3.1) has a positive solution $(u(\lambda),v(\lambda))$ for $\lambda\in(0,\delta)$ in this case.

Next we assume that $h(0)=0$, and $h'(0)>0$ for each of f_1, f_2, g_1 and g_2. Then the linearized operator at $(\lambda,0,0)$ is

$$\begin{aligned}(3.7)\qquad F_{(u,v)}(\lambda,0,0)\begin{pmatrix}\Phi\\ \Psi\end{pmatrix}&=\begin{pmatrix}\Delta\Phi+\lambda[f_2'(0)\Phi+f_1'(0)\Psi]\\ \Delta\Psi+\lambda[g_1'(0)\Phi+g_2'(0)\Psi]\end{pmatrix}\\ &=\begin{pmatrix}\Delta\Phi\\ \Delta\Psi\end{pmatrix}+\lambda\begin{pmatrix}f_2'(0)&f_1'(0)\\ g_1'(0)&g_2'(0)\end{pmatrix}\begin{pmatrix}\Phi\\ \Psi\end{pmatrix}\\ &=\begin{pmatrix}\Delta\Phi\\ \Delta\Psi\end{pmatrix}+\lambda J\begin{pmatrix}\Phi\\ \Psi\end{pmatrix},\end{aligned}$$

where $J=\begin{pmatrix}f_2'(0)&f_1'(0)\\ g_1'(0)&g_2'(0)\end{pmatrix}$. Since $h'(0)>0$ for each $h=f_i,g_i$, then all entries of matrix J are positive. Therefore by using the Perron–Frobenius theorem (see [23, Theorem 5.3.1]), there exists a positive principal eigenvalue χ_J and the corresponding eigenvector $(1,k)^{\mathrm T}$ of J for some $k>0$, such that $(\varphi_1,k\varphi_1)^{\mathrm T}$ is a positive eigenvector of $F_{(u,v)}(\lambda_*,0,0)$ where $\lambda_*=\lambda_1/\chi_J$. Similarly, the adjoint operator of $F_{(u,v)}(\lambda_J,0,0)$, that is $\Delta+\lambda J^{\mathrm T}$, has the same principal eigenvalue λ_J, and the corresponding eigenvector $(\varphi_1,k_*\varphi_1)^{\mathrm T}$, where k_* is a positive constant.

Hence when $\lambda=\lambda_*=\lambda_1/\chi_J$, $F_{(u,v)}(\lambda,0,0)$ is not invertible and $\lambda=\lambda_*$ is a potential bifurcation point. More precisely, the null space $N(F_{(u,v)}(\lambda_*,0,0))=\operatorname{span}\{(\varphi_1,k\varphi_1)\}$ is one-dimensional. Suppose that $(h_1,h_2)^{\mathrm T}\in R(F_{(u,v)}(\lambda_*,0,0))$, then there exist $(\phi,\psi)\in X$ such that

$$(3.8)\qquad F_{(u,v)}(\lambda_*,0,0)\begin{pmatrix}\phi\\ \psi\end{pmatrix}=\begin{pmatrix}\Delta\phi\\ \Delta\psi\end{pmatrix}+\lambda_*\begin{pmatrix}f_2'(0)&f_1'(0)\\ g_1'(0)&g_2'(0)\end{pmatrix}\begin{pmatrix}\phi\\ \psi\end{pmatrix}=\begin{pmatrix}h_1\\ h_2\end{pmatrix}.$$

Consider the adjoint eigenvalue equation:

$$(3.9)\qquad \Delta\begin{pmatrix}\phi_1\\ \psi_1\end{pmatrix}+\lambda_*J^{\mathrm T}\begin{pmatrix}\phi_1\\ \psi_1\end{pmatrix}=\begin{pmatrix}\Delta\phi_1\\ \Delta\psi_1\end{pmatrix}+\lambda_*\begin{pmatrix}f_2'(0)&g_1'(0)\\ f_1'(0)&g_2'(0)\end{pmatrix}\begin{pmatrix}\phi_1\\ \psi_1\end{pmatrix}=0,$$

where $(\phi_1,\psi_1)^{\mathrm T}=(\varphi_1,k_*\varphi_1)^{\mathrm T}$. Inner-producting the system (3.8) by (ϕ_1,ψ_1), the system (3.9) by (ϕ,ψ), integrating over Ω and subtracting, we obtain

$$(3.10)\qquad \int_\Omega(h_1\phi_1+h_2\psi_1)\,dx=\int_\Omega(h_1\varphi_1+k_*h_2\varphi_1)\,dx=0.$$

Hence $(h_1,h_2)^{\mathrm T}\in R(F_{(u,v)}(\lambda_*,0,0))$ if and only if (3.14) holds, which implies that the codimension of $(R(F_{(u,v)}(\lambda_*,0,0))$ is one.

