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APPLICATIONS OF WEIGHTED MAPS TO PERIODIC PROBLEMS OF AUTONOMOUS DIFFERENTIAL EQUATIONS

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Abstract. In this paper we present a new approach for solving the problem of the existence of closed trajectories for autonomous differential equations without the uniqueness property. To this aim, we are using a special class of set-valued maps, called weighted carriers or weighted maps.

Introduction

In this paper we are interested in the existence of solutions of the following problem

$$\text{(P)}\qquad \begin{cases} \dot u(s) = f(u(s)) & \text{for almost all } s\in[0,T],\\ u(s)\in M\times S^1 & \text{for all } s\in[0,T],\\ u(0)=u(t_0) & \text{for some } 0<t_0\leq T,\end{cases}$$

where $M\subset\mathbb{R}^n$ is closed and contractible, $f\colon M\times S^1\to\mathbb{R}^{n+2}$ is continuous and $T>0$ (additional assumptions on $M\subset\mathbb{R}^n$ and f will be specified later). A solution u of the problem (P) will be called a *closed trajectory* or a *periodic solution* (1).

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(1) Throughout this paper by a solution to a differential equation $\dot u=f(u)$ we mean an absolutely continuous function u that satisfies the equation $\dot u(s)=f(u(s))$ almost everywhere.

The above problem for smooth maps f and smooth manifolds M has been treated in the following papers [9], [11]. It should be noted that this problem requires extreme caution because the counterexample has been provided by F.B. Fuller in [17] (see also Figure 8 in this paper). Namely, he constructed a nonvanishing vector field in a solid 4-dimensional torus $D_3(0,1) \times S^1$ with no closed trajectories ([2]).

Recall that the above problem in the case when $M \times S^1$ is replaced by any compact set K with $\chi(K) \neq 0$ has been studied by many authors (see for example [5], [31]), where $\chi(K)$ is the Euler characteristic defined by the formula $\chi(K) := \sum_{i \geq 0} (-1)^i \dim H_i(K;\mathbb{Q})$ (where $H_*(\,\cdot\,;\mathbb{Q})$ denotes the singular homology functor with rational coefficients). Notice that the methods discussed in the mentioned papers cannot be applied in our case since $\chi(M \times S^1) = 0$.

The main aim of this paper is to give sufficient conditions under which the problem (P) admits a solution. It turned out that there was a need to apply set-valued weighted carriers introduced by G. Darbo and further developed by several authors as G. Conti, J. Pejsachowicz and R. Skiba ([11], [26], [29], [35], [36]). We should say a few words why set-valued weighted maps play an important role in our considerations. Let X be a metric space and let $\Pi\colon X \times \mathbb{R} \to X$ be a flow ([3]). Consider $Y \subset X$. Let $Y_0 := \{y \in Y \mid \text{there exists } t > 0 \text{ such that } \Pi(y,t) \notin Y\}$. Let $\tau\colon Y_0 \to [0,\infty)$ be defined by $\tau(y) := \sup\{t \geq 0 \mid \Pi(\{y\} \times [0,t]) \subset Y\}$. Recall that the above map, for example, is used to prove the Ważewski principle. In general, the above function is not continuous. Therefore, to solve the problem (P) we replaced the function τ by the following set-valued map $\varphi\colon Y_0 \multimap [0,\infty)$ defined as follows $\varphi(y) := \{t \geq 0 \mid \Pi(y,t) \in \operatorname{bd} Y\}$. It turns out that under our assumptions the latter map is well-defined and belongs to the class of set-valued weighted carriers. That is why we use weighted maps in our considerations.

It should be noted that this article is strongly motivated by the papers [9], [11] in which the problem (P) is also considered. But in [9], [11] the authors assumed that the right-hand side of (P) is at least of class C^1. In this article we reject this assumption which in turn implies that this problem is more involved.

This article is organized as follows. After this Introduction it consists of seven sections. The first section is devoted to some preliminaries. Whereas the second section contains a slightly modified construction of the intersection index (comp. [16] and [11]) which is much more useful and convenient in our

([2]) It should be noted that in the paper [17] the solutions of differential equations are of class C^1 but it is not hard to see that all results obtained in [17] are also true in the case of abolutely continuous solutions.

([3]) By a flow we mean a continuous function $\Pi\colon X \times \mathbb{R} \to X$ satisfying the following conditions: (a) $\Pi(x,0) = x$ for all $x \in X$, and (b) $\Pi(x,t+s) = \Pi(\Pi(s,x),t)$ for all $t,s \in \mathbb{R}$ and $x \in X$.

studies. In the third section we recall some basic definitions and facts concerning weighted carriers. Furthermore, we prove that set-valued maps which appear in the study of the problem (P) belong to the class of weighed maps. For more information about weighted carriers we refer the reader to [29] and [35]. In the next section we will present the main results of this paper. Namely, we prove that under some assumptions on M and f the problem (P) admits a solution. The fifth section concerns also the problem (P) but on manifolds. We show that this assumption allows us to formulate easily verifiable conditions ensuring the existence of closed trajectories. In the short sixth section we provide some comments about possible extensions and applications of the results obtained in this article. Section 7 contains for the reader's convenience some technical proofs of results from Section 3.

Summing up this Introduction, the main results of this paper are contained in Theorems 4.13, 4.17 and 5.11. As far as the author knows, this is the first time that periodic results for differential equations without uniqueness property have been obtained by means of a set-valued weighted analysis.

1. Preliminaries

We start with some notations which will be used in this article. Throughout the paper by a *space* we mean a metric space, by a *pair of spaces* – a pair (X,A), where X is a space and $A \subset X$; any space X is identified with the pair $(X,\emptyset)$; all single-valued maps between spaces are considered to be continuous. Let (X,d) be a metric space. Given $Y \subset X$ and $A \subset Y$, by $\mathrm{int}_Y A$, $\mathrm{cl}_Y A$ and $\mathrm{bd}_Y A$ we denote the interior, the closure and the boundary of A in Y, respectively, while $\mathrm{int} A$, $\mathrm{cl} A$ and $\mathrm{bd}\, A$ denote the interior, the closure and the boundary of A in X. For any $\varepsilon > 0$,

$$B(A,\varepsilon) := \{x \in X \mid \mathrm{dist}(x,A) < \varepsilon\}, \quad D(A,\varepsilon) := \{x \in X \mid \mathrm{dist}(x,A) \le \varepsilon\},$$

where $\mathrm{dist}(x,A) := \inf\{d(x,a) \mid a \in A\}$ is the distance of $x \in X$ from A. In particular, by $D_n(x,r)$ (resp. $B_n(x,r)$) we will denote the closed (resp. open) ball around $x \in \mathbb{R}^n$ of radius $r > 0$, $n \ge 1$. The Euclidean norm and the scalar product in $\mathbb{R}^n$ are denoted by $|\cdot|$ and $\langle\cdot\,,\cdot\rangle$, respectively.

By $d_H(A_1,A_2)$ we shall denote the Hausdorff separation between two nonempty compact subsets A_1 and A_2 of X defined by $d_H(A_1,A_2) := \sup_{a\in A_1} \mathrm{dist}(a,A_2)$. It is well-known that $d_H(A_1,A_2) < \varepsilon$ if and only if $A_1 \subset B(A_2,\varepsilon)$.

Now we recall some notions of nonsmooth analysis (see [10]). Let $M \subset \mathbb{R}^n$ be a nonempty closed set. A function $d_M\colon \mathbb{R}^n \to [0,\infty)$ defined by $d_M(x) := \inf\{|x-y| \mid y \in M\}$ is called *the distance function* to M. For any $x \in M$ we put

$$T_M(x) := \left\{ v \in \mathbb{R}^n \;\middle|\; \liminf_{h\to 0^+} \frac{d_M(x+hv)}{h} = 0 \right\}, \tag{1.1}$$

where $T_M(x)$ is called the *Bouligand contingent cone* to M at x.

We will say that $f\colon M \to \mathbb{R}^n$ is *tangent* if $f(x) \in T_M(x)$ for all $x \in M$ and in this case we will write $f \in \operatorname{Vect}(M)$. Given a closed subset $M \subset \mathbb{R}^n$, the subset

$$TM = \{(x,v) \in M \times \mathbb{R}^n \mid v \in T_M(x)\}$$

of $\mathbb{R}^n \times \mathbb{R}^n$ is called *the tangent bundle* of M. Observe that $f \in \operatorname{Vect}(M)$ induces the following continuous map $Tf\colon M \to TM$ given by $(Tf)(x) := (x, f(x))$.

Recall two properties of the Bouligand cone which will be used in this paper (see [4]):

- If $M = M_1 \times M_2$, then $T_{M_1\times M_2}(x_1,x_2) = T_{M_1}(x_1) \times T_{M_2}(x_2)$ and $T(M_1 \times M_2) = TM_1 \times TM_2$.
- If M is a smooth manifold without boundary, then $T_M(x) = T_xM$, where T_xM stands for the tangent space of M at x.

Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function. The generalized directional derivative of f at x in the direction $v \in \mathbb{R}^n$ in the sense of Clarke is defined as follows

$$f^\circ(x;v) = \limsup_{\substack{y\to x\\ h\to 0^+}} \frac{f(y+hv)-f(y)}{h}.$$

The *generalized gradient* of f at x is defined by

$$\partial f(x) := \{p \in \mathbb{R}^n \mid \langle p,u\rangle \le f^\circ(x;u) \text{ for all } u \in \mathbb{R}^n\}.$$

Recall that if f is C^1, then $\partial f(x) = \{\nabla f(x)\}$. Following [5], [12] we recall the notion of a *strictly regular set*. Assume that $f\colon \operatorname{Dom}(f) \to \mathbb{R}$ is a locally Lipschitz function, where the domain $\operatorname{Dom}(f)$ is open in $\mathbb{R}^n$. We put

$$M := \{x \in \operatorname{Dom}(f) \mid f(x) \le 0\}. \tag{1.2}$$

Notice that M need not be closed in $\mathbb{R}^n$. We will say that M is represented by f.

Definition 1.1 ([5], [12]). We say that the set M given by (1.2) (represented by a locally Lipschitz function $f\colon \operatorname{Dom}(f) \to \mathbb{R}$) is said to be *strictly regular* if

(a) M is closed,
(b) there is a neighbourhood U of M such that $\inf_{y\in U\setminus M} |||\partial f(y)||| > 0$, where

$$|||\partial f(y)||| := \inf_{p\in\partial f(y)} |p|.$$

In what follows, we will need the following property of strictly regular sets.

Remark 1.2. It is easily seen that if $M \subset \mathbb{R}^n$ is strictly regular, then so is $M \times \mathbb{R}$. Indeed, let $f: \mathrm{Dom}(f) \to \mathbb{R}$ be a locally Lipschitz function and let $M := \{x \in \mathrm{Dom}(f) \mid f(x) \le 0\}$. Observe that

$$M \times \mathbb{R} = \{(x, z) \in \mathrm{Dom}(f) \times \mathbb{R} \mid \widetilde{f}(x, z) \le 0\},$$

where $\widetilde{f}: \mathrm{Dom}(\widetilde{f}) \to \mathbb{R}$ is defined by $\widetilde{f}(x, z) := f(x)$ for all $(x, z) \in \mathrm{Dom}(\widetilde{f}) := \mathrm{Dom}(f) \times \mathbb{R}$. Moreover, $\widetilde{f}^\circ((x, z); (u, w)) = f^\circ(x; u)$ for all $(x, z) \in \mathrm{Dom}(\widetilde{f})$ and $(u, w) \in \mathbb{R}^n \times \mathbb{R}$ and hence

$$\partial\widetilde{f}(x, z) = \partial f(x) \times \{0\}.$$

From this it follows that

$$\inf_{y \in U - M} |||\partial f(y)||| > 0 \Leftrightarrow \inf_{(y,z) \in (U-M) \times \mathbb{R}} |||\partial\widetilde{f}(y, z)||| > 0,$$

which implies that $M \times \mathbb{R}$ is strictly regular.

Notice that the class of strictly regular sets is quite large. Below we shall provide a few examples of strictly regular sets (see [5] or [12]):

- Any closed convex subset K of $\mathbb{R}^n$, represented by d_K, is strictly regular.
- A proximate retract $K \subset \mathbb{R}^n$ is strictly regular ([4]).
- Any closed manifold $K \subset \mathbb{R}^n$ of class C^2 is strictly regular.
- If a compact subset $M \subset \mathbb{R}^n$ is strictly regular, then M is a neighbourhood retract in $\mathbb{R}^n$ ([5]).

The terminology and results from the algebraic topology which are used here are quoted from the books [16] and [22], [39]. In particular, $H_*(X, Y; G)$ (resp. $\check{H}_*(X, Y; G)$) denotes the singular homology group (resp. the Čech homology group) of a pair (X, Y) with coefficients in a group G. A compact space X will be called acyclic (resp. positively acyclic) if $\check{H}_*(X; \mathbb{Q}) = \check{H}_*(\mathrm{pt}; \mathbb{Q})$ (resp. $\check{H}_i(X; \mathbb{Q}) = \check{H}_i(\mathrm{pt}; \mathbb{Q})$) for all $i \ge 1$), where pt is a one-point space and $\mathbb{Q}$ denotes the set of rational numbers.

Given a metric space (X, d), by $C(X, \mathbb{R}^n)$ we denote the space of all bounded and continuous functions $f: X \to \mathbb{R}^n$ from X to $\mathbb{R}^n$ equipped with the norm $\|f\|_C := \sup_{x \in X} |f(x)|$.

Recall that a space X is an absolute neighbourhood retract (or ANR) if for every space Y and a homeomorphic embedding $i: X \to Y$ of X onto a closed subset $i(X) \subset Y$ there is an open neighbourhood U of $i(X)$ in Y and a function $r: U \to i(X)$ such that $r(y) = y$ for all $y \in i(X)$. In particular, any compact convex set is an ANR (see [7]).

([4]) Following [30] we say that a closed subset $K \subset \mathbb{R}^n$ is a proximate retract if there exists a neighbourhood U of K and a retraction $r: U \to K$ such that $|r(x) - x| = d_K(x)$.

We finish this section by recalling some definitions and facts from the theory of set-valued maps. Given two spaces X and Y, by a set-valued map (denoted by the symbol $\multimap$) $\varphi\colon X \multimap Y$ we mean a transformation which assigns to any $x \in X$ a nonempty compact set $\varphi(x) \subset Y$. A set-valued map $\varphi\colon X \multimap Y$ is upper semicontinuous (written usc) if, given an open subset $V \subset Y$, the set $\{x \in X \mid \varphi(x) \subset V\}$ is open. A map φ is compact if $\varphi(X) = \bigcup_{x\in X} \varphi(x)$ is relatively compact. If $\varphi\colon X \multimap Y$ is usc and $K \subset X$ is compact, then $\varphi(K)$ is compact. If $f\colon X \to Y$ (resp. $\varphi\colon X \multimap Y$) is a map (resp. set-valued map), then $\mathrm{Gr}(f)$ (resp. $\mathrm{Gr}(\varphi)$) stands for the graph of f (resp. of φ), i.e.

$$\mathrm{Gr}(f) := \{(x, f(x)) \mid x \in X\}, \qquad \mathrm{Gr}(\varphi) := \{(x, y) \in X \times Y \mid y \in \varphi(x)\}.$$

We refer the reader to the book [21] which is a comprehensive source of set-valued maps.

2. Intersection index

In this section we are going to give a construction of an intersection index. But in our approach we also use some ideas from the construction of the fixed point index for single-valued maps due to Dold and Granas (see [16] and [22]). To be brief, we shall present a slightly modified version of the intersection index given in [11], which will turn out to be very useful in our considerations. In this section we will use the singular homology functor with integer coefficients $\mathbb{Z}$, which will be omitted from the notation.

Now we are going to define a fundamental class O_K of $H_1(\mathbb{R}, \mathbb{R} - K)$. Let $K \subset \mathbb{R}$ be compact. Since $H_1(\mathbb{R}, \mathbb{R} - 0) = \mathbb{Z}$, we can choose one of the two possible generators once and for all and call it by O. Since K is compact, there exists $r > 0$ such that $K \subset B_1(0, r)$.

