**T**opological **M**ethods in **N**onlinear **A**nalysis
Volume 41, No. 2, 2013, 229–253



# RATE OF CONVERGENCE OF GLOBAL ATTRACTORS OF SOME PERTURBED REACTION-DIFFUSION PROBLEMS

José M. Arrieta — Flank D.M. Bezerra — Alexandre N. Carvalho

Abstract. In this paper we treat the problem of the rate of convergence of attractors of dynamical systems for some autonomous semilinear parabolic problems. We consider a prototype problem, where the diffusion $a_0(\,\cdot\,)$ of a reaction-diffusion equation in a bounded domain $\Omega$ is perturbed to $a_\varepsilon(\,\cdot\,)$. We show that the equilibria and the local unstable manifolds of the perturbed problem are at a distance given by the order of $\|a_\varepsilon - a_0\|_\infty$. Moreover, the perturbed nonlinear semigroups are at a distance $\|a_\varepsilon - a_0\|_\infty^\theta$ with $\theta < 1$ but arbitrarily close to 1. Nevertheless, we can only prove that the distance of attractors is of order $\|a_\varepsilon - a_0\|_\infty^\beta$ for some $\beta < 1$, which depends on some other parameters of the problem and may be significantly smaller than 1. We also show how this technique can be applied to other more complicated problems.

## 1. Introduction

As a sequel of the studies carried out for last forty years on attractors for dissipative dynamical systems in infinite dimensional spaces, we investigate the

2010 *Mathematics Subject Classification.* 35B41, 37L30, 37L45.

*Key words and phrases.* Attractors, continuity, uniform exponential attraction, rate of convergence.

Research of the first named author partially supported by grants MTM2009-07549 and MTM2012-31298, MEC; GR35/10-A Grupo 920894, UCM-BSCH; and PHB2006-0003-PC, MEC, Spain.

Research of the second named author partially supported by grants CAPES BEX-1356/08-9 and FAPESP 06/57231-0, Brazil.

Research of the third named author partially supported by grants CNPq 302022/2008-2, CAPES/DGU 267/2008 and FAPESP 2008/55516-3, Brazil.

rate of convergence of the attractors of some gradient problems under (singular) perturbation. This is done using the work of [10], [11] on regular attractor or its extensions in [14]. Our aim is to obtain the rate of convergence of attractors in terms of the rate of convergence of the semigroups and the later in terms of the parameters in the corresponding models.

Consider the prototype semilinear parabolic problems of the form

$$\begin{cases} u^\varepsilon_t - \operatorname{div}(a_\varepsilon(x)\nabla u^\varepsilon) = f(u^\varepsilon), & x \in \Omega, \quad t > 0, \\ u^\varepsilon(t,x) = 0, & x \in \partial\Omega, \ t > 0, \\ u^\varepsilon(0,x) = u^\varepsilon_0(x), \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded smooth domain, $\varepsilon \in [0,1]$ is a parameter and $f\colon \mathbb{R} \to \mathbb{R}$ is continuously differentiable and is a dissipative nonlinearity; that is

$$\limsup_{|s|\to+\infty} \frac{f(s)}{s} \leq 0. \tag{1.2}$$

The parameter $\varepsilon$ represents the fact that, as $\varepsilon$ goes to zero, the diffusivity $a_\varepsilon$ converges to $a_0$ uniformly in $\Omega$. With this, the difference $\|a_\varepsilon - a_0\|_\infty$ will be our measure for the study of proximity between the perturbed and limit attractor.

The analysis will be carried out for the model problem (1.1) but it applies to many other (singular) perturbation problems as seen in Section 8. Such type of problems have been considered in [20], where the author studies the linear theory, in [8], where the authors study the upper semicontinuity of attractors, and in [13], where the lower semicontinuity of attractors is proved. In none of these cases the authors are concerned with the rate at which the attractors approach one another. This is the aim of our work.

In appropriate functional spaces, we will see that problem (1.1) can be written as

$$\begin{cases} u^\varepsilon_t + A_\varepsilon u^\varepsilon = f_\varepsilon(u^\varepsilon), \quad t > 0, \\ u^\varepsilon(0) = u^\varepsilon_0, \end{cases} \tag{1.3}$$

where $f_\varepsilon$ also denotes the Nemyt'skiĭ operator associated to $f$, $0 \leq \varepsilon \leq 1$.

Definition 1.1. The equilibrium solutions of (1.3) are the solutions which are independent of time; that is, the solutions of the elliptic problems

$$A_\varepsilon u^\varepsilon - f_\varepsilon(u^\varepsilon) = 0, \quad \varepsilon \in [0,1]. \tag{1.4}$$

Denote by $\mathcal{E}_\varepsilon$ the set of solutions to (1.4), $\varepsilon \in [0,1]$.

Definition 1.2. We say that an equilibrium $u_*^\varepsilon$ of (1.3) is hyperbolic if the spectrum $\sigma(A_\varepsilon - f_\varepsilon'(u_*^\varepsilon))$ of $A_\varepsilon - f_\varepsilon'(u_*^\varepsilon)$ is disjoint from the imaginary axis.

Now we are prepared to describe the program that we will follow to prove the continuity with rate of the attractors. It is divided in seven parts as follows:

(1) First we study the rate of convergence of $A_\varepsilon^{-1}$ to $A_0^{-1}$. This step defines the parameter that will be used for the study of the rate of convergence of the the nonlinear dynamics of (1.3);
(2) We use the information obtained in (1) to study the rate of convergence of the resolvent operators $(\lambda + A_\varepsilon)^{-1}$ to $(\lambda + A_0)^{-1}$ (in some sector). We also show the rate of convergence of the resolvent for operators of the form $\lambda + A_\varepsilon + V_0$ and of the form $\lambda + A_\varepsilon + V_\varepsilon$, where $V_\varepsilon, V_0$ are potentials and $V_\varepsilon$ converges to $V_0$ with rate equal to the one obtained for the convergence of $A_\varepsilon^{-1}$ to $A_0^{-1}$;
(3) With the convergence of the resolvent operators we will prove the rate of convergence of the equilibrium points. Writing the stationary problems as a fixed point problem; that is, $u_\varepsilon$ is an equilibrium for (1.3) if and only if $u_\varepsilon = A_\varepsilon^{-1} f_\varepsilon(u_\varepsilon)$ (respectively, $u_0 = A_0^{-1} f_0(u_0)$) we obtain the convergence of equilibria from the convergence of resolvents;
(4) From the convergence of resolvent operators $(\lambda + A_\varepsilon)^{-1}$ to $(\lambda + A_0)^{-1}$ with rate, we show the convergence of the linear semigroups $e^{-A_\varepsilon t}$ to $e^{-A_0 t}$ with rate. Using the *variation of constants formula* we show the convergence of nonlinear semigroups with rate;
(5) We show the rate of convergence of equilibria assuming hyperbolicity of the limiting equilibria;
(6) Using that the local unstable manifold is given as a graph we show the rate of convergence of the local unstable manifolds;
(7) Using the results in [10], [17], we obtain uniform exponential attraction and rate of convergence for the attractors.

Variants of this agenda have been proved to be successful when addressing the continuity of attractors in different examples of singularly perturbed problems. Here, there are some important additions to consider the continuity with rate at each step. One very important step is the continuity with rate of equilibria which will lead to the continuity with rate of the resolvents of the linearized operators, linearized semigroups and local unstable manifolds.

We remark that the rate of convergence (and attraction) of local unstable manifolds is a lot better than the rate of convergence (and attraction) of the global attractors. An important open question is to establish when it is possible to obtain rates of convergence (attraction) for the global attractor which are the same as the rate of convergence (attraction) of the local unstable manifolds, see [9] for some results in this direction.