Next we verify that $F_{\lambda(u,v)}(\lambda_*,0,0)(\varphi_1,k\varphi_1)^{\mathrm{T}} \notin R(F_{(u,v)}(\lambda_*,0,0))$. Indeed

$$F_{\lambda(u,v)}(\lambda_*,0,0)\begin{pmatrix}\varphi_1\\ k\varphi_1\end{pmatrix} = J\begin{pmatrix}1\\ k\end{pmatrix}\varphi_1 = \chi_J\begin{pmatrix}1\\ k\end{pmatrix}\varphi_1. \tag{3.11}$$

But

$$0 = \chi_J\int_\Omega (1+kk_*)\varphi_1^2\,dx > 0, \tag{3.12}$$

hence from (3.10), $F_{\lambda(u,v)}(\lambda_*,0,0)[\varphi_1,k\varphi_1]^{\mathrm{T}} \notin R(F_{(u,v)}(\lambda_*,0,0))$.

Applying a bifurcation from simple eigenvalue theorem of Crandall–Rabinowitz [4], we conclude that $(\lambda_*,0,0)$ is a bifurcation point for (3.1), and the nontrivial solutions of $F(\lambda,u,v)=(0,0)$ near $(\lambda_*,0,0)$ are in form of

$$\{(\lambda(s),u(s),v(s)) : s\in(-\delta,\delta)\} \quad \text{where } u(s)=s\varphi_1+o(s),\ v(s)=ks\varphi_1+o(s).$$

From the stability of positive solutions, each positive solution is stable thus nondegenerate.

We claim that (3.1) has no positive solution when $\lambda\le\lambda_*$. We assume that (u,v) is a positive solution of (3.1) and recall that $(\varphi_1,k_*\varphi_1)$ satisfies

$$\Delta\begin{pmatrix}\varphi_1\\ k_*\varphi_1\end{pmatrix} + \lambda_*\begin{pmatrix}f_2'(0) & g_1'(0)\\ f_1'(0) & g_2'(0)\end{pmatrix}\begin{pmatrix}\varphi_1\\ k_*\varphi_1\end{pmatrix} = 0, \tag{3.13}$$

Multiplying the system (3.1) by $(\varphi_1,k_*\varphi_1)$, the system (3.13) by (u,v), integrating over Ω and subtracting, and by using $(B2)$ and $h(0)=0$ for $h=f_i,g_i$, we obtain

$$\begin{aligned}\lambda_*&\int_\Omega[(f_2'(0)u+f_1'(0)v)\varphi_1+(g_1'(0)u+g_2'(0)v)k_*\varphi_1]\,dx\\ &=\lambda\int_\Omega[(f_1(v)+f_2(u))\varphi_1+(g_1(u)+g_2(v))k_*\varphi_1]\,dx\\ &<\lambda\int_\Omega[(f_2'(0)u+f_1'(0)v)\varphi_1+(g_1'(0)u+g_2'(0)v)k_*\varphi_1]\,dx.\end{aligned} \tag{3.14}$$

Hence (3.1) has no positive solution when $\lambda\le\lambda_*$, and the bifurcating solution $(\lambda(s),u(s),v(s))$ must satisfy $\lambda(s)>\lambda_*$ for $s\in(0,\delta)$. Hence the curve $\{(\lambda(s),u(s),v(s)) : s\in(0,\delta)\}$ can also be parameterized as $(\lambda,u_\lambda,v_\lambda)$ for $\lambda\in(\lambda_*,\lambda_*+\delta)$. Since any positive solution is stable, then with implicit function theorem, we can extend this curve to a largest $\lambda^*\le\infty$.