Consider the following diagram

$$H_1(\mathbb{R}, \mathbb{R} - K) \xleftarrow{i_*} H_1(\mathbb{R}, \mathbb{R} - B_1(0, r)) \xrightarrow{j_*} H_1(\mathbb{R}, \mathbb{R} - 0)$$

in which i, j are the respective inclusions. Since j_* is an isomorphism (see [2, Lemma 10.2.12]), we can define O_K as follows

$$O_K = (i_* \circ (j_*)^{-1})(O). \tag{2.1}$$

It should be noted that the above definition does not depend on the choice of $r > 0$. Indeed, this follows from the fact that the following diagram

$$\begin{array}{ccccc} H_1(\mathbb{R}, \mathbb{R} - K) & \longleftarrow & H_1(\mathbb{R}, \mathbb{R} - B_1(0, r)) & \longrightarrow & H_1(\mathbb{R}, \mathbb{R} - 0) \\ \Big\downarrow{\scriptstyle \mathrm{id}} & & \Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{id}} \\ H_1(\mathbb{R}, \mathbb{R} - K) & \longleftarrow & H_1(\mathbb{R}, \mathbb{R} - B_1(0, \widetilde{r})) & \longrightarrow & H_1(\mathbb{R}, \mathbb{R} - 0) \end{array}$$

is commutative, where the unlabelled arrows are induced by the inclusions and $\widetilde{r} > r$.

Observe that if U is an open set with $K \subset U \subset \mathbb{R}$, then the excision property of the singular homology implies that the induced homomorphism $i_*: H_1(U, U - K) \to H_1(\mathbb{R}, \mathbb{R} - K)$ is an isomorphism. Thus one can put

$$O_K^U := i_*^{-1}(O_K). \tag{2.2}$$

The following lemma is easy to prove.

Lemma 2.1 (see [11] or [21, Chapter 1]). *Let $K \subset K_1 \subset V \subset U$, where K, K_1 are compact and U, V are open subsets of $\mathbb{R}$. Let $k: (V, V - K_1) \to (U, U - K)$ be the inclusion. Then $k_*(O_{K_1}^V) = O_K^U$.*

Let E be a subspace of $\mathbb{R}^n$, let L be a closed subset of E such that $H_1(E, E - L) \neq 0$ and let U be an open subset of $\mathbb{R}$. We put

$$C(U, E; L) := \{f: U \to E \mid f^{-1}(L) \text{ is compact}\}. \tag{2.3}$$

Now we are ready to define the following concept:

Definition 2.2. Under the above assumptions, we define the intersection index $i(f, U)$ of $f \in C(U, E; L)$ by

$$i(f, U) := f_*(O_K^U),$$

where $f_*: H_1(U, U - K) \to H_1(E, E - L)$ is the homomorphism induced by f and $K := f^{-1}(L)$.

Now we shall present two lemmas which will be needed in the proof of the main properties of the intersection index.

Lemma 2.3. *Given $f \in C(U, E; L)$, assume that F is a compact set such that $K := f^{-1}(L) \subset F$ and $F \subset U$. Then*

$$\widetilde{f}_*(O_F^U) = f_*(O_K^U),$$

where $\widetilde{f}: (U, U - F) \to (E, E - L)$ is induced by f.

Proof. It follows easily from Lemma 2.1 and the fact that the following diagram

$$\begin{array}{ccc} H_1(U, U - K) & \xrightarrow{f_*} & H_1(E, E - L) \\ {\scriptstyle i_*}\uparrow & & \uparrow{\scriptstyle \mathrm{id}} \\ H_1(U, U - F) & \xrightarrow[\widetilde{f}_*]{} & H_1(E, E - L) \end{array}$$

is commutative, where $i: (U, U - F) \to (U, U - K)$ is the inclusion. □

Lemma 2.4 (see [8, Appendix B]). *Let U_1 and U_2 be two disjoint open subsets of $\mathbb{R}$ and let $F \subset \mathbb{R}$ be closed such that $F \subset U_1 \cup U_2$. Then the following diagram is commutative:*

$$\begin{array}{ccc}
H_1(\mathbb{R}, \mathbb{R} - F) & \xrightarrow{(i_{1*}, i_{1*})} & H_1(\mathbb{R}, \mathbb{R} - F_1) \oplus H_1(\mathbb{R}, \mathbb{R} - F_2) \\
{\scriptstyle j_*}\uparrow & & \uparrow{\scriptstyle j_{1*} \oplus j_{2*}} \\
H_1(U, U - F) & \xleftarrow[h_{1*} + h_{2*}]{} & H_1(U_1, U_1 - F_1) \oplus H_1(U_2, U_2 - F_2)
\end{array}$$

where all the homomorphisms in the above diagram are induced by the inclusions and $F_i := F \cap U_i$, for $i = 1, 2$, and $U := U_1 \cup U_2$.

The intersection index satisfies the following properties:

Proposition 2.5. *Let $f \in C(U, E; L)$ and let $K := f^{-1}(L)$.*

(a) (Existence) *If $i(f, U) \neq 0$, then $K \neq \emptyset$.*

(b) (Excision and Additivity) *If $K \subset \bigcup_{i=1}^{k} U_i$, where U_i, $1 \leq i \leq k$, are disjoint open subsets of U, then*

$$i(f, U) = \sum_{i=1}^{k} i(f|U_i, U_i).$$

(c) (Homotopy invariance) *Let $h \colon U \times [0, 1] \to E$ be a continuous function. If*

$$K_h := \{x \in U \mid h(x, t) \in L \text{ for some } t \in [0, 1]\}$$

is compact, then $i(h_0, U) = i(h_1, U)$.

Proof. (a) Existence. Suppose on the contrary that $K = \emptyset$. Then we have $H_1(U, U - K) = H_1(U, U) = 0$ and, in view of (2.2), $O_K^U = 0$. Thus, taking into account Definition 2.2, we deduce that $i(f, U) = 0$, a contradiction.

(b) Excision and Additivity. The proof will be divided into two steps.

Step 1 (Excision). We assume that $K \subset U_0$, where $U_0 \subset U$. Consider the following commutative diagram:

$$\begin{array}{ccccc}
H_1(\mathbb{R}, \mathbb{R} - K) & \xleftarrow{\simeq} & H_1(U, U - K) & \xrightarrow{f_*} & H_1(E, E - L) \\
{\scriptstyle \mathrm{id}}\uparrow & & \uparrow & \nearrow{\scriptstyle (f|U_0)_*} & \\
H_1(\mathbb{R}, \mathbb{R} - K) & \xleftarrow[\simeq]{} & H_1(U_0, U_0 - K) & &
\end{array}$$

where the unlabelled arrows are induced by the inclusions. Now from the above diagram it follows immediately that $i(f, U) = i(f, U_0)$, as required.

Step 2 (Additivity). We assume that $K \subset \bigcup_{i=1}^{k} U_i$, where U_i, $1 \leq i \leq k$, are disjoint open subsets of U. In addition, without loss of generality we can assume that $k = 2$. Let $K_i := K \cap U_i$, $1 \le i \le 2$. By Step 1, we may replace U by $U_1 \cup U_2$. Now, it suffices to prove that the following diagram is commutative:

$$\begin{array}{ccc}
H_1(\mathbb{R}, \mathbb{R} - 0) & \xrightarrow{(\mathrm{id}_*, \mathrm{id}_*)} & H_1(\mathbb{R}, \mathbb{R} - 0) \oplus H_1(\mathbb{R}, \mathbb{R} - 0) \\
\simeq \uparrow & & \uparrow \simeq \\
H_1(\mathbb{R}, \mathbb{R} - B_1(0,r)) & \xrightarrow{(\mathrm{id}_*, \mathrm{id}_*)} & H_1(\mathbb{R}, \mathbb{R} - B_1(0,r)) \oplus H_1(\mathbb{R}, \mathbb{R} - B_1(0,r)) \\
\downarrow & & \downarrow \\
H_1(\mathbb{R}, \mathbb{R} - K) & \xrightarrow{(i_{1*}, i_{2*})} & H_1(\mathbb{R}, \mathbb{R} - K_1) \oplus H_1(\mathbb{R}, \mathbb{R} - K_2) \\
\simeq \uparrow & & \uparrow \simeq \\
H_1(U, U - K) & \xleftarrow{h_{1*} + h_{2*}} & H_1(U_1, U_1 - K_1) \oplus H_1(U_2, U_2 - K_2) \\
\downarrow f_* & \swarrow f_{1*} + f_{2*} & \\
H_1(E, E - L) & &
\end{array}$$

in which, except for f and its restrictions, all the homomorphisms are induced by the inclusions. Indeed, let us observe that the lower square of the above diagram commutes by Lemma 2.4, while the commutativity of the remaining squares and the lower triangle is obvious, which completes the proof of the additivity property.

(c) Homotopy invariance. First, let us observe that $(h_t)^{-1}(L) \subset K_h$ for every $t \in [0,1]$. Consider now the following diagram:

$$H_1(\mathbb{R}, \mathbb{R} - K_h) \xleftarrow{j_*} H_1(U, U - K_h) \xrightarrow{(h_t)_*} H_1(E, E - L),$$

for any $t \in [0,1]$. Then, by Definition 2.2 and Lemma 2.3, we obtain

$$i(h_t, U) = (\widetilde{h}_t)_*(O^U_{K_t}) = (h_t)_*(O^U_{K_h}) \tag{2.4}$$

for all $t \in [0,1]$, where

$$(\widetilde{h}_t)_* \colon H_1(U, U - K_t) \to H_1(E, E - L)$$

is induced by h_t and $K_t := h_t^{-1}(L)$. From the homotopy invariance of the singular homology functor it follows that

$$(h_0)_* = (h_1)_*. \tag{2.5}$$

Consequently, taking into account (2.4) and (2.5), we get $i(h_0, U) = i(h_1, U)$, which completes the proof. □

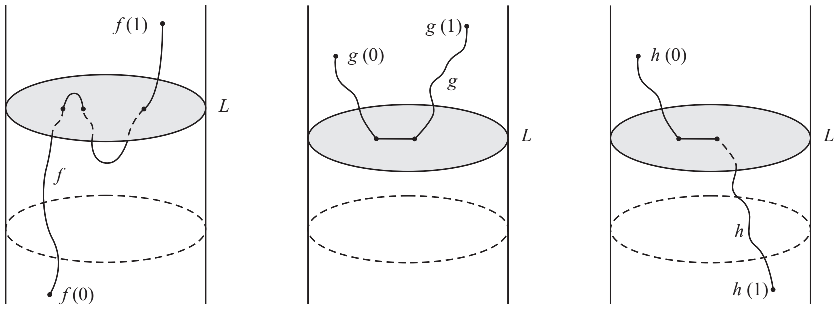


Figure 1. $i(f,(0,1)) = 1$, $i(g,(0,1)) = 0$, $i(h,(0,1)) = -1$

We finish this section by illustrating Definition 2.2 by putting $E = D_2(0,1) \times \mathbb{R}$ and $L = D_2(0,1) \times \{c\}$ (see Figure 1).

3. Weighted carriers

In this section, we shall survey the most important properties of set-valued weighted carriers which will be used in the sequel. For a complete description of the theory of set-valued weighted carriers we refer the reader to the monograph [35] (see also: [11], [26]–[29], [32], [24], [25]).

In what follows, we shall use the following notation. Given any set-valued map $\Phi\colon X \multimap Y$, we put

$$D(\Phi) = \{(V,x) \mid V \text{ is an open subset of Y and } \Phi(x) \cap \operatorname{bd} V = \emptyset\}.$$

We begin with the following two definitions.

Definition 3.1. An usc set-valued map $\Phi\colon X \multimap Y$ with compact values is said to be a weighted carrier if there exists a function $I_{\mathrm{wloc}}\colon D(\Phi) \to \mathbb{Q}$ satisfying the following three conditions:

(a) (Existence) If $I_{\mathrm{wloc}}(\Phi,V,x) \neq 0$, then $\Phi(x) \cap V \neq \emptyset$.
(b) (Local invariance) For every $(V,x) \in D(\Phi)$ there exists an open neighbourhood U_x of x such that, for all $\widetilde{x} \in U_x$,

$$I_{\mathrm{wloc}}(\Phi,V,x) = I_{\mathrm{wloc}}(\Phi,V,\widetilde{x}).$$

(c) (Additivity) If $\Phi(x) \cap V \subset \bigcup_{i=1}^{k} V_i$, where V_i, $1 \leq i \leq k$, are open disjoint subsets of V, then

$$I_{\mathrm{wloc}}(\Phi,V,x) = \sum_{i=1}^{k} I_{\mathrm{wloc}}(\Phi,V_i,x).$$

Remark 3.2. (a) The additivity property in the case of $k = 1$ will be called the excision property.

(b) It is easy to see that a function $I_{\mathrm{wloc}}\colon D(\Phi) \to \mathbb{Q}$ defined by $I_{\mathrm{wloc}}(\Phi, V, x) = 0$ for all $(V, x) \in D(\Phi)$ satisfies all the conditions of Definition 3.1. But this example is trivial and it will not be interesting for us and we will always try to look for a nontrivial function I_{wloc} for Φ.

Definition 3.3. Let $\Phi\colon X \multimap Y$ be a weighted carrier and let X be a connected space. Then the number

$$I_w(\Phi) := I_{\mathrm{wloc}}(\Phi, Y, x_0)$$

is said to be the weighted index of Φ, where $x_0 \in X$ is a fixed point.

Proposition 3.4 (see [35, Proposition 3.2.4]). *Let $\Phi_2\colon Y \multimap Z$ and $\Phi_1\colon X \multimap Y$ be two weighted carriers. If Φ_1 is a set-valued map with connected values, then $\Phi_2 \circ \Phi_1\colon X \multimap Z$ is a weighted carrier, where $I_{\mathrm{wloc}}\colon D(\Phi_2 \circ \Phi_1) \to \mathbb{Q}$ is given by*

$$I_{\mathrm{wloc}}(\Phi_2 \circ \Phi_1, U, x) := I_{\mathrm{wloc}}(\Phi_2, U, y),$$

where $(U, x) \in D(\Phi_2 \circ \Phi_1)$ and $y \in \Phi_1(x)$ is any fixed point. In particular, if X and Y are connected, then $I_w(\Phi_2 \circ \Phi_1) = I_w(\Phi_2)$.

Below we shall present a number of examples of weighted carriers.

Example 3.5. It is easy to see that if a set-valued map $\Phi\colon X \multimap Y$ is usc with compact and connected values, then Φ is a weighted carrier. Indeed, it suffices to define a function $I_{wloc}\colon D(\Phi) \to \mathbb{Q}$ as follows

$$I_{\mathrm{wloc}}(\Phi, V, x) := \begin{cases} 1 & \text{if } \Phi(x) \cap V \neq \emptyset, \\ 0 & \text{if } \Phi(x) \cap V = \emptyset, \end{cases}$$

for any $(V, x) \in D(\Phi)$. In particular, if $\Phi\colon \mathbb{R} \multimap \mathbb{R}$ is defined by $\Phi(x) = [-x, x]$,

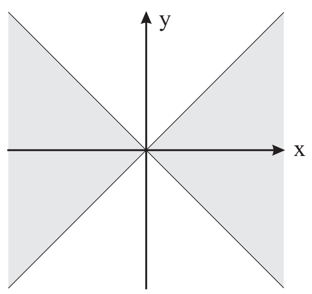


Figure 2. A graph of a set-valued map $\Phi\colon \mathbb{R} \multimap \mathbb{R}$

then $D(\Phi)=\{(V,x)\mid V\subset\mathbb{R}$ is open and $[-x,x]\subset\mathbb{R}-\operatorname{bd}V\}$ and

$$I_{\rm wloc}(\Phi,V,x)=\begin{cases}1 & \text{if } x\in V,\\ 0 & \text{if } x\notin V.\end{cases}$$

Example 3.6. Let X be a compact ANR and let $f\colon X\times[0,1]\to X$ be a continuous function with the Lefschetz number $\lambda(f_0)\neq 0$ of f_0, where $f_0(\,\cdot\,)=f(\,\cdot\,,0)$. Then an usc set-valued map $\Phi\colon[0,1]\multimap X$ defined by $\Phi(t)=\{x\in X\mid f_t(x):=f(x,t)=x\}$, for all $t\in[0,1]$, is a weighted carrier. Indeed, it suffices to define a nontrivial function $I_{\rm wloc}\colon D(\Phi)\to\mathbb{Q}$ by

$$I_{\rm wloc}(\Phi,U,t):=\operatorname{ind}(f_t,U,X),$$

where $\operatorname{ind}(f_t,U,X)$ denotes the fixed point index for single-valued maps (for more information on the fixed point index for single-valued maps see [22]).

