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ANALYTIC ROBUSTNESS OF PARAMETER-DEPENDENT PERTURBATIONS OF DIFFERENCE EQUATIONS

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Abstract. We establish the robustness of nonuniform exponential dichotomies under sufficiently small analytic parameter-dependent perturbations. We also show that the stable and unstable subspaces of the exponential dichotomies depend analytically on the parameter.

1. Introduction

We consider nonautonomous linear difference equations

$$v_{m+1} = A_m v_m + B_m(\lambda) v_m \tag{1.1}$$

in a finite-dimensional space, where $\lambda \mapsto B_m(\lambda)$ is analytic for each $m \in \mathbb{Z}$. Assuming that the unperturbed dynamics

$$v_{m+1} = A_m v_m \tag{1.2}$$

has a nonuniform exponential dichotomy, we establish the existence of nonuniform exponential dichotomies for equation (1.1) provided that the the maps $B_m(\lambda)$ are sufficiently small. Namely, we assume that there exists a constant $\delta > 0$ such that

$$\|B_m(\lambda)\| \le \delta e^{-3\varepsilon |m|}$$

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for every $m \in \mathbb{Z}$, for some appropriate constant $\varepsilon > 0$. The above property is called the robustness of the dichotomy. Moreover, we show that the stable and unstable subspaces associated to equation (1.1) are analytic in λ.

The notion of exponential dichotomy, introduced by Perron in [10], plays a central role in a large part of the theory of dynamical systems, and the study of robustness has a long history. In particular, the problem was discussed by Massera and Schäffer [7] (building on earlier work of Perron [10]; see also [8]), Coppel [5], and in the case of Banach spaces by Dalec'kiĭ and Kreĭn [6]. For more recent works we refer to [4], [9], [11], [12] and the references therein. We note that all these works consider only uniform exponential dichotomies. This is a very stringent notion for the dynamics, and it is of interest to look for more general types of hyperbolic behavior. In particular, when all Lyapunov exponents are nonzero the linear dynamics in (1.2) has a nonuniform exponential dichotomy. We refer to the books [1], [2] for detailed related discussions.

2. Setup

We describe in this section the standing assumptions in the paper. Given a sequence $(A_m)_{m\in\mathbb{Z}}$ of $l \times l$ matrices with real entries, we consider the difference equation

$$v_{m+1} = A_m v_m, \quad m \in \mathbb{Z}.$$

For each $m, n \in \mathbb{Z}$ with $m \geq n$, we have $v_m = \mathcal{A}(m,n)v_n$, where

$$\mathcal{A}(m,n) = \begin{cases} A_{m-1} \dots A_n & \text{if } m > n, \\ \mathrm{Id} & \text{if } m = n. \end{cases}$$

We say that the sequence $(A_m)_{m\in\mathbb{Z}}$ admits a *nonuniform exponential dichotomy* if:

(a) there exist projections $P_m : \mathbb{R}^l \to \mathbb{R}^l$ for each $m \in \mathbb{Z}$ satisfying

$$\mathcal{A}(m,n)P_n = P_m\mathcal{A}(m,n), \quad m \geq n,$$

such that the map

$$\overline{\mathcal{A}}(m,n) := \mathcal{A}(m,n)|\ker P_n : \ker P_n \to \ker P_m$$

is invertible for each $m \geq n$;

(b) there exist constants $a, D, \varepsilon > 0$ such that

$$\begin{aligned} \|\mathcal{A}(m,n)P_n\| &\leq De^{-a(m-n)+\varepsilon|n|}, \quad m \geq n, \\ \|\mathcal{A}(m,n)Q_n\| &\leq De^{-a(n-m)+\varepsilon|n|}, \quad m \leq n, \end{aligned} \tag{2.1}$$

where $Q_m = \mathrm{Id} - P_m$ for each $m \in \mathbb{Z}$, and where

$$\mathcal{A}(m,n) = \overline{\mathcal{A}}(n,m)^{-1}|\ker P_n, \quad m \leq n.$$

Given a nonuniform exponential dichotomy, for each $n \in \mathbb{Z}$ we define the stable and unstable subspaces by

$$E_n = P_n(\mathbb{R}^l) \quad \text{and} \quad F_n = Q_n(\mathbb{R}^l).$$

Now let M_l and $\widetilde{M_l}$ be respectively the sets of $l \times l$ matrices with real and complex entries. We denote by $\mathcal{H}$ the space of continuous functions $L\colon \Delta \to M_l$, where

$$\Delta = \{(\lambda_1, \dots, \lambda_p) \in \mathbb{R}^p : |\lambda_i| \le 1 \text{ for } i = 1, \dots, p\},$$

admitting a continuous extension $\widetilde{L}\colon \widetilde{\Delta} \to \widetilde{M_l}$ which is holomorphic in the interior of the polydisk

$$\widetilde{\Delta} = \{(\lambda_1, \dots, \lambda_p) \in \mathbb{C}^p : |\lambda_i| \le 1 \text{ for } i = 1, \dots, p\}.$$

We equip the space $\mathcal{H}$ with the norm

$$\|L\| := \sup\{\|L(\lambda)\| : \lambda \in \Delta\}.$$

We also consider parameterized perturbations of a nonuniform exponential dichotomy. Namely, given a sequence $(B_m)_{m\in\mathbb{Z}} \subset \mathcal{H}$, we assume that there is a constant $\delta > 0$ such that

$$\|B_m(\lambda)\| \le \delta e^{-3\varepsilon|m|} \tag{2.2}$$

for every $m \in \mathbb{Z}$ and $\lambda \in \operatorname{int}\Delta$. In particular, it follows from Cauchy's integral formula for the first derivative that

$$\|B'_m(\lambda)\| \le \delta e^{-3\varepsilon|m|} \quad \text{for every } m \in \mathbb{Z} \text{ and } \lambda \in \operatorname{int}\Delta.$$

Given $n \in \mathbb{Z}$ and $v_n = (\xi, \eta) \in E_n \times F_n$, we denote by

$$(x_m, y_m) = (x_m(n, v_n, \lambda), y_m(n, v_n, \lambda))$$

the sequence obtained from the difference equation

$$v_{m+1} = A_m v_m + B_m(\lambda) v_m, \quad m \in \mathbb{Z} \tag{2.3}$$

with $v_m = (x_m, y_m)$. One can easily verify that

$$x_m = \mathcal{A}(m,n)\xi + \sum_{l=n}^{m-1} P_m \mathcal{A}(m, l+1) B_l(\lambda)(x_l, y_l), \tag{2.4}$$

$$y_m = \mathcal{A}(m,n)\eta + \sum_{l=n}^{m-1} Q_m \mathcal{A}(m, l+1) B_l(\lambda)(x_l, y_l), \tag{2.5}$$

for $m \ge n$, with analogous identities for $m \le n$.

3. Analytic robustness and strategy of the proof

We want to show that if the sequence $(A_m)_{m\in\mathbb{Z}}$ admits a nonuniform exponential dichotomy, then for each λ the same happens with the sequence $(A_m v_m + B_m(\lambda))_{m\in\mathbb{Z}}$, in fact with stable and unstable subspaces varying analytically with λ. The following is our main result.

Theorem 3.1. *If the sequence $(A_m)_{m\in\mathbb{Z}}$ admits a nonuniform exponential dichotomy satisfying*

$$-a + 2\varepsilon < 0, \tag{3.1}$$

and $\lambda \mapsto B_m(\lambda)$, for $m \in \mathbb{Z}$, are analytic functions satisfying (2.2)*, then provided that δ is sufficiently small, the sequence $(A_m + B_m(\lambda))_{m\in\mathbb{Z}}$ admits a nonuniform exponential dichotomy for each $\lambda \in \Delta$, with the same constant a, and with the constant ε replaced by 2ε. Moreover, the stable and unstable subspaces E_n^λ and F_n^λ of these dichotomies are analytic in λ.*

Now we describe the strategy of the proof. We note that some of the arguments are inspired in our work [3], although now for analytic perturbations and for exponential dichotomies on the whole line. For simplicity, we consider only the stable subspaces E_n^λ, although the discussion for the unstable subspaces would be entirely analogous. Being a vectorspace, E_n^λ must be the graph of a linear operator. Moreover, one should expect that E_n^λ is close to E_n provided that the functions B_m are sufficiently small. This leads us to look for each space E_n^λ as a graph over E_n. More precisely, we look for linear operators $\Phi_{n,\lambda}\colon E_n \to F_n$ such that

$$E_n^\lambda = \operatorname{graph}(\mathrm{Id}_{E_n} + \Phi_{n,\lambda}), \quad n \in \mathbb{Z}. \tag{3.2}$$

The unstable subspaces F_n^λ are obtained in a similar manner. Namely, we look for linear operators $\Psi_{n,\lambda}\colon F_n \to E_n$ such that

$$F_n^\lambda = \operatorname{graph}(\mathrm{Id}_{F_n} + \Psi_{n,\lambda}), \quad n \in \mathbb{Z}. \tag{3.3}$$

We have

$$E_m = \mathcal{A}(m,n)E_n, \quad m \ge n.$$

A corresponding identity must hold for the spaces E_m^λ, replacing $\mathcal{A}(m,n)$ by

$$\mathcal{A}_\lambda(m,n) = \begin{cases} C_{m-1}\dots C_n & \text{if } m > n, \\ \mathrm{Id} & \text{if } m = n, \end{cases}$$

where $C_k = A_k + B_k(\lambda)$ for each $k \in \mathbb{Z}$. Namely, we must have

$$E_m^\lambda = \mathcal{A}_\lambda(m,n)E_n^\lambda, \quad m \ge n. \tag{3.4}$$

This means that given $(x_n, y_n) \in E_n^\lambda$, the sequence (x_m, y_m) obtained from (2.4) and (2.5) must satisfy $(x_m, y_m) \in E_m^\lambda$ for every $m \geq n$. On the other hand, by (3.2), the point (x_m, y_m) can be written in the form

$$(x_m, \Phi_{m,\lambda} x_m) = (\mathrm{Id}_{E_m} + \Phi_{m,\lambda}) x_m,$$

and thus, the pair of equations (2.4)–(2.5) is equivalent to

$$x_m = \mathcal{A}(m,n)x_n + \sum_{l=n}^{m-1} P_m \mathcal{A}(m, l+1) B_l(\lambda)(\mathrm{Id}_{E_l} + \Phi_{l,\lambda}) x_l, \tag{3.5}$$

and

$$\Phi_{m,\lambda} x_m = \mathcal{A}(m,n)\Phi_{n,\lambda} x_n + \sum_{l=n}^{m-1} Q_m \mathcal{A}(m, l+1) B_l(\lambda)(\mathrm{Id}_{E_l} + \Phi_{l,\lambda}) x_l. \tag{3.6}$$

Given linear operators $\Phi_{m,\lambda}$, for each $n \in \mathbb{Z}$, the first equation defines recursively linear operators W_l^n such that $W_n^n = \mathrm{Id}_{E_n}$ and $x_l = W_l^n x_n$. Substituting in (3.6), we obtain

$$\Phi_{m,\lambda} W_m^n = \mathcal{A}(m,n)\Phi_{n,\lambda} + \sum_{l=n}^{m-1} Q_m \mathcal{A}(m, l+1) B_l(\lambda)(\mathrm{Id}_{E_l} + \Phi_{l,\lambda}) W_l^m. \tag{3.7}$$

The strategy of the proof of Theorem 3.1 is to show that equation (3.7) has a unique solution $\Phi = (\Phi_{m,\lambda})_{(m,\lambda)\in\mathbb{Z}\times\Delta}$ in an appropriate space. The main difficulties are the nonuniform exponential behavior of the original dichotomy, and the dependence of the operators W_l^n on Φ. A similar approach can be applied to obtain unstable subspaces. On the other hand, one of the main advantages of our approach is that we are able to show in a more or less direct manner that the unique operators $\Psi_{m,\lambda}$ and $\Phi_{m,\lambda}$ in (3.2) and (3.3) are analytic in λ, and thus, the same happens with the subspaces E_n^λ and F_n^λ. The proof requires considering simultaneously additional equations related to the formal derivatives of (3.7) and of the corresponding identity for the operators $\Psi_{m,\lambda}$ with respect to λ.

4. Proof of Theorem 3.1

We first describe the class of functions where we look for the operators $\Phi_{n,\lambda}$. Given a constant $\kappa < 1$, let $\mathcal{X}$ be the space of parameterized sequences $\Phi = (\Phi_{n,\lambda})_{n\in\mathbb{Z},\lambda\in\Delta}$ of linear operators $\Phi_{n,\lambda}\colon E_n \to F_n$ such that

$$\|\Phi\| := \sup\{\|\Phi_{n,\lambda}\| e^{\varepsilon|n|} : (n,\lambda) \in \mathbb{Z}\times\Delta\} \leq \kappa,$$
$$C_{\lambda\mu}(\Phi) := \sup\{\|\Phi_{n,\lambda} - \Phi_{n,\mu}\| e^{\varepsilon|n|} : (n,\lambda) \in \mathbb{Z}\} \leq \kappa\|\lambda - \mu\|$$

for each $\lambda, \mu \in \Delta$. When equipped with the distance

$$d(\Phi, \Psi) = \sup\{\|\Phi_{n,\lambda} - \Phi_{n,\mu}\| e^{\varepsilon|n|} : (n,\lambda) \in \mathbb{Z}\times\Delta\},$$

the space $\mathfrak{X}$ becomes a complete metric space. Given $\Phi \in \mathfrak{X}$ and $\lambda \in \Delta$, for each $n \in \mathbb{Z}$ we consider the vector space

$$E_n^\lambda = \operatorname{graph}(\operatorname{Id}_{E_n} + \Phi_{n,\lambda}) = \{(\xi, \Phi_{n,\lambda}\xi) : \xi \in E_n\}.$$

Step 1. Construction of stable subspaces. Due to the required invariance in (3.4), we must solve the system of equations (3.5)–(3.6). For that we introduce two sequences of linear operators, one related to each equation, whose fixed points are solutions of these equations.