Let $\Gamma=\{(\lambda,u_\lambda,v_\lambda) : \lambda_*<\lambda<\lambda^*\}$. We show that (u_λ,v_λ) is strictly increasing in λ for $\lambda\in(\lambda_*,\lambda^*)$. In fact, $(\partial u_\lambda/\partial\lambda, \partial v_\lambda/\partial\lambda)$ satisfies the equation:

$$\begin{aligned}F_{(u,v)}(\lambda,u,v)\begin{pmatrix}\dfrac{\partial u_\lambda}{\partial\lambda}\\ \dfrac{\partial v_\lambda}{\partial\lambda}\end{pmatrix} &= \begin{pmatrix}\Delta\dfrac{\partial u_\lambda}{\partial\lambda}\\ \Delta\dfrac{\partial v_\lambda}{\partial\lambda}\end{pmatrix} + \lambda\begin{pmatrix}f_2'(u) & f_1'(v)\\ g_1'(u) & g_2'(v)\end{pmatrix}\begin{pmatrix}\dfrac{\partial u_\lambda}{\partial\lambda}\\ \dfrac{\partial v_\lambda}{\partial\lambda}\end{pmatrix}\\ &= -\begin{pmatrix}f_1(v)+f_2(u)\\ g_1(u)+g_2(v)\end{pmatrix},\end{aligned} \tag{3.15}$$

hence $(\partial u_\lambda/\partial\lambda, \partial v_\lambda/\partial\lambda) > 0$ from the maximum principle (Lemma 2.1(c)) and the fact that $\mu_1((\lambda, u_\lambda, v_\lambda)) > 0$ from stability of positive solutions. We claim that $\lambda^* = \infty$. Suppose not, then $\lambda^* < \infty$, and $\|(u_\lambda, v_\lambda)\|_X < \infty$, then one can show that the curve Γ can be extended to $\lambda = \lambda^*$ from some standard elliptic estimates, then from implicit function theorem, Γ can be extended beyond $\lambda = \lambda^*$, which is a contradiction; if $\lambda^* < \infty$, and $\|(u_\lambda, v_\lambda)\|_X = \infty$, a contradiction can be derived with the solution curve can not blow-up at a finite λ^*(see similar arguments for scalar equation in [24]). Hence we must have $\lambda^* = \infty$.

If there is another positive solution for some $\lambda > \lambda_*$, then the arguments above show this solution also belongs to a solution curve defined for $\lambda \in (\lambda_*, \infty)$, and the solutions on the curve are increasing in λ, but the nonexistence of positive solutions for $\lambda < \lambda_*$ and the local bifurcation at $\lambda = \lambda_*$ excludes the possibility of another solution curve. Hence the positive solution is unique for all $\lambda > \lambda_*$.□

We remark that nonlinearity $h(x)$ satisfying (B1), (B2) and (B3) appears very often in applied problems such as ecological studies and chemical reactions. For example, the Michaelis–Menten type functions $h(x) = ax/(1+bx)$ for $a, b > 0$ and $h(x) = 1 - e^{-ax}$ for $a > 0$, see [25].

4. Application: logistic type system

In this section, we consider the following semilinear elliptic system:

$$
\begin{cases}
\Delta u + \lambda(au - h_1(u) + f_1(v)) = 0, & x \in \Omega,\\
\Delta v + \lambda(bv - h_2(v) + g_1(u)) = 0, & x \in \Omega,\\
u(x) = v(x) = 0, & x \in \partial\Omega,
\end{cases}
\tag{4.1}
$$

where $a > 0$, $b > 0$. Suppose that each of the functions f_1, g_1, h_1 and h_2 is a smooth real-valued function defined on $\mathbb{R}^+$. Moreover, we assume that each of $f_1(v)$ and $g_1(u)$ still satisfies (B1) and (B2) as defined in Section 3; and for $i = 1, 2$, $h_i(x)$ satisfies:

(H1) $h_i(0) = h_i'(0) = 0$;
(H2) $h_i'(x) \ge 0$, $(h_i(x)/x)' \ge 0$, for all $x \ge 0$;
(H3) There exists a function $h_3 \in C^1(\mathbb{R}^+)$ such that for $u, v \ge 0$ and $u+v \ge 1$,

$$h_1(u) + h_2(v) \ge h_3(u+v) \quad \text{and} \quad \lim_{x\to\infty} \frac{h_3(x)}{x} = \infty.$$