In what follows, we shall make use of the following space:

$$C_V([a,b],\mathbb{R}^m):=\{f\in C([a,b],\mathbb{R}^m)\mid f(a)\in V,\ f(b)\in\mathbb{R}^m-{\rm cl}V\},\tag{3.1}$$

where $[a,b]\subset\mathbb{R}$ and V is an open subset of $\mathbb{R}^m$. The space $C_V([a,b],\mathbb{R}^m)$ is equipped with the following metric $d(f,g):=\max\limits_{t\in[a,b]}|f(t)-g(t)|$. Furthermore, one can prove that $C_V([a,b],\mathbb{R}^m)$ is an ANR.

Lemma 3.7. *Under the above assumptions, $C_V([a,b],\mathbb{R}^m)$ is an* ANR.

Proof. Since $C([a,b],\mathbb{R}^m)$ is a linear space, and hence an ANR, it suffices to show that $C_V([a,b],\mathbb{R}^m)$ is an open subset of $C([a,b],\mathbb{R}^m)$. To this end, fix $f\in C_V([a,b],\mathbb{R}^m)$. Then $f(a)\in V$ and $f(b)\in\mathbb{R}^m-{\rm cl}V$. Since V and $\mathbb{R}^m-{\rm cl}V$ are open, it follows that there exists $\varepsilon>0$ such that $B(f(a),\varepsilon)\subset V$ and $B(f(b),\varepsilon)\subset\mathbb{R}^m-{\rm cl}V$. Thus

$$B(f,\varepsilon):=\{g\in C([a,b],\mathbb{R}^m)\mid \|f-g\|_C<\varepsilon\}\subset C_V([a,b],\mathbb{R}^m),$$

which completes the proof. □

The following two results will be of crucial importance for our further considerations.

Lemma 3.8. *Under the above assumptions, a set-valued map $\mathbb{P}\colon C_V([a,b],\mathbb{R}^m)\multimap(a,b)$ defined by*

$$\mathbb{P}(f):=\{t\in[a,b]\mid f(t)\in\operatorname{bd}V\}\tag{3.2}$$

is usc with compact values.

Proof. First, define a set-valued map $\mathbb{P}_0\colon C_V([a,b],\mathbb{R}^m)\multimap[a,b]$ by $\mathbb{P}_0(f):=\mathbb{P}(f)$ for all $f\in C_V([a,b],\mathbb{R}^m)$. It is clear that the upper semicontinuity of $\mathbb{P}_0$ implies that $\mathbb{P}$ is usc. Therefore it is enough to prove that $\mathbb{P}_0$ is usc. Since

$\mathbb{P}_0(C_V([a,b],\mathbb{R}^m)) \subset [a,b]$, it suffices to show that the graph $\mathrm{Gr}(\mathbb{P}_0)$ of $\mathbb{P}_0$ is closed (see [21, Proposition 14.5]). For this purpose, take a sequence $(f_n,t_n) \in \mathrm{Gr}(\mathbb{P}_0)$ such that

$$(f_n,t_n) \xrightarrow{n\to\infty} (f_0,t_0) \in C_V([a,b],\mathbb{R}^m) \times [a,b].$$

We have to prove that $f_0(t_0) \in \mathrm{bd}\, V$. Let $\varepsilon > 0$. Then there exists $n_0 > 0$ such that for any $n \geq n_0$ one has

$$|f_0(t_0) - f_0(t_{n_0})| < \varepsilon/2 \quad \text{and} \quad |f_0(s) - f_n(s)| < \varepsilon/2 \tag{3.3}$$

for all $s \in [a,b]$. Consequently, in view of (3.3), we get

$$|f_0(t_0) - f_{n_0}(t_{n_0})| < \varepsilon.$$

Thus $f_0(t_0) \in B(\mathrm{bd}\, V, \varepsilon)$. Since ε was arbitrary, this shows (recall that $\mathrm{bd}\, V$ is closed) that $f_0(t_0) \in \mathrm{bd}\, V$. This completes the proof. □

Proposition 3.9. *A set-valued map* $\mathbb{P}\colon C_V([a,b],\mathbb{R}^m) \multimap (a,b)$ *defined in* (3.2) *is a weighted carrier.*

Proof. We have proved in Lemma 3.8 that $\mathbb{P}$ is usc. Now, let $I_{\mathrm{wloc}}\colon D(\mathbb{P}) \to \mathbb{Z}$ be defined by the formula:

$$I_{\mathrm{wloc}}(\mathbb{P},U,f) := i(f|U,U), \tag{3.4}$$

for any $(U,f) \in D(\mathbb{P}) := \{(U,f) \mid f \in C_V([a,b],\mathbb{R}^m),\ U \subset (a,b),\ \mathbb{P}(f) \cap \mathrm{bd}\, U = \emptyset\}$ ([5]). First, observe that if $(U,f) \in D(\mathbb{P})$, then $f|U \in C(U,\mathbb{R}^m;\mathrm{bd}\, V)$ (see (2.3)). Now, we shall prove that such a function $I_{\mathrm{wloc}}\colon D(\mathbb{P}) \to \mathbb{Z}$ satisfies all the conditions of Definition 3.1.

Existence. If $I_{\mathrm{wloc}}(\mathbb{P},U,f) \neq 0$, then $i(f|U,U) \neq 0$. Consequently, Proposition 2.5 implies that $f^{-1}(\mathrm{bd}\, V) \cap U \neq \emptyset$, which proves that $\mathbb{P}(f) \cap U \neq \emptyset$.

Local invariance. Let $(U,f) \in D(\mathbb{P})$. We are to prove that there exists $r > 0$ such that

$$I_{\mathrm{wloc}}(\mathbb{P},U,f) = I_{\mathrm{wloc}}(\mathbb{P},U,g) \tag{3.5}$$

for all $g \in B(f,r) = \{g \in C_V([a,b],\mathbb{R}^m) \mid d(f,g) < r\}$. Let

$$\varepsilon_0 := \min_{x \in \mathrm{bd}\, U} \mathrm{dist}(f(x),\mathrm{bd}\, V) > 0. \tag{3.6}$$

Before proceeding further, we need to state the following lemma.

([5]) Just in case, we recall that the boundary of U is taken with respect to $\mathbb{R}$.

Lemma 3.10 ([22, Chapter 11]). *Let X be a compact* ANR *and let Y be an* ANR. *In addition, let $f\colon X \to Y$ be a continuous function and let $\varepsilon > 0$. Then there exists $\delta_f > 0$ such that for any continuous map $g\colon X \to Y$ with $d(f(x), g(x)) < \delta_f$, for all $x \in X$, there exists a continuous map $h\colon X \times [0,1] \to Y$ such that*

(a) $h(x,0) = f(x)$, $h(x,1) = g(x)$, *for all $x \in X$,*
(b) $\operatorname{diam}(h(\{x\} \times [0,1])) < \varepsilon$, *for all $x \in X$,*

where $\operatorname{diam}(h(\{x\} \times [0,1])) := \sup\{d(h(x,t_1), h(x,t_2)) \mid t_1, t_2 \in [0,1]\}$.

Let δ_f be as in Lemma 3.10 for $\varepsilon_0/2$ and f (where $X = [a,b]$ and $Y = \mathbb{R}^m$). We claim that it is enough to put $r := \delta_f$. To see this, choose any function $g \in B(f,r)$. Then, by Lemma 3.10, there exists a homotopy $h\colon [a,b]\times[0,1] \to \mathbb{R}^m$ such that $h(x,0) = f(x)$, $h(x,1) = g(x)$ and $h(\,\cdot\,,t) \in B(f,\varepsilon_0/2)$, for all $t \in [0,1]$. Let us observe that

$$\{x \in \operatorname{cl} U \mid h(x,t) \in \operatorname{bd} V \text{ for some } t \in [0,1]\} \cap \operatorname{bd} U = \emptyset.$$

Indeed, otherwise, there exists $x_0 \in \operatorname{bd} U$ such that $h(x_0,t_0) \in \operatorname{bd} V$ for some $t_0 \in [0,1]$. Moreover, one has

$$|\operatorname{dist}(f(x), \operatorname{bd} V) - \operatorname{dist}(h(x,t_0), \operatorname{bd} V)| \le |f(x) - h(x,t_0)|, \tag{3.7}$$

for all $x \in [a,b]$. Consequently, taking into account (3.6)–(3.7), one obtains

$$\varepsilon_0 \le \operatorname{dist}(f(x_0), \operatorname{bd} V) \le \varepsilon_0/2,$$

a contradiction. Therefore, by the homotopy invariance of the intersection index, one obtains

$$I_{\mathrm{wloc}}(\mathbb{P}, U, f) = i(f|U, U) = i(g|U, U) = I_{\mathrm{wloc}}(\mathbb{P}, U, g),$$

which proves (3.5) as required.

Additivity. This condition follows immediately from the additivity property of the intersection index. □

Remark 3.11. Let U_0 and U_1 be two disjoint nonempty connected subsets of $\mathbb{R}^m$. Then from the long exact sequence of the pair $(\mathbb{R}^m, U_0 \cup U_1)$ for the singular homology functor it follows that

$$H_1(\mathbb{R}^m, U_0 \cup U_1) = \mathbb{Z}.$$

Furthermore, any continuous function $\sigma\colon [0,1] \to \mathbb{R}^m$ with $\sigma(0) \in U_0$ and $\sigma(1) \in U_1$ belongs to the group of relative 1-cycles $\mathbf{Z}_1(\mathbb{R}^m, U_0 \cup U_1)$ and the homology class $[\sigma]$ of σ generates $H_1(\mathbb{R}^m, U_0 \cup U_1)$. What is more, if τ is another continuous function with $\tau(0) \in U_0$ and $\tau(1) \in U_1$, then $[\sigma] = [\tau]$. Therefore we will identify this homology class $[\sigma]$ with the generator $1 \in \mathbb{Z}$.

Now we are able to prove the following important lemma.

Lemma 3.12. *Let* $\mathbb{P}\colon C_V([a,b],\mathbb{R}^m) \multimap (a,b)$ *and* $I_{wloc}\colon D(\mathbb{P}) \to \mathbb{Q}$ *be given by* (3.2) *and* (3.4), *respectively. Let* $V \subset \mathbb{R}^m$ *be open and connected such that* $\mathbb{R}^m - \operatorname{cl} V$ *is connected. Then*

(a) $I_{\rm wloc}(\mathbb{P},(a,b),f)$ *is the generator of* $H_1(\mathbb{R}^m, \mathbb{R}^m - \operatorname{bd} V)$, *for any* $f \in C_V([a,b],\mathbb{R}^m)$;

(b) $I_{\rm wloc}(\mathbb{P},(a,b),f) = I_{\rm wloc}(\mathbb{P},(a,b),g)$, *for all* $f, g \in C_V([a,b],\mathbb{R}^m)$, *and hence* $I_w(\mathbb{P}) = 1$.

Proof. Fix $f \in C_V([a,b],\mathbb{R}^m)$. Let $K := f^{-1}(\operatorname{bd} V) \subset (a,b)$ and let $r\colon \mathbb{R} \to [a,b]$ be a retraction such that $r(x) = a$ for $x \le a$ and $r(x) = b$ for $x \ge b$. Let $\mathbb{R}^m - \operatorname{bd} V = U_0 \cup U_1$, where $U_0 := V$ and $U_1 := \mathbb{R}^m - \operatorname{cl} V$ Consider the following commutative diagram:

$$
\begin{array}{ccccc}
H_1(\mathbb{R},\mathbb{R}-K) & \xrightarrow[\simeq]{r_*} & H_1([a,b],[a,b]-K) & \xrightarrow{f_*} & H_1(\mathbb{R}^m,\mathbb{R}^m-\operatorname{bd} V) \\
\uparrow{\scriptstyle \rm id} & & \uparrow{\scriptstyle \simeq} & \nearrow{\scriptstyle (f|(a,b))_*} & \\
H_1(\mathbb{R},\mathbb{R}-K) & \xleftarrow[\simeq]{} & H_1((a,b),(a,b)-K). & &
\end{array} \tag{3.8}
$$

Let $c := 2\max\{|a|,|b|\}$. Let $\sigma\colon [0,1] \to \mathbb{R}$ be any continuous function such that $\sigma(0) = -c$ and $\sigma(1) = c$. Then $O_K = [\sigma] \in H_1(\mathbb{R},\mathbb{R}-K)$ and

$$
I_{\rm wloc}(\mathbb{P},(a,b),f) = i(f, f|(a,b)) = (f|(a,b))_*(O_K^{(a,b)}).
$$

In addition, one has

$$
(f_* \circ r_*)(O_K) = [f \circ r \circ \sigma]. \tag{3.9}
$$

By Remark 3.11, $H_1(\mathbb{R}^m,\mathbb{R}^m - \operatorname{bd} V) = \mathbb{Z}$ and $[f \circ r \circ \sigma]$ is the generator of $H_1(\mathbb{R}^m,\mathbb{R}^m - \operatorname{bd} V)$ since $(f \circ r \circ \sigma)(0) = f(r(-c)) = f(a) \in U_0$ and $(f \circ r \circ \sigma)(1) = f(r(c)) = f(b) \in U_1$. Hence, taking into account (3.8)–(3.9), we get that

$$
I_{\rm wloc}(\mathbb{P},(a,b),f) = [f \circ r \circ \sigma]
$$

is the generator of $H_1(\mathbb{R}^m,\mathbb{R}^m - \operatorname{bd} V) = \mathbb{Z}$, which completes the proof of (a). As concerns (b), if $f, g \in C_V([a,b],\mathbb{R}^m)$, then

$$
I_{\rm wloc}(\mathbb{P},(a,b),f) = [f \circ r \circ \sigma] \overset{\text{Remark 3.11}}{=\!=\!=\!=} [g \circ r \circ \sigma] = I_{\rm wloc}(\mathbb{P},(a,b),g).
$$

Now we shall present some very important fact which may not be true in $\mathbb{R}^n$ for $n > 1$.

Lemma 3.13. *If* $A \subset \mathbb{R}$ *is compact, then* $\check{H}_k(A;\mathbb{Q}) = 0$ *for* $k \ge 1$.

Proof. Let $B(A,\varepsilon) := \{x \in \mathbb{R} \mid \operatorname{dist}(x,A) < \varepsilon\}$. Since $B(A,\varepsilon) \subset \mathbb{R}$, we infer that it can be represented as a finite (disjoint) sum of convex sets. Hence we deduce that $D(A,\varepsilon)$ is also a finite (disjoint) sum of convex sets. Consequently, it follows that $\check{H}_k(D(A,\varepsilon);\mathbb{Q}) = 0$, for $k \geq 1$. Now from this we arrive at the conclusion of lemma, since

$$\check{H}_*(A;\mathbb{Q}) = \varprojlim \check{H}_*(D(A,1/n);\mathbb{Q})$$

(see [22, Chapter 20]). □

4. Main results

In this section we will prove the existence of solutions for the problem (P). To this aim we need some preliminary results. We start with some remarks and lemmas. Before we do it, we will introduce some notations. Let $p:\mathbb{R} \to S^1$ be a covering map defined by $p(t) = (\cos(t), \sin(t))$, for $t \in \mathbb{R}$. Let $\mathrm{id} \times p: M \times \mathbb{R} \to M \times S^1$ and $T_0: \mathbb{R}^{n+1} \times \mathbb{R} \to \mathbb{R}^{n+2}$ be given by

$$(\mathrm{id} \times p)(x,y) := (x, p(y)) \quad \text{and} \quad T_0((x,u),y) := (x, u(-\sin(y), \cos(y))).$$

In what follows, we shall make use of the following two projections: $\mathrm{pr}_1: X_1 \times X_2 \to X_1$ and $\mathrm{pr}_2: X_1 \times X_2 \to X_2$.