We first introduce linear operators related to equation (3.5). Given $\Phi \in \mathfrak{X}$, $n \in \mathbb{Z}$ and $\lambda \in \Delta$, we consider the linear operators $W_{m,\lambda}^n = W_{m,\Phi,\lambda}^n \colon E_n \to F_n$ determined recursively by the identities

$$W_{m,\lambda}^n = P_m\mathcal{A}(m,n) + \sum_{l=n}^{m-1} P_m\mathcal{A}(m,l+1)B_l(\lambda)(\operatorname{Id}_{E_l} + \Phi_{l,\lambda})W_{l,\lambda}^n \tag{4.1}$$

for $m \geq n$, setting $W_{n,\lambda}^n = \operatorname{Id}_{E_n}$. We note that for $x_n = \xi \in E_n$, the sequence $x_m = W_{m,\lambda}^n x_n = W_{m,\lambda}^n \xi$ is the solution of equation (2.4) with $y_l = \Phi_{j,\lambda} x_l$ for $l \geq n$. Equivalently, it is a solution of equation (3.5).

Now we rewrite equation (3.7) in an equivalent form.

Lemma 4.1. *For any sufficiently small δ, given $\Phi \in \mathfrak{X}$ and $\lambda \in \Delta$ the following properties are equivalent:*

(a) *for every $n \in \mathbb{Z}$ and $m \geq n$,*

$$\begin{aligned}\Phi_{m,\lambda}W_{m,\lambda}^n = {}&\mathcal{A}(m,n)\Phi_{n,\lambda}\\ &+ \sum_{l=n}^{m-1} Q_m\mathcal{A}(l+1,m)^{-1}B_l(\lambda)(\operatorname{Id}_{E_l} + \Phi_{l,\lambda})W_{l,\lambda}^n;\end{aligned} \tag{4.2}$$

(b) *for every $n \in \mathbb{Z}$,*

$$\Phi_{n,\lambda} = -\sum_{l=n}^{\infty} Q_n\mathcal{A}(l+1,n)^{-1}B_l(\lambda)(\operatorname{Id}_{E_l} + \Phi_{l,\lambda})W_{l,\lambda}^n. \tag{4.3}$$

Proof. We first show that the series in (4.3) is well-defined. Setting $\widetilde{D} = De^{a+\varepsilon}$, by (2.1) and (2.2), we obtain

$$\begin{aligned}\sum_{l=n}^{\infty}&\|Q_n\mathcal{A}(l+1,n)^{-1}B_l(\lambda)(W_{l,\lambda}^n + \Phi_{l,\lambda}W_{l,\lambda}^n)\|e^{\varepsilon|n|}\\ &\leq (1+\kappa)\delta\widetilde{D}\sum_{l=n}^{\infty} e^{-a(l-n)+\varepsilon|l|-3\varepsilon|l|+\varepsilon|n|}\|W_{l,\lambda}^n\|\\ &\leq 2\,\delta\widetilde{D}\sum_{l=n}^{\infty} e^{-(a-\varepsilon)(l-n)-\varepsilon|l|}\|W_{l,\lambda}^n\|.\end{aligned} \tag{4.4}$$

By (4.1), for each $m \geq n$, we have

$$\|W^n_{m,\lambda}\| \leq De^{-a(m-n)+\varepsilon|n|} + (1+\kappa)\,\delta\widetilde{D}\sum_{l=n}^{m-1} e^{-a(m-l)-2\varepsilon|l|}\|W^n_{l,\lambda}\|. \tag{4.5}$$

Setting $\Upsilon = \sup_{m\geq n}(e^{a(m-n)}\|W^n_{m,\lambda}\|)$, for each l we obtain

$$\Upsilon \leq De^{\varepsilon|n|} + 2\delta\widetilde{D}\,\Upsilon\sum_{l=n}^{m} e^{-2\varepsilon|l|} \leq De^{\varepsilon|n|} + 2\delta\widetilde{D}\,\Upsilon\,\Gamma_{2\varepsilon},$$

where

$$\Gamma_c := \sum_{l\in\mathbb{Z}} e^{-c|l|} = \frac{1+e^{-c}}{1-e^{-c}}$$

for each $c > 0$. Taking δ sufficiently small so that $2\delta\widetilde{D}\Gamma_{2\varepsilon} < 1/2$ (independently of n), we obtain $\Upsilon \leq 2De^{\varepsilon|n|}$, and hence

$$\|W^n_{m,\lambda}\| \leq 2De^{-a(m-n)+\varepsilon|n|}. \tag{4.6}$$

By (4.4), this implies that

$$\begin{aligned}
&\sum_{l=n}^{\infty}\|Q_n\mathcal{A}(l+1,n)^{-1}B_l(\lambda)(W^n_{l,\lambda} + \Phi_{l,\lambda}W^n_{l,\lambda})\|e^{\varepsilon|n|} \\
&\quad\leq 4\,\delta D\widetilde{D}\sum_{l=n}^{\infty} e^{-2a(l-n)+\varepsilon|n|-\varepsilon|l|} \\
&\quad\leq 4\,\delta D\widetilde{D}\sum_{l=n}^{\infty} e^{-(2a-\varepsilon)(l-n)} \leq 4\,\delta D\widetilde{D}\Gamma_{2a-\varepsilon} \leq \kappa,
\end{aligned} \tag{4.7}$$

provided that δ is sufficiently small. Now we assume that identity (4.2) holds. It is equivalent to

$$\begin{aligned}
\Phi_{n,\lambda} =\, &Q_n\mathcal{A}(m,n)^{-1}\Phi_{m,\lambda}W^n_{m,\lambda} \\
&- \sum_{l=n}^{m-1} Q_n\mathcal{A}(l+1,n)^{-1}B_l(\lambda)(\mathrm{Id}_{E_l} + \Phi_{l,\lambda})W^n_{l,\lambda}.
\end{aligned} \tag{4.8}$$

By the second inequality in (2.1) and (4.6), for each positive $m \geq n$ we have

$$\begin{aligned}
\|Q_n\mathcal{A}(m,n)^{-1}\Phi_{m,\lambda}W^n_{m,\lambda}\| &\leq 2D\widetilde{D}\kappa e^{-a(m-n)+\varepsilon|m|}e^{-\varepsilon|m|}e^{-a(m-n)+\varepsilon|n|} \\
&\leq 2D\widetilde{D}\kappa e^{-2a(m-n)+\varepsilon|n|}.
\end{aligned}$$

Since $a > 0$, letting $m \to \infty$ in (4.8) we obtain identity (4.3). Conversely, let us assume that identity (4.3) holds. Then

$$\begin{aligned}\mathcal{A}(m,n)\Phi_{n,\lambda} &+ \sum_{l=n}^{m-1} Q_m \mathcal{A}(l+1,m)^{-1} B_l(\lambda)(\mathrm{Id}_{E_l} + \Phi_{l,\lambda}) W^n_{l,\lambda} \\ &= -\sum_{l=n}^{\infty} Q_m \mathcal{A}(l+1,m)^{-1} B_l(\lambda)(\mathrm{Id}_{E_l} + \Phi_{l,\lambda}) W^n_{l,\lambda} \\ &\quad + \sum_{l=n}^{m-1} Q_m \mathcal{A}(l+1,m)^{-1} B_l(\lambda)(\mathrm{Id}_{E_l} + \Phi_{l,\lambda}) W^n_{l,\lambda} \\ &= -\sum_{l=m}^{\infty} Q_m \mathcal{A}(l+1,m)^{-1} B_l(\lambda)(\mathrm{Id}_{E_l} + \Phi_{l,\lambda}) W^n_{l,\lambda}\end{aligned}$$

for each $m \geq n$. Since $W^n_{l,\lambda} = W^m_{l,\lambda} W^n_{m,\lambda}$, it follows from (4.3) with n replaced by m that (4.2) holds for each $m \geq n$. □

For each $\lambda \in \Delta$ and $m \geq n$, we denote

$$U_\lambda(m,n) = (\mathrm{Id}_{E_m} + \Phi_{m,\lambda}) W^n_{m,\lambda}.$$

We claim that

$$\begin{aligned}U_\lambda(m,n) = {} & \mathcal{A}(m,n)P_n + \sum_{l=n}^{m-1} \mathcal{A}(m,l+1) P_{l+1} B_l(\lambda) U_\lambda(l,n) \\ & - \sum_{l=m}^{\infty} \mathcal{A}(m,l+1) Q_{l+1} B_l(\lambda) U_\lambda(l,n).\end{aligned} \tag{4.9}$$

Indeed, it follows from (3.4)–(3.6) that

$$\begin{aligned}U_\lambda(m,n) = {} & (\mathrm{Id}_{E_m} + \Phi_{m,\lambda}) W^n_{m,\lambda} = W^n_{m,\lambda} + \Phi_{m,\lambda} W^n_{m,\lambda} \\ = {} & \mathcal{A}(m,n)P_n + \sum_{l=n}^{m-1} \mathcal{A}(m,l+1) P_{l+1} B_l(\lambda) U_\lambda(l,n) \\ & - \mathcal{A}(m,n) \sum_{l=n}^{\infty} \mathcal{A}(n,l+1) Q_{l+1} B_l(\lambda) U_\lambda(l,n) \\ & + \sum_{l=n}^{m-1} Q_m \mathcal{A}(m,l+1) B_l(\lambda) U_\lambda(l,n) \\ = {} & \mathcal{A}(m,n)P_n + \sum_{l=n}^{m-1} \mathcal{A}(m,l+1) P_{l+1} B_l(\lambda) U_\lambda(l,n) \\ & - \sum_{l=n}^{\infty} \mathcal{A}(m,l+1) Q_{l+1} B_l(\lambda) U_\lambda(l,n)\end{aligned} \tag{4.10}$$

$$+\sum_{l=n}^{m-1} Q_m \mathcal{A}(m,l+1)B_l(\lambda)U_\lambda(l,n)$$

$$= \mathcal{A}(m,n)P_n + \sum_{l=n}^{m-1} \mathcal{A}(m,l+1)P_{l+1}B_l(\lambda)U_\lambda(l,n)$$

$$- \sum_{l=m}^{\infty} \mathcal{A}(m,l+1)Q_{l+1}B_l(\lambda)U_\lambda(l,n).$$

For each $\lambda \in \Delta$, let $I = \{(m,n) \in \mathbb{Z} \times \mathbb{Z} : m \geq n\}$ and consider the Banach space $\mathcal{C}_\lambda = \{U_\lambda : I \to M_l : \|U_\lambda\| < +\infty\}$ with the norm

$$\|U_\lambda\| = \sup\{\|U_\lambda(m,n)\| e^{a(m,n)-\varepsilon|n|} : (m,n) \in I\}.$$

Lemma 4.2. *For any sufficiently small δ, there is a unique $U_\lambda \in \mathcal{C}_\lambda$ satisfying* (4.9) *for each $(m,n) \in I$. Moreover, for each $\xi \in \mathbb{R}^l$ the sequence $x_m = U_\lambda(m,n)\xi$, $m \geq n$ is a solution of equation* (2.3).