Our main result in this section is as follows:

Theorem 4.1. *Assume that each of $f_1(v)$ and $g_1(u)$ satisfies* (B1) *and* (B2), *$f_1(0) = g_1(0) = 0$, and $h_i(x)$ (for $i = 1, 2$) satisfies* (H1)–(H3). *Then there exists $\lambda_* > 0$ such that,* (4.1) *has no positive solution when $\lambda \le \lambda_*$;* (4.1) *has a unique positive solution $(u(\lambda), v(\lambda))$ for $\lambda > \lambda_*$, and $\|u(\lambda) + v(\lambda)\|_\infty \le K$, where the constant K depends only on a, b, $g_1'(0)$, $f_1'(0)$ and h_3. Moreover,*

$\Sigma = \{(\lambda, u(\lambda), v(\lambda)) : \lambda > \lambda_*\}$ *is a smooth curve and* Σ *is unbounded in the positive* λ *direction.*

PROOF. A local bifurcation analysis using bifurcation from simple eigenvalue theorem similar to the one in the proof of Theorem 3.1 can be carried out at $\lambda = \lambda_*$ and we omit the details. We can get a similar analysis at the bifurcation point $(\lambda_*, 0, 0)$, where $\lambda_* = \lambda_1/\chi_J$, χ_J is the positive principal eigenvalue of the matrix

$$J_1 = \begin{pmatrix} a & f_1'(0) \\ g_1'(0) & b \end{pmatrix}.$$

Note that from $f_1(0) = g_1(0) = 0$, and (B2), we must have $f_1'(0) > 0$ and $g_1'(0) > 0$. Let (u, v) be a positive solution of (4.1) and let $(\varphi_1, k_*'\varphi_1)^{\mathrm{T}}$ satisfy

$$\Delta \begin{pmatrix} \varphi_1 \\ k_*'\varphi_1 \end{pmatrix} + \lambda_* \begin{pmatrix} a & g_1'(0) \\ f_1'(0) & b \end{pmatrix} \begin{pmatrix} \varphi_1 \\ k_*'\varphi_1 \end{pmatrix} = 0, \tag{4.2}$$

where $(1, k_*')^{\mathrm{T}}$ is the corresponding positive eigenvector of J_1^T with eigenvalue χ_J. Inner-producting the system (4.1) by $(\varphi_1, k_*'\varphi_1)$, the system (4.2) by (u, v), integrating over Ω and subtracting, and by using (B2) and (H2), we obtain that

$$\begin{aligned} \lambda_* &\int_\Omega [(au + f_1'(0)v)\varphi_1 + (g_1'(0)u + bv)k_*'\varphi_1]\, dx \\ &= \lambda \int_\Omega [(au - h_1(u) + f_1(v))\varphi_1 + (bv - h_2(v) + g_2(u))k_*'\varphi_1]\, dx \\ &< \lambda \int_\Omega [(au + f_1'(0)v)\varphi_1 + (bv + g_2'(0)v)k_*'\varphi_1]\, dx. \end{aligned} \tag{4.3}$$

Hence (4.1) has no positive solution when $\lambda \le \lambda_*$.

Next we claim that there exists a positive constant K which depends only on a, $g_1'(0)$, $f_1'(0)$, b and h_3 such that any positive solution (u, v) of (4.1) satisfies $\|u+v\|_\infty \le K$. Actually, adding the equation of u and the equation of v in (4.1), owing to (H3), we get that, if $x \in \Omega$ and $u(x) + v(x) \ge 1$, then

$$\begin{aligned} -\Delta(u+v) &= \lambda[au + g_1(u) + bv + f_1(v) - h_1(u) - h_2(v)] \\ &\le \lambda[au + g_1'(0)u + bv + f_1'(0)v - h_1(u) - h_2(v)] \\ &\le \lambda[M(u+v) - h_1(u) - h_2(v)] \le \lambda[M(u+v) - h_3(u+v)], \end{aligned} \tag{4.4}$$

where $M = \max\{(a+g_1'(0)), (f_1'(0)+d)\}$. Because of $\lim_{x\to\infty} \frac{h_3(x)}{x} = \infty$, then there exists $K > 0$ such that for $x > K$, $Mx - h_3(x) < 0$. By the maximum principle, we obtain $\|u+v\|_\infty \le K$.