Remark 4.1. From now on we will assume that $M \subset \mathbb{R}^n$ (represented by a locally Lipschitz function $f:\mathrm{Dom}(f) \to \mathbb{R}$) is a contractible and strictly regular ANR.

Now we shall prove a lifting lemma which will be used in our further considerations.

Lemma 4.2 (Lifting lemma). *Let $f: M \times S^1 \to \mathbb{R}^{n+2}$ be continuous and tangent. Then there exists a continuous and tangent map $\widetilde{f}: M \times \mathbb{R} \to \mathbb{R}^{n+1}$ such that the following diagram:*

$$\begin{array}{ccc} M \times \mathbb{R} & \xrightarrow{(\widetilde{f},\mathrm{pr}_2)} & \mathbb{R}^{n+1} \times \mathbb{R} \\ {\scriptstyle \mathrm{id}\times p}\downarrow & & \downarrow{\scriptstyle T_0} \\ M \times S^1 & \xrightarrow[f]{} & \mathbb{R}^{n+2} \end{array} \tag{4.1}$$

commutes.

Proof. Let $f(x,y) = (f_1(x,y), f_2(x,y))$, where $f_1(x,y) \in \mathbb{R}^n$, $f_2(x,y) \in \mathbb{R}^2$, $x \in \mathbb{R}^n$ and $y \in S^1$. Then it suffices to define $\widetilde{f}$ as follows

$$\widetilde{f} := (f_1 \circ (\mathrm{id} \times p), \widetilde{f}_2 \circ (\mathrm{id} \times p)),$$

where $\widetilde{f}_2(x,y) := \langle f_2(x,y), y^{\perp} \rangle$ and $y^{\perp} = (y_1,y_2)^{\perp} = (-y_2, y_1)$. □

Remark 4.3. A function $\widetilde{f}$ satisfying (4.1) will be called a lift of f. Furthermore, it is easily seen that if f is bounded, then $\widetilde{f}$ is also a bounded map.

Let $N \subset \mathbb{R}^k$ be a strictly regular set and let $g \in \mathrm{Vect}(N)$ be bounded. Consider the Cauchy problem:

$$\begin{cases} \dot{u}(t) = g(u(t)), \\ u(0) = z_0 \in N. \end{cases}$$

In what follows by

$$S_g(z) := \{u \mid \dot{u}(t) = g(u(t)) \text{ a.e. on } [0,\infty),\ u(0) = z,\ u([0,\infty)) \subset N\}$$

we will denote the set of all solutions to the Cauchy problem, for all $z \in N$. The following lemma will be useful in the sequel.

Lemma 4.4 ([5]). *If $g \in \mathrm{Vect}(N)$ is bounded, then $S_g(z)$ is a compact R_δ-set in $C_u([0,\infty),\mathbb{R}^k)$* ([6]).

Remark 4.5. (a) It is well-known that the map $S_g\colon N \multimap C_u([0,\infty),\mathbb{R}^k)$ is usc (see [4], [15], [18]).

(b) It should be noted that in the paper [5] it was only proved that the set of all solutions restricted to $[0,T]$ (denoted by $S_g^T(z)$, for any $z \in N$) is a compact R_δ-set. However, by using the technique of inverse systems, one can extend this result to the case where all solutions are defined on $[0,\infty)$ (see [1], [20]). What is more, Lemma 4.4 is also true for maps $g\colon N \to \mathbb{R}^k$ having *a sublinear growth*, i.e. such that there is $c > 0$ with $|g(z)| \le c(1+|z|)$ for all $z \in N$. This follows from the fact that, for any $T > 0$, by using the Gronwall inequality ([15, p. 52]) one can prove that g can be replaced by a bounded map $\overline{g}$ such that $S_{\overline{g}}^T(z) = S_g^T(z)$, for any $z \in N$.

(c) Consider a map $\Pi_g\colon N \times [0,\infty) \multimap N$ given by the formula

$$\Pi_g(x,t) := \{u(t) \mid u(\,\cdot\,) \in S_g(x)\}.$$

Then Π_g is an usc set-valued map with compact values satisfying the following conditions:

- $\Pi_g(x,0) = \{x\}$;
- $\Pi_g(\Pi_g(x,s),t) = \Pi_g(x,s+t)$, for $s,t \in [0,\infty)$.

In what folows, we will call Π_g a set-valued semiflow.

Definition 4.6. A compact subset $K \subset N$ is called an attractor for a vector field $g \in \mathrm{Vect}(N)$ if $d_H(\Pi_g(x,t),K) \to 0$ as $t \to \infty$, for every $x \in N$. A vector field $g \in \mathrm{Vect}(N)$ is said to be of compact attraction (written $g \in \mathrm{Vect}_c(N)$) if g has a compact attractor.

([6]) Recall here that a nonempty compact subset C of a metric space X is called an R_δ-set if it is the intersection of a decreasing family of compact contractible sets $C_n \subset X$ (see [23]). In particular, an R_δ-set is connected. $C_u([0,\infty),\mathbb{R}^k)$ stands for the Fréchet space of all continuous maps $[0,\infty) \to \mathbb{R}^k$ with the topology of almost uniform convergence.

Remark 4.7. Notice that if K is a compact attractor for $g \in \mathrm{Vect}(N)$, then any compact set K' containing K is also a compact attractor.

Let $f \in \mathrm{Vect}(M \times S^1)$ and let $\widetilde{f} \in \mathrm{Vect}(M \times \mathbb{R})$ be a lift of f. Fix $x_0 \in M$, $y_0 \in S^1$ and $\widetilde{y}_0 \in p^{-1}(y_0)$. Then one can consider two Cauchy problems:

$$\text{(CP0)} \qquad \begin{cases} \dot{u}(t) = f(u(t)), \\ u(0) = (x_0, y_0), \end{cases}$$

$$\text{(CP1)} \qquad \begin{cases} \dot{u}(t) = \widetilde{f}(u(t)), \\ u(0) = (x_0, \widetilde{y}_0). \end{cases}$$

We shall prove that there exists a connection between problem (CP0) and (CP1) which will be used in order to solve problem (P). To be precise, we will prove that $S_{\widetilde{f}}(x_0, \widetilde{y}_0) \subset C_u([0,\infty), \mathbb{R}^{n+1})$ is homeomorphic to $S_f(x_0, y_0) \subset C_u([0,\infty), \mathbb{R}^{n+2})$.

Since $p\colon \mathbb{R} \to S^1$ is a covering map, it follows in view of the lifting theorem that for any map $u\colon [0,\infty) \to M \times S^1$ there exists a unique map $\widetilde{u}\colon [0,\infty) \to M \times \mathbb{R}$ such that the following diagram commutes (see [Span66]):

$$\text{(4.2)} \qquad \begin{array}{ccc} & & M \times \mathbb{R} \\ & \overset{\widetilde{u}}{\nearrow} & \Big\downarrow {\scriptstyle \mathrm{id} \times p} \\ [0,\infty) & \xrightarrow[u]{} & M \times S^1 \end{array}$$

and $u(0) = (x_0, y_0)$, $\widetilde{u}(0) = (x_0, \widetilde{y}_0)$. It is clear that the above diagram induces the following:

$$\text{(4.3)} \qquad \begin{array}{ccc} & & T(M \times \mathbb{R}) \\ & \overset{T[\widetilde{u}]}{\nearrow} & \Big\downarrow {\scriptstyle T} \\ [0,\infty) & \xrightarrow[T[u]]{} & T(M \times S^1) \end{array}$$

where T, $T[\widetilde{u}]$ and $T[u]$ ([7]) are given by

$$T((x,y),(u,v)) = ((x,(\cos(y),\sin(y))), (u, v(-\sin(y),\cos(y)))),$$
$$T[\widetilde{u}](t) = (\widetilde{u}(t), \dot{\widetilde{u}}(t)), \quad T[u](t) = (u(t), \dot{u}(t)).$$

Now we are ready to prove the following proposition.

Proposition 4.8. *Under the above assumptions, a function $\mathcal{S}\colon S_f(x_0, y_0) \to S_{\widetilde{f}}(x_0, \widetilde{y}_0)$ defined by $\mathcal{S}(u) := \widetilde{u}$ is a bijection.*

Proof. First we will prove that

$$\text{(4.4)} \qquad u \in S_f(x_0, y_0) \Leftrightarrow \widetilde{u} \in S_{\widetilde{f}}(x_0, \widetilde{y}_0),$$

([7]) For simplicity, later in this paper, we will denote $T[\widetilde{u}]$ and $T[u]$ by $\dot{\widetilde{u}}$ and $\dot{u}$, respectively.

where u and $\widetilde{u}$ satisfy (4.2). To this aim, it suffices to observe that (4.4) follows directly from the following commutative diagram:

$$\begin{array}{ccccccc}
 & & T(M\times S^1) & \xleftarrow{\;Tf\;} & M\times S^1 & & \\
 & {\scriptstyle \dot{u}}\nearrow & \big\uparrow{\scriptstyle T} & & \big\uparrow{\scriptstyle \mathrm{id}\times p} & \nwarrow{\scriptstyle u} & \\
{[0,\infty)} & \xrightarrow[\;\dot{\widetilde{u}}\;]{} & T(M\times\mathbb{R}) & \xleftarrow[\;T\widetilde{f}\;]{} & M\times\mathbb{R} & \xleftarrow[\;\widetilde{u}\;]{} & [0,\infty)
\end{array}$$

and the fact that for any point $(x,y)\in M\times\mathbb{R}$ a function $T_{(x,y)}\colon\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}^n\times\mathbb{R}^2$ induced by T, i.e.

$$T_{(x,y)}(u,v) := (u, v(-\sin(y),\cos(y))),$$

is an isomorphism. Notice that the commutativity of the left and right triangle in the above diagram follows from (4.2) and (4.3), respectively. Recall that the left triangle is well-defined for almost all $t\in[0,\infty)$. Thus (4.4) implies that $\mathcal{S}\colon S_f(x_0,y_0)\to S_{\widetilde{f}}(x_0,\widetilde{y}_0)$ is well-defined. Finally, the surjectivity of $\mathcal{S}$ follows from (4.4), but the injectivity of $\mathcal{S}$ follows easily from (4.2). This completes the proof. $\square$

Remark 4.9. It is easy to see that $\mathcal{S}^{-1}\colon S_{\widetilde{f}}(x_0,\widetilde{y}_0)\to S_f(x_0,y_0)$ is given by $\mathcal{S}^{-1}(\widetilde{u}) := (\mathrm{id}\times p)\circ\widetilde{u}$ for all $\widetilde{u}\in S_{\widetilde{f}}(x_0,\widetilde{y}_0)$. Since $C_u([0,\infty),\mathbb{R}^m)$, for any $m\in\mathbb{N}$, is endowed with the topology of almost uniform convergence, it follows that $\mathcal{S}^{-1}$ is continuous, and hence $\mathcal{S}$ is continuous because $S_{\widetilde{f}}(x_0,\widetilde{y}_0)$ is compact in $C_u([0,\infty),\mathbb{R}^{n+1})$.

From the above considerations it follows that the following diagram is commutative:

(4.5)
$$\begin{array}{ccc}
M\times\mathbb{R}\times[0,\infty) & \xrightarrow{\;\Pi_{\widetilde{f}}\;}\!\!\circ & M\times\mathbb{R} \\
\big\downarrow{\scriptstyle \mathrm{id}\times p\times\mathrm{id}} & & \big\downarrow{\scriptstyle \mathrm{id}\times p} \\
M\times S^1\times[0,\infty) & \xrightarrow[\;\Pi_f\;]{}\!\!\circ & M\times S^1.
\end{array}$$

The next lemma explains what properties of f are inherited by a lift $\widetilde{f}$ of f.

Lemma 4.10. *Let $f\in\mathrm{Vect}(M\times S^1)$. Then*:

(a) *if f has a sublinear growth (resp. f is bounded), then $\widetilde{f}$ has also a sublinear growth (resp. $\widetilde{f}$ is also bounded)*;

(b) *if $f\in\mathrm{Vect}_c(M\times S^1)$, then there exists a compact set $\widetilde{K}\subset M$ such that*

$$d_H(\mathrm{pr}_1(\Pi_{\widetilde{f}}((x,0),t)),\widetilde{K}) \xrightarrow{t\to\infty} 0,$$

for every $x\in M$.

Proof. The assertion (a) follows from the following calculations:

$$\begin{aligned}|\widetilde{f}(x,y)|^2 &= |f_1(x,p(y)),\widetilde{f}_2(x,p(y))|^2 = |f_1(x,p(y))|^2 + |\widetilde{f}_2(x,p(y))|^2\\ &= |f_1(x,p(y))|^2 + \langle f_2(x,p(y)), p(y)^{\perp}\rangle^2\\ &\le |f_1(x,p(y))|^2 + |f_2(x,p(y))|^2|p(y)|^2 = |f_1(x,p(y))|^2 + |f_2(x,p(y))|^2\\ &= |f(x,p(y))|^2 \le c^2(1+|(x,p(y))|)^2 \le c^2(1+|x|+|p(y)|)^2\\ &\le c^2(1+|x|+1+|y|)^2 \le (2c)^2(1+|(x,y)|)^2,\end{aligned}$$

where $(x,y) \in M \times S^1$. Since $|\widetilde{f}(x,y)|^2 \le |f(x,p(y))|^2$, it follows that the boundedness of f implies the boundedness of $\widetilde{f}$. As for (b), let K be a compact attractor for $f \in \mathrm{Vect}(M \times S^1)$. Without loss of generality we can assume that $K = K' \times S^1$, where K' is a compact subset of M. Let $\widetilde{K} := \mathrm{pr}_1(K) = K'$. Then, taking into account the diagram (4.5), one obtains

$$\begin{gathered}\Pi_f((x,(1,0)),t) \subset B(K,\varepsilon) \Rightarrow \mathrm{pr}_1(\Pi_f((x,(1,0)),t)) \subset B(\mathrm{pr}_1(K),\varepsilon),\\ \mathrm{pr}_1(\Pi_f((x,(1,0)),t)) = \mathrm{pr}_1(\Pi_{\widetilde{f}}((x,0),t)), \quad B(\mathrm{pr}_1(K),\varepsilon) = B(\widetilde{K},\varepsilon),\end{gathered}$$

which implies that

$$d_H(\Pi_f((x,(1,0)),t),K) \xrightarrow{t\to\infty} 0 \Rightarrow d_H(\mathrm{pr}_1(\Pi_{\widetilde{f}}((x,0),t)),\widetilde{K}) \xrightarrow{t\to\infty} 0. \qquad \square$$

Remark 4.11. A set of vector fields $\widetilde{f} \in \mathrm{Vect}(M \times \mathbb{R})$ satisfying condition (b) from Lemma 4.10 will be denoted by the symbol $\mathrm{Vect}_{wc}(M \times \mathbb{R})$, while a set $\widetilde{K}$ from the above lemma will be called a weak attractor for $\widetilde{f}$.

It should be noted that without additional assumptions on M nothing can be said about the structure of solutions to (P).

Example 4.12 (see [5]). Let $M = M_{-1} \cup M_1 \subset \mathbb{R}^2$, where

$$M_i := \{(x,y) \in \mathbb{R}^2 \mid (x-i)^2 + y^2 = 1\}.$$

Let $f\colon M \times S^1 \to \mathbb{R}^4$ be defined by

$$f((x,y),z) = \begin{cases} ((y,1-x),0) & \text{if } (x,y) \in M_1,\\ ((-y,1+x),0) & \text{if } (x,y) \in M_{-1}.\end{cases}$$

It is easy to see that for all $((x,y),z) \in M \times S^1$, $f((x,y),z) \in T_{M\times S^1}((x,y),z)$ and that the set $S_f^T((0,0),0)$ (for any $T > 0$) is disconnected, and hence it is not an R_δ-set.

Now we are going to prove the result which is closely related (by Proposition 4.8) to problem (P) – see also Theorem 4.17.