Proof. For each $\lambda \in \Delta$, we consider the operator L defined for each $U_\lambda \in \mathcal{C}_\lambda$ by

$$(LU_\lambda)_\lambda(m,n) = \mathcal{A}(m,n)P_n + \sum_{k=n}^{m-1} \mathcal{A}(m,k+1)P_{k+1}B_k(\lambda)U(k,n)$$

$$- \sum_{k=m}^{\infty} \mathcal{A}(m,k+1)Q_{k+1}B_k(\lambda)U_\lambda(k,n).$$

We have

$$\sum_{k=m}^{\infty} \|\mathcal{A}(m,k+1)Q_{k+1}B_k(\lambda)U_\lambda(k,n)\| \leq \widetilde{D}\delta e^{-a(m-n)+\varepsilon|n|}\|U_\lambda\| \sum_{k=m}^{\infty} e^{-2a\ (k-m)}$$

$$\leq \frac{\widetilde{D}\delta}{1-e^{-2a}}\, e^{-a\ (m-n)+\varepsilon|n|}\|U_\lambda\| < +\infty.$$

Therefore, $(LU_\lambda)_\lambda(m,n)$ is well defined, and we obtain

$$\|(LU_\lambda)_\lambda(m,n)\| \leq \|\mathcal{A}(m,n)P_n\| \tag{4.11}$$

$$+ \sum_{k=n}^{m-1} \|\mathcal{A}(m,k+1)P_{k+1}\| \cdot \|B_k(\lambda)\| \cdot \|U_\lambda(k,n)\|$$

$$+ \sum_{k=m}^{\infty} \|\mathcal{A}(m,k+1)Q_{k+1}\| \cdot \|B_k(\lambda)\| \cdot \|U_\lambda(k,n)\|$$

$$\leq De^{-a\ (m-n)+\varepsilon|n|} + \widetilde{D}\delta e^{-a\ (m-n)+\varepsilon|n|}\|U_\lambda\| \sum_{k=n}^{m-1} e^{-2\varepsilon|k|}$$

$$+ \widetilde{D}\delta e^{-a\ (m-n)+\varepsilon|n|}\|U_\lambda\| \sum_{k=m}^{\infty} e^{-2a(k-m)}$$

$$\leq D\delta e^{-a\ (m-n)+\varepsilon|n|} + \frac{\widetilde{D}\delta}{1-e^{-2\varepsilon}}\, e^{-a(m-n)+\varepsilon|n|}\|U_\lambda\| + \frac{\widetilde{D}\delta}{1-e^{-2a}}\, e^{-a\ (m-n)+\varepsilon|n|}\|U_\lambda\|.$$

This implies that

$$\|(LU_\lambda)_\lambda\| \leq D + \widetilde{D}\delta\bigg(\frac{1}{1-e^{-2\varepsilon}} + \frac{1}{1-e^{-2a}}\bigg)\|U_\lambda\| < +\infty,$$

and hence, we have a well defined operator $L\colon \mathcal{C}_\lambda \to \mathcal{C}_\lambda$. Proceeding in a similar manner to that in (4.11) we also obtain

$$\|(LU_{1,\lambda})_\lambda - (LU_{2,\lambda})_\lambda\| \leq \widetilde{D}\delta\bigg(\frac{1}{1-e^{-2\varepsilon}} + \frac{1}{1-e^{-2a}}\bigg)\|U_{1,\lambda} - U_{2,\lambda}\| \tag{4.12}$$

for every $U_{1,\lambda}, U_{2,\lambda} \in \mathcal{C}_\lambda$. Therefore, for any sufficiently small δ the operator L is a contraction, and there exists a unique $U_\lambda \in \mathcal{C}_\lambda$ such that $LU_\lambda = U_\lambda$. Finally,

$$\begin{aligned} &U_\lambda(m,n) - \mathcal{A}(m,n)U_\lambda(n,n) = \mathcal{A}(m,n)P_n - \mathcal{A}(m,n)P_n \\ &+ \sum_{k=n}^{m-1}\mathcal{A}(m,k+1)P_{k+1}B_k(\lambda)U_\lambda(k,n) + \sum_{k=n}^{m-1}\mathcal{A}(m,k+1)Q_{k+1}B_k(\lambda)U_\lambda(k,n) \\ &\qquad = \sum_{k=n}^{m-1}\mathcal{A}(m,k+1)B_k(\lambda)U_\lambda(k,n) \end{aligned}$$

for each $m \geq n$. This completes the proof of the lemma. □

Lemma 4.3. *For any sufficiently small δ, we have*

$$U_\lambda(m,l)U_\lambda(l,n) = U_\lambda(m,n) \quad \textit{for every } m \geq l \geq n.$$

Proof. We first note that

$$\begin{aligned} U_\lambda(m,l)U_\lambda(l,n) = {}&\mathcal{A}(m,n)P_n + \sum_{k=n}^{l-1}\mathcal{A}(m,k+1)P_{k+1}B_k(\lambda)U_\lambda(k,n) \\ &+ \sum_{k=l}^{m-1}\mathcal{A}(m,k+1)P_{k+1}B_k(\lambda)U_\lambda(k,l)U_\lambda(l,n) \\ &- \sum_{k=m}^{\infty}\mathcal{A}(m,k+1)Q_{k+1}B_k(\lambda)U_\lambda(k,l)U_\lambda(l,n). \end{aligned}$$

Given $n \in \mathbb{Z}$ and $\lambda \in \Delta$, let $I_n = \{(m,l) \in \mathbb{Z}\times\mathbb{Z} : m \geq l \geq n\}$, and consider the Banach space $\mathcal{C}_{n,\lambda} = \{H_\lambda\colon I_n \to M_l : \|H_\lambda\|_n < +\infty\}$ with the norm

$$\|H_\lambda\|_n = \sup\{\|H_\lambda(m,l)\|e^{-2\varepsilon|m|} : (m,l) \in I_n\}.$$

Writing

$$h_\lambda(m,l) = U_\lambda(m,l)U_\lambda(l,n) - U_\lambda(m,n)$$

for $m \geq l$ (with n fixed), we obtain $L_1 h_\lambda = h_\lambda$, where

$$\begin{aligned}(L_1 H_\lambda)(m,l) = &\sum_{k=l}^{m-1} \mathcal{A}(m,k+1)P_{k+1}B_k(\lambda)H_\lambda(k,n)\\ &- \sum_{k=m}^{\infty} \mathcal{A}(m,k+1)Q_{k+1}B_k(\lambda)H_\lambda(k,n).\end{aligned}$$

for each $H_\lambda \in \mathcal{C}_{n,\lambda}$ and $(m,l) \in I_n$. Now we observe that

$$\begin{aligned}&\sum_{k=l}^{m-1} \|\mathcal{A}(m,k+1)P_{k+1}\| \cdot \|B_k(\lambda)\| \cdot \|H_\lambda(k,n)\|\\ &\qquad + \sum_{k=m}^{\infty} \|\mathcal{A}(m,k+1)Q_{k+1}\| \cdot \|B_k(\lambda)\| \cdot \|H_\lambda(k,n)\|\\ &\qquad \leq \frac{\widetilde{D}\delta}{e^a-1}\|H_\lambda\|_n + \frac{\widetilde{D}\delta}{1-e^{-a}}\|H_\lambda\|_n = \widetilde{D}\delta\frac{1+e^{-a}}{1-e^{-a}}\|H_\lambda\|_n.\end{aligned}$$

This shows that $(L_1 H_\lambda)(m,l)$ is well defined, and that

$$\|L_1 H_\lambda\|_n \leq \widetilde{D}\delta\,\frac{1+e^{-a}}{1-e^{-a}}\|H_\lambda\|_n < +\infty.$$

Hence, we obtain an operator $L_1\colon \mathcal{C}_{n,\lambda} \to \mathcal{C}_{n,\lambda}$. Moreover, one can show in a similar manner that for each $H_{1,\lambda}, H_{2,\lambda} \in \mathcal{C}_{n,\lambda}$ and $m \geq l$,

$$\|L_1 H_{1,\lambda} - L_1 H_{2,\lambda}\|_n \leq \widetilde{D}\delta\,\frac{1+e^{-a}}{1-e^{-a}}\|H_{1,\lambda} - H_{2,\lambda}\|_n.$$

Therefore, for any sufficiently small δ the operator L_1 is a contraction, and hence there exists a unique $H_\lambda \in \mathcal{C}_{n,\lambda}$ such that $L_1 H_\lambda = H_\lambda$. Since $0 \in \mathcal{C}_{n,\lambda}$ also satisfies this identity, we have $H_\lambda = 0$. Now we show that $h_\lambda \in \mathcal{C}_{n,\lambda}$. Indeed, it follows from Lemma 4.2 that

$$\begin{aligned}\|U_\lambda(m,l)U_\lambda(l,n)\| &\leq \|U_\lambda(m,l)\| \cdot \|U_\lambda(l,n)\| \leq \|U_\lambda\|^2 e^{-a(m-n)+\varepsilon(|l|+|n|)}\\ &\leq \|U_\lambda\|^2 e^{(2\varepsilon-a)(m-n)}e^{2\varepsilon|m|} \leq \|U\|^2 e^{2\varepsilon|m|},\\ \|U_\lambda(m,n)\| &\leq \|U_\lambda\| e^{-a(m-n)+\varepsilon|n|}\\ &\leq \|U_\lambda\| e^{(\varepsilon-a)(m-n)+}e^{\varepsilon|m|} \leq \|U_\lambda\| e^{2\varepsilon|m|}\end{aligned}$$

for $m \geq l \geq n$. This shows that $h_\lambda \in \mathcal{C}_{n,\lambda}$, and by the uniqueness of the fixed point of L_1 we conclude that $h_\lambda = 0$. □

Step 2. Construction of unstable subspaces. Now we describe the class of functions where we look for the operators $\Psi_{n,\lambda}$ (see (3.3)). Let $\mathcal{Y}$ be the space of parameterized sequences $\Psi = (\Psi_{n,\lambda})_{n\in\mathbb{Z},\lambda\in\Delta}$ of linear operators $\Psi_{n,\lambda}\colon F_n \to E_n$ such that

$$\sup\{\|\Psi_{n,\lambda}\|e^{\varepsilon|n|} : (n,\lambda) \in \mathbb{Z}\times\Delta\} \le \kappa,$$
$$\sup\{\|\Psi_{n,\lambda}-\Psi_{n,\mu}\|e^{\varepsilon|n|} : n \in \mathbb{Z}\} \le \kappa\,\|\lambda-\mu\|,$$

for each $\lambda,\mu \in \Delta$. Given $\Psi \in \mathfrak{Y}$ and $\lambda \in \Delta$, for each $n \in \mathbb{Z}$ we consider the vector spaces

$$F_n^\lambda = \mathrm{graph}\,(\mathrm{Id}_{F_n} + \Psi_{n,\lambda}).$$

Given $\Psi \in \mathfrak{Y}$, $n \in \mathbb{Z}$ and $\lambda \in \Delta$, we consider the linear operators $Y^n_{m,\lambda} = Y^n_{m,\Psi,\lambda}\colon F_n \to E_n$ determined recursively by the identities

$$Y^n_{m,\lambda} = Q_m\mathcal{A}(m,n) - \sum_{l=m}^{n-1} Q_m\mathcal{A}(m,l+1)B_l(\lambda)(\mathrm{Id}_{E_l}+\Psi_{l,\lambda})Y^n_{l,\lambda}$$

for $m \le n$, setting $Y^n_{n,\lambda} = \mathrm{Id}_{F_n}$. We note that for $y_n = \nu \in F_n$, the sequence

$$y_m = Y^n_{m,\lambda}y_n = Y^n_{m,\lambda}\nu$$

is the solution of equation (2.4) with $x_l = \Psi_{j,\lambda}y_l$ for $l \le n$. Equivalently, it is a solution of equation (3.5).

The following is a version of Lemma 4.1 for the operators $\Psi_{m,\lambda}$ and it can be obtained in an analogous manner.

Lemma 4.4. *For any sufficiently small δ, given $\Psi \in \mathfrak{Y}$ and $\lambda \in \Delta$ the following properties are equivalent:*

(a) *for every $n \in \mathbb{Z}$ and $m \le n$,*

$$\Psi_{m,\lambda}Y^n_{m,\lambda} = \mathcal{A}(m,n)\Psi_{n,\lambda} - \sum_{l=m}^{n-1} P_m\mathcal{A}(l+1,m)^{-1}B_l(\lambda)(\mathrm{Id}_{E_l}+\Psi_{l,\lambda})Y^n_{l,\lambda};$$

(b) *for every $n \in \mathbb{Z}$,*

$$\Psi_{n,\lambda} = \sum_{-\infty}^{n} P_n\mathcal{A}(l+1,n)^{-1}B_l(\lambda)(\mathrm{Id}_{E_l}+\Psi_{l,\lambda})Y^n_{l,\lambda}.$$

Now set $V_\lambda(m,n) = (\mathrm{Id}_{E_m}+\Psi_{m,\lambda})Y^n_{m,\lambda}$. Proceeding in a similar manner to that in (4.10) we obtain

$$\begin{aligned}V_\lambda(m,n) = \mathcal{A}(m,n)Q_n &+ \sum_{k=-\infty}^{m-1}\mathcal{A}(m,k+1)P_{k+1}B_k(\lambda)V_\lambda(k,n)\\ &- \sum_{k=m}^{n-1}\mathcal{A}(m,k+1)Q_{k+1}B_k(\lambda)V_\lambda(k,n).\end{aligned}\tag{4.13}$$

For each $\lambda \in \Delta$, let $J = \{(m,n) \in \mathbb{Z} \times \mathbb{Z} : m \le n\}$, and consider the Banach space $\mathcal{D}_\lambda = \{V_\lambda\colon J \to M_l : \|V_\lambda\| < +\infty\}$ with the norm

$$\|V_\lambda\| = \sup\{\|V_\lambda(m,n)\| e^{-a(m-n)-\varepsilon|n|} : (m,n) \in J\}.$$

LEMMA 4.5. *For any sufficiently small δ, there is a unique $V_\lambda \in \mathcal{D}_\lambda$ satisfying* (4.13) *for each $(m,n) \in J$. Moreover, for each $\xi \in \mathbb{R}^l$ the sequence $x_m = V_\lambda(m,n)\xi$, $m \le n$ is a solution of equation* (2.3).