This paper is organized as follows. In Section 2 we state the main result of the paper; in Section 3 we study the convergence of the operators $A_\varepsilon^{-1}$ to $A_0^{-1}$; in Section 4 we study the convergence or equilibria; in Section 5 we study some important properties of the Nemytskiĭ operators $f^e\colon H_0^1(\Omega) \to L^2(\Omega)$ involved and the convergence of linearizations of the operators $A_\varepsilon + (f^e)'(u_\varepsilon)$ when $u_\varepsilon$ converges to $u_0$ in $H_0^1(\Omega)$; in Section 6 we study the rate of convergence of equilibria and, finally, in Section 7 we study the rate of convergence and attraction of local unstable manifolds of equilibria.

## 2. Statement of the results

Let us consider (1.1) and assume that $a_\varepsilon\colon \Omega \subset \mathbb{R}^N \to \mathbb{R}$ is a bounded function in $\Omega$, satisfying $0 < m_0 \leq a_\varepsilon(x) \leq M_0$, for all $x \in \Omega$ and $0 < \varepsilon \leq \varepsilon_0$. Define the operator $A_\varepsilon\colon \mathcal{D}(A_\varepsilon) \subset L^2(\Omega) \to L^2(\Omega)$ by

$$\begin{aligned} &A_\varepsilon u := -\operatorname{div}(a_\varepsilon(x)\nabla u), \quad u \in \mathcal{D}(A_\varepsilon) \\ &\qquad \text{where } \mathcal{D}(A_\varepsilon) := \{u \in H^1(\Omega) : -\operatorname{div}(a_\varepsilon(x)\nabla u) \in L^2(\Omega),\ u = 0 \text{ on } \partial\Omega\}. \end{aligned}$$

It is well known that $A_\varepsilon$ is positive selfadjoint operator with compact resolvent, $\varepsilon \in [0, \varepsilon_0]$. Hence, we can define the fractional power spaces $X_\varepsilon^\alpha$ associated with the operators $A_\varepsilon$, $\varepsilon \in [0, \varepsilon_0]$ and $0 \leq \alpha \leq 1$, where $X_\varepsilon^0 = L^2(\Omega)$, $X_\varepsilon^1 = \mathcal{D}(A_\varepsilon)$ and $X_\varepsilon^{1/2} = H_0^1(\Omega)$ with the inner product

$$\langle \phi, \psi \rangle_{H_0^1(\Omega)} := \int_\Omega a_\varepsilon(x) \nabla\phi \nabla\psi \, dx.$$

From the bounds of $a_\varepsilon$ stated above, the norm associated to all of these inner products are all uniformly equivalent to the standard $H_0^1(\Omega)$ norm

$$\|u\|_{H_0^1(\Omega)} := \bigg( \int_\Omega |\nabla u|^2 \, dx \bigg)^{1/2}$$

which is the one we will be using. Also note that if $a_\varepsilon$ is smooth then $X_\varepsilon^1 = \mathcal{D}(A_\varepsilon) = H^2(\Omega) \cap H_0^1(\Omega)$, although for general $a_\varepsilon \in L^\infty(\Omega)$ this characterization of $\mathcal{D}(A_\varepsilon)$ does not hold.

Denote by $\{e^{-A_\varepsilon t} : t \geq 0\}$ the analytic semigroup generated by $-A_\varepsilon$, $0 \leq \varepsilon \leq 1$. The following holds:

$$\begin{aligned} \|e^{-A_\varepsilon t}\|_{\mathcal{L}(X_\varepsilon^\alpha, X_\varepsilon^\beta)} &\leq C\, t^{\alpha-\beta} e^{-\omega t}, && t > 0, \\ \|(\lambda + A_\varepsilon)^{-1}\|_{\mathcal{L}(L^2(\Omega))} &\leq \frac{C}{1+|\lambda|}, && \lambda \in \Sigma_\phi, \\ \|(\lambda + A_\varepsilon)^{-1}\|_{\mathcal{L}(H_0^1(\Omega))} &\leq \frac{C}{1+|\lambda|}, && \lambda \in \Sigma_\phi, \end{aligned} \tag{2.1}$$

for constants $C \geq 1$ and $\omega > 0$ independent of $\varepsilon$.

Now we give conditions under which the problem (1.3) is locally well posed. To that end we must impose some growth restrictions on $f$. In fact, we assume that, if $N = 2$, for every $\eta > 0$ there is $c_\eta > 0$ such that

$$|f(u) - f(v)| \leq c_\eta \Big( e^{\eta|u|^2} + e^{\eta|v|^2} \Big) |u - v|, \quad \text{for all } u, v \in \mathbb{R}$$

and if $N \geq 3$, there is a $\rho < 4/(N-2)$ and a constant $c = c(\rho) > 0$ such that

$$|f(u) - f(v)| \leq c|u - v|(|u|^\rho + |v|^\rho + 1), \quad \text{for all } u, v \in \mathbb{R}.$$

We remark that, if the coefficient $a_\varepsilon$ is smooth $\rho$ can be taken equal to $4/(N-2)$.

Under these assumptions problem (1.3), $\varepsilon \in [0, \varepsilon_0]$ is locally well posed in $H^1_0(\Omega)$ (see [1], [6]). Moreover, under standard dissipative conditions like (1.2), we have that solutions are globally defined. That is, for any $u^\varepsilon_0 \in H^1_0(\Omega)$ and $\varepsilon \in [0, \varepsilon_0]$, there is a unique $u^\varepsilon(\,\cdot\,, u^\varepsilon_0) \in C([0,\infty), H^1_0(\Omega)) \cap C^1((0,\infty), H^1_0(\Omega))$ with $u^\varepsilon(t, u^\varepsilon_0) \in D(A_\varepsilon)$ for all $t > 0$ which satisfies (1.3) (see [1]) and

$$u(t, u^\varepsilon_0) = e^{-A_\varepsilon t} u^\varepsilon_0 + \int_0^t e^{-A_\varepsilon (t-s)} f^e(u(s, u^\varepsilon_0))\, ds, \quad t \geq 0.$$

Therefore, we can define in $H^1_0(\Omega)$ the semigroup $\{T_\varepsilon(t) : t \geq 0\}$ associated with (1.3) by $T_\varepsilon(t) u^\varepsilon_0 = u^\varepsilon(t, u^\varepsilon_0)$, $t \geq 0$. To simplify the notation we will denote the solution $u^0(t, u^0_0)$ by $u(t, u_0)$.

The existence of attractors and uniform bounds for semigroups $\{T_\varepsilon(t) : t \geq 0\}$ associated with (1.3), $\varepsilon \in [0, \varepsilon_0]$, are also established in [7]. In fact we have

Theorem 2.1. *The semigroup* $\{T_\varepsilon(t) : t \geq 0\}$ *associated with* (1.3)*,* $\varepsilon \in [0, \varepsilon_0]$*, has a global attractor* $\mathcal{A}_\varepsilon$ *in* $H^1_0(\Omega)$*. Furthermore*

$$\sup_{\varepsilon \in [0,\varepsilon_0]} \sup_{w \in \mathcal{A}_\varepsilon} \|w\|_{H^1_0(\Omega)} < \infty \quad \textit{and} \quad \sup_{\varepsilon \in [0,\varepsilon_0]} \sup_{w \in \mathcal{A}_\varepsilon} \|w\|_{L^\infty(\Omega)} < \infty.$$

Once the uniform bound in $L^\infty(\Omega)$ for the attractors has been obtained, we may perform a cutoff to the nonlinerity so that the new nonlinearity is globally Lipschitz and globally bounded with bounded derivatives up to second order (in particular the dissipativity condition (1.2) also holds for the new nonlinearity), it coincides with the original one in a $L^\infty$-neighbourhood of all the attractors and is strictly dissipative outside this neighbourhood. This guarantees that the system with the new nonlinearities have attractors which coincide exactly with the original ones. With this in mind we can state one of the main results in this paper.