Theorem 4.13. *Assume that $M \subset \mathbb{R}^n$ is a contractible and strictly regular* ANR*. Let $\widetilde{f} \in \mathrm{Vect}_{wc}(M \times \mathbb{R})$ be a lift of a bounded and continuous* (*or a continuous map with a sublinear growth*) *vector field $f \in \mathrm{Vect}_c(M \times S^1)$. If*

(4.6) *there exists $T > 0$ such that $\gamma(T) \in M \times (2\pi, \infty)$*

for all $\gamma \in S_{\widetilde{f}}(M \times \{0\})$ ([8]),

then there exists $x_0 \in M$ and $\gamma \in S_{\widetilde{f}}(x_0, 0)$ such that $\gamma(t_0) = (x_0, 2\pi)$, for some $t_0 \in (0, T)$ (*see Figure* 3).

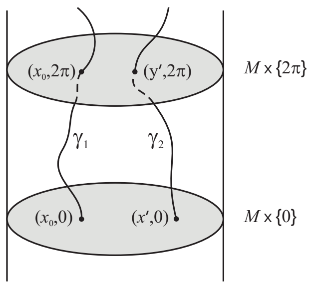


Figure 3. A trajectory γ_1 satisfies the assertion of Theorem 4.13

Proof. Consider the following diagram:

$$M \xrightarrow{\ \mathbb{S}\ } C_0 \xrightarrow{\ \Delta\ } C_0 \times C_0 \xrightarrow{\mathbb{P}\times \mathrm{id}} (0,T) \times C_0 \xrightarrow{\ \lambda\ } M \times \mathbb{R} \xrightarrow{\ \mathrm{pr}_1\ } M,$$

where $\mathbb{P}$ is given by (3.2), $C_0 := C_V([0,T], \mathbb{R}^{n+1})$ (see (3.1) for $V := \mathbb{R}^n \times (-\infty, 2\pi)$) and $\mathbb{S}$, Δ, $\mathbb{P} \times \mathrm{id}$ and λ are defined as follows

$$\begin{aligned} \mathbb{S}(x) &:= S_{\widetilde{f}}(x,0), & \Delta(x) &= (x,x), \\ (\mathbb{P} \times \mathrm{id})(x,y) &= (\mathbb{P}(x), y), & \lambda(t,h) &= h(t). \end{aligned} \tag{4.7}$$

Let us define $\Phi\colon M \multimap M$ by

$$\Phi := (\mathrm{pr}_1 \circ \lambda) \circ ((\mathbb{P} \times \mathrm{id}) \circ (\Delta \circ \mathbb{S})). \tag{4.8}$$

Now, let us observe that if $\mathrm{Fix}(\Phi) \neq \emptyset$, then there exist $x_0 \in M$ and a trajectory $\gamma \in \mathbb{S}(x_0)$ such that

$$\gamma(t_0) = (x_0, 2\pi), \quad \text{for some } t_0 \in (0,T).$$

Consequently, it suffices to show that $\mathrm{Fix}(\Phi) \neq \emptyset$. For this purpose, we shall make use of Theorem 7.8 from Appendix. First observe that Lemma 4.4, Remark 4.5 and Example 3.5 imply that $\mathbb{S}$ is a weighted carrier with acyclic values

([8]) If $Z \subset M \times S^1$ (resp. $Z \subset M \times \mathbb{R}$), then we put $S_f(Z) := \bigcup_{z \in Z} S_f(z)$ $\big($resp. $S_{\widetilde{f}}(Z) := \bigcup_{z \in Z} S_{\widetilde{f}}(z)\big)$.

and $I_w(\mathbb{S}) = 1$. Of course, Δ, pr_1 and λ are weighted carriers with $I_w(\Delta) = 1$, $I_w(\mathrm{pr}_1) = 1$ and $I_w(\lambda) = 1$, respectively. Furthermore, Propositions 3.9 and 7.3 and Lemmas 3.12 and 3.13 imply that $\mathbb{P} \times \mathrm{id}$ is a weighted carrier with positively acyclic values (recall from algebraic topology that the Cartesian product of two positively acyclic sets is also positively acyclic) and $I_w(\mathbb{P} \times \mathrm{id}) = 1$. Consequently, we deduce from Proposition 3.4 and Lemma 3.13 that $(\mathbb{P} \times \mathrm{id}) \circ (\Delta \circ \mathbb{S})$ is a weighted carrier with positively acyclic values and $I_w((\mathbb{P} \times \mathrm{id}) \circ (\Delta \circ \mathbb{S})) = 1$. Thus we have proved that Φ has the following decomposition:

$$\Phi = f \circ \Psi \in \mathrm{CA_W}(M)$$

(see Appendix), where $f := \mathrm{pr}_1 \circ \lambda$ and $\Psi := (\mathbb{P} \times \mathrm{id}) \circ (\Delta \circ \mathbb{S})$. What is more, taking into account Lemma 3.7, and since the Cartesian product of ANRs is an ANR, we infer that $C_0 \times C_0$, $(0, T) \times C_0$ and $M \times \mathbb{R}$ are ANRs.

Now we have to prove that the set-valued map $\Phi\colon M \multimap M$ defined as in (4.8) has a compact attractor. To this aim, we are going to prove it in a few steps.

Claim 1. If $C \subset M$ is a compact subset of M, then there exists $0 < \varepsilon < T$ such that $\mathbb{P}(\mathbb{S}(C)) \subset [\varepsilon, T)$.

Indeed, assume on the contrary that $\mathbb{P}(\mathbb{S}(C)) \cap [0, \varepsilon] \neq \emptyset$, for any $0 < \varepsilon < T$. Then there exists a sequence $\varepsilon_m \to 0$ and a sequence $\gamma_m \in \mathbb{S}(C)$ such that $\mathbb{P}(\gamma_m) \cap [0, \varepsilon_m] \neq \emptyset$. Since $\mathbb{S}(C)$ is compact, we can assume without loss of generality that a sequence γ_m converges to some point $\gamma \in \mathbb{S}(C)$. In particular, γ_m converges uniformly to γ on $[0, T]$. Furthermore, for any γ_m there exists $0 < t_m \leq \varepsilon_m$ such that $\gamma_m(t_m) \in \mathrm{bd}\, V$. It is easily seen that $\gamma_m(t_m) \to \gamma(0)$ as $m \to \infty$, which implies that $\gamma(0) \in \mathrm{bd}\, V = \mathbb{R}^n \times \{2\pi\}$. This contradicts the fact that $\gamma(0) \in \mathbb{R}^n \times \{0\}$.

Claim 2. For any $x \in M$ there exists $\varepsilon > 0$ such that for all $\gamma \in \mathbb{S}(x)$ and for all $m \in \mathbb{N}$:

$$t_m(\gamma) := \inf\{t \in [0, \infty) \mid \gamma(t) \in \mathrm{bd}\, V_m\} \geq m\varepsilon,$$

where $V_m := \mathbb{R}^n \times (-\infty, 2m\pi)$.

Indeed, fix $x \in M$. Let us put

$$C := \mathrm{cl}(\mathrm{pr}_1(\Pi_{\widetilde{f}}(\{(x, 0)\} \times [0, \infty)))) \subset M.$$

We will show that C is compact. To this aim, let $\widetilde{K} \subset M$ be a weak global attractor for $\widetilde{f}$ and take $D(\widetilde{K}, \delta) \subset M$, where $\delta > 0$. Then Lemma 4.10 implies that there exists $t_0 > 0$ such that $\mathrm{pr}_1(\Pi_{\widetilde{f}}(\{(x, 0)\} \times [t_0, \infty))) \subset B(\widetilde{K}, \delta)$. Hence, one has

$$C \subset \mathrm{pr}_1(\Pi_{\widetilde{f}}(\{(x, 0)\} \times [0, t_0])) \cup D(\widetilde{K}, \delta) =: K_0.$$

Since C is closed and K_0 is compact, we infer that C is compact. Now observe that, in view of Claim 1, there exists $\varepsilon > 0$ such that $t_1(\gamma) \geq \varepsilon$ for any $\gamma \in \mathbb{S}(C)$.

Fix $\gamma \in \mathbb{S}(C)$ and assume by the induction hypothesis that $t_m(\gamma) \geq m\varepsilon$. We are to prove that

$$t_{m+1}(\gamma) \geq (m+1)\varepsilon.$$

For this purpose, define $\widetilde{\gamma}\colon [0,\infty) \to M \times \mathbb{R}$ by

$$\widetilde{\gamma}(t) := (\mathrm{pr}_1(\gamma(t + t_m(\gamma))), \mathrm{pr}_2(\gamma(t + t_m(\gamma))) - 2m\pi).$$

Then $\widetilde{\gamma}(0) \in C \times \{0\}$. Consequently, since

$$\widetilde{f}(x,y) = \widetilde{f}(x, y + 2k\pi)$$

for all $k \in \mathbb{Z}$ and $(x,y) \in M \times \mathbb{R}$, we deduce that $\widetilde{\gamma} \in \mathbb{S}(C)$. Furthermore,

$$\widetilde{\gamma}(t) \in \mathrm{bd}\, V_1 \Leftrightarrow \gamma(t + t_m(\gamma)) \in \mathrm{bd}\, V_{m+1}. \tag{4.9}$$

Hence,

$$t_{m+1}(\gamma) = t_m(\gamma) + t_1(\widetilde{\gamma}) \geq m\varepsilon + \varepsilon = (m+1)\varepsilon,$$

which completes the proof of Claim 2.

Claim 3. For all $x \in M$ there exists $\varepsilon > 0$ such that

$$\Phi^m(x) \subset \mathrm{pr}_1(\Pi_{\widetilde{f}}(\{(x,0)\} \times [m\varepsilon, \infty))), \tag{4.10}$$

for all $m \in \mathbb{N}$, where Φ^m denotes the m-th iterate of Φ. Indeed, this assertion follows from the fact that for any $x \in M$ there exists $\varepsilon > 0$ such that

$$\begin{aligned}
\Phi^m(x) &= \Phi(\Phi^{m-1}(x)) \\
&\overset{\text{Claim 1}}{=\!=} \{y \in M \mid \exists \gamma \in S_{\widetilde{f}}(\Phi^{m-1}(x) \times \{0\}) \\
&\qquad \text{such that } \gamma(t) = (y, 2\pi) \text{ for some } t \in [\varepsilon, \infty)\} \\
&\overset{\text{Claim 2}}{=\!=} \{y \in M \mid \exists \gamma \in S_{\widetilde{f}}(x, 0) \\
&\qquad \text{such that } \gamma(t) = (y, 2m\pi) \text{ for some } t \in [m\varepsilon, \infty)\},
\end{aligned}$$

for all $m \in \mathbb{N}$.

Claim 4. $\Phi\colon M \multimap M$ has a compact attractor. Indeed, let $\widetilde{K} \subset M$ be a weak attractor for $\widetilde{f}$. Fix $x \in M$ and $\delta > 0$. Then

$$d_H(\mathrm{pr}_1(\Pi_{\widetilde{f}}((x,0),t)), \widetilde{K}) \to 0 \quad \text{as } t \to \infty.$$

Thus there exists $\widetilde{t} > 0$ such that

$$d_H(\mathrm{pr}_1(\Pi_{\widetilde{f}}((x,0),t)), \widetilde{K}) < \delta,$$

for all $t \geq \widetilde{t}$, which implies that

$$\mathrm{pr}_1(\Pi_{\widetilde{f}}(\{(x,0)\} \times [\widetilde{t}, \infty))) \subset B(\widetilde{K}, \delta). \tag{4.11}$$

Now, taking into account (4.10) and (4.11), we deduce that there exists m_0 such that

$$\Phi^m(x) \subset B(\widetilde{K}, \delta)$$

for all $m \geq m_0$, which proves that $\widetilde{K}$ is an attractor for Φ.

Finally, Theorem 7.8 implies the assertion of Theorem 4.13. □

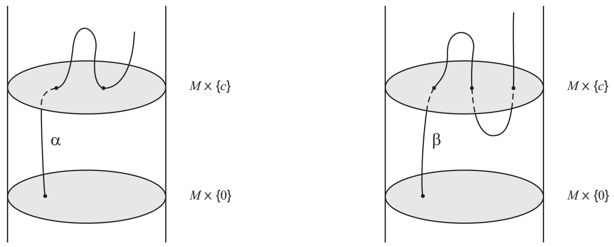


Figure 4. A trajectory β is transversal to $M \times \{c\}$ but α is not transversal to $M \times \{c\}$

Remark 4.14. The main difficulty in the proof of Theorem 4.13 is that a given trajectory starting from $M \times \{0\}$ and passing through $M \times \{c\}$ need not be transversal to $M \times \{c\}$ (see Figures 4 and 5), which implies that the so-called exit function $\tau\colon S_{\widetilde{f}}(M \times \{0\}) \to [0, \infty)$ given by $\tau(\gamma) = \sup\{t \geq 0 \mid \gamma(t) \in M \times (-\infty, c]\}$ is only upper semicontinuous, i.e. for any $r \in \mathbb{R}$, the set $\{\gamma \in S_{\widetilde{f}}(M \times \{0\}) \mid \tau(\gamma) < r\}$ is open in $S_{\widetilde{f}}(M \times \{0\})$ (see [3]). Therefore we had to modify the definition of the exit function in order to prove Theorem 4.13.

Moreover, in our situation it may happen that a given trajectory can stay in the section $M \times \{c\}$ for some time.

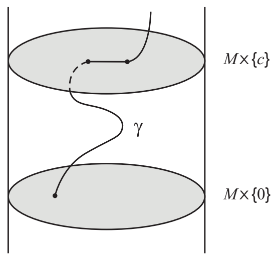


Figure 5. The trajectory γ is not transversal to $M \times \{c\}$

From the proof of Theorem 4.13 we obtain the following corollary.

Corollary 4.15. *Let $M \subset \mathbb{R}^n$ be a contractible compact and strictly regular set and let $g \in \mathrm{Vect}(M \times \mathbb{R})$ be a bounded continuous map (or a continuous map with a sublinear growth), $c > 0$. If*

there exists $T > 0$ such that $\gamma(T) \in M \times (c, \infty)$ for all $\gamma \in S_g(M \times \{0\})$, *then there exists* $x_0 \in M$ *and* $\gamma \in S_g(x_0, 0)$ *such that* $\gamma(t_0) = (x_0, c)$ *for some* $t_0 \in (0, T)$.

Remark 4.16. It should be noted that if in Theorem 4.13 we replace 2π by -2π, then one can prove the following assertion: if there exists $T > 0$ such that $\gamma(T) \in M \times (-\infty, -2\pi)$ for all $\gamma \in S_{\widetilde{f}}(M \times \{0\})$, then there exists $x_0 \in M$ and $\gamma \in S_{\widetilde{f}}(x_0, 0)$ such that $\gamma(t_0) = (x_0, -2\pi)$ for some $t_0 \in (0, T)$. The proof of this fact goes without any essential changes.

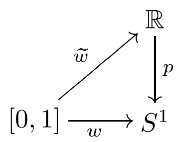
Now we prove the next of the main results of the paper.

Theorem 4.17. *Let* $M \subset \mathbb{R}^n$ *be a contractible strictly regular* ANR *and let* $f \in \mathrm{Vect}_c(M \times S^1)$ *be a continuous and bounded map. If a lift* $\widetilde{f}$ *of* f *satisfies* (4.6) *in Theorem* 4.13, *then Problem* (P) *admits a solution* $u \in S_f(M \times \{(1, 0)\})$ *which generates* $\pi_1(M \times S^1)$.