PROOF. For each $\lambda \in \Delta$, we consider the operator M defined for each $V_\lambda \in \mathcal{D}_\lambda$ by

$$\begin{aligned}(MV_\lambda)(m,n) = {}&\mathcal{A}(m,n)Q_n + \sum_{k=-\infty}^{m-1} \mathcal{A}(m,k+1)P_{k+1}B_k(\lambda)V_\lambda(k,n)\\ &- \sum_{k=m}^{n-1} \mathcal{A}(m,k+1)Q_{k+1}B_k(\lambda)V_\lambda(k,n).\end{aligned}$$

We have:

$$\begin{aligned}&\|\mathcal{A}(m,n)Q_n\| + \sum_{k=-\infty}^{m-1} \|\mathcal{A}(m,k+1)P_{k+1}\| \cdot \|B_k(\lambda)\| \cdot \|V_\lambda(k,n)\|\\ &\quad + \sum_{k=m}^{n-1} \|\mathcal{A}(m,k+1)Q_{k+1}\| \cdot \|B_k(\lambda)\| \cdot \|V_\lambda(k,n)\|\\ &\le De^{a(m-n)+\varepsilon|n|} + \widetilde{D}\delta e^{a(m-n)+\varepsilon|n|}\|V\| \sum_{k=-\infty}^{m-1} e^{-2a\ (m-k)}\\ &\quad + \widetilde{D}\delta e^{a(m-n)+\varepsilon|n|}\|V_\lambda\| \sum_{k=m}^{n-1} e^{-2\varepsilon k}\\ &\le De^{a(m-n)+\varepsilon|n|} + \frac{\widetilde{D}\delta}{e^{2a}-1}\, e^{a(m-n)+\varepsilon|n|}\|V_\lambda\|\\ &\quad + \frac{\widetilde{D}\delta}{1-e^{-2\varepsilon}}\, e^{a(m-n)+\varepsilon|n|}\|V_\lambda\|.\end{aligned} \tag{4.14}$$

This implies that $(MV_\lambda)(m,n)$ is well defined, and that

$$\|MV_\lambda\| \le D + \delta\widetilde{D}\left(\frac{1}{e^{2a}-1} + \frac{1}{1-e^{-2\varepsilon}}\right)\|V\| < +\infty. \tag{4.15}$$

Hence, we obtain a well defined operator $M_\lambda\colon \mathcal{D}_\lambda \to \mathcal{D}_\lambda$. Proceeding in a similar manner to that in (4.14) we also obtain

$$\|MV_{1,\lambda} - MV_{2,\lambda}\| \le \delta\widetilde{D}\left(\frac{1}{e^{2a}-1} + \frac{1}{1-e^{-2\varepsilon}}\right)\|V_{1,\lambda} - V_{2,\lambda}\|$$

for every $V_{1,\lambda}, V_{2,\lambda} \in \mathcal{D}_\lambda$. Therefore, for any sufficiently small δ the operator M is a contraction, and there is a unique $V \in \mathcal{D}_\lambda$ such that $MV_\lambda = V_\lambda$. Moreover,

$$\begin{aligned}
V_\lambda(n,n) - \mathcal{A}(n,m)V_\lambda(m,n) &= Q_n + \sum_{k=-\infty}^{n-1} \mathcal{A}(n,k+1)P_{k+1}B_k(\lambda)V_\lambda(k,n) \\
&\quad - \mathcal{A}(n,m)\mathcal{A}(m,n)Q_n - \mathcal{A}(n,m) \sum_{k=-\infty}^{m-1} \mathcal{A}(m,k+1)P_{k+1}B_k(\lambda)V_\lambda(k,n) \\
&\quad + \mathcal{A}(n,m) \sum_{k=m}^{n-1} \mathcal{A}(m,k+1)Q_{k+1}B_k(\lambda)V_\lambda(k,n) \\
&= Q_n + \sum_{k=-\infty}^{m-1} \mathcal{A}(n,k+1)P_{k+1}B_k(\lambda)V_\lambda(k,n) \\
&\quad + \sum_{k=m}^{n-1} \mathcal{A}(n,k+1)P_{k+1}B_k(\lambda)V_\lambda(k,n) \\
&\quad - Q_n - \sum_{k=-\infty}^{m-1} \mathcal{A}(n,k+1)P_{k+1}B_k(\lambda)V_\lambda(k,n) \\
&\quad + \sum_{k=m}^{n-1} \mathcal{A}(n,k+1)Q_{k+1}B_k(\lambda)V_\lambda(k,n) \\
&= \sum_{k=m}^{n-1} \mathcal{A}(n,k+1)P_{k+1}B_k(\lambda)V_\lambda(k,n) + \sum_{k=m}^{n-1} \mathcal{A}(n,k+1)Q_{k+1}B_k(\lambda)V_\lambda(k,n) \\
&= \sum_{k=m}^{n-1} \mathcal{A}(n,k+1)B_kV_\lambda(k,n)
\end{aligned}$$

for each $m \le n$. This completes the proof of the lemma. □

Lemma 4.6. *For any sufficiently small δ, we have*

$$V_\lambda(m,l)V_\lambda(l,n) = V_\lambda(m,n) \quad \textit{for every } m \le l \le n.$$

Proof. We have

$$\begin{aligned}
V_\lambda(m,l)V_\lambda(l,n) = \mathcal{A}(m,n)Q_n &- \sum_{k=l}^{n-1} \mathcal{A}(m,k+1)Q_{k+1}B_k(\lambda)V_\lambda(k,n) \\
&+ \sum_{k=-\infty}^{m-1} \mathcal{A}(m,k+1)P_{k+1}B_k(\lambda)V_\lambda(k,l)V_\lambda(l,n) \\
&- \sum_{k=m}^{l-1} \mathcal{A}(m,k+1)Q_{k+1}B_k(\lambda)V_\lambda(k,l)V_\lambda(l,n).
\end{aligned}$$

Given $n \in \mathbb{Z}$ and $\lambda \in \Delta$, let $J_n = \{(m,l) \in \mathbb{Z}\times\mathbb{Z} : m \le l \le n\}$, and consider the Banach space $\mathcal{D}_{n,\lambda} = \{\overline{H}_\lambda : J_n \to M_l : \|\overline{H}_\lambda\|_n < +\infty\}$ with the norm

$$\|\overline{H}_\lambda\|_n = \sup\{\|\overline{H}_\lambda(m,l)\|e^{-2\varepsilon|m|} : (m,l) \in J_n\}.$$

Writing

$$\overline{h}_\lambda(m,n) = V_\lambda(m,l)V_\lambda(l,n) - V_\lambda(m,n)$$

for $m \le l \le n$ (with n fixed), we obtain $M_1\overline{h}_\lambda = \overline{h}_\lambda$, where

$$\begin{aligned}(M_1\overline{H}_\lambda)(m,l) = &\sum_{k=-\infty}^{m-1} \mathcal{A}(m,k+1)P_{k+1}B_k(\lambda)H_\lambda(k,n)\\ &- \sum_{k=m}^{l-1} \mathcal{A}(m,k+1)Q_{k+1}B_k(\lambda)H_\lambda(k,n)\end{aligned}$$

for each $\overline{H}_\lambda \in \mathcal{D}_{n,\lambda}$ and $(m,l) \in J_n$. Proceeding in a similar manner to that in the proof of Lemma 4.3, one can show that 0 is the unique fixed point of M_1 in $\mathcal{D}_{n,\lambda}$, and since $\overline{h}_\lambda \in \mathcal{D}_{n,\lambda}$ we conclude that $\overline{h}_\lambda = 0$. □

Now we characterize the bounded solutions of equation (2.3).

Step 3. Characterization of bounded solutions.

Lemma 4.7. *Given $n \in \mathbb{Z}$, if $(y_m)_{m\ge n} \subset \mathbb{R}^l$ is a bounded sequence satisfying equation* (2.3) *with $y_n = \xi$, then*

$$\begin{aligned}y_m = &\mathcal{A}(m,n)P_n\xi + \sum_{k=n}^{m-1} \mathcal{A}(m,k+1)P_{k+1}B_k(\lambda)y_k\\ &- \sum_{k=m}^{\infty} \mathcal{A}(m,k+1)Q_{k+1}B_k(\lambda)y_k.\end{aligned} \tag{4.16}$$

Proof. For each $m \ge n$, we have

$$P_m y_m = \mathcal{A}(m,n)P_n\xi + \sum_{k=n}^{m-1} \mathcal{A}(m,k+1)P_{k+1}B_k(\lambda)y_k, \tag{4.17}$$

$$Q_m y_m = \mathcal{A}(m,n)Q_n\xi + \sum_{k=n}^{m-1} \mathcal{A}(m,k+1)Q_{k+1}B_k(\lambda)y_k. \tag{4.18}$$

The identity (4.18) can be written in the form

$$Q_n\xi = \mathcal{A}(n,m)Q_m y_m - \sum_{k=n}^{m-1} \mathcal{A}(n,k+1)Q_{k+1}B_k(\lambda)y_k. \tag{4.19}$$

Since y_m is bounded, we have

$$\|\mathcal{A}(n,m)Q_m y_m\| \le CDe^{-a(m-n)+\varepsilon|m|},$$

where $C = \sup\{\|y_m\| : m \geq n\} < +\infty$. Therefore, taking limits on both sides of (4.19) when $m \to +\infty$, we obtain

$$Q_n\xi = -\sum_{k=n}^{\infty} \mathcal{A}(n, k+1)Q_{k+1}B_k(\lambda)y_k. \tag{4.20}$$

Replacing (n, ξ) by (m, y_m) in (4.20) and adding the resulting identity to (4.19) we obtain (4.16). □

Lemma 4.8. *Given $n \in \mathbb{Z}$, if $(y_m)_{m \leq n} \subset \mathbb{R}^l$ is a bounded sequence satisfying equation* (2.3) *with $y_n = \xi$, then*

$$\begin{aligned} y_m = &\mathcal{A}(m, n)Q_n\xi + \sum_{k=-\infty}^{m} \mathcal{A}(m, k+1)P_{k+1}B_k(\lambda)y_k \\ &- \sum_{k=m}^{n-1} \mathcal{A}(m, k+1)Q_{k+1}B_k(\lambda)y_k. \end{aligned}$$

Proof. For each $m \leq n$, we have

$$P_n\xi = \mathcal{A}(n, m)P_m y_m + \sum_{k=m}^{n-1} \mathcal{A}(n, k+1)P_{k+1}B_k(\lambda)y_k, \tag{4.21}$$

$$Q_n\xi = \mathcal{A}(n, m)Q_m y_m + \sum_{k=m}^{n-1} \mathcal{A}(n, k+1)Q_{k+1}B_k(\lambda)y_k. \tag{4.22}$$

Since the sequence $(y_m)_{m \leq n}$ is bounded, we have

$$\|\mathcal{A}(n, m)P_m y_m\| \leq C\widetilde{D}e^{-a(n-m)+\varepsilon|m|},$$

where $C = \sup\{\|y_m\| : m \leq n\} < +\infty$. Since $a > \varepsilon$ (see (3.1)), taking limits in (4.21) when $m \to -\infty$, we obtain

$$P_n\xi = \sum_{k=-\infty}^{n} \mathcal{A}(n, k+1)P_{k+1}B_k(\lambda)y_k.$$

Replacing (n, ξ) by (m, y_m) in this identity we finally obtain

$$P_m y_m = \sum_{k=-\infty}^{m} \mathcal{A}(m, k+1)P_{k+1}B_k(\lambda)y_k. \tag{4.23}$$

On the other hand, by (4.22),

$$Q_m y_m = \mathcal{A}(m, n)Q_n\xi - \sum_{k=m}^{n-1} \mathcal{A}(m, k+1)Q_{k+1}B_k(\lambda)y_k. \tag{4.24}$$

Adding (4.23) and (4.24) yields the desired dentity. □

Step 4. Invariance of the stable and unstable subspaces

Now we set $E_m^\lambda = \operatorname{Im} U_\lambda(m,m)$ and $F_m^\lambda = \operatorname{Im} V_\lambda(m,m)$.