Theorem 2.2. *Let* $\{T_\varepsilon(t) : t \geq 0\}$ *be the gradient nonlinear semigroup associated with* (1.3) *and let* $\mathcal{A}_\varepsilon \subset H^1_0(\Omega)$ *be its global attractor,* $\varepsilon \in [0,1]$. *Then, there is a* $\varepsilon_0 > 0$ *such that:*

(i) *For each* $0 < \theta < 1/2$, *there are constants* $L > 0$ *and* $c > 0$ *such that*

$$\|T_\varepsilon(t)u^\varepsilon - T_0(t)u\|_{H^1_0(\Omega)} \leq ce^{Lt}t^{-1/2-2\theta}(\|u^\varepsilon - u\|_{H^1_0(\Omega)} + \|a_\varepsilon - a_0\|^{2\theta}_\infty), \tag{2.2}$$

*for all* $t > 0$, $\varepsilon \in [0,\varepsilon_0]$.

(ii) *If all equilibrium points* $\mathcal{E}_0 = \{u^{1,0}_*, \ldots, u^{\mathbf{n},0}_*\}$ *of* (1.3) *with* $\varepsilon = 0$ *are hyperbolic* (*hence there are only a finitely many of them*), *the semigroup* $\{T_\varepsilon(t) : t \geq 0\}$ *has a set of exactly* $n$ *equilibria,* $\mathcal{E}_\varepsilon = \{u^{1,\varepsilon}_*, \ldots, u^{\mathbf{n},\varepsilon}_*\}$, *all of them hyperbolic, satisfying:*

$$\|u^{i,\varepsilon}_* - u^i_*\|_{H^1_0(\Omega)} \leq C\|a_\varepsilon - a_0\|_\infty, \quad 1 \leq i \leq n,$$

(iii) *There is a* $\delta_0 > 0$ *such that, if*

$$W^u_{\delta_0}(u^{i,\varepsilon}_*) = \{w \in W^u(u^{i,\varepsilon}_*) : \|w - u^{i,\varepsilon}_*\|_{H^1_0(\Omega)} < \delta_0\}$$

(*here* $W^u(u^{i,\varepsilon}_*)$ *denotes the unstable manifold of the equilibrium* $u^{i,\varepsilon}_*$), *there is a* $C_\theta > 0$ *such that for all* $i = 1, \ldots, n$,

$$\mathrm{dist}(W^u_{\delta_0}(u^{i,\varepsilon}_*), W^u_{\delta_0}(u^{i,0}_*)) + \mathrm{dist}(W^u_{\delta_0}(u^{i,0}_*), W^u_{\delta_0}(u^{i,\varepsilon}_*)) \leq C_\theta\|a_\varepsilon - a_0\|^{2\theta}_\infty.$$

(iv) *There is a* $\delta_1 > 0$, *small enough, such that for each* $i = 1, \ldots, n$ *and each* $w \in H^1_0(\Omega)$ *with* $\|w - u^{i,\varepsilon}_*\|_{H^1_0(\Omega)} \leq \delta_1$ *we have*

$$\mathrm{dist}(T_\varepsilon(t)w, W^u_{\delta_1}(u^{i,\varepsilon}_*)) \leq Me^{-\varrho_1(t-t_0)}\mathrm{dist}(T_\varepsilon(t_0)(w), W^u_{\delta_1}(u^{i,\varepsilon}_*))$$

*as long as* $\|T_\varepsilon(t)(w) - u^{i,\varepsilon}_*\|_{H^1_0(\Omega)} \leq \delta_1$, *where* $\varrho_1 > 0$ *and* $M \leq 1$ *are independent of* $\varepsilon$.

*In addition, there is a* $\varrho \in (0, \varrho_1]$ (*in general strictly smaller than* $\varrho_1$) *such that*

(a) *Given* $B \subset H^1_0(\Omega)$ *bounded, there is* $c = c(B) > 0$ *such that*

$$\mathrm{dist}(T_\varepsilon(t)B_0, \mathcal{A}_\varepsilon) \leq c\,e^{-\varrho t}.$$

(b) *There is* $c > 0$ *such that*

$$\mathrm{dist}(\mathcal{A}_\varepsilon, \mathcal{A}_0) + \mathrm{dist}(\mathcal{A}_0, \mathcal{A}_\varepsilon) \leq c\,\|a_\varepsilon - a_0\|^{2\theta\varrho/(\varrho+L)}_\infty$$

*where* $\mathrm{dist}(A,B) := \sup_{x\in A}\inf_{y\in B}\|x-y\|_{H^1_0(\Omega)}$ *is the Hausdorff semi-distance.*

Part (i) of Theorem 2.2 is proved in Theorem 5.2, part (ii) is proved in Theorem 6.3, parts (iii) and (iv) are proved in Theorem 7.1. The remaining parts (a) and (b) follow from part (i) and (iv) and from the results in and [10], [11], [14]. The constant $\varrho$ can be any positive number smaller than the minimum, over all equilibria, of the constant $\beta$ in Theorem 7.1 and $L$ is given in Theorem 2.2.

## 3. Resolvent convergence

In this section we show the convergence of the resolvent operators $A_\varepsilon^{-1}$ to $A_0^{-1}$ and we establish that the rate of this convergence is $\|a_\varepsilon - a_0\|_\infty$.

Lemma 3.1. *For $f \in L^2(\Omega)$ and $\varepsilon \in [0, \varepsilon_0]$, let $u^\varepsilon$ solution of the problem:*

$$\begin{cases} -\mathrm{div}(a_\varepsilon(x)\nabla u) = f, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{3.1}$$

*Then, there is a constant $C$, independent of $\varepsilon$, such that $\|u^\varepsilon\|_{H_0^1(\Omega)} \le C\|f\|_{L^2(\Omega)}$,*

$$\|u^\varepsilon - u\|_{H_0^1(\Omega)} \le C\|f\|_{L^2(\Omega)} \cdot \|a_\varepsilon - a_0\|_\infty. \tag{3.2}$$

Proof. Both conclusions of the lemma are easily obtained from the weak formulation of (3.1). Recall that, $u_\varepsilon \in H_0^1(\Omega)$ is weak solution for the problem (3.1) if and only if it satisfies the following identity:

$$\int_\Omega a_\varepsilon \nabla u_\varepsilon \nabla \phi \, dx = \int_\Omega f\phi \, dx, \quad \text{for all } \phi \in H_0^1(\Omega). \tag{3.3}$$

The function $\phi$ in (3.3) is called a test function.

The proof of the estimate $\|u^\varepsilon\|_{H_0^1(\Omega)} \le C\|f\|_{L^2(\Omega)}$ follows easily using $u_\varepsilon$ as test function in the weak formulation (3.3) of (3.1), noticing that $0 < m_0 \le a_\varepsilon(x)$, for all $x \in \Omega$ and using Poincaré's inequality.