Proof. From Theorem 4.13 it follows that there exist $x_0 \in M$, $t_0 \in (0, T)$ and $\widetilde{u} \in S_{\widetilde{f}}(x_0, 0)$ such that

$$\widetilde{u}(0) = (x_0, 0) \quad \text{and} \quad \widetilde{u}(t_0) = (x_0, 2\pi). \tag{4.12}$$

Then, by Proposition 4.8, $u := S^{-1}(\widetilde{u}) \in S_f(x_0, (1, 0))$ and $u(0) = u(t_0)$. Let $\widetilde{t} := \inf\{t > 0 \mid \widetilde{u}(t) = (x_0, 2\pi)\} > 0$. For simplicity one can assume that $\widetilde{t} = 1$. We will show that the homotopy class $[u]$ of $u\colon [0, 1] \to M \times S^1$ generates $\pi_1(M \times S^1)$. Since $\pi_1(M \times S^1) = \pi_1(M) \times \pi_1(S^1) \simeq \pi_1(S^1)$, we infer that it suffices to prove that $[u_2 := \mathrm{pr}_2 \circ u]$ is a generator of $\pi_1(S^1)$. To this end, recall that a homomorphism $h\colon \pi_1(S^1) \to \mathbb{Z}$ given by $h([w]) = \widetilde{w}(1)$ is an isomorphism (see [39]), where $\widetilde{w}$ is a lift of w, i.e. the following diagram



is commutative with $w(0) = (1, 0)$ and $\widetilde{w}(0) = 0$. Since h is an isomorphism, it follows that a loop $[w] \in \pi_1(S^1)$ is a generator of $\pi_1(S^1)$ if and only if $\widetilde{w}(1) = 2\pi$. Thus in view of (4.12) one gets that $h([u_2]) = 2\pi$. This completes the proof. □

5. Differential equations on manifolds

Now we are going to show that in the case when N is a manifold, we are able to express the assumptions of Theorem 4.16 in the language of differential forms. For this purpose, we need to introduce the following concepts.

Recall that if N is an m-dimensional manifold of class C^2, then a one-form $\omega\colon N \to TN^*$ has the following form $\omega(x) = \sum_{i=1}^{m} a_i(x)dx^i$, where $a_i\colon N \to \mathbb{R}$ are functions of class C^2. Furthermore, in this paper we will assume that a one-form

ω is at least closed, i.e. $d\omega = 0$. Moreover, recall that if a path $\gamma\colon [a,b] \to N$ is of class C^1, then

$$\oint_\gamma \omega = \int_a^b \langle \omega(\gamma(s)), \dot\gamma(s)\rangle\, ds = \int_a^b \left(\sum_{i=1}^m a_i(\gamma(s))\dot\gamma_i(s)\right) ds,$$

where the γ_i are the coordinates of γ in N. Let notice that the right-hand side of the above formula makes sense even if γ is absolute continuous. Therefore in our paper we will integrate one-forms ω on absolute continuous paths γ.

In this section by $M \subset \mathbb{R}^n$ we shall denote a k-dimensional closed and contractible manifold of class C^2. By $\omega_{S^1}\colon M\times S^1 \to T(M\times S^1)^*$ we shall understand a one-form defined by

$$\omega_{S^1} = dx^{k+1}. \tag{5.1}$$

Remark 5.1. Notice that ω_{S^1} is a generator of $H^1_d(M \times S^1;\mathbb{R}) \simeq \mathbb{R}$, where H^1_d denotes de Rham cohomology group with coefficients in $\mathbb{R}$. Moreover, if $h\colon N \to M\times S^1$ is a diffeomorphism, then the pullback $h^*\omega_{S^1}$ of ω_{S^1} is a generator of $H^1_d(N;\mathbb{R})$ (see also Remark 5.14).

Remark 5.2. Let $\gamma\colon [0,t] \to M \times S^1$ be a path with $\gamma(0) = (x_0,(1,0))$ and let $\widetilde\gamma\colon [0,t] \to M \times \mathbb{R}$ be a path such that $\gamma = (\mathrm{id} \times p) \circ \widetilde\gamma$ and $\widetilde\gamma(0) = (x_0, 0)$. Let $\gamma = (\gamma_1,\dots,\gamma_k,\gamma_{k+1})$. Then $\gamma_{k+1} = p \circ \widetilde\gamma_{k+1}$. Since the derivative $\dot p\colon T_s\mathbb{R} \to T_{p(s)}S^1$ of p at $s \in \mathbb{R}$ is the identity map, we infer that $\dot\gamma_{k+1}(s) = \dot{\widetilde\gamma}_{k+1}(s)$ for all $s \in \mathbb{R}$. Now we are ready to make the following calculations:

$$\begin{aligned}\oint_\gamma \omega_{S^1} &= \int_0^t \langle \omega_{S^1}(\gamma(s)), \dot\gamma(s)\rangle\, ds = \int_0^t \dot\gamma_{k+1}(s)\, ds \\ &= \int_0^t \dot{\widetilde\gamma}_{k+1}(s)\, ds = \widetilde\gamma_{k+1}(t) - \widetilde\gamma_{k+1}(0) = \widetilde\gamma_{k+1}(t).\end{aligned} \tag{5.2}$$

Remark 5.3. In what follows by $\omega_l\colon M \times S^1 \to T(M \times S^1)^*$ we will denote a one-form which has the following decomposition:

$$\omega_l = \omega_e + l \cdot \omega_{S^1}, \tag{5.3}$$

where $\omega_e\colon M \times S^1 \to T(M \times S^1)^*$ is an exact one-form and ω_{S^1} is as in (5.1), $l \in \mathbb{R}$. Since ω_e is exact, it follows that there exists a differentiable function $g\colon M \times S^1 \to \mathbb{R}^{n+2}$ such that $\omega_e = dg$. We will say that ω_e is a *bounded one-form* provided g is bounded, i.e. there exists a constant M_ω such that $|g(z)| \le M_\omega$ for all $z \in M \times S^1$. It is clear that if M is compact, then ω_e is bounded. Furthermore, we will say that ω_l is bounded provided ω_e is bounded.

Definition 5.4 (see also [9]). Let $f \in \mathrm{Vect}(N)$. A one-form $\omega\colon N \to TN^*$ is said to be a *Lyapunov form* with respect to f if there exists $c > 0$ such that

$$\langle \omega(x), f(x)\rangle > c, \quad \text{for all } x \in N. \tag{5.4}$$

Remark 5.5. Observe that if N is compact, then (5.4) one can replace by $\langle\omega(x), f(x)\rangle > 0$ for all $x \in N$.

Let us observe that Condition (5.4) can be checked pointwise without knowing the trajectories of f just like in the Lyapunov theory. The following example illustrates the concept of a Lyapunov form.

Example 5.6 (see [9]). Let $0 < \beta < 1 < \alpha$ and $\varepsilon > 0$. Consider the following system on N:

$$\begin{cases} \dot{x} = x(1 - x - \alpha y - \beta z) + \varepsilon, \\ \dot{y} = y(1 - \beta x - y - \alpha z) + \varepsilon, \\ \dot{z} = z(1 - \alpha x - \beta y - z) + \varepsilon. \end{cases}$$

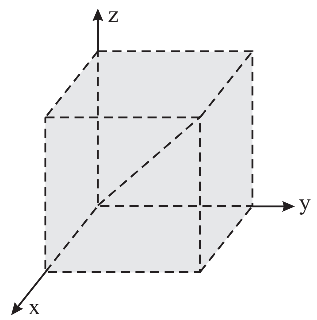


Figure 6. $N := \{(x, y, z) \in \mathbb{R}^3 \mid 0 < x, y, z < 1\} - \{(x, y, z) \in \mathbb{R}^3 \mid x = y = z > 0\}$

It is not hard to see that N is diffeomorphic to $\mathbb{R}^2 \times S^1$. Then the following closed one-form

$$\omega(x, y, z) := \frac{(z - y)dx + (x - z)dy + (y - x)dz}{(z - y)^2 + (x - z)^2 + (y - x)^2}$$

satisfies the condition: $\langle\omega(x, y, z), f(x, y, z)\rangle > 0$, for all $(x, y, z) \in N$, where

$$f(x, y, z) = (x(1-x-\alpha y-\beta z)+\varepsilon, y(1-\beta x-y-\alpha z)+\varepsilon, z(1-\alpha x-\beta y-z)+\varepsilon).$$

Indeed, it follows from the following calculations:

$$\begin{aligned} &\langle((z - y)^2 + (x - z)^2 + (y - x)^2)\omega(x, y, z), f(x, y, z)\rangle \\ &\quad = x(y(1 - \beta x - y - \alpha z) + \varepsilon) - y(x(1 - x - \alpha y - \beta z) + \varepsilon) \\ &\qquad + y(z(1 - \alpha x - \beta y - z) + \varepsilon) - z(y(1 - \beta x - y - \alpha z) + \varepsilon) \\ &\qquad + z(x(1 - x - \alpha y - \beta z) + \varepsilon) - x(z(1 - \alpha x - \beta y - z) + \varepsilon) \\ &\quad = (1 - \beta)(x^2y + y^2z + z^2x) + (\alpha - 1)(x^2z + y^2x + z^2y) + 3xyz(\beta - \alpha) \\ &\quad = (1 - \beta)(x^2y + y^2z + z^2x - 3xyz) + (\alpha - 1)(x^2z + y^2x + z^2y - 3xyz). \end{aligned}$$

But the inequality of arithmetic and geometric means implies that $x^2y + y^2z + z^2x - 3xyz > 0$ and $x^2z + y^2x + z^2y - 3xyz > 0$ for $(x, y, z) \in N$. Thus we deduce the desired inequality.

The following result explains the relationship between a periodic orbit and a Lyapunov form.

Proposition 5.7 ([9]). *If γ is an asymptotically stable* (9) *orbit of a smooth vector field $f\colon \mathbb{R}^n \to \mathbb{R}^n$, then there exists a smooth positively invariant n-dimensional submanifold $\gamma \subset M \subset \mathbb{R}^n$, homeomorphic to $D_n(0,1) \times S^1$, and a Lyapunov form ω (with respect to f).*

Now we are going to establish the most important properties of Lyapunov forms.

Lemma 5.8. *Let $f \in \mathrm{Vect}(M \times S^1)$. If ω_l is a bounded Lyapunov form with respect to f, then $l \neq 0$ and there exists $c > 0$ such that*

$$\oint_{\gamma_t} \omega_l > tc,$$

for all $\gamma \in S_f(M \times S^1)$ and $t > 0$ (10). *In particular, there exists $T > 0$ such that*

$$\oint_{\gamma_T} \omega_l > 2(\pi|l| + M_\omega),$$

for all $\gamma \in S_f(M \times S^1)$.

Proof. Let $c > 0$ be such that $\langle \omega_l(x), f(x) \rangle > c$ for all $x \in M \times S^1$. Fix $\gamma \in S_f(M \times S^1)$. Then

$$\oint_{\gamma_t} \omega_l = \int_0^t \langle \omega_l(\gamma_t(s)), \dot{\gamma}_t(s) \rangle\, ds = \int_0^t \langle \omega_l(\gamma_t(s)), f(\gamma_t(s)) \rangle\, ds > \int_0^t c\, ds = tc,$$

for any $t > 0$. Now we are going to show that $l \neq 0$. To this aim, assume on the contrary that $l = 0$. Then

$$\omega_l(x) = \omega_e(x) = \sum_{i=1}^{k+1} a_i(x)\, dx^i,$$

where $a_i\colon M \times S^1 \to \mathbb{R}$ are functions of class C^2. Since ω_l is an exact form, there exists a differentiable function $g\colon M \times S^1 \to \mathbb{R}$ such that

$$\frac{\partial g(x)}{\partial x_i} = a_i(x)$$

(9) Recall that a compact set $K \subset \mathbb{R}^n$ is said to be asymptotically stable if K is stable (i.e. for every neighbourhood V of K, there exists a neighbourhood V' of K such that $\Pi_f(V' \times \{t\}) \subset V$ for all $t \geq 0$, where Π_f denotes a flow generated by f) and attracts points locally (i.e. there exists a neighbourhood W of K such that K attracts each point in W).

(10) Given $\gamma \in S_f(M \times S^1)$, by γ_t we will denote the restriction of γ to $[0, t]$.

for $i = 1, \ldots, k+1$ and there exists $M_\omega > 0$ such that $|g(x)| < M_\omega$ for all $x \in M \times S^1$. Finally, one has

$$\begin{aligned}
(5.5)\quad tc < \oint_{\gamma_t} \omega_l &= \int_0^t \langle \omega_l(\gamma_t(s)), \dot\gamma_t(s)\rangle\, ds = \int_0^t \langle \omega_e(\gamma_t(s)), \dot\gamma_t(s)\rangle\, ds \\
&= \int_0^t \bigg(\sum_{i=1}^{k+1} a_i(\gamma_t(s))\dot\gamma_{t\,i}(s)\bigg)\, ds = \int_0^t \bigg(\sum_{i=1}^{k+1} \frac{\partial g}{\partial x_i}(\gamma_t(s))\dot\gamma_{t\,i}(s)\bigg)\, ds \\
&= \int_0^t \frac{d(g\circ\gamma_t)(s)}{ds}\, ds = g(\gamma_t(t)) - g(\gamma_t(0)) < 2M_\omega.
\end{aligned}$$

Hence we get that $tc < 2M_\omega$ for all $t > 0$, which implies that $c \le 0$. This contradicts the fact that $c > 0$. Finally, it is easy to that there exists $T > 0$ such that $Tc > 2(\pi|l| + M_\omega)$. This completes the proof. □

Lemma 5.9. *If $\omega\colon M \times S^1 \to T(M \times S^1)^*$ is a closed one-form, then there exists $l \in \mathbb{R}$ such that $\omega = \omega_l$. If additionally ω_l is a bounded Lyapunov form with respect to $f \in \operatorname{Vect}(M \times S^1)$, then $l \ne 0$.*

Proof. This follows from the fact that $H^1_d(M \times S^1; \mathbb{R}) \simeq \mathbb{R}$, where $H^1_d(M \times S^1; \mathbb{R})$ denotes de Rham cohomology group with coefficients in $\mathbb{R}$ ([11]). Finally, Lemma 5.8 implies that $l \ne 0$, which completes the proof. □

Remark 5.10. From now on we will say that a closed one-form $\omega\colon M \times S^1 \to T(M \times S^1)^*$ is bounded if the corresponding one-form ω_l is bounded.

Theorem 5.11. *Assume that $f \in \operatorname{Vect}_c(M \times S^1)$ is bounded. If there exists a bounded Lyapunov form $\omega\colon M \times S^1 \to T(M \times S^1)^*$ for f, then problem* (P) *admits a solution which generates $\pi_1(M \times S^1)$.*

Proof. Let $\widetilde{f}\colon M \times \mathbb{R} \to \mathbb{R}^{n+1}$ be a lift of f. Lemma 5.9 implies that there exists a closed one-form ω_l such that $\omega = \omega_l$ and $l \ne 0$. Without loss of generality we can assume that $l > 0$ (see Remark 4.16). From Lemma 5.8 it follows that there exists $T > 0$ such that

$$\oint_{\gamma_T} \omega_l > 2(\pi l + M_\omega),$$

for all $\gamma \in S_f(M \times \{(1,0)\})$, where γ_T denotes the restriction of γ to $[0,T]$. Fix $\gamma \in S_f(M \times \{(1,0)\})$. Let $\widetilde\gamma \in S_{\widetilde f}(M \times \{0\})$ be a lift of γ, i.e. satisfying the condition $\gamma = (\mathrm{id} \times p) \circ \widetilde\gamma$. Now, reasoning as in (5.2) and (5.5), we obtain

$$\begin{aligned}
2(\pi l + M_\omega) &< \oint_{\gamma_T} \omega_l = \oint_{\gamma_T} \omega_e + l \oint_{\gamma_T} \omega_{S^1} \\
&= g(\gamma_T(T)) - g(\gamma_T(0)) + l\widetilde\gamma_{k+1}(T) < 2M_\omega + l\widetilde\gamma_{k+1}(T).
\end{aligned}$$

([11]) A different proof can be found in [9].

Thus $2\pi < \widetilde{\gamma}_{k+1}(T)$ and, consequently, we get that $\widetilde{\gamma}(T) \in M \times (2\pi, \infty)$. Therefore $\widetilde{f}$ satisfies the condition (4.6). Consequently, our conclusion follows directly from Theorem 4.16, which completes the proof. □

In particular, we get following corollary which has been proved in [9] and [11].