LEMMA 4.9. *For each $m \in \mathbb{Z}$ and $\lambda \in \Delta$, we have*

$$E_m^\lambda = \mathcal{A}_\lambda(m,n)E_n^\lambda \quad \textit{and} \quad F_m^\lambda = \mathcal{A}_\lambda(m,n)F_n^\lambda.$$

PROOF. By Lemma 4.2, for each $\xi \in \mathbb{R}^l$ the sequence $m \mapsto U_\lambda(m,n)\xi$, $m \geq n$ is a solution of equation (2.4) with initial condition at time n equal to $U_\lambda(n,n)\xi$. Therefore, $U_\lambda(m,n) = \mathcal{A}_\lambda(m,n)U_\lambda(n,n)$, where $\mathcal{A}_\lambda(m,n)$ is the cocycle associated to equation (2.3). Hence, by Lemma 4.3,

$$\begin{aligned}\mathcal{A}_\lambda(m,n)E_n^\lambda &= \operatorname{Im} U_\lambda(m,n)\\ &= \operatorname{Im}\,(U_\lambda(m,m)U_\lambda(m,n)) = U_\lambda(m,m)\operatorname{Im} U_\lambda(m,n) \subset E_m^\lambda\end{aligned}$$

for each $m \geq n$. Similarly, by Lemma 4.5, the sequence $m \mapsto V_\lambda(m,n)\xi$, $m \leq n$ is a solution of equation (2.4), and hence,

$$V_\lambda(n,n) = \mathcal{A}_\lambda(n,m)V_\lambda(m,n). \tag{4.25}$$

This implies that

$$\begin{aligned}F_n^\lambda &= \mathcal{A}_\lambda(n,m)\operatorname{Im} V_\lambda(m,n)\\ &= \mathcal{A}_\lambda(n,m)\operatorname{Im}\,(V_\lambda(m,m)V_\lambda(m,n)) \subset \mathcal{A}_\lambda(n,m)F_m^\lambda\end{aligned}$$

for each $m \leq n$. For the reverse inclusions we use the characterization of bounded solutions in Lemmas 4.7 and 4.8. Take $x \in E_m^\lambda$ and $y \in \mathcal{A}_\lambda(m,n)^{-1}x$. Then $\mathcal{A}_\lambda(k,n)y = \mathcal{A}_\lambda(k,m)x$ for each $k \geq m$. Since $x \in E_m^\lambda = \operatorname{Im} U_\lambda(m,m)$, we have $x \in U_\lambda(m,m)z$ for some $z \in \mathbb{R}^l$, and hence,

$$\mathcal{A}_\lambda(k,n)y = \mathcal{A}_\lambda(k,m)U_\lambda(m,m)z = U_\lambda(k,m)z.$$

This shows that the sequence $[n,+\infty) \cap \mathbb{Z} \ni k \mapsto \mathcal{A}_\lambda(k,n)y$ is bounded, and it follows from Lemma 4.7 that $\mathcal{A}_\lambda(k,n)y = U_\lambda(k,n)w$ for some $w \in \mathbb{R}^l$. In particular,

$$y = \mathcal{A}_\lambda(n,n)y = U_\lambda(n,n)w \in E_n^\lambda.$$

Therefore, $x = \mathcal{A}_\lambda(m,n)y \in \mathcal{A}_\lambda(m,n)E_n^\lambda$, and we obtain $E_m^\lambda \subset \mathcal{A}_\lambda(m,n)E_n^\lambda$. This establishes the first identity in the lemma. For the second identity, take $x \in \mathcal{A}_\lambda(n,m)F_m^\lambda$ and $y \in \mathcal{A}_\lambda(n,m)^{-1}x$. Then $k \mapsto V_\lambda(k,m)y$, $k \leq m$ is a bounded sequence satisfying equation (2.3), and hence,

$$(-\infty,n] \cap \mathbb{Z} \ni k \mapsto \begin{cases} V_\lambda(k,m)y, & k \leq m,\\ \mathcal{A}_\lambda(n,m)V_\lambda(m,m)y, & m \leq k \leq n\end{cases}$$

is also a bounded sequence satisfying the equation. Hence, it follows from Lemma 4.8 that $V_\lambda(k,m)y = V_\lambda(k,n)z$, $k \le m$ for some $z \in \mathbb{R}^l$. In particular, by (4.25),

$$x = \mathcal{A}_\lambda(n,m)V_\lambda(m,m)y = \mathcal{A}_\lambda(n,m)V_\lambda(m,n)z = V_\lambda(n,n)z.$$

Therefore, $x \in F_n^\lambda$, and hence $\mathcal{A}_\lambda(n,m)F_m^\lambda \subset F_n^\lambda$. □

Now we show that the perturbed dynamics is invertible along the subspaces F_m^λ. Since $V_\lambda(n,n)^2 = V_\lambda(n,n)$, restricting identity (4.25) to F_n^λ yields

$$\mathrm{Id}_{F_n^\lambda} = V_\lambda(n,n)|F_n^\lambda = \mathcal{A}_\lambda(n,m)V_\lambda(m,n)|F_n^\lambda.$$

Therefore, the operator $\mathcal{A}_\lambda(n,m)|F_m^\lambda$ is invertible, with

$$(\mathcal{A}_\lambda(n,m)|F_m^\lambda)^{-1} = V_\lambda(m,n)|F_n^\lambda.$$

It follows from Lemma 4.9 that

$$\begin{aligned} \mathcal{A}_\lambda(m,n)|E_n^\lambda &= U_\lambda(m,n)|E_n^\lambda\colon E_n^\lambda \to E_m^\lambda, \quad m \ge n,\\ (\mathcal{A}_\lambda(n,m)|F_m^\lambda)^{-1}* &= V_\lambda(m,n)|F_n^\lambda\colon F_n^\lambda \to F_m^\lambda, \quad m \le n. \end{aligned}$$

Therefore, since $U_\lambda \in \mathcal{C}_\lambda$, we obtain

$$\|\mathcal{A}_\lambda(m,n)|E_n^\lambda\| \le Ke^{-a\ (m-n)+\varepsilon|n|}, \quad m \ge n, \tag{4.26}$$

and since $V_\lambda \in \mathcal{D}_\lambda$, we obtain

$$\|(\mathcal{A}_\lambda(n,m)|F_m^\lambda)^{-1}\| \le Ke^{-a\ (n-m)+\varepsilon|n|}, \quad m \le n, \tag{4.27}$$

for some constant $K < 0$.

Step 5. Construction of projections. Set $S_{n,\lambda} = U_\lambda(n,n) + V_\lambda(n,n)$.

Lemma 4.10. *For any sufficiently small δ, the operator $S_{n,\lambda}$ is invertible for every $n \in \mathbb{Z}$ and $\lambda \in \Delta$.*

Proof. We have

$$\begin{aligned} S_{n,\lambda} = P_n &- \sum_{k=n}^{\infty} \mathcal{A}(n,k+1)Q_{k+1}B_k(\lambda)U_\lambda(k,n)\\ &+ Q_n + \sum_{k=-\infty}^{n-1} \mathcal{A}(n,k+1)P_{k+1}B_k(\lambda)V_\lambda(k,n), \end{aligned}$$

and hence

$$\begin{aligned} S_{n,\lambda} - \mathrm{Id} = &- \sum_{k=n}^{\infty} \mathcal{A}(n,k+1)Q_{k+1}B_k(\lambda = U_\lambda(k,n)\\ &+ \sum_{k=-\infty}^{n-1} \mathcal{A}(n,k+1)P_{k+1}B_k(\lambda)V_\lambda(k,n). \end{aligned}$$

By Lemmas 4.2 and 4.5, we obtain

$$\begin{aligned}\|S_{n,\lambda}-\mathrm{Id}\| &\le \sum_{k=n}^{\infty}\|\mathcal{A}(n,k+1)Q_{k+1}\|\cdot\|B_k(\lambda)\|\cdot\|U_\lambda(k,n)\| \\ &\quad+\sum_{k=-\infty}^{n-1}\|\mathcal{A}(n,k+1)P_{k+1}\|\cdot\|B_k(\lambda)\|\cdot\|V_\lambda(k,n)\| \\ &\le \widetilde{D}\delta\|U_\lambda\|\sum_{k=n}^{\infty}e^{-2a\ (k-n)}+\widetilde{D}\delta\|V_\lambda\|\sum_{k=-\infty}^{n-1}e^{-2a(k-n)} \\ &\le \widetilde{D}\delta\biggl(\frac{\|U_\lambda\|}{1-e^{-2a}}+\frac{\|V_\lambda\|}{e^{2a}-1}\biggr).\end{aligned}$$

Moreover, it follows from (4.12) and (4.15) that

$$\|U_\lambda\|\le D\Big/\biggl(1-\widetilde{D}\delta\biggl(\frac{1}{1-e^{-2\varepsilon}}+\frac{1}{1-e^{-2a}}\biggr)\biggr)$$

and

$$\|V_\lambda\|\le D\Big/\biggl(1-\widetilde{D}\delta\biggl(\frac{1}{e^{2a}-1}+\frac{1}{1-e^{-2\varepsilon}}\biggr)\biggr).$$

This yields the desired statement. □

Lemma 4.11. *Provided that δ is sufficiently small, we have $E_m^\lambda\oplus F_m^\lambda=\mathbb{R}^l$ for each $\lambda\in\Delta$ and $m\in\mathbb{Z}$.*

Proof. Let $\xi\in E_m^\lambda\cap F_m^\lambda$. By (4.27), for each $m\ge n$ we have

$$\frac{1}{K}\,e^{a\ (m-n)-\varepsilon|m|}\|\xi\|\le\|\mathcal{A}_\lambda(m,n)\xi\|\le Ke^{-a\ (m-n)+\varepsilon|n|}\|\xi\|.$$

Since $\varepsilon<a$ (see (3.1)), this implies that $\xi=0$, and hence, $E_m^\lambda\cap F_m^\lambda=\{0\}$. On the other hand, since $S_{m,\lambda}$ is invertible, we have

$$\mathbb{R}^l=S_{m,\lambda}\mathbb{R}^l=\operatorname{Im}U_\lambda(m,m)+\operatorname{Im}V_\lambda(m,m)=E_m^\lambda+F_m^\lambda.$$

This concludes the proof of the lemma. □

By Lemma 4.11, for each $m\in\mathbb{Z}$ any vector $x\in\mathbb{R}^l$ can be written uniquely in the form $x=y_m+z_m$ with $y_m\in E_m^\lambda$ and $z_m\in F_m^\lambda$. Hence, one can define projections P_m^λ and Q_m^λ by $P_m^\lambda x=y_m$ and $Q_m^\lambda x=z_m$. The following statement is an immediate consequence of Lemmas 4.9 and 4.11.

Lemma 4.12. *Provided that δ is sufficiently small, we have*

$$P_m^\lambda\widehat{\mathcal{A}}(m,n)=\mathcal{A}(m,n)P_n^\lambda\quad\textit{for each } m\ge n.$$

Step 6. Upper bounds for the projections. Set

$$\alpha_m^\lambda=\inf\{\|x-y\|:x\in E_m^\lambda,\ y\in F_m^\lambda,\ \|x\|=\|y\|=1\}.$$

Lemma 4.13. *Provided that δ is sufficiently small, there exists a constant $c > 0$ such that*

$$\alpha_m^\lambda \ge c e^{-\varepsilon|m|}, \quad m \in \mathbb{Z}. \tag{4.28}$$

Proof. Given $x \in E_m^\lambda$ and $y \in F_m^\lambda$, there exist $\overline{x} \in E_m$ and $\overline{y} \in F_m$ such that

$$x = U_\lambda(m,m)\overline{x} = (\mathrm{Id} + G_{E,m})\overline{x} \quad \text{and} \quad y = V_\lambda(m,m)\overline{y} = (\mathrm{Id} + G_{F,m})\overline{y},$$

where

$$G_{E,m} = -\sum_{k=m}^{\infty} \mathcal{A}(m,k+1)Q_{k+1}B_k(\lambda)U_\lambda(k,m)$$

and

$$G_{F,m} = \sum_{k=-\infty}^{m-1} \mathcal{A}(m,k+1)P_{k+1}B_k(\lambda)V_\lambda(k,m).$$

We have

$$\begin{aligned}\|G_{E,m}\|e^{\varepsilon|m|} &\le \sum_{k=m}^{\infty} \|\mathcal{A}(m,k+1)Q_{k+1}\| \cdot \|B_k(\lambda)\| \cdot \|U_\lambda(k,m)\| \\ &\le D\delta\|U_\lambda\| \sum_{k=m}^{\infty} e^{-2(\varepsilon-a)(k+1-m)} \le \mu\|U_\lambda\|,\end{aligned}$$

where $\mu = D\delta/(1 - e^{-2(\varepsilon-a)})$, and hence $\|G_{E,m}\| \le \mu\|U_\lambda\|e^{-\varepsilon|m|}$. Similarly,

$$\begin{aligned}\|G_{F,m}\|e^{\varepsilon|m|} &\le \sum_{k=0}^{m-1} \|\mathcal{A}(m,k+1)P_{k+1}\| \cdot \|B_k(\lambda)\| \cdot \|V_\lambda(k,m)\| \\ &\le D\delta\|V_\lambda\| \sum_{k=0}^{m-1} e^{-2(\varepsilon-a)(m-k-1)} \le \mu\|V_\lambda\|,\end{aligned}$$

and hence $\|G_{F,m}\| \le \mu\|V_\lambda\|e^{-\varepsilon|m|}$. Therefore,

$$(1 - \mu\|U_\lambda\|e^{-\varepsilon|m|})\|\overline{x}\| \le \|x\| \le (1 + \mu\|U_\lambda\|e^{-\varepsilon|m|})\|\overline{x}\|, \tag{4.29}$$

$$(1 - \mu\|V_\lambda\|e^{-\varepsilon|m|})\|\overline{y}\| \le \|y\| \le (1 + \mu\|V_\lambda\|e^{-\varepsilon|m|})\|\overline{y}\|. \tag{4.30}$$

On the other hand, setting $n = m$ in (2.1), we obtain

$$\|P_m\| \le De^{\varepsilon|m|} \quad \text{and} \quad \|Q_m\| \le De^{\varepsilon|m|}.$$