Now we prove (3.2). Using $u_\varepsilon - u_0 \in H_0^1(\Omega)$ twice as test functions in (3.1), we obtain that

$$\begin{aligned} \int_\Omega a_\varepsilon \nabla u_\varepsilon (\nabla u_\varepsilon - \nabla u_0)\, dx &= \int_\Omega f(u_\varepsilon - u_0)\, dx, \\ \int_\Omega a_0 \nabla u_0 (\nabla u_\varepsilon - \nabla u_0)\, dx &= \int_\Omega f(u_\varepsilon - u_0)\, dx. \end{aligned}$$

Subtracting these expressions we arrive at

$$\int_\Omega a_\varepsilon \nabla u_\varepsilon (\nabla u_\varepsilon - \nabla u_0)\, dx = \int_\Omega a_0 \nabla u_0 (\nabla u_\varepsilon - \nabla u_0)\, dx$$

which can be rewritten as

$$\int_\Omega a_\varepsilon |\nabla u_\varepsilon - \nabla u_0|^2 \, dx = \int_\Omega (a_0 - a_\varepsilon)\nabla u_0 (\nabla u_\varepsilon - \nabla u_0)\, dx. \tag{3.4}$$

From (3.4), it follows that

$$\|\nabla u_\varepsilon - \nabla u_0\|^2_{L^2(\Omega)} \le m_0^{-1}\|a_0 - a_\varepsilon\|_{L^\infty(\Omega)}\|\nabla u_0\|_{L^2(\Omega)}\|\nabla u_\varepsilon - \nabla u_0\|_{L^2(\Omega)}, \tag{3.5}$$

where we have used that $0 < m_0 \le a_\varepsilon(x)$ for all $x \in \Omega$. With the aid of Poincaré's inequality we obtain from (3.5) that

$$\|u_\varepsilon - u_0\|^2_{H^1(\Omega)} \le C\|a_0 - a_\varepsilon\|_{L^\infty(\Omega)}\|\nabla u_0\|_{L^2(\Omega)}\|u_\varepsilon - u_0\|_{H^1(\Omega)}.$$

Using $\|\nabla u_0\|_{L^2(\Omega)} \le C\|f\|_{L^2(\Omega)}$ we have that

$$\|u_\varepsilon - u_0\|_{H^1(\Omega)} \le C\|a_0 - a_\varepsilon\|_{L^\infty(\Omega)}\|f\|_{L^2(\Omega)}$$

and the result is proved. □

As an immediate corollary of the previous result, we can prove,

Corollary 3.2. *The operators $A_\varepsilon^{-1}: L^2(\Omega) \to H_0^1(\Omega)$ are uniformly bounded, converge in the uniform operator topology to $A_0^{-1}: L^2(\Omega) \to H_0^1(\Omega)$ and, for $0 \le \varepsilon \le \varepsilon_0$,*

$$\|A_\varepsilon^{-1}\|_{\mathcal{L}(L^2(\Omega),H_0^1(\Omega))} \le C, \tag{3.6}$$

$$\|A_\varepsilon^{-1} - A_0^{-1}\|_{\mathcal{L}(L^2(\Omega),H_0^1(\Omega))} \le C\|a_\varepsilon - a_0\|_{L^\infty(\Omega)} \tag{3.7}$$

*for a constant $C$ independent of $\varepsilon$.*

The uniform convergence of the operators imply the convergence of their spectra (that is, eigenvalues and eigenfunctions). As a matter of fact, the following result holds (see [19] and also [16]),

Proposition 3.3. *The following statements hold:*

(a) *If $\mu_0 \in \sigma(-A_0)$, then exists a sequence $\varepsilon_n \to 0$ and $\{\mu_n\}$, with $\mu_n \in \sigma(-A_{\varepsilon_n})$, $n \in \mathbb{N}$ such that $\mu_n \to \mu_0$ as $n \to \infty$;*

(b) *If for some sequences $\varepsilon_n \to 0$ and $\mu_n \to \mu_0$ as $n \to \infty$, with $\mu_n \in \sigma(-A_{\varepsilon_n})$, $n \in \mathbb{N}$, then $\mu_0 \in \sigma(-A_0)$;*

(c) *If $\gamma$ is a closed rectifiable simple curve with trace $\{\gamma\}$ contained in $\rho(A_0)$, there exists $\varepsilon_\gamma > 0$ such that $\{\gamma\} \subset \rho(A_\varepsilon)$ for all $\varepsilon \in (0, \varepsilon_\gamma)$. For $\mu \notin \{\gamma\}$ define the spectral projection*

$$Q_\varepsilon(\mu) = \frac{1}{2\pi i}\int_\gamma (\lambda + A_\varepsilon)^{-1} d\lambda.$$

*Clearly $Q_\varepsilon(\mu)$ is compact and, consequently,* $\mathrm{rank}(Q_\varepsilon(\mu)) < \infty$. *If $\mu \notin \{\gamma\}$ and $W(\mu, -A_\varepsilon) = Q_\varepsilon(\mu)(L^2(\Omega))$, $0 \le \varepsilon \le \varepsilon_\gamma$, there exists $\varepsilon_\mu \in (0, \varepsilon_\gamma)$ such that* $\dim W(\mu, -A_\varepsilon) = \dim W(\mu, -A_0)$ *for all $0 < \varepsilon \le \varepsilon_\mu$;*

(d) *If $u \in W(\mu_0, -A_0)$, there are sequences $\{\varepsilon_n\}$ with $\varepsilon_n \xrightarrow{n\to\infty} 0$ and $\{u^{\varepsilon_n}\}$ with $u^{\varepsilon_n} \in W(\mu_0, -A_{\varepsilon_n})$ such that $u^{\varepsilon_n} \xrightarrow{n\to\infty} u$;*

(e) *Suppose that $u_n \in W(\mu, -A_{\varepsilon_n})$ and $\|u_n\|_{L^2(\Omega)} = 1$ for each $n \in \mathbb{N}$. If $\varepsilon_n \xrightarrow{n\to\infty} 0$, then $\{u_n\}$ has a convergent subsequence with limit belonging to $W(\mu_0, -A_0)$.*

In particular, since all operators are selfadjoint we have that $\sigma(A_\varepsilon) \subset (-\infty, \alpha]$ for some $\alpha < 0$ and in particular, the set $\Sigma_\phi = \{\lambda \in \mathbb{C} : |\arg \lambda| \le \phi\}$, $\phi \in (\pi/2, \pi)$ is contained in the resolvent set of $A_\varepsilon$ for all $\varepsilon \in [0, \varepsilon_0]$. In particular, we have

Lemma 3.4. *For each $\phi$ with $\pi/2 < \phi < \pi$ we have a constant $C = C(\phi)$ such that*

$$\sup_{\mu\in\Sigma_\phi} \|(\mu + A_\varepsilon)^{-1} - (\mu + A_0)^{-1}\|_{\mathcal{L}(L^2(\Omega),H_0^1(\Omega))} \le C\|a_\varepsilon - a_0\|_\infty.$$

Proof. It is easy to see that

$$(\mu + A_\varepsilon)^{-1} - (\mu + A_0)^{-1} = (I - \mu(\mu + A_\varepsilon)^{-1})(A_\varepsilon^{-1} - A_0^{-1})(I - \mu(\mu + A_0)^{-1}).$$

From this statement, Corollary 3.2 and (2.1), we easily can prove that there is a constant $C > 0$ (independent of $\varepsilon$ and of $\mu \in \Sigma_\phi$) such that

$$\|(\mu + A_\varepsilon)^{-1} - (\mu + A_0)^{-1}\|_{\mathcal{L}(L^2(\Omega),H_0^1(\Omega))} \le C\|a_\varepsilon - a_0\|_\infty. \qquad \square$$

The spectral projections behave continuously as seen in the following result

Proposition 3.5. *The family of operators $Q_\varepsilon(\mu_0)\colon L^2(\Omega) \to L^2(\Omega)$ converges uniformly to $Q_0(\mu_0)\colon L^2(\Omega) \to L^2(\Omega)$ as $\varepsilon \to 0$. Moreover,*

$$\begin{aligned} \|Q_\varepsilon(\mu_0) - Q_0(\mu_0)\|_{\mathcal{L}(L^2(\Omega),H_0^1(\Omega))} &\le C\|a_\varepsilon - a_0\|_\infty, \\ \|A_\varepsilon Q_\varepsilon(\mu_0) - A_0 Q_0(\mu_0)\|_{\mathcal{L}(L^2(\Omega),H_0^1(\Omega))} &\le C\|a_\varepsilon - a_0\|_\infty, \end{aligned}$$

*where $C > 0$ independent of $\varepsilon$.*

Proof. Note that

$$Q_\varepsilon(\mu_0) - Q_0(\mu_0) = \frac{1}{2\pi i}\int_\gamma [(\lambda + A_\varepsilon)^{-1} - (\lambda + A_0)^{-1}]\,d\lambda$$

and using Lemma 3.4, we obtain

$$\|Q_\varepsilon(\mu_0) - Q_0(\mu_0)\|_{\mathcal{L}(L^2(\Omega),H_0^1(\Omega))} \le C\|a_\varepsilon - a_0\|_\infty.$$

which proves the result. $\square$

If $\mu_0$ is an isolated eigenvalue for $A_0$, we may define $Q_\varepsilon(\mu_0)$ as above and it follows from Proposition 3.3 that there is a $\mu_\varepsilon$ which is an eigenvalue of $A_\varepsilon$ such that $\mu_\varepsilon \xrightarrow{\varepsilon\to 0} \mu_0$. Hence $Q_\varepsilon(\mu_0) = Q_\varepsilon(\mu_\varepsilon)$.