Corollary 5.12. *If $f \in \mathrm{Vect}(D_n(0,1) \times S^1)$ and there exists a Lyapunov form $\omega\colon D_n(0,1) \times S^1 \to T(D_n(0,1) \times S^1)^*$ for f, then there exists a nontrivial and noncontractible periodic orbit in $D_n(0,1) \times S^1$, where $n \geq 1$.*

However, if $n = 1$, then we have the following stronger result:

Theorem 5.13 (Poincaré–Bendixson). *Assume that $N \subset \mathbb{R}^2$ is diffeomorphic to $D_1(0,1) \times S^1$. Let $f\colon N \to \mathbb{R}^2$ be of class C^1 pointing inward on $\mathrm{bd}\, N$. If f has no equilibria, then f has a periodic orbit which is not contractible in N.*

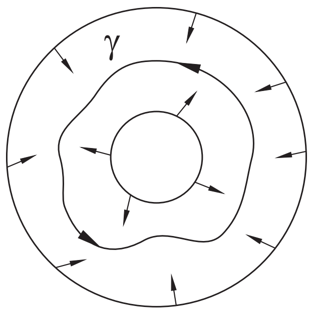


Figure 7. An illustration of Poincaré–Bendixson theorem

Unfortunately the above result is not true in higher dimensions without additional assumptions. Namely, F.B. Fuller [17] has constructed a nonvanishing vector field $f \in \mathrm{Vect}(D_2(0,1) \times S^1)$ which has no periodic and noncontractible trajectory in the torus $D_2(0,1) \times S^1$ (see [17]). Briefly speaking, his construction (see Figure 8) has the property that any trajectory starting from the section $\mathbf{S}_1$ is attracting by the section $\mathbf{S}_2$ which in turn implies that such a trajectory does not reach the area between two sections $\mathbf{S}_1$ and $\mathbf{S}_2$, and therefore there exists no closed trajectory generating $\pi_1(D_2(0,1) \times S^1)$. It is a main reason why we have to assume the existence of a Lyapunov form on $M \times S^1$. However, it should be noted that in this case there are closed and contractible trajectories. Namely, they appear only in Section $\mathbf{S}_2$.

It should be noted that the problem (P) defined on $M \times S^1$ (where M is assumed to be a manifold of class C^2) can be considered on any manifold N which is diffeomorphic to $M \times S^1$. For example $M \times S^1$ can be diffeomorphic to the space drawn on Figures 6, 8 and 9.

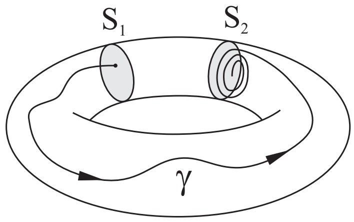


Figure 8. Fuller's construction

Remark 5.14. Consider $f\colon N \to TN$ and a closed one-form $\omega\colon N \to TN^*$. Assume additionally that $h\colon M \times S^1 \to N$ is a diffeomorphism ([12]) between two manifolds N and $M \times S^1$. Then the following two diagrams:

$$\begin{array}{ccc} M \times S^1 & \xrightarrow{\ \widetilde{f}\ } & T(M \times S^1) \\ {\scriptstyle h}\downarrow & & \uparrow{\scriptstyle D(h^{-1})} \\ N & \xrightarrow[f]{} & TN \end{array} \qquad \begin{array}{ccc} M \times S^1 & \xrightarrow{\ \widetilde{\omega}\ } & T(M \times S^1)^* \\ {\scriptstyle h}\downarrow & & \uparrow{\scriptstyle (Dh)^*} \\ N & \xrightarrow[\omega]{} & TN^* \end{array}$$

induce $\widetilde{f}$ and $\widetilde{\omega}$, where Dh (resp. $D(h^{-1})$) stands for the derivative map of h (resp. h^{-1}). Another words, $\widetilde{\omega}$ is the pull-back of a differential form ω, i.e. $\widetilde{\omega} = h^*\omega$. Moreover, one has

$$\begin{aligned} \langle \widetilde{\omega}(x), \widetilde{f}(x) \rangle &= \langle (Dh)^*(\omega(h(x))), D(h^{-1})(f(h(x))) \rangle \\ &= \langle \omega(h(x)), Dh\left(D(h^{-1})(f(h(x)))\right) \rangle \\ &= \langle \omega(h(x)), f(h(x)) \rangle = \langle \omega(y), f(y) \rangle \end{aligned}$$

for $x \in M \times S^1$ and $y = h(x) \in N$. Hence we infer that ω is a Lyapunov form if and only if $\widetilde{\omega}$ is a Lyapunov form. Furthermore, it is not hard to see that $\gamma \in S_{\widetilde{f}}(M \times S^1)$ if and only if $h \circ \gamma \in S_f(N)$ and $\widetilde{f} \in \mathrm{Vect}_c(M \times S^1)$ if and only if $f \in \mathrm{Vect}_c(N)$. Finally, we will say that $\omega\colon N \to TN^*$ is a *bounded one-form* if the pull-back $h^*\omega$ is a bounded one-form.

Thus, from Remark 5.14 and Theorem 5.11 we get the following corollary.

Corollary 5.15. *Let $h\colon M \times S^1 \to N$ be a diffeomorphism and let $f \in \mathrm{Vect}_c(N)$ be bounded. If there exists a bounded Lyapunov form $\omega\colon N \to TN^*$ for f, then there exists a nontrivial and noncontractible periodic orbit in N.*

Now we would like to provide some examples illustrating our results presented in this article. In addition, the first example shows how one can follow in other cases in order to find a closed trajectory.

([12]) By a diffeomorphism between X and Y we will understand a homeomorphism $h\colon X \to Y$ such that h and h^{-1} are of class C^2.

EXAMPLE 5.16. Consider the following system of differential equations on N (see Figure 9):

$$\begin{cases} \dot{x} = -yg_1(x,y,z) + xg_2(x,y,z), \\ \dot{y} = xg_1(x,y,z) + yg_2(x,y,z), \\ \dot{z} = \alpha(x,y,z), \end{cases} \tag{5.6}$$

where $g_1\colon N \to \mathbb{R}$, $g_2\colon N \to \mathbb{R}$ and $\alpha\colon N \to \mathbb{R}$ are continuous and bounded and satisfy the following conditions:

- there exists $c_1 > 0$ such that $\alpha(x,y,z) < 0$ for $(x,y,z) \in N$ with $z \geq c_1$ and there exists $c_2 < 0$ such that $\alpha(x,y,z) > 0$ for $(x,y,z) \in N$ with $z \leq c_2$;
- $g_2(x,y,z) > 0$ for $(x,y,z) \in N$ with $x^2 + y^2 = r^2$ and $g_2(x,y,z) < 0$ for $(x,y,z) \in N$ with $x^2 + y^2 = R^2$;
- there exists $m > 0$ such that $g_1(x,y,z) \geq m$ for all $(x,y,z) \in N$.

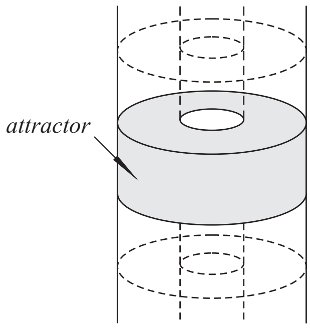


FIGURE 9. $N := \{(x,y,z) \in \mathbb{R}^3 \mid r^2 \leq x^2 + y^2 \leq R^2,\ z \in \mathbb{R}\}$

Let $f\colon N \to \mathbb{R}^3$ be given by

$$f(x,y,z) = (-yg_1(x,y,z) + xg_2(x,y,z), xg_1(x,y,z) + yg_2(x,y,z), \alpha(x,y,z)),$$

where N is defined as follows: let $\omega\colon N \to TN^*$ be a closed one-form defined by

$$\omega(x,y,z) = \frac{-y\,dx + x\,dy}{x^2 + y^2} - 0\,dz.$$

Then

$$\langle \omega(x,y,z), f(x,y,z)\rangle = g_1(x,y,z) \geq m > 0, \quad \text{for all } (x,y,z) \in N.$$

It is easy to see that if $h\colon M \times S^1 \to N$ is a diffeomorphism given by

$$h(x,z,\alpha) = (x\cos(\alpha), x\sin(\alpha), z),$$

where $M = [r,R] \times \mathbb{R}$ and $S^1 = [0,2\pi]/\{0,2\pi\}$ (we identify 0 with 2π), then

$$h^*\left(\frac{-ydx + xdy}{x^2+y^2}\right) = d\alpha = \omega_{S^1}.$$

Thus ω is a bounded Lyapunov form. Now we will show that $f \in \mathrm{Vect}(N)$. For this purpose it suffices to show that f points inward on the boundary of N. But this holds if and only if the following two inequalities are satisfied:

$$\langle (f_1(x,y,z), f_2(x,y,z)), (x,y)\rangle \geq 0 \quad (\text{resp. } \leq 0)$$
$$\text{if } x^2+y^2 = r^2 \quad (\text{resp. } x^2+y^2 = R^2).$$

Since

$$\begin{aligned}
&\langle (f_1(x,y,z), f_2(x,y,z)), (x,y)\rangle \\
&= \langle (-yg_1(x,y,z) + xg_2(x,y,z), xg_1(x,y,z) + yg_2(x,y,z)), (x,y)\rangle \\
&= (x^2+y^2)g_2(x,y,z),
\end{aligned}$$

it follows that the above inequalities hold true. Now we will show that

$$N_0 := \{(x,y,z) \in N \mid c_2 \leq z \leq c_1\}$$

is a compact attractor for f (see Figure 9). First of all, since the vector field f is tangent on the set N_0, i.e. $f(x,y,z) \in T_{N_0}(x,y,z)$, it follows that N_0 is positively invariant with respect to a set-valued semiflow generated by (5.6). Furthermore, for any trajectory $[0,\infty) \ni t \mapsto (x(t),y(t),z(t))$ starting from a point $(x_0,y_0,z_0) \in N - N_0$ there exists $t_0 > 0$ such that $(x(t_0),y(t_0),z(t_0)) \in N_0$. To this aim, suppose on the contrary, that $(x(t),y(t),z(t)) \notin N_0$ for all $t \geq 0$. Let $L\colon N \to \mathbb{R}$ be given by $L(x,y,z) = z^2$. Then a function $[0,\infty) \ni t \overset{\gamma}{\longmapsto} L(x(t),y(t),z(t))$ is absolute continuous and hence the derivative of γ exists for almost all $t \geq 0$. It is easy to see that if $\dot\gamma(t)$ exists, then

$$\dot\gamma(t) = 2z(t)\alpha(x(t),y(t),z(t)) < 0. \tag{5.7}$$

Consequently, the function L is nonincreasing along the trajectory γ. Hence if $t > s$, then

$$z(t)^2 = L(x(t),y(t),z(t)) \leq L(x(s),y(s),z(s)) = z(s)^2, \tag{5.8}$$

which implies that $|z(t)| \xrightarrow{t\to\infty} c$. There are two possibilities:

(i) $z(t) > c_1$ for all $t \geq 0$, or
(ii) $z(t) < c_2$ for all $t \geq 0$.

It suffices to consider the first case (in the second case the reasoning is similar). In the case (i) we have $z(t) \xrightarrow{t\to\infty} c \geq c_1$. Then

$$(x(t),y(t),z(t)) \in N_{c_1} := \{(x,y,z) \in N \mid c_1 \leq |z| \leq z_0\},$$

for all $t \geq 0$. Let $M_{z_0} := \max_{(x,y,z)\in N_{c_1}} \alpha(x,y,z) < 0$. Hence $\dot{\gamma}(t) \leq 2z_0 M_{z_0} < 0$, for almost all $t \geq 0$. Now taking into account the Fundamental Theorem of Calculus for absolute continuous functions we obtain

$$L(x(t),y(t),z(t)) - L(x_0,y_0,z_0) = \int_0^t \frac{d}{ds} L(x(s),y(s),z(s))\,ds$$
$$= \int_0^t \dot{\gamma}(s) ds \leq \int_0^t 2z_0 M_{z_0}\,ds \leq 2tz_0 M_{z_0}.$$

Consequently,

$$L(x(t),y(t),z(t)) \leq L(x_0,y_0,z_0) + 2tz_0 M_{z_0}. \tag{5.9}$$

This follows that there exists $t_0 > 0$ such that $L(x_0,y_0,z_0) + 2t_0 z_0 M_{z_0} < 0$ and hence we deduce that $L(x(t_0),y(t_0),z(t_0)) < 0$, which contradicts the fact that the function L is nonnegative. Now we are to prove that for every $(x_0,y_0,z_0) \in N$ one has

$$d_H(\Pi_f((x_0,y_0,z_0),t),N_0) \xrightarrow{t\to\infty} 0. \tag{5.10}$$

First, observe that if $(x_0,y_0,z_0) \in N_0$, then any $\gamma \in S_f(x_0,y_0,z_0)$ satisfies the following condition: $\gamma(t) \in N_0$ for all $t \geq 0$. Consequently, we deduce that $d_H(\Pi_f((x_0,y_0,z_0),t),N_0) = 0$ for all $t \geq 0$. On the other hand, since L is always nonincreasing along each part of the trajectory of (5.6) included in $N - N_0$, we infer that if $(x_0,y_0,z_0) \in N - N_0$, then there exists $t_0 > 0$ (depending on $(x_0,y_0,z_0) \in N - N_0$) such that $\gamma(t) \in N_0$ for all $\gamma \in S_f(x_0,y_0,z_0)$ and $t \geq t_0$ (for instance, if $z_0 > c_1$, then, in view of (5.9), it suffices to put $t_0 := (z_0^2 - c_1^2)/(2z_0|M_{z_0}|)$, which implies that $d_H(\Pi_f((x_0,y_0,z_0),t),N_0) = 0$ for all $t \geq t_0$. This proves (5.10).

Consequently, we have proved that N_0 is an attractor for f. Finally, Corollary 5.15 implies that there exists a nontrivial periodic orbit. Since the space N_0 is an attractor for f, it follows that a periodic orbit is contained in N_0.

Example 5.17. In particular one can consider the following system of differential equations:

$$\begin{cases} \dot{x} = -y + x(1 - x^2 - 2y^2), \\ \dot{y} = x + y(1 - x^2 - 2y^2), \\ \dot{z} = -2\operatorname{sign}(z)\sqrt{|z|} + x^2 - y^4, \end{cases}$$

where $N := \{(x,y,z) \in \mathbb{R}^3 \mid 4^{-1} \leq x^2 + y^2 \leq 4,\ z \in \mathbb{R}\}$ and $\omega\colon N \to TN^*$ is defined by

$$\omega(x,y,z) = \frac{-y\,dx + x\,dy}{x^2 + y^2} - 0\,dz. \tag{5.11}$$

Thus, reasoning as in the previous example, we deduce that there exists a nontrivial periodic orbit in N.

Example 5.18. Consider the following system of differential equations on N:

$$\begin{cases} \dot{x} = -yg_1(x,y,z) + xg_2(x,y,z), \\ \dot{y} = xg_1(x,y,z) + yg_2(x,y,z), \\ \dot{z} = (g_1(x,y,z) - d)(z^2+1), \end{cases}$$

where $N := \{(x,y,z) \in \mathbb{R}^3 \mid r^2 \le x^2+y^2 \le R^2\}$, $0 < r < R$, $g_1 \colon N \to \mathbb{R}$ and $g_2 \colon N \to \mathbb{R}$ are continuous and bounded and satisfy the following conditions:

- $g_1(x,y,z) < d$ if $z > c_1$ and $g_1(x,y,z) > d$ if $z < -c_2$, where $c_1, c_2, d > 0$;
- $g_2(x,y,z) > 0$ for $(x,y,z) \in N$ with $x^2+y^2 = r^2$;
- $g_2(x,y,z) < 0$ for $(x,y,z) \in N$ with $x^2+y^2 = R^2$.