From the assumptions, $au - h_1(u) + f_1(v)$, $dv - h_2(v) + g_1(u)$ satisfy (A_1). Hence from Theorem 2.3, any positive solution of (4.1) is stable. Thus, we can extend the solution branch Σ for $\lambda > \lambda_*$. By using the global bifurcation theorem of Rabinowitz [20], we can conclude that either Σ is unbounded or Σ contains another bifurcation point $(\lambda^*, 0, 0)$ with $\lambda^* \ne \lambda_*$. But latter case cannot happen as $\lambda = \lambda_*$ is the only λ so that the corresponding linearized operator has a positive eigenvector. Hence Σ is unbounded. Since all positive solutions (u, v) of (4.1) are uniformly bounded for $\lambda > \lambda_*$, then Σ must be unbounded in λ direction. Similar to the proof of Theorem 3.1, we obtain that (4.1) has a unique positive solution for any $\lambda > \lambda_*$. □

Example 4.2. Consider

$$\begin{cases} \Delta u + \lambda(au - u^p + cv) = 0, & x \in \Omega, \\ \Delta v + \lambda(bv - v^q + du) = 0, & x \in \Omega, \\ u(x) = v(x) = 0, & x \in \partial\Omega, \end{cases} \tag{4.5}$$

where $a, b, c, d > 0$, and $p, q > 1$. Then (4.5) is the classical cooperative logistic system when $p = q = 2$.

It is easy to see that the system (4.5) satisfies the conditions (B1), (B2), (H1) and (H2). We only need to verify the condition (H3). In fact, when $p = q = 2$, $u^2 + v^2 \geq (u + v)^2/2$, thus $h_3(u + v) = (u + v)^2/2$. When $p \neq q$, without loss of generality, we assume that $1 < q \leq p$. Consider the function $j(u) = u^p + (1 - u)^q$, then for $u \in [0, 1]$, $j(u)$ achieves a global minimum value $m_* > 0$ at some $u_* \in (0, 1)$. Thus $u^p + v^q \geq m_*$ for any $u + v = 1$, $u, v \geq 0$. This implies that for any $u, v \geq 0$ and $u+v \geq 1$, define $V = u+v$, then $u/V+v/V = 1$, and $(u/V)^p + (v/V)^q \geq m_*$. It follows that

$$\frac{u^p}{V^q} + \frac{v^q}{V^q} \geq \frac{u^p}{V^p} + \frac{v^q}{V^q} \geq m_*.$$

Hence we can define $h_3(u + v) = m_*(u + v)^q$ which satisfies (H3). Therefore, Theorem 4.1 implies the existence and uniqueness result for the positive solutions of (4.5), when $\lambda > \lambda_1/\chi_J$ and χ_J is the principal eigenvalue of the positive matrix

$$J = \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

References

[1] J. Busca and B. Sirakov, *Symmetry results for semilinear elliptic systems in the whole space*, J. Differential Equations **163** (2000), 41–56.

[2] Z.-Y. Chen, J.-L. Chern, J. Shi and Y.-L. Tang, *On the uniqueness and structure of solutions to a coupled elliptic system*, J. Differential Equations **249** (2010), 3419–3442.

[3] J.-L. Chern, Y.-L. Tang, Ch.-Sh. Lin and J. Shi, *Existence, uniqueness and stability of positive solutions to sublinear elliptic systems*, Proc. Roy. Soc. Edinburgh Sect. A **141** (2011), 45–64.

[4] M.G. Crandall and P.H. Rabinowitz, *Bifurcation from simple eigenvalues*, J. Funct. Anal. **8** (1971), 321–340.

[5] R. Cui, Y. Wang and J. Shi, *Uniqueness of the positive solution for a class of semilinear elliptic systems*, Nonlinear Anal. **67** (2007), 1710–1714.

[6] R. Cui, J. Shi and Y. Wang, *Existence and uniqueness of positive solutions for a class of semilinear elliptic systems*, Acta Math. Sin. (English Ser.) **27** (2011), 1079–1090.