From the convergence results above, it is easy to see that $\|Q_\varepsilon(\mu_\varepsilon)Q_0(\mu_0) - Q_0(\mu_0)\| \le C\|a_\varepsilon - a_0\|_\infty$ and that $Q_\varepsilon(\mu_\varepsilon)Q_0(\mu_0)$ is an isomorphism between $R(Q_0(\mu_0))$ and $R(Q_\varepsilon(\mu_\varepsilon))$.

Corollary 3.6. *If $\lambda_\varepsilon$ is an eigenvalue of $A_\varepsilon$, $0 \le \varepsilon \le \varepsilon_0$ and $N(\lambda_0 - A) = R(Q_0(\lambda_0))$, then, for some $C > 0$,*

$$|\lambda_\varepsilon - \lambda_0| \le C\|a_\varepsilon - a_0\|_\infty.$$

Proof. Using the comments preceding this corollary, the Proposition 3.5 and the above estimate, we have that, for each $\varepsilon$ suitably small, there exists

$u_\varepsilon \in R(Q_0)$, $\|u_\varepsilon\| = 1$ such that $Q_\varepsilon u_\varepsilon$ is an eigenvector of $A_\varepsilon$ associated to $\lambda_\varepsilon$ and

$$|\lambda_\varepsilon - \lambda_0| \leq \|\lambda_\varepsilon Q_0 u_\varepsilon - \lambda_\varepsilon Q_\varepsilon u_\varepsilon\| + \|\lambda_\varepsilon Q_\varepsilon u_\varepsilon - \lambda_0 Q_0 u_\varepsilon\|_{H^1_0(\Omega)} \leq C\|a_\varepsilon - a_0\|_\infty. \qquad \square$$

## 4. Rate of convergence of resolvents of linearized operators

In this section we obtain the rate of convergence of the resolvents of operators which corresponds to linearizations of (1.3) arround equlibria. The next theorem will be used for the convergence of equilibria of the problem (1.3) when $\varepsilon \to 0$. For this return to the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ assume that $f$ is $C^2$ bounded with bounded derivatives up to second order.

Lemma 4.1. *Let $u_* \in H^1_0(\Omega)$, $\theta\colon \Omega \to [0,1]$ a measurable function and $\delta > 0$, then for any $u, v \in H^1_0(\Omega)$ with $\|u - u_*\|_{H^1_0(\Omega)} < \delta$ and $\|u - u_*\|_{H^1_0(\Omega)} < \delta$ we have:*

$$\Omega \ni x \xmapsto{\gamma_\delta} \gamma_\delta(x) := f'((1-\theta(x))u(x) - \theta(x)v(x)) - f'(u_*(x)) \in \mathbb{R}$$

*verifies the properties: $\gamma_\delta \in L^\infty(\Omega)$, $\gamma_\delta \in L^2(\Omega)$ with $\lim_{\delta\to 0} \|\gamma_\delta\|_{L^2(\Omega)} = 0$, and therefore, $\gamma_\delta \in L^p(\Omega)$ with $\lim_{\delta\to 0}\|\gamma_\delta\|_{L^p(\Omega)} = 0$ for $p \in [1,\infty)$. Note that $\gamma_\delta$ also depends on the functions $u$ and $v$.*

Proof. The properties of $f$ ensure that $\gamma_\delta \in L^\infty(\Omega)$ with $\|\gamma_\delta\|_\infty \leq C$, where $C > 0$, independent of $\varepsilon$, and therefore, $\gamma_\delta \in L^p(\Omega)$ for all $p \in [1,\infty)$. Left us shown that $\lim_{\delta\to 0}\|\gamma_\delta\|_{L^2(\Omega)} = 0$.

For each $x \in \Omega$, there is $0 \leq \rho(x) \leq 1$ such that

$$\begin{aligned}
\|\gamma_\delta\|^2_{L^2(\Omega)} &= \int_\Omega |f'((1-\theta(x))u(x) - \theta(x)v(x)) - f'(u_*(x))|^2\,dx \\
&\leq L^2_{f'}\int_\Omega |(1-\theta(x))u(x) - \theta(x)v(x) - u_*(x)|^2\,dx \leq 2L^2_{f'}\delta^2,
\end{aligned}$$

and therefore, $\lim_{\delta\to 0}\|\gamma_\delta\|_{L^2(\Omega)} = 0$. The rest of the proof follows by Hölder's inequality. $\square$

Lemma 4.2. *Denoting by $f^e\colon H^1_0(\Omega) \to L^2(\Omega)$ the Nemyt'skiĭ map associated to $f$; that is, $f^e(u)(x) = f(u(x))$, $x \in \Omega$, then $f^e$ is Fréchet continuously differentiable. Moreover, if $u^\varepsilon \xrightarrow{H^1_0(\Omega)} u^0$ and $0 \notin \sigma(A_0 - (f^e)'(u^0))$, then $(f^e)'(u^\varepsilon) \circ A_\varepsilon^{-1}$ converges to $(f^e)'(u^0) \circ A_0^{-1}$ in the uniform operator topology of $\mathcal{L}(L^2(\Omega))$.*

Proof. Using that $H^1_0(\Omega) \hookrightarrow L^{2n/(n-2)}(\Omega)$, $n \geq 3$, and Hölder's inequality with exponents $n/2$ and $n/(n-2)$, for each $x \in \Omega$, there is $0 < \theta(x) < 1$, such that

$$\begin{aligned}\|f^e(u) - f^e(v) \ - (f^e)'(u)(u-v)\|^2_{L^2(\Omega)} \\ &\leq \bigg(\int_\Omega |\gamma_{u,v}(x)|^n\,dx\bigg)^{2/n}\bigg(\int_\Omega |u(x)-v(x)|^{2n/(n-2)}\,dx\bigg)^{(n-2)/n} \\ &\leq \|\gamma_{u,v}\|^2_{L^n(\Omega)}\|u-v\|_{L^{2n/(n-2)}(\Omega)} \leq c\|\gamma_{u,v}\|^2_{L^n(\Omega)}\|u-v\|_{H^1_0(\Omega)}\end{aligned}$$

where $\gamma_{u,v}(x) = f'((1-\theta(x))u(x)+\theta(x)v(x)) - f'(u(x))$, $c > 0$. Using Lemma 4.1 with $u_* = u$,

$$\lim_{\delta\to 0}\frac{\|f^e(u) - f^e(v) - (f^e)'(u)(u-v)\|_{L^2(\Omega)}}{\|u-v\|_{H^1_0(\Omega)}} = 0$$

and $f^e\colon H^1_0(\Omega) \to L^2(\Omega)$ is Fréchet differentiable with

$$((f^e)'(u)v)(x) = f'(u(x))v(x), \quad u \in H^1_0(\Omega).$$

Let us prove now that $(f^e)'\colon H^1_0(\Omega) \to \mathcal{L}(H^1_0(\Omega), L^2(\Omega))$ is continuous. If $u_n \xrightarrow{H^1_0(\Omega)} u$, then

$$\begin{aligned}\|(f^e)'(u_n) - (f^e)'(u_0)\|_{\mathcal{L}(H^1_0(\Omega),L^2(\Omega))} \\ &= \sup_{\substack{u\in H^1_0(\Omega)\\ \|u\|=1}}\bigg(\int_\Omega |f'(u_n(x))u(x) - f'(u_0(x))u(x)|^2\,dx\bigg)^{1/2}.\end{aligned}$$