In this example we define $\omega \colon N \to TN^*$ as follows:

$$\omega(x,y,z) = \frac{-y\,dx + x\,dy}{x^2+y^2} - \frac{1}{1+z^2}\,dz.$$

Then $\langle \omega(x,y,z), f(x,y,z)\rangle = d > 0$, where

$$f(x,y,z) = (-yg_1(x,y,z)+xg_2(x,y,z), xg_1(x,y,z)+yg_2(x,y,z), g_1(x,y,z)-d).$$

Notice that a one-form $\omega_0 \colon N \to TN^*$ given by

$$\omega_0(x,y,z) = \frac{-y\,dx + x\,dy}{x^2+y^2} - 0\,dz$$

does not work in this case because

$$\langle \omega_0(x,y,z), f(x,y,z)\rangle = g_1(x,y,z)$$

and $g_1(x,y,z)$ can take the value zero for some $(x,y,z) \in N$. Let $h \colon M \times S^1 \to N$ be a diffeomorphism as in Example 5.16. Then

$$h^*\omega = d\alpha - \frac{1}{1+z^2}\,dz = \omega_{S^1} + \omega_e,$$

where $\omega_{S^1}(x,z,\alpha) = d\alpha$ and $\omega_e(x,z,\alpha) = (-1)/(1+z^2)dz$. Since

$$\frac{1}{1+z^2} = \frac{d(\arctan(z))}{dz}$$

and $\arctan(z)$ is bounded, it follows that ω is a bounded one-form (see Remark 5.14). Now following Example 5.16 one can prove that $f \in \mathrm{Vect}_c(N)$. Thus from Corollary 5.15 we deduce that there exists a nontrivial periodic orbit in N which generates $\pi_1(N)$.

6. Comments

In this section we will make some comments about possible extensions and applications of the results obtained in this paper.

- The methods presented in Section 2 suggests that one can obtain an extension of the well-known Ważewski principle to some cases in which not all egress points are strict egress points (see a survey paper on the Ważewski retract method [19] and Introduction in this paper).
- By using the standard methods from the theory of set-valued maps one can extend all results obtained in this paper to the case of differential inclusions (see [21]).
- Example 5.6 suggests that the technique of Lyapunov forms can be applied in the mathematical theory of persistence (see [38]).

7. Appendix

In the last section we have collected some definitions and facts from the theory of weighted maps which are used in this article. For more information about this class of set-valued maps we refer the reader to the textbook [35] and the papers [11], [26]–[29], [32], [24], [25].

From now on we will assume the all considered spaces are connected ANRs. Given a map $\Phi\colon (X,X_0) \multimap (Y,Y_0)$ we denote by $\Phi_X\colon X \multimap Y$ and $\Phi_{X_0}\colon X_0 \multimap Y_0$ the evident maps defined by Φ (if $X_0=\emptyset$, then we will identify Φ_X with Φ).

We put ([13]):

$$\begin{aligned}
\mathrm{W}((X,X_0),(Y,Y_0)) &:= \{\Psi\colon (X,X_0) \multimap (Y,Y_0) \mid \Psi \text{ is a weighted carrier} \\
&\qquad \text{with } I_w(\Psi)\neq 0\},\\
\mathrm{A_W}((X,X_0),(Y,Y_0)) &:= \{\Psi \in \mathrm{W}((X,X_0),(Y,Y_0)) \mid \check{H}_k(\Psi(x);\ \mathbb{Q})=0 \\
&\qquad \text{for all } k\geq 1 \ \text{ for all } x\in X\},\\
\mathrm{C}((X,X_0),(Y,Y_0)) &:= \{f\colon (X,X_0)\to (Y,Y_0) \mid f \text{ is continuous} \\
&\qquad \text{with } I_w(f)=1\},\\
\mathrm{CA}_W(X,X_0) &:= \{\Phi \mid \Phi = f\circ\Psi,\ f\in \mathrm{C}((Y,Y_0),(X,X_0)),\\
&\qquad \Psi\in \mathrm{A}_W((X,X_0),(Y,Y_0))\}.
\end{aligned}$$

Definition 7.1. Let $\Psi\colon X \multimap Y$ and $\mathrm{id}\colon Z\to Z$ be two weighted carriers. Let $(x,z)\in X\times Z$ be an arbitrary point and let U be an open subset of $Y\times Z$ such that $(\Psi(x)\times \mathrm{id}(z))\cap \mathrm{bd}\, U=\emptyset$. Then $I_{wloc}\colon D(\Psi\times\mathrm{id})\to\mathbb{Q}$ is defined as follows

$$I_{\mathrm{wloc}}(\Psi\times\mathrm{id},U,(x,z)) := I_{\mathrm{wloc}}(\Psi,\mathrm{pr}(U_z),x) \ ([14]), \tag{7.1}$$

([13]) If $X_0=\emptyset$, then we will write $\mathrm{CA}_W(X)$ instead of $\mathrm{CA}_W(X,X_0)$ and so on.

where $U_z := U \cap (Y \times \{z\})$.

Remark 7.2. Let us observe that $\mathrm{pr}(U_z)$ is an open subset of a space Y. Moreover, since

$$(\Psi(x) \times \{z\}) \cap \mathrm{bd}\,(\mathrm{pr}(U_z)) = \emptyset \quad \text{and} \quad \mathrm{bd}_{Y\times\{z\}} U_z \subset \mathrm{bd}\, U,$$

we conclude that $(\Psi(x) \times \{z\}) \cap \mathrm{bd}_{Y\times\{z\}}\, U_z = \emptyset$. Hence,

$$\begin{aligned}\Psi(x) \cap \mathrm{bd}\,(\mathrm{pr}(U_z)) &= \mathrm{pr}(\Psi(x) \times \{z\}) \cap \mathrm{pr}(\mathrm{bd}_{Y\times\{z\}} U_z)\\ &= \mathrm{pr}((\Psi(x) \times \{z\}) \cap \mathrm{bd}_{Y\times\{z\}} U_z) = \emptyset.\end{aligned}$$

Consequently, the right-hand side of (7.1) is well-defined.

Proposition 7.3. *Let $\Psi\colon X \multimap Y$ and $\mathrm{id}\colon Z \to Z$ be two weighted carriers. Then a function $I_{\mathrm{wloc}}\colon D(\Psi \times \mathrm{id}) \to \mathbb{Q}$ defined as in* (7.1) *satisfies all the conditions of Definition* 3.1. *In particular, if X is connected, then $I_w(\Psi\times\mathrm{id}) = I_w(\Psi)$.*

Proof. Let $(x, z) \in X \times Z$ and let U be an open subset of $Y \times Z$ with $(\Psi(x) \times \mathrm{id}(z)) \cap \mathrm{bd}\, U = \emptyset$.

Existence. Assume that $I_{\mathrm{wloc}}(\Psi\times\mathrm{id}, U, (x, z)) \neq 0$. Then, by (7.1), we obtain

$$I_{\mathrm{wloc}}(\Psi, \mathrm{pr}(U_z), x) \neq 0.$$

Consequently, $\Psi(x) \cap \mathrm{pr}(U_z) \neq \emptyset$ and

$$\emptyset \neq (\Psi(x)\times\mathrm{id}(z))\cap(\mathrm{pr}(U_z)\times\mathrm{id}(z)) = (\Psi(x)\times\mathrm{id}(z))\cap U_z \subset (\Psi(x)\times\mathrm{id}(z))\cap U,$$

as required.

Local invariance. Let

$$(\Psi(x) \times \mathrm{id}(z)) \cap U = F_x \times \{z\}, \quad (\Psi(x) \times \mathrm{id}(z)) \cap ((X \times Y) \setminus \overline{U}) = F'_x \times \{z\}.$$

Then $F_x \cup F'_x = \Psi(x)$ and $F_x \cap F'_x = \emptyset$. Moreover, the compactness of F_x and F'_x implies that there exist open subsets $V_x, V'_x \subset Y$ and $V_z \subset Z$ such that

$$F_x \times \{z\} \subset V_x \times V_z \subset U, \tag{7.2}$$

$$F'_x \times \{z\} \subset V'_x \times V_z \subset (Y \times Z) \setminus \overline{U}. \tag{7.3}$$

Since $\Psi \times \mathrm{id}$ is usc, it follows that there exist open sets W_x and W_z such that

$$x \in W_x, z \in W_z, \quad \Psi(\widetilde{x}) \times \mathrm{id}(\widetilde{z}) \subset V_x \times V_z \cup V'_z \times V_z, \tag{7.4}$$

for all $(\widetilde{x}, \widetilde{z}) \in W_x \times W_z$. In addition, from the local invariance property of I_{wloc} for Ψ it follows that there exists an open neighbourhood $B(x, \varepsilon)$ of a point x such that

$$I_{\mathrm{wloc}}(\Psi, \mathrm{pr}(U_z), x) = I_{\mathrm{wloc}}(\Psi, \mathrm{pr}(U_z), \widetilde{x}), \tag{7.5}$$

([14]) In this section by pr we denote the projection of the Cartesian product of two spaces on the first factor.

for all $\widetilde{x} \in B(x,\varepsilon)$. Now we will show that the following equality holds

$$I_{\mathrm{wloc}}(\Psi \times \mathrm{id}, U, (x,z)) = I_{\mathrm{wloc}}(\Psi \times \mathrm{id}, U, (\widetilde{x},\widetilde{z})).$$

for all $(\widetilde{x},\widetilde{z}) \in (B(x,\varepsilon) \cap W_x) \times W_z$. For this purpose, fix a point $(\widetilde{x},\widetilde{z}) \in (B(x,\varepsilon) \cap W_x) \times W_z$. Then, taking into account (7.2)–(7.4), we obtain

$$(\Psi(x) \times \{z\}) \cap U_z = (\Psi(x) \times \{z\}) \cap (U \cap (Y \times \{z\})) \subset V_x \times \{z\} \subset U_z, \tag{7.6}$$

$$(\Psi(\widetilde{x}) \times \{\widetilde{z}\}) \cap U_{\widetilde{z}} = (\Psi(\widetilde{x}) \times \{\widetilde{z}\}) \cap (U \cap (Y \times \{\widetilde{z}\})) \subset V_x \times \{\widetilde{z}\} \subset U_{\widetilde{z}}, \tag{7.7}$$

where $U_z = U \cap (Y \times \{z\})$ and $U_{\widetilde{z}} = U \cap (Y \times \{\widetilde{z}\})$. Consequently,

$$\begin{aligned}
\Psi(x) \cap \mathrm{pr}(U_z) &= \mathrm{pr}(\Psi(x) \times \{z\}) \cap \mathrm{pr}(U_z) \\
&= \mathrm{pr}(\Psi(x) \times \{z\} \cap U_z) \overset{(7.6)}{\subset} \mathrm{pr}(V_x \times \{z\}) = V_x, \\
\Psi(\widetilde{x}) \cap \mathrm{pr}(U_{\widetilde{z}}) &= \mathrm{pr}(\Psi(\widetilde{x}) \times \{\widetilde{z}\}) \cap \mathrm{pr}(U_{\widetilde{z}}) \\
&= \mathrm{pr}(\Psi(\widetilde{x}) \times \{\widetilde{z}\} \cap U_{\widetilde{z}}) \overset{(7.7)}{\subset} \mathrm{pr}(V_x \times \{\widetilde{z}\}) = V_x.
\end{aligned}$$

Hence from the excision property of I_{wloc} for Ψ it follows that

$$I_{\mathrm{wloc}}(\Psi, \mathrm{pr}(U_z), \widetilde{x}) = I_{\mathrm{wloc}}(\Psi, V_x, \widetilde{x}) = I_{\mathrm{wloc}}(\Psi, \mathrm{pr}(U_{\widetilde{z}}), \widetilde{x}). \tag{7.8}$$

Finally,

$$\begin{aligned}
I_{\mathrm{wloc}}(\Psi \times \mathrm{id}, U, (x,z)) &= I_{\mathrm{wloc}}(\Psi, \mathrm{pr}(U_z), x) \overset{(7.5)}{=} I_{\mathrm{wloc}}(\Psi, \mathrm{pr}(U_z), \widetilde{x}) \\
&\overset{(7.8)}{=} I_{\mathrm{wloc}}(\Psi, \mathrm{pr}(U_{\widetilde{z}}), \widetilde{x}) = I_{\mathrm{wloc}}(\Psi \times \mathrm{id}, U, (\widetilde{x},\widetilde{z})),
\end{aligned}$$

as desired.

Additivity. Let $\Psi(x) \times \mathrm{id}(z) \cap U \subset \bigcup_{i=1}^{k} U^i \subset U$, where U^i, for $i = 1, \ldots, k$, are open subsets of U and $U^i \cap U^j = \emptyset$ for $i \neq j$. Since

$$\Psi(x) \cap \mathrm{pr}(U_z) = \mathrm{pr}((\Psi(x) \times \{z\}) \cap U_z) \subset \bigcup_{i=1}^{k} \mathrm{pr}(U_z^i) \subset \mathrm{pr}(U_z),$$

we deduce from the additivity property of I_{wloc} for Ψ that

$$\begin{aligned}
I_{\mathrm{wloc}}(\Psi \times \mathrm{id}, U, (x,z)) &= I_{\mathrm{wloc}}(\Psi, \mathrm{pr}(U_z), x) \\
&= \sum_{i=1}^{k} I_{\mathrm{wloc}}(\Psi, \mathrm{pr}(U_z^i), x) = \sum_{i=1}^{k} I_{\mathrm{wloc}}(\Psi \times \mathrm{id}, U^i, (x,z)),
\end{aligned}$$

as required. Finally, let us observe that

$$I_w(\Psi \times \mathrm{id}) = I_{\mathrm{wloc}}(\Psi \times \mathrm{id}, Y \times Z, (x,z)) = I_{\mathrm{wloc}}(\Psi, Y, x) = I_w(\Psi),$$

which completes the proof. □

Definition 7.4. An usc set-valued map $\Phi: X \multimap Y$ is called *locally compact* provided each $x \in X$ has a neighbourhood U_x such that the restriction $\Psi|U_x: U_x \multimap Y$ is compact.

Definition 7.5. Let $\Phi: X \multimap X$ be locally compact. We say that

(a) Φ has a compact attractor provided there exists a compact set $K \subset X$ such that for every open neighbourhood U of K in X and for every $x \in X$ there exists a natural number n_x such that $\Phi^n(x) \subset U$ for every $n \geq n_x$.

(b) Φ is called a *compact absorbing contraction* if there exists an open subset X_0 of X satisfying: (1) $\Phi(X_0) \subset X_0$, (2) $\Phi|X_0: X_0 \multimap X_0$ is a compact map, (3) for every $x \in X$ there exists n_x such that $\Phi^{n_x}(x) \subset X_0$ (written $\Phi \in \mathrm{CAC}(X, X_0)$).

Lemma 7.6 (see [21, Chapter IV]). *If $\Phi: X \multimap X$ has a compact attractor, then Φ is a compact absorbing contraction.*

Notice that for any map $\Phi \in \mathrm{CAC}(X, X_0) \cap \mathrm{CA_W}(X, X_0)$, using the methods developed in [35], one can define the Lefschetz numbers $\Lambda(\Phi)$, $\Lambda(\Phi_X)$, $\Lambda(\Phi_{X_0}) \in \mathbb{Q}$ which have all the expected properties of the Lefschetz number for single-valued maps (see [35, Chapter 4], [29] and [22, Chapter V]). In particular, if X is a contractible ANR and $\Phi \in \mathrm{CAC}(X, X_0) \cap \mathrm{CA_W}(X, X_0)$, then

$$\Lambda(\Phi) = 0 \quad \text{and} \quad \Lambda(\Phi_X) = \Lambda(\Phi_{X_0}) = I_w(\Phi_X) \neq 0. \tag{7.9}$$

The proof of this fact is analogous as in the case of single-valued maps (see [22, Chapter V] for single-valued maps).

Theorem 7.7 ([35, Corollary 4.5.17]). *If X is an* ANR *and $\Phi \in \mathrm{CA_W}(X)$ is compact with $\Lambda(\Phi) \neq 0$, then* $\mathrm{Fix}(\Phi) \neq \emptyset$.

The above theorem can be extended to the case when Φ is not compact.

Theorem 7.8. *If X is a contractible* ANR *and*

$$\Phi \in \mathrm{CAC}(X, X_0) \cap \mathrm{CA_W}(X, X_0),$$

then $\mathrm{Fix}(\Phi) \neq \emptyset$.

Proof. It follows directly from (7.9) and Theorem 7.7. □

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