Now we recall that (see for example [2])

$$\frac{1}{\|P_m\|} \le \alpha_m \le \frac{2}{\|P_m\|} \quad \text{and} \quad \frac{1}{\|Q_m\|} \le \alpha_m \le \frac{2}{\|Q_m\|} \tag{4.31}$$

for each $m \in \mathbb{Z}$, where $\alpha_m = \inf\{\|x - y\| : x \in E_m, y \in F_m,\ \|x\| = \|y\| = 1\}$. Therefore

$$\alpha_m \ge \frac{1}{D} e^{-\varepsilon|m|}, \quad m \in \mathbb{Z}.$$

Since

$$\left\| \frac{\overline{x}}{\|\overline{x}\|} - \frac{\overline{y}}{\|\overline{y}\|} \right\| \le \frac{\|(\overline{x}-\overline{y})\|\overline{y}\| + \overline{y}(\|\overline{y}\| - \|\overline{x}\|)}{\|\overline{x}\| \cdot \|\overline{y}\|} \le \frac{2}{\|\overline{x}\|}\|\overline{x}-\overline{y}\|,$$

it follows from (4.29) and (4.30) that

$$\begin{aligned}
\|x-y\| &= \|\overline{x} - \overline{y} + G_{E,m}\overline{x} - G_{F,m}\overline{y}\| \\
&\ge \|\overline{x}-\overline{y}\| - \|G_{E,m}\| \cdot \|\overline{x}\| - \|G_{F,m}\| \cdot \|\overline{y}\| \\
&\ge \frac{\|\overline{x}\|}{2} \left\| \frac{\overline{x}}{\|\overline{x}\|} - \frac{\overline{y}}{\|\overline{y}\|} \right\| - \frac{\|G_{E,m}\|}{1-\mu\delta\|U_\lambda\|e^{-\varepsilon|m|}}\,\|x\| \\
&\quad - \frac{\|G_{F,m}\|}{1-\mu\delta\|V_\lambda\|e^{-\varepsilon|m|}}\|y\| \\
&\ge \frac{\|x\|}{2(1+\delta\mu\|U_\lambda\|e^{-\varepsilon|m|})} \left\| \frac{\overline{x}}{\|\overline{x}\|} - \frac{\overline{y}}{\|\overline{y}\|} \right\| \\
&\quad - \frac{\delta\mu\|U_\lambda\|e^{-\varepsilon|m|}}{1-\delta\mu\|U_\lambda\|e^{-\varepsilon|m|}}\,\|x\| - \frac{\delta\mu\|V_\lambda\|e^{-\varepsilon|m|}}{1-\delta\mu\|V_\lambda\|e^{-\varepsilon|m|}}\,\|y\|.
\end{aligned}$$

Taking the infimum over all vectors x, y with $\|x\| = \|y\| = 1$, we obtain

$$\begin{aligned}
\alpha_m^\lambda &\ge \frac{1}{2(1+\delta\mu\|U_\lambda\|e^{-\varepsilon|m|})}\,\alpha_m - \frac{\delta\mu\|U_\lambda\|e^{-\varepsilon|m|}}{1-\delta\mu\|U_\lambda\|e^{-\varepsilon|m|}}\|x\| \\
&\quad - \frac{\delta\mu\|V_\lambda\|e^{-\varepsilon|m|}}{1-\delta\mu\|V_\lambda\|e^{-\varepsilon|m|}}\|y\| \\
&\ge \frac{e^{-\varepsilon|m|}}{4D(1+\delta\mu\|U_\lambda\|)} - \frac{\delta\mu\|U\|e^{-a|m|}}{1-\delta\mu\|U_\lambda\|} - \frac{\delta\mu\|V\|e^{-\varepsilon|m|}}{1-\delta\mu\|V_\lambda\|}.
\end{aligned}$$

This yields inequality (4.28). □

Step 7. Exponential dichotomies for the perturbation. In a similar manner to that in (4.31), we have

$$\frac{1}{\|P_m^\lambda\|} \le \alpha_m^\lambda \le \frac{2}{\|P_m^\lambda\|} \quad \text{and} \quad \frac{1}{\|Q_m^\lambda\|} \le \alpha_m^\lambda \le \frac{2}{\|Q_m^\lambda\|},$$

and hence it follows from Lemma 4.13 that

$$\|P_m^\lambda\| \le \frac{2}{\alpha_m^\lambda} \le \frac{2}{c}\,e^{\varepsilon|m|} \quad \text{and} \quad \|Q_m^\lambda\| \le \frac{2}{\alpha_m^\lambda} \le \frac{2}{c}\,e^{\varepsilon|m|} \tag{4.32}$$

for each $m \in \mathbb{Z}$. Since

$$\begin{aligned}
\|\mathcal{A}_\lambda(m,n)P_n^\lambda\| &\le \|\mathcal{A}_\lambda(m,n)|\widehat{E}_n^\lambda\| \cdot \|P_n^\lambda\|, && m \ge n, \\
\|\mathcal{A}_\lambda(m,n)Q_n^\lambda\| &\le \|(\mathcal{A}_\lambda(n,m)|F_m^\lambda)^{-1}\| \cdot \|Q_n^\lambda\|, && m \le n,
\end{aligned}$$

the existence of a nonuniform exponential dichotomy follows readily from (4.32) together with inequalities (4.26) and (4.27).

Step 8. Construction of auxiliary operators. In order to show that the stable and unstable subspaces E_n^λ and F_n^λ are analytic in λ we first construct auxiliary operators. By Lemma 4.13, equation (3.7) is equivalent to equation (4.3). This motivates the introduction of linear operators $A(\Phi)_{n,\lambda}\colon E_n \to F_n$ for each $\Phi \in \mathfrak{X}$, $n \in \mathbb{Z}$ and $\lambda \in \Delta$ by

$$A(\Phi)_{n,\lambda} = -\sum_{l=n}^{\infty} Q_n \mathcal{A}(l+1,n)^{-1} B_l(\lambda)(\mathrm{Id}_{E_l} + \Phi_{l,\lambda}) W_{l,\lambda}^n, \tag{4.33}$$

where $W_{l,\lambda}^n\colon E_n \to E_l$ are the linear operators given by (4.1). We note that solving equation (4.2) is equivalent to find a fixed point Φ of the sequence of linear operators $\Phi \mapsto A(\Phi)_{n,\lambda}$. We first establish some auxiliary properties.

Lemma 4.14. *For any sufficiently small δ, the operator A is well defined, and $A(\mathfrak{X}) \subset \mathfrak{X}$.*

Proof. Repeating estimates in the proof of Lemma 4.13, one can show that the operator A is well defined. Moreover, it follows from (4.7) that $\|A(\Phi)\| \le \kappa$. Moreover, writing $W_{l,\lambda}^n = W_{l,\lambda}$ and $W_{l,\mu}^n = W_{l,\mu}$, by (4.6) we have

$$\begin{aligned}
b_l &:= \|B_l(\lambda)(\mathrm{Id}_{E_l} + \Phi_{l,\lambda}) W_{l,\lambda} - B_l(\mu)(\mathrm{Id}_{E_l} + \Phi_{l,\mu}) W_{l,\mu}\| \\
&\le \|B_l(\lambda) - B_l(\mu)\| \cdot \|W_{l,\lambda}\|(1 + \|\Phi_{l,\lambda}\|) \\
&\quad + \|B_l(\lambda)\| \cdot \|W_{l,\lambda} - W_{l,\mu}\|(1 + \|\widetilde{\Phi}_{l,\lambda}\|) \\
&\quad + \|B_l(\mu)\| \cdot \|W_{l,\mu}\| \cdot \|\Phi_{l,\lambda} - \Phi_{l,\mu}\| \\
&\le 4\,\delta D e^{-2\varepsilon|l|}\|\lambda - \mu\| e^{a(l-n)+\varepsilon|n|} + 2\,\delta e^{-2\varepsilon|l|}\|W_{l,\lambda} - W_{l,\mu}\| \\
&\quad + 2\delta e^{-3\varepsilon|l|} D e^{a(l-n)+\varepsilon|n|} \kappa e^{-\varepsilon|l|}\|\lambda - \mu\| \\
&\le 6\,\delta D e^{a(l-n)+\varepsilon|n|-3\varepsilon|l|}\|\lambda - \mu\| + 2\,\delta e^{-3\varepsilon|l|}\|W_{l,\lambda} - W_{l,\mu}\|.
\end{aligned} \tag{4.34}$$

Therefore

$$\begin{aligned}
\|W_{l,\lambda} - W_{l,\mu}\| \le \sum_{l=n}^{m-1} \|P_m \mathcal{A}(m,l+1)\| b_l \\
&\le 6\,\delta D\widetilde{D} e^{-a(m-n)+\varepsilon|n|}\|\lambda - \mu\| \sum_{l=n}^{m-1} e^{-2\varepsilon|l|} \\
&\quad + 2\,\delta \widetilde{D} \sum_{l=n}^{m-1} e^{-a(m-l)-2\varepsilon|l|}\|W_{l,\lambda} - W_{l,\mu}\| \\
&\le 6\,\delta D\widetilde{D}\Gamma_{2\varepsilon} e^{-a(m-n)+\varepsilon|n|}\|\lambda - \mu\| \\
&\quad + 2\,\delta \widetilde{D} e^{-a(m-n)} \sum_{l=n}^{m-1} e^{a(l-n)-2\varepsilon|l|}\|W_{l,\lambda} - W_{l,\mu}\|.
\end{aligned}$$

Setting $\Upsilon_l = e^{a(l-n)}\|W_{l,\lambda} - W_{l,\mu}\|$, it follows from this inequality that

$$\Upsilon_m \leq 6\,\delta D\widetilde{D}\Gamma_{2\varepsilon}e^{\varepsilon|n|}\|\lambda-\mu\| + 2\,\delta\widetilde{D}\sum_{l=n}^{m-1}\Upsilon_l e^{-2\varepsilon|l|},$$

and hence

$$\Upsilon \leq 6\,\delta D\widetilde{D}\Gamma_{2\varepsilon}e^{\varepsilon|n|}\|\lambda-\mu\| + 2\,\delta\widetilde{D}\Gamma_{2\varepsilon}\Upsilon,$$

where $\Upsilon = \sup\{\Upsilon_m : m \geq n\}$. Taking δ sufficiently small, so that $2\,\delta\widetilde{D}\Gamma_{2\varepsilon} < 1/2$, we obtain

$$\Upsilon \leq 12\,\delta D\widetilde{D}\Gamma_{2\varepsilon}e^{\varepsilon|n|}\|\lambda-\mu\|,$$

which yields

$$\|W_{l,\lambda} - W_{l,\mu}\| \leq 12\,\delta D\widetilde{D}\Gamma_{2\varepsilon}e^{-a(m-n)+\varepsilon|n|}\|\lambda-\mu\|.$$

Hence, it follows from (4.34) that

$$b_l \leq \delta K e^{-a(l-n)+\varepsilon|n|}e^{-3\varepsilon|l|}\|\lambda-\mu\|,$$

for some positive constant K. Proceeding as in (4.6), since $-a+\varepsilon < 0$ we obtain

$$\begin{aligned}\|A(\Phi)_{n,\lambda} - A(\Phi)_{n,\mu}\|e^{\varepsilon|n|} &\leq \sum_{l=n}^{\infty}\|Q_n\mathcal{A}(l+1,n)^{-1}\|\,b_l e^{\varepsilon|n|}\\ &\leq \delta K\widetilde{D}\|\lambda-\mu\|\sum_{l=n}^{\infty}e^{-a(l-n)+\varepsilon|l|}e^{-a(l-n)+\varepsilon|n|-3\varepsilon|l|+\varepsilon|n|}\\ &\leq \delta KD\|\lambda-\mu\|\sum_{l=n}^{\infty}e^{-(2a-\varepsilon)(l-n)+\varepsilon(l-n)} = \delta KD\Gamma_{2a-\varepsilon}\|\lambda-\mu\|,\end{aligned}$$

and hence, $C_{\lambda\mu}(A(\Phi)) \leq \kappa$ provided that δ is sufficiently small. This shows that $A(\mathfrak{X}) \subset \mathfrak{X}$. □

Now we consider the space $\mathfrak{F}$ of sequences $U = (U_{n,\lambda})_{n\in\mathbb{Z},\lambda\in\Delta}$ of linear operators $U_{n,\lambda}\colon E_n \to F_n$ such that

$$\|U\| := \sup\{\|U_{n,\lambda}\| : (n,\lambda) \in \mathbb{Z}\times\Delta\} \leq 1. \tag{4.35}$$

One can easily verify that $\mathfrak{F}$ is a complete metric space with this norm. We also define operators $B(\Phi,U)_{n,\lambda}$ for each $(\Phi,U) \in \mathfrak{X}\times\mathfrak{F}$, $n \in \mathbb{Z}$, and $\lambda \in \Delta$ by

$$\begin{aligned}B(\Phi,U)_{n,\lambda} = -\sum_{l=n}^{\infty}Q_n\mathcal{A}(l+1,n)^{-1}[B_l(\lambda)(Z^n_{l,\lambda} + \Phi_{l,\lambda}Z^n_{l,\lambda} + U_{l,\lambda}W^n_{l,\lambda})\\ + B'_l(\lambda)(\mathrm{Id}_{E_l} + \Phi_{l,\lambda})W^n_{l,\lambda}],\end{aligned} \tag{4.36}$$

where $W^n_{l,\lambda}\colon E_n \to E_l$ are the linear operators given by (4.1), and where $Z^n_{m,\lambda} = Z^n_{m,\Phi,U,\lambda}\colon E_n \to E_m$ are linear operators determined recursively by the identities

$$(4.37)\quad Z^m_{n,\lambda} = \sum_{l=n}^{m-1} P_m \mathcal{A}(m,l+1)[B_l(\lambda)(Z^n_{l,\lambda} + \Phi_{l,\lambda} Z^n_{l,\lambda} + U_{l,\lambda} W^n_{l,\lambda}) + B'_l(\lambda)(\mathrm{Id}_{E_l} + \Phi_{l,\lambda})W^n_{l,\lambda}],$$

for $m > n$, setting $Z^n_{n,\lambda} = 0$. We observe that by thecontinuity of the functions $\Phi_{l,\lambda}$ and $U_{l,\lambda}$ on λ, the functions $\lambda \mapsto W^n_{l,\lambda}$ and $\lambda \mapsto Z^n_{l,\lambda}$ are also continuous.