Since $f''$ is bounded and $H^1_0(\Omega) \hookrightarrow L^{2n/(n-2)}(\Omega)$, Hölder's inequality with exponent $n/2$ and $n/(n-2)$, implies

$$\begin{aligned}\|f'(u_n(\,\cdot\,))u(\,\cdot\,) - f'(u_0(\,\cdot\,)u(\,\cdot\,)\|_{L^{n/(n-2)}(\Omega)} \\ &= C\|u_n - u_0\|_{L^{2n/(n-2)}}\|u\|_{L^{2n/(n-2)}(\Omega)} \leq C\|u_n-u_0\|_{H^1_0(\Omega)}\|u\|_{H^1_0(\Omega)}.\end{aligned}$$

which implies the continuity. Also, since $\sup_{s\in\mathbb{R}}|f'(s)| < \infty$, we have that

$$\|(f^e)'(u_n)u - (f^e)'(u_0)u\|_{L^{2n/(n-2)}(\Omega)} \leq C\|u\|_{H^1_0(\Omega)}.$$

The proof of continuous differentiability now follows by interpolation (for $n = 3, 4$ the first estimate is sufficient, the last estimate is used for $n > 4$).

To show the last part of the lemma, assume it is not true. Then, there is a sequence $\varepsilon_n \xrightarrow{n\to\infty} 0$, $\delta > 0$, $\{v_{\varepsilon_n}\}$ in $L^2(\Omega)$ with $\|v_{\varepsilon_n}\|_{L^2(\Omega)} = 1$ such that $\|(f^e)'(u^{\varepsilon_n})A^{-1}_{\varepsilon_n}v_{\varepsilon_n} - (f^e)'(u^0)A^{-1}_0 v_{\varepsilon_n}\| \geq \delta$ and (using Corollary 3.2) $\|A^{-1}_{\varepsilon_n}v_{\varepsilon_n} - A^{-1}_0 v_{\varepsilon_n}\|_{H^1_0(\Omega)} \xrightarrow{n\to\infty} 0$. That leads to a contradiction with the continuity that we have just proved. This completes the proof of the lemma. $\square$

Lemma 4.3. *Assume $u_\varepsilon \to u_0$ in $H^1_0(\Omega)$ and that $0 \notin \sigma(A_0 - f'(u_0))$. Then there is a $\varepsilon_0 > 0$ such that, for any $0 \le \theta < 1$, the sequence of operators*

$$\{(A_\varepsilon)^\theta (A_\varepsilon - f'(u^\varepsilon))^{-1} : 0 \le \varepsilon \le \varepsilon_0\}$$

*is uniformly bounded in $\mathcal{L}(L^2(\Omega))$ and $(A_\varepsilon)^\theta (A_\varepsilon - f'(u^\varepsilon))^{-1}$ converges to*

$$(A_0)^\theta (A_0 - f'(u_0))^{-1}$$

*in the uniform operator topology in $\mathcal{L}(L^2(\Omega))$.*

Proof. Note that $A^i_\varepsilon (A_\varepsilon - f'(u_\varepsilon))^{-1} = A^{i-1}_\varepsilon (I - f'(u_\varepsilon) A^{-1}_\varepsilon)^{-1}$ and apply Corollary 3.2 to conclude its continuity, for $i = 1, 2$. The proof now follows by interpolation with the aid of Theorem 1.4.4 in [18]. □

## 5. Rate of convergence of the linear and nonlinear semigroups

Since the operators $A_\varepsilon$, $\varepsilon \in [0, \varepsilon_0]$ are selfadjoint and $A^{-1}_\varepsilon$ converges uniformly to $A^{-1}_0$, then for each $\alpha < \lambda^0_1$, the first eigenvalue of $A_0$, there are $\varepsilon_\alpha > 0$ and $M_\alpha > 0$, independent of $\varepsilon \in [0, \varepsilon_\alpha]$, such that

$$\|e^{-A_\varepsilon t}\|_{\mathcal{L}(L^2(\Omega), H^1_0(\Omega))} \le M_\alpha e^{-\alpha t} t^{-1/2}, \quad t > 0,\ \varepsilon \in [0, \varepsilon_\alpha]. \tag{5.1}$$

Theorem 5.1. *If $0 < \theta \le 1/2$ and $\alpha < \lambda^0_1$, there exists $C_\alpha > 0$ such that*

$$\|e^{-A_\varepsilon t} - e^{-A_0 t}\|_{\mathcal{L}(L^2(\Omega), H^1_0(\Omega))} \le C_\alpha e^{-\alpha t} \|a_\varepsilon - a_0\|^{2\theta}_\infty t^{-1/2-\theta}$$

*for all $t > 0$ and for all $0 \le \varepsilon \le \varepsilon_0$.*

Proof. Considering the linear semigroup

$$e^{-A_\varepsilon t} = \frac{1}{2\pi i} \int_\Gamma e^{\mu t} (\mu + A_\varepsilon)^{-1}\, d\mu, \quad 0 \le \varepsilon \le \varepsilon_0,$$

where $\Gamma$ is the boundary of sector $\Sigma_{-\omega,\phi} = \{\mu \in \mathbb{C} : |\arg(\mu + \omega)| \le \phi\}$ with $\pi/2 < \phi < \pi$ oriented in such a way that the imaginary part of $\mu$ increases as $\mu$ runs in $\Gamma$.

Follows (5.1) the estimative

$$\begin{aligned} \|e^{-A_\varepsilon t} - e^{-A_0 t}\|&_{\mathcal{L}(L^2(\Omega), H^1_0(\Omega))} \\ &\le \|e^{-A_\varepsilon t}\|_{\mathcal{L}(L^2(\Omega), H^1_0(\Omega))} + \|e^{-A_0 t}\|_{\mathcal{L}(L^2(\Omega), H^1_0(\Omega))} \le M e^{-\alpha t} t^{-1/2}. \end{aligned} \tag{5.2}$$

On the other hand, using the Lemma 3.4, we have

$$\|e^{-A_\varepsilon t} - e^{-A_0 t}\|_{\mathcal{L}(L^2(\Omega), H^1_0(\Omega))} \le C e^{-\alpha t} \|a_\varepsilon - a_0\|_\infty t^{-1}. \tag{5.3}$$

Interpolating the expressions (5.2) and (5.3) with $1 - 2\theta$ and $2\theta$, respectively, we have

$$\|e^{-A_\varepsilon t} - e^{-A_0 t}\|_{\mathcal{L}(L^2(\Omega), H^1_0(\Omega))} \le C e^{-\alpha t} \|a_\varepsilon - a_0\|^{2\theta}_\infty t^{-1/2-\theta}.$$ □

THEOREM 5.2. *Let $u^\varepsilon, u \in H_0^1(\Omega)$ and $0 < \theta < 1/2$, then there are positive constants $c$ and $L$ such that*

$$\|T_\varepsilon(t)u^\varepsilon - T_0(t)u\|_{H_0^1(\Omega)} \leq ce^{Lt}t^{-1/2-\theta}(\|u^\varepsilon - u\|_{H_0^1(\Omega)} + \|a_\varepsilon - a_0\|_\infty^{2\theta}) \quad \text{for all } t \geq 0.$$