[7] R. Dalmasso, *Existence and uniqueness of positive solutions of semilinear elliptic systems*, Nonlinear Anal. **39** (2000), 559–568.

[8] ———, *Existence and uniqueness of positive radial solutions for the Lane–Emden system*, Nonlinear Anal. **57** (2004), 341–348.

[9] D.D. Hai, *Existence and uniqueness of solutions for quasilinear elliptic systems*, Proc. Amer. Math. Soc. **133** (2005), 223–228.

[10] ———, *Uniqueness of positive solutions for semilinear elliptic systems*, J. Math. Anal. Appl. **313** (2006), 761–767.

[11] D.D. Hai and R. Shivaji, *An existence result on positive solutions for a class of semilinear elliptic systems*, Proc. Roy. Soc. Edinburgh Sect. A **134** (2004), 137–141.

[12] T. Kato, *Perturbation Theory for Linear Operators*, Reprint of the 1980 edition, Classics in Mathematics, Springer–Verlag, Berlin, 1995.

[13] Ph. Korman, *Global solution curves for semilinear systems*, Math. Methods Appl. Sci. **25** (2002), 3–20.

[14] Ph. Korman, Y. Li and T. Ouyang, *Exact multiplicity results for boundary value problems with nonlinearities generalising cubic*, Proc. Roy. Soc. Edinburgh Sect. A **126** (1996), 599–616.

[15] ———, *An exact multiplicity result for a class of semilinear equations*, Comm. Partial Differential Equations **22** (1997), 661–684.

[16] Ph. Korman and J. Shi, *On Lane–Emden type systems*, Discrete Contin. Dyn. Syst. (2005), 510–517.

[17] L. Ma and B. Liu, *Symmetry results for decay solutions of elliptic systems in the whole space*, Adv. Math. **225** (2010), 3052–3063.

[18] T. Ouyang and J. Shi, *Exact multiplicity of positive solutions for a class of semilinear problem*, J. Differential Equations **146** (1998), 121–156.

[19] ———, *Exact multiplicity of positive solutions for a class of semilinear problem* II, J. Differential Equations **158** (1999), 94–151.

[20] P.H. Rabinowitz, *Some global results for nonlinear eigenvalue problems*, J. Func. Anal. **7** (1971), 487–513.

[21] J. Serrin and H. Zou, *Existence of positive solutions of the Lane–Emden system*, Atti Sem. Mat. Fis. Univ. Modena **46** (1998), 369–380.

[22] ———, *Existence of positive entire solutions of elliptic Hamiltonian systems*, Comm. Partial Differential Equations **23** (1998), 577–599.

[23] D. Serre, *Matrices: theory and applications*, Graduate Texts in Mathematics, vol. 216, Springer, Berlin, 2002.

[24] J. Shi, *Blow up points of solution curves for a semilinear problem*, Topol. Methods Nonlinear Anal **15** (2000), 251–266.

[25] J. Shi and R. Shivaji, *Global bifurcation for concave semipositon problems*, Advances in Evolution Equations: Proceedings in honor of J.A. Goldstein's 60th birthday (G.R. Goldstein, R. Nagel and S. Romanelli, eds.), Marcel Dekker, Inc., New York, Basel, 2003, pp. 385–398.

[26] ———, *Exact multiplicity of positive solutions to cooperative elliptic systems*, Preprint (2009).

[27] G. Sweers, *Strong positivity in $C(\overline{\Omega})$ for elliptic systems*, Math. Z. **209** (1992), 251–271.

[28] W.C. Troy, *Symmetry properties in systems of semilinear elliptic equations*, J. Differential Equations **42** (1981), 400–413.

Manuscript received November 21, 2011

Renhao Cui, Ping Li and Yunwen Wang
Y.Y. Tseng Functional Analysis Research Center
and School of Mathematical Sciences
Harbin Normal University
Harbin, Heilongjiang, 150025, P.R. CHINA

E-mail address: renhaocui@gmail.com, wangyuwen1950@yahoo.com.cn

Junping Shi
Department of Mathematics
College of William and Mary
Williamsburg, Virginia, 23187-8795, USA

E-mail address: shij@math.wm.edu