Lemma 4.15. *For any sufficiently small δ, the operator B is well defined, and $B(\mathcal{X} \times \mathcal{F}) \subset \mathcal{F}$.*

Proof. Set

$$(4.38)\quad C = \sum_{l=n}^{\infty} \|Q_n \mathcal{A}(l+1,n)^{-1}\| \cdot \|B_l(\lambda)(Z^n_{l,\lambda} + \Phi_{l,\lambda} Z^n_{l,\lambda}) + B_l(\lambda) U_{l,\lambda} W^n_{l,\lambda} + B'_l(\lambda)(\mathrm{Id}_{E_l} + \Phi_{l,\lambda}) W^n_{l,\lambda}\|.$$

It follows from (4.3), (4.6) and (4.35) that

$$(4.39)\quad \begin{aligned} C &\le (1+\kappa)\,\delta \widetilde{D} \sum_{l=n}^{\infty} e^{-a(l-n))+\varepsilon|l|-3\varepsilon|l|} \|Z^n_{l,\lambda}\| \\ &\quad + 2(2+\kappa)\,\delta D\widetilde{D} \sum_{l=n}^{\infty} e^{-a(l-n)+\varepsilon|l|-3\varepsilon|l|-a(l-n)+\varepsilon|n|} \\ &\le 2\,\delta\widetilde{D} \sum_{l=n}^{\infty} e^{-a(l-n)-2\varepsilon|l|} \|Z^n_{l,\lambda}\| + 6\,\delta D\widetilde{D} \sum_{l=n}^{\infty} e^{-(2a-\varepsilon)(l-n)}, \end{aligned}$$

where in the last inequality we have used that $-a+\varepsilon < 0$. On the other hand, by (4.35) and (4.37), we have

$$\begin{aligned} \|Z^n_{m,\lambda}\| &\le (1+\kappa)\,\delta\widetilde{D} \sum_{l=n}^{m-1} e^{-a(m-l)-2\varepsilon|l|} \|Z^n_{l,\lambda}\| \\ &\quad + 2(2+\kappa)\,\delta D\widetilde{D} \sum_{l=n}^{m-1} e^{-a(m-l)-2\varepsilon|l|+a(l-n)+\varepsilon|n|}. \end{aligned}$$

Setting $\Upsilon_m = e^{a(m-n)} \|Z^n_{m,\lambda}\|$, we obtain

$$\begin{aligned} \Upsilon_m &\le 2\,\delta\widetilde{D} \sum_{l=n}^{m-1} e^{-2\varepsilon|l|} \Upsilon_l + 6\,\delta D\widetilde{D} \sum_{l=n}^{m-1} e^{-2\varepsilon|l|+\varepsilon|n|} \\ &\le 2\,\delta\widetilde{D} \sum_{l=n}^{m-1} e^{-2\varepsilon|l|} \Upsilon_l + 6\,\delta D\widetilde{D} e^{\varepsilon|n|} \sum_{l=n}^{m-1} e^{-2\varepsilon|l|}, \end{aligned}$$

and hence

$$\Upsilon \le 2\,\delta\,\Gamma_{2\varepsilon} \Upsilon + 6\,\delta D\widetilde{D} \Gamma_{2\varepsilon} e^{\varepsilon|n|},$$

where $\Upsilon = \sup\{\Upsilon_m : m \geq n\}$. Thus, taking δ sufficiently small so that $2\delta \widetilde{D}\Gamma_{2\varepsilon} \leq 1/2$, we obtain

$$\Upsilon \leq 12\,\delta D\widetilde{D}\Gamma_{2\varepsilon}e^{\varepsilon|n|},$$

and hence,

$$\|Z^n_{m,\lambda}\| \leq 12\,\delta D\widetilde{D}\Gamma_{2\varepsilon}e^{-a(m-n)+\varepsilon|n|}. \tag{4.40}$$

By (4.39) and (4.40), we obtain

$$\begin{aligned} C &\leq 24\,\delta^2 D\widetilde{D}^2\Gamma_{2\varepsilon}\sum_{l=n}^{\infty} e^{-2a(l-n)+\varepsilon|n|-2\varepsilon|l|} + 6\,\delta D\widetilde{D}\sum_{l=n}^{\infty} e^{2(2a-\varepsilon)(l-n)} \\ &\leq 24\,\delta^2 D\widetilde{D}^2\Gamma_{2\varepsilon}\sum_{l=n}^{\infty} e^{-(2a-\varepsilon)(l-n)} + 6\,\delta D\widetilde{D}\Gamma_{2a-\varepsilon} \\ &\leq 24\,\delta^2 D\widetilde{D}^2\Gamma_{2\varepsilon}\Gamma_{2a-\varepsilon} + 6\,\delta D\widetilde{D}\Gamma_{2a-\varepsilon} \leq 1, \end{aligned} \tag{4.41}$$

provided that δ is sufficiently small. This shows that $B(\Phi, U)_n$ is well defined for each n, and that $\|B(\Phi, U)\| \leq 1$. Therefore $B(\mathfrak{X} \times \mathfrak{F}) \subset \mathfrak{F}$. □

Now we define a map $S\colon \mathfrak{X} \times \mathfrak{F} \to \mathfrak{X} \times \mathfrak{F}$ by

$$S(\Phi, U) = (A(\Phi), B(\Phi, U)).$$

By Lemmas 4.14 and 4.15, the map S is well defined, and $S(\mathfrak{X} \times \mathfrak{F}) \subset \mathfrak{X} \times \mathfrak{F}$.

Lemma 4.16. *For any sufficiently small δ, the map S is a contraction.*

Proof. Given $\Phi, \Psi \in \mathfrak{X}$, set $W_{l,\Phi} = W^n_{l,\Phi,\lambda}$ and $W_{l,\Psi} = W^n_{l,\Psi,\lambda}$. We have

$$\begin{aligned} \|A(\Phi)_{n,\lambda} - A(\Psi)_{n,\lambda}\|\, e^{\varepsilon|\rho(n)|} &\leq D\sum_{l=n}^{\infty} e^{-a(l-n)+\varepsilon|l|+\varepsilon|n|} \\ &\quad \times \|B_l(\lambda)(W_{l,\Phi} - W_{l,\Psi}) + B_l(\lambda)(\Phi_{l,\lambda}W_{l,\Phi} - \Psi_{l,\lambda}W_{l,\Psi})\| \\ &\leq D\delta\sum_{l=n}^{\infty} e^{-a(l-n)-2\varepsilon|l|+\varepsilon|n|} \\ &\quad \times (\|W_{l,\Phi} - W_{l,\Psi}\| + \|\Phi_{l,\lambda}W_{l,\Phi} - \Psi_{l,\lambda}W_{l,\Psi}\|) \\ &\leq D\delta\sum_{l=n}^{\infty} e^{-(a-\varepsilon)(l-n)-\varepsilon|l|}\big(\|W_{l,\Phi} - W_{l,\Psi}\| \\ &\quad + \|\Phi_{l,\lambda}\|\cdot\|W_{l,\Phi} - W_{l,\Psi}\| + \|\Phi_{l,\lambda} - \Psi_{l,\lambda}\|\cdot\|W_{l,\Psi}\|\big) \\ &\leq D\delta\sum_{l=n}^{\infty} e^{-(a-\varepsilon)(l-n)-\varepsilon|l|} \\ &\quad \times [(1+\kappa)\|W_{l,\Phi} - W_{l,\Psi}\| + \|\Phi - \Psi\|\cdot\|W_{l,\Psi}\|e^{-\varepsilon|l|}]. \end{aligned} \tag{4.42}$$

In an analogous manner to that in (4.5) and using (4.6), we obtain

$$\begin{aligned}\|W_{m,\Phi}-W_{m,\Psi}\| &\le (1+\kappa)\,\delta\widetilde{D}\sum_{l=n}^{m-1} e^{-a(m-l))+\varepsilon|l|-3\varepsilon|l|}\|W_{l,\Phi}-W_{l,\Psi}\| \\ &\quad + 2\,\delta D\widetilde{D}\|\Phi-\Psi\|\sum_{l=n}^{m-1} e^{-a(m-l)+\varepsilon|l|-3\varepsilon|l|}e^{a(l-n)+\varepsilon|n|-\varepsilon|l|} \\ &\le 2\,\delta\widetilde{D}e^{-a(m-n)}\sum_{l=n}^{m-1} e^{a(l-n)-2\varepsilon|l|}\|W_{l,\Phi}-W_{l,\Psi}\| \\ &\quad + 2\,\delta D\widetilde{D}e^{-a(m-n)}\|\Phi-\Psi\|\sum_{l=n}^{\infty} e^{-3\varepsilon|l|+\varepsilon|n|} \\ &= 2\,\delta\widetilde{D}e^{-a(m-n)}\sum_{l=n}^{m-1} e^{a(l-n)-2\varepsilon|l|}\|W_{l,\Phi}-W_{l,\Psi}\| \\ &\quad + 2\,\delta D\widetilde{D}e^{-a(m-n)}\|\Phi-\Psi\|e^{\varepsilon|n|}\sum_{l=n}^{\infty} e^{-3\varepsilon|l|}.\end{aligned}$$

Setting $\Upsilon_m = e^{a(m-n)}\|W_{m,\Phi}-W_{m,\Psi}\|$, yields

$$\Upsilon_m \le 2\,\delta D\widetilde{D}\Gamma_{3\varepsilon}e^{\varepsilon|n|}\|\Phi-\Psi\| + 2\,\delta\widetilde{D}\sum_{l=n}^{m-1} e^{-2\varepsilon|l|}\Upsilon_l.$$

Therefore

$$\Upsilon \le 2\delta D\widetilde{D}\Gamma_{3\varepsilon}e^{\varepsilon|n|}\|\Phi-\Psi\| + 2\delta D\Gamma_{2\varepsilon}\Upsilon,$$

where $\Upsilon = \sup_{m\ge n}\Upsilon_m$. Taking δ sufficiently small so that $2\,\delta\widetilde{D}\Gamma_{2\varepsilon} < 1/2$ (independently of n) yields

$$\Upsilon \le 4\,\delta D^2\Gamma_{3\varepsilon}e^{\varepsilon|n|}\|\Phi-\Psi\|,$$

and thus,

$$\|W_{m,\Phi}-W_{m,\Psi}\| \le 4\,\delta D\widetilde{D}\Gamma_{3\varepsilon}\|\Phi-\Psi\|e^{-a(m-n)+\varepsilon|n|}. \tag{4.43}$$

Introducing the estimates (4.6) and (4.43) in (4.42) we obtain

$$\begin{aligned}&\|A(\Phi)_{n,\lambda} - A(\Psi)_{n,\lambda}\|e^{\varepsilon|n|} \\ &\quad\le \delta K'\|\Phi-\Psi\|\sum_{l=n}^{\infty} e^{-(a-\varepsilon)(l-n)-a(l-n)-\varepsilon|l|+\varepsilon|n|} \\ &\quad\le \delta K'\|\Phi-\Psi\|\sum_{l=n}^{\infty} e^{-2(a-\varepsilon)(l-n)} = \delta K'\Gamma_{2(a-\varepsilon)}\|\Phi-\Psi\|,\end{aligned} \tag{4.44}$$

for some constant $K' > 0$, provided that $\delta \le 1$. Moreover, given $\Phi,\Psi \in \mathcal{X}$, $U,V \in \mathcal{F}$, and $\lambda \in \Delta$, set

$$Z_{l,\Phi,U} = Z^n_{l,\Phi,U,\lambda} \quad\text{and}\quad Z_{l,\Psi,V} = Z^n_{l,\Psi,V,\lambda}.$$

We have

$$\begin{aligned}(4.45)\qquad &\|B(\Phi,U)_{n,\lambda}-B(\Psi,V)_{n,\lambda}\|\\ &\le D\sum_{l=n}^{\infty}e^{-a(l-n)+\varepsilon|l|}\|B_l(\lambda)[Z_{l,\Phi,U}+\Phi_{l,\lambda}Z_{l,\Phi,U}\\ &\quad+U_{l,\lambda}W_{l,\Phi}-Z_{l,\Psi,V}-\Psi_{l,\lambda}Z_{l,\Psi,V}-V_{l,\lambda}W_{l,\Psi}]\\ &\quad+B_l'(\lambda)[W_{l,\Phi}+\Phi_{l,\lambda}W_{l,\Phi}-W_{l,\Psi}-\Psi_{l,\lambda}W_{l,\Psi}]\|\\ &\le \delta D\sum_{l=n}^{\infty}e^{-a(l-n)-2\varepsilon|l|}[(1+\kappa)\|Z_{l,\Phi,U}-Z_{l,\Psi,V}\|\\ &\quad+\|\Phi_{l,\lambda}-\Psi_{l,\lambda}\|(\|Z_{l,\Phi,U}\|+\|W_{l,\Phi}\|)\\ &\quad+\|U_{l,\lambda}-V_{l,\lambda}\|\cdot\|W_{l,\Phi}\|\\ &\quad+\|W_{l,\Phi}-W_{l,\Psi}\|(1+\|V_{l,\lambda}\|+\|\Psi_{l,\lambda}\|)].\end{aligned}$$