PROOF. For $t \geq 0$, $u_\varepsilon, u \in H_0^1(\Omega)$,

$$T_\varepsilon(t)u^\varepsilon = e^{-A_\varepsilon t}u^\varepsilon + \int_0^t e^{-A_\varepsilon(t-s)} f(T_\varepsilon(t)u^\varepsilon)\, ds, \quad 0 \leq \varepsilon \leq \varepsilon_0,$$

and we have that

$$\begin{aligned}\|T_\varepsilon(t)u^\varepsilon - T_0(t)u\|_{H_0^1(\Omega)} &\leq \|e^{-A_\varepsilon t}u^\varepsilon - e^{-A_0 t}u\|_{H_0^1(\Omega)} \\ &\quad + \int_0^t \|e^{-A_\varepsilon(t-s)}f(T_\varepsilon(s)u^\varepsilon) - e^{-A_0(t-s)}f(T_0(s)u)\|_{H_0^1(\Omega)}\, ds.\end{aligned}$$

From Theorem 5.1 and (5.1)

$$\|e^{-A_\varepsilon t}u^\varepsilon - e^{-A_0 t}u\|_{H_0^1(\Omega)} \leq Mt^{-1/2-\theta}\|u^\varepsilon - u\|_{H_0^1(\Omega)} + C\|a_\varepsilon - a_0\|_\infty^{2\theta} t^{-1/2-\theta}. \tag{5.4}$$

and

$$\begin{aligned}&\int_0^t \|e^{-A_\varepsilon(t-s)}f(T_\varepsilon(s)u^\varepsilon) - e^{-A_0(t-s)}f(T_0(s)u)\|_{H_0^1(\Omega)}\, ds \\ &\qquad \leq ML_f \int_0^t (t-s)^{-1/2} e^{-\alpha(t-s)} \|T_\varepsilon(s)u^\varepsilon - T_0(s)u\|_{H_0^1(\Omega)}\, ds \\ &\qquad\quad + C_0\|a_\varepsilon - a_0\|_\infty^{2\theta} \int_0^t (t-s)^{-1/2-\theta} e^{-\alpha(t-s)}\, ds.\end{aligned}$$

Consequently, from (5.4),

$$\begin{aligned}\|T_\varepsilon(t)u^\varepsilon - T_0(t)u\|_{H_0^1(\Omega)} &\leq C(\|u^\varepsilon - u\|_{H_0^1(\Omega)} + \|a_\varepsilon - a_0\|_\infty^{2\theta})t^{-1/2-\theta}e^{-\alpha t} \\ &\quad + ML_f \int_0^t (t-s)^{-1/2} e^{-\alpha(t-s)} \|T_\varepsilon(s)u^\varepsilon - T_0(s)u\|_{H_0^1(\Omega)}\, ds\end{aligned}$$

and using the singular Gronwall's inequality (see [18, Chapter 07]), there is a constant $L > 0$

$$\|T_\varepsilon(t)u^\varepsilon - T_0(t)u\|_{H_0^1(\Omega)} \leq Ce^{Lt}t^{-1/2-\theta}(\|u^\varepsilon - u\|_{H_0^1(\Omega)} + \|a_\varepsilon - a_0\|_\infty^{2\theta}). \quad \square$$

## 6. Rate of convergence of equilibria and of linearizations

We start proving the upper semicontinuity of the family of equilibria.

PROPOSITION 6.1. *The family $\{\mathcal{E}_\varepsilon : \varepsilon \in [0, \varepsilon_0]\}$ is upper semicontinuous at $\varepsilon = 0$.*

Proof. Note that $\mathcal{E}_\varepsilon \subset \mathcal{A}_\varepsilon$ and therefore

$$\sup\{\|u^\varepsilon\|_{H_0^1(\Omega)} : u^\varepsilon \in \mathcal{E}_\varepsilon,\ \varepsilon \in [0,\varepsilon_0]\} < \infty$$

and that $f\colon H_0^1(\Omega) \to L^2(\Omega)$ is bounded. If $u^\varepsilon \in \mathcal{E}_\varepsilon$, we have that $u^\varepsilon = A_\varepsilon^{-1} f(u^\varepsilon)$ and the result follows from the uniform convergence of $A_\varepsilon^{-1}$ to $A_0^{-1}$. $\square$

The proof of lower semicontinuity requires additional assumptions. We need to assume that the equilibrium points of (1.3) are stable under perturbation. This stability under perturbation will be given by the hyperbolicity.

Proposition 6.2. *If all equilibrium points of* (1.3) *are isolated, then there are only a finite number of them. Any hyperbolic equilibrium point $u^*$ of* (1.3) *is isolated.*

Proof. Since $\mathcal{E}_\varepsilon$ is compact we only need to prove that hyperbolic equilibria are isolated. We note that $u \in \mathcal{E}_\varepsilon$ is a solution of (1.4) if and only if $u_*$ is a fixed point of

$$\Psi(u) := (A_\varepsilon - f'(u_*))^{-1}(f(u) - f'(u_*)u).$$

If we show that, for some $\delta > 0$, $\Psi\colon \overline{B}_\delta(u_*) \to \overline{B}_\delta(u_*)$ is a contraction, where $\overline{B}_\delta(u_*) := \{u \in H_0^1(\Omega) : \|u - u^*\|_{H_0^1(\Omega)} \leq \delta\}$, then $u_*$ is the only element in $\overline{B}_\delta(u_*) \cap \mathcal{E}_\varepsilon$ and, consequently, isolated. In fact, let $\delta > 0$ and $u, v \in \overline{B}_\delta(u_*)$, using the Lemma 4.1, we have

$$\|\Psi(u) - \Psi(v)\|_{H_0^1(\Omega)} \leq C\|f(u) - f(v) - f'(u_*)(u - v)\|_{L^2(\Omega)}.$$

We remark that, from Lemma 4.3, $C$ is independent of $\varepsilon$ for all $\varepsilon$ suitably small. Now, note that

$$\begin{aligned} \|f(u) - f(v) - f'(u_*)(u - v)\|^2_{L^2(\Omega)} \\ = \int_\Omega |\gamma_\delta(x)|^2 |(u(x) - v(x))|^2\,dx \leq \|\gamma_\delta\|^2_{L^n(\Omega)}\|u - v\|^2_{L^{2n/(n-2)}} \end{aligned}$$

and

$$\begin{aligned} \|\Psi(u) - \Psi(v)\|_{H_0^1(\Omega)} &\leq \|\gamma_\delta\|_{L^n(\Omega)}\|u - v\|_{L^{2n/(n-2)}(\Omega)} \\ &\leq C\delta\|u - v\|_{L^{2n/(n-2)}(\Omega)} \leq C\delta\|u - v\|_{H_0^1(\Omega)}. \end{aligned}$$

Thus, choosing $\delta$ such that $C\delta < 1/2$, we have $\Psi$ is a contraction. Note that, if $v \in \overline{B}_\delta(u_*)$, then $\|\Psi(v) - u_*\|_{H_0^1(\Omega)} = \|\Psi(v) - \Psi(u_*)\|_{H_0^1(\Omega)} \leq c\,\|v - u_*\|_{H_0^1(\Omega)} < \delta$, for some constant $0 \leq c < 1$, this shows that $\Psi(\overline{B}_\delta(u_*)) \subset \overline{B}_\delta(u_*)$; that is, that $\Psi$ has a unique fixed point in $\overline{B}_\delta(u_*)$. $\square$

We are going to study now the convergence properties of resolvent operators of the form $(A_\varepsilon + V_\varepsilon)^{-1}$ to $(A_0 + V_0)^{-1}$, where $V_\varepsilon$ converges to $V_0$ in a sense to be specified. We need to perform this study since we want to compare the resolvent operators of the linearization around equilibria.

Having this in mind, let us consider the following setting for the potentials.