Using (4.6), (4.40) and (4.43), we obtain

$$\begin{aligned}(4.46)\quad &\|Z_{m,\Phi,U}-Z_{m,\Psi,V}\|\\ &\le \delta\widetilde{D}\sum_{l=n}^{\infty}e^{-a(m-l)-2\varepsilon|l|}[(1+\kappa)\|Z_{l,\Phi,U}-Z_{l,\Psi,V}\|\\ &\quad+\|\Phi_{l,\lambda}-\Psi_{l,\lambda}\|(\|Z_{l,\Phi,U}\|+\|W_{l,\Phi}\|)+\|U_{l,\lambda}-V_{l,\lambda}\|\cdot\|W_{l,\Phi}\|\\ &\quad+\|W_{l,\Phi}-W_{l,\Psi}\|(1+\|V_{l,\lambda}\|+\|\Psi_{l,\lambda}\|)]\\ &\le \delta\widetilde{D}\sum_{l=n}^{m-1}e^{-a(m-l)-2\varepsilon|l|}[(1+\kappa)\|Z_{l,\Phi,U}-Z_{l,\Psi,V}\|\\ &\quad+(2+\kappa)\|W_{l,\Phi}-W_{l,\Psi}\|+\|\Phi_{l,\lambda}-\Psi_{l,\lambda}\|(\|Z_{l,\Phi,U}\|+\|W_{l,\Phi}\|)\\ &\quad+\|U_{l,\lambda}-V_{l,\lambda}\|\cdot\|W_{l,\Phi}\|]\\ &\le \delta\widetilde{D}\sum_{l=n}^{m-1}e^{-a(m-l)-2\varepsilon|l|}[(1+\kappa)\|Z_{l,\Phi,U}-Z_{l,\Psi,V}\|\\ &\quad+\delta\widetilde{D}K_0e^{-a(l-n)+\varepsilon|n|}(\|\Phi-\Psi\|+\|U-V\|],\end{aligned}$$

for some positive constant K_0, provided that $\delta\le 1$. Setting

$$\Upsilon_m=e^{a(m-n)}\|Z_{m,\Phi,U}-Z_{m,\Psi,V}\|,$$

we obtain

$$\Upsilon_m\le\delta\widetilde{D}K_0(\|\Phi-\Psi\|+\|U-V\|)e^{\varepsilon|n|}\sum_{l=n}^{m-1}e^{-2\varepsilon|l|}+2\,\delta\widetilde{D}\sum_{l=n}^{m-1}e^{-2\varepsilon|l|}\Upsilon_l.$$

Therefore

$$\Upsilon\le\delta\widetilde{D}K_0\Gamma_{2\varepsilon}e^{\varepsilon|n|}(\|\Phi-\Psi\|+\|U-V\|)+2\,\delta\widetilde{D}\Gamma_{2\varepsilon}\Upsilon,$$

where $\Upsilon = \sup_{m\ge n} \Upsilon_m$. Taking δ sufficiently small so that $2\delta\widetilde{D}\Gamma_{2\varepsilon} < 1/2$ (independently of n) we obtain

$$\Upsilon \le \delta K'' e^{\varepsilon|n|}(\|\Phi-\Psi\| + \|U-V\|)$$

for some constant $K''>0$, and hence

$$\|Z_{m,\Phi,U} - Z_{m,\Psi,V}\| \le \delta K''(\|\Phi-\Psi\| + \|U-V\|)e^{-a(m-n)+\varepsilon|n|}. \tag{4.47}$$

Proceeding as in (4.46), by (4.40) and (4.47) it follows from (4.45) that

$$\begin{aligned} \|B(\Phi,U)_{n,\lambda} &- B(\Psi,V)_{n,\lambda}\| \\ &\le \delta K'' D(\|\Phi-\Psi\| + \|U-V\|)\sum_{l=n}^{\infty} e^{-(2a-\varepsilon)(l-n)} \\ &\quad + \delta\widetilde{D}K_0(\|\Phi-\Psi\| + \|U-V\|)\sum_{l=n}^{\infty} e^{-(2a-\varepsilon)(l-n)} \\ &\le \delta L(\|\Phi-\Psi\| + \|U-V\|), \end{aligned} \tag{4.48}$$

for some positive constant L, provided that $\delta\le 1$. By (4.44) and (4.48), for any sufficiently small δ the operator S is a contraction. □

Step 9. Analytic dependence of the stable and unstable subspaces. We proceed with the proof of the theorem. We first observe that by Lemma 4.16 there exists a unique pair $(\overline{\Phi},\overline{U}) \in \mathcal{X}\times\mathcal{F}$ such that $S(\overline{\Phi},\overline{U}) = (\overline{\Phi},\overline{U})$. Since the operators $\Phi\mapsto A(\Phi)_{n,\lambda}$ are contractions (see (4.44)), $\overline{\Phi}$ is the unique sequence in $\mathcal{X}$ such that

$$A(\overline{\Phi})_{n,\lambda} = \overline{\Phi}_{n,\lambda} \quad \text{for every } n\in\mathbb{Z},\ \lambda\in\Delta.$$

In other words, $\overline{\Phi}$ is the unique solution of equation (4.8), and thus, by Lemma 4.13, also of equation (4.2). Together with (4.1) this implies that if $\xi\in E_n$, then

$$m\mapsto (W^n_{m,\lambda}\xi, \overline{\Phi}_{m,\lambda}W^n_{m,\lambda}\xi)$$

is a solution of the pair of equations (3.5) and (3.6). This means that (3.4) holds. To establish the uniqueness, let Φ be another sequence for which (3.4) holds. If $\xi\in E_n$, then

$$(\xi,\Phi_{n,\lambda}\xi)\in E^\lambda_n \quad\text{and}\quad \mathcal{A}_\lambda(m,n)(\xi,\Phi_{n,\lambda}\xi)\in E^\lambda_m.$$

Therefore, if (x_m,y_m) is the solution of equation (2.3) with $x_n=\xi$ and $y_n = \Phi_{n,\lambda}\xi$, then $y_m = \Phi_{m,\lambda}x_m$ for $m\ge n$. This shows that (3.5) and (3.6) hold. We note that the sequence $x_m = W^n_{m,\lambda}\xi$ satisfies (4.1) and that (3.7) holds. Hence, $\Phi=\overline{\Phi}$. It remains to obtain the last statement in the theorem, that is, the analytic dependence of the maps $\lambda\mapsto\Phi_{n,\lambda}$. We first establish an auxiliary statement. We recall that $A(\Phi)_{n,\lambda}$ and $B(\Phi,U)_{n,\lambda}$ are given respectively by (4.33) and (4.36).

Lemma 4.17. *Given $\Phi \in \mathcal{X}$, if $\lambda \mapsto \Phi_{n,\lambda}$ is analytic and $U_{n,\lambda} = d\Phi_{n,\lambda}/d\lambda$ for each $n \in \mathbb{Z}$, then $\lambda \mapsto A(\Phi)_{n,\lambda}$ is analytic for every $n \in \mathbb{Z}$ and*

$$\frac{d}{d\lambda} A(\Phi)_{n,\lambda} = B(\Phi, U)_{n,\lambda} \tag{4.49}$$

for every $n \in \mathbb{Z}$ and $\lambda \in \Delta$.

Proof. If the map $\lambda \mapsto \Phi_{n,\lambda}$ is analytic and $U_{n,\lambda} = d\Phi_{n,\lambda}/d\lambda$ for each $n \in \mathbb{Z}$, then clearly the linear operators $W^n_{m,\lambda}$ and $Z^n_{m,\lambda}$ in (4.1) and (4.37) satisfy

$$Z^n_{m,\lambda} = \frac{d}{d\lambda} W^n_{m,\lambda}$$

for each $m \geq n$ and $\lambda \in \Delta$. This implies that

$$B(\Phi, U)_{n,\lambda} = -\sum_{l=n}^{\infty} \frac{\partial}{\partial\lambda}[Q_n \mathcal{A}(l+1,n)^{-1} B_l(\lambda)(W^n_{l,\lambda} + \Phi_{l,\lambda} W^n_{l,\lambda})] \tag{4.50}$$

for every $n \in \mathbb{Z}$ and $\lambda \in \Delta$. Now we observe that by (4.7) and (4.41) (see also (4.38)) the series defining $A(\Phi)_{n,\lambda}$ and $B(\Phi, U)_{n,\lambda}$ converge uniformly in λ. This allows one to interchange the series with the derivatives in (4.50) to obtain (4.49). □

Now we consider the pair $(\Phi^1, U^1) = (0, 0) \in \mathcal{X} \times \mathcal{F}$. Clearly,

$$U^1_{n,\lambda} = \frac{d}{d\lambda} \Phi^1_{n,\lambda} \quad \text{for every } n \in \mathbb{Z} \text{ and } \lambda \in \Delta.$$

We define recursively a sequence $(\Phi^m, U^m) \in \mathcal{X} \times \mathcal{F}$ by

$$(\Phi^{m+1}, U^{m+1}) = S(\Phi^m, U^m) = (A(\Phi^m), B(\Phi^m, U^m)).$$

Given $m \in \mathbb{N}$, if $\lambda \mapsto \Phi^m_{n,\lambda}$ is analytic for each $n \in \mathbb{Z}$, and

$$U^m_{n,\lambda} = \frac{d}{d\lambda} \Phi^m_{n,\lambda}$$

for every $n \in \mathbb{Z}$ and $\lambda \in \Delta$, then it follows from Lemma 4.17 that $\lambda \mapsto \Phi^{m+1}_{n,\lambda}$ is analytic for each $n \in J$, and that

$$\frac{d}{d\lambda} \Phi^{m+1}_{n,\lambda} = \frac{d}{d\lambda} A(\Phi^m)_{n,\lambda} = B(\Phi^m, U^m)_{n,\lambda} = U^{m+1}_{n,\lambda} \tag{4.51}$$

for every $n \in \mathbb{Z}$ and $\lambda \in \Delta$. Furthermore, if $(\overline{\Phi}, \overline{U})$ is the unique fixed point of the contraction map S in $\mathcal{X} \times \mathcal{F}$, then for each $n \in \mathbb{Z}$ and $\lambda \in \Delta$ the sequences $(\Phi^m_{n,\lambda})_{m\in\mathbb{N}}$ and $(U^m_{n,\lambda})_{m\in\mathbb{N}}$ converge uniformly respectively to $\overline{\Phi}_{n,\lambda}$ and $\overline{U}_{n,\lambda}$. Now we observe that if a sequence f_m of analytic functions converges uniformly, and the sequence of its derivatives f'_m also converges uniformly, then the limit of f_m is analytic, and its derivative is the limit of f'_m. Thus, it follows from (4.51) that each function $\lambda \mapsto \overline{\Phi}_{n,\lambda}$ is analytic, and that

$$\frac{d}{d\lambda}\,\overline{\Phi}_{n,\lambda} = \overline{U}_{n,\lambda} \quad \text{for every } n \in \mathbb{Z} \text{ and } \lambda \in \Delta.$$

The analytic dependence of the spaces F_n^λ on λ can be obtained in a similar manner. This completes the proof of the theorem.

References

[1] L. Barreira and Ya. Pesin, *Nonuniform Hyperbolicity*, Encyclopedia of Mathematics and Its Applications, vol. 115, Cambridge University Press, 2007.

[2] L. Barreira and C. Valls, *Stability of Nonautonomous Differential Equations*, Lect. Notes in Math., vol. 1926, Springer, 2008.

[3] ______, *Smooth robustness of parameterized perturbations of exponential dichotomies*, J. Differential Equations **249** (2010), 2021–2043.

[4] S.-N. Chow and H. Leiva, *Existence and roughness of the exponential dichotomy for skew-product semiflow in Banach spaces*, J. Differential Equations **120** (1995), 429–477.

[5] W. Coppel, *Dichotomies and reducibility*, J. Differential Equations **3** (1967), 500–521.

[6] Ju. Dalec'kiĭ and M. Kreĭn, *Stability of Solutions of Differential Equations in Banach Space*, Translations of Mathematical Monographs, vol. 43, Amer. Math. Soc., 1974.

[7] J. Massera and J. Schäffer, *Linear differential equations and functional analysis* I, Ann. of Math. (2) **67** (1958), 517–573.

[8] ______, *Linear Differential Equations and Function Spaces*, Pure and Applied Mathematics, vol. 21, Academic Press, 1966.

[9] R. Naulin and M. Pinto, *Admissible perturbations of exponential dichotomy roughness*, Nonlinear Anal. **31** (1998), 559–571.

[10] O. Perron, *Die Stabilitätsfrage bei Differentialgleichungen*, Math. Z. **32** (1930), 703–728.

[11] V. Pliss and G. Sell, *Robustness of exponential dichotomies ininfinite-dimensional dynamical systems*, J. Dynam. Differential Equations **11** (1999), 471–513.

[12] L. Popescu, *Exponential dichotomy roughness on Banach spaces*, J. Math. Anal. Appl. **314** (2006), 436–454.

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