(H) $V_\varepsilon \in L^\infty(\Omega)$, $0 \le \varepsilon \le 1$ potential which satisfy that $|V_\varepsilon| \le a$ for some $a > 0$ and such that

$$\|V_\varepsilon - V_0\|_{H_0^1(\Omega)} \le \|a_\varepsilon - a_0\|_\infty.$$

The convergence of resolvents of $A_\varepsilon + V_0$ follows from the convergence of resolvents of $A_\varepsilon$ and the lemma below whose proof is immediate.

LEMMA 6.3. *The operator $A_\varepsilon + V_0$, $0 \le \varepsilon \le \varepsilon_0$, satisfies the follows identity*

$$\begin{aligned}(A_\varepsilon + V_0)^{-1} &- (A_0 + V_0)^{-1} \\ &= [I - (A_\varepsilon + V_0)^{-1}V_0](A_\varepsilon^{-1} - A_0^{-1})[I - V_0(A_0 + V_0)^{-1}].\end{aligned} \tag{6.1}$$

We can show now

THEOREM 6.4. *Let $u_*^0$ be a hyperbolic equilibrium of* (1.3) *with $\varepsilon = 0$ and $0 \notin \sigma(A_0 - f'(u_*^0))$. Then, there is $\overline{\varepsilon} > 0$ and $\delta > 0$ such that the problem* (1.3) *has exactly one equilibrium solution $u_*^\varepsilon \in \overline{B}_\delta(u_*^0) := \{u \in H_0^1(\Omega) : \|u - u_*^0\|_{H_0^1(\Omega)} \le \delta\}$ for $\varepsilon \in (0, \overline{\varepsilon}]$. Furthermore, $\|u_*^\varepsilon - u_*^0\|_{H_0^1(\Omega)} \le C\|a_\varepsilon - a_0\|_\infty$ for some $C > 0$.*

PROOF. Note that the hyperbolicity of $u_*^0$ means that $\sigma(A_\varepsilon - f'(u_*^0))$ is disjoint from the imaginary axis. Using Lemma 4.3 with $\theta = 1/2$ we obtain $M > 0$ such that

$$\|(A_\varepsilon - f'(u_*^0))^{-1}\|_{\mathcal{L}(L^2(\Omega), H_0^1(\Omega))} \le M \qquad \text{for all } 0 < \varepsilon \le \varepsilon_0.$$

Note that, $u^\varepsilon$ is a solution of (1.4) if and only if $u^\varepsilon$ is a fixed point of the map

$$\omega \to \Psi_\varepsilon(\omega) := (A_\varepsilon - f'(u_*^0))^{-1}(f(\omega) - f'(u_*^0)\omega).$$

From Lemma 4.3, $A_\varepsilon^{1/2}(A_\varepsilon - f'(u_*^0))^{-1}$ converges uniformly to $A_0^{1/2}(A_0 - f'(u_*^0))^{-1}$. This implies that,

$$\Psi_\varepsilon(u_*^0) \to \Psi_0(u_*^0), \quad \text{in } H_0^1(\Omega),$$

since

$$\Psi_\varepsilon(u_*^0) = (A_\varepsilon - f'(u_*^0))^{-1}(f(u_*^0) - f'(u_*^0)u_*^0) = (A_\varepsilon - f'(u_*^0))^{-1}(A_0 - f'(u_*^0))u_*^0$$

and similarly $\Psi_0(u_*^0) = (A_0 - f'(u_*^0))^{-1}(A_0 - f'(u_*^0))u_*^0$.

Now, we prove that there exists $\delta > 0$ and $\overline{\varepsilon} \in (0, \varepsilon_0]$ such that $\Psi_\varepsilon$ is contraction of $\overline{B}_\delta^\varepsilon(u_*^0) = \{u^\varepsilon \in H_0^1(\Omega) : \|u^\varepsilon - u_*^0\|_{H_0^1(\Omega)} < \delta\}$ into itself, uniformly in $(0, \overline{\varepsilon}]$. First, we see that $\Psi_\varepsilon$ is a contraction map. For this, let $u^\varepsilon$ and $v^\varepsilon$ in $\overline{B}_\delta^\varepsilon(u_*^0)$ and note that

$$\begin{aligned}\|\Psi_\varepsilon(u^\varepsilon) &- \Psi_\varepsilon(v^\varepsilon)\|_{H_0^1(\Omega)} \\ &= \|(A_\varepsilon - f'(u_*^0))^{-1}[f(u^\varepsilon) - f(v^\varepsilon) - f'(u_*^0)(u^\varepsilon - v^\varepsilon)]\|_{H_0^1(\Omega)} \\ &\le \|(A_\varepsilon - f'(u_*^0))^{-1}\|_{\mathcal{L}(L^2(\Omega), H_0^1(\Omega))}\|f(u^\varepsilon) - f(v^\varepsilon) - f'(u_*^0)(u^\varepsilon - v^\varepsilon)\|_{L^2(\Omega)}\end{aligned}$$

and using Lemma 4.1 and Theorem 4.2, we obtain

$$\|\Psi_\varepsilon(u^\varepsilon)-\Psi_\varepsilon(v^\varepsilon)\|_{H^1_0(\Omega)}\le C\delta\|u-v\|_{H^1_0(\Omega)}.$$

Thus, choosing $\delta$ such that $C\delta<1/2$, we obtain that $\Psi_\varepsilon$ is a contraction. To show that $\Psi_\varepsilon(\overline{B}^\varepsilon_\delta(u^0_*))\subset\overline{B}^\varepsilon_\delta(u^0_*)$, note that, if $u^\varepsilon\in\overline{B}^\varepsilon_\delta(u^0_*)$, then

$$\|\Psi_\varepsilon(u^\varepsilon)-u^0_*\|_{H^1_0(\Omega)}\le\frac{1}{2}\|u^\varepsilon-u^0_*\|_{H^1_0(\Omega)}+\|\Psi_\varepsilon(u^0_*)-u^0_*\|_{H^1_0(\Omega)}.$$

It follows from Lemma 4.3 that there exists $\overline{\varepsilon}>0$ such that $\|\Psi_\varepsilon(u^0_*)-u^0_*\|_{H^1_0(\Omega)}\le\delta/2$, and for any $u^\varepsilon\in\overline{B}^\varepsilon_\delta(u^0_*)$, we have

$$\|\Psi_\varepsilon(u^\varepsilon)-u^0_*\|_{H^1_0(\Omega)}\le\delta.$$

Thus, $\Psi_\varepsilon\colon\overline{B}^\varepsilon_\delta(u^0_*)\to\overline{B}^\varepsilon_\delta(u^0_*)$ is a contraction, for any $\varepsilon\in(0,\varepsilon_0]$. Hence, there exists a fixed point of $\Psi_\varepsilon$ in $\overline{B}^\varepsilon_\delta(u^0_*)$, which we shall call $u^\varepsilon_*$.

Finally, we estimate the distance $u^\varepsilon_*-u^0_*$ in terms of the difference $\|a_\varepsilon-a_0\|_\infty$.

Observe that $u^*_\varepsilon=\Psi_\varepsilon(u^*_\varepsilon)$ and $u^*_0=\Psi_0(u^*_0)$. If we denote by $V_0=f'(u^0_*)$, we have

$$\begin{aligned}\|u^\varepsilon_*-u^0_*\|_{H^1_0}\le{}&\|((A_\varepsilon+V_0)^{-1}-(A_0+V_0)^{-1})[f(u^\varepsilon_*)+V_0u^\varepsilon_*]\\&+(A_0+V_0)^{-1}[f(u^\varepsilon_*)-f(u^0_*)+V_0(u^\varepsilon_*-u^0_*)]\|_{H^1_0}\\\le{}&\|((A_\varepsilon+V_0)^{-1}-(A_0+V_0)^{-1})\|_{\mathcal{L}(L^2,H^1_0)}\|f(u^\varepsilon_*)+V_0u^\varepsilon_*\|_{L^2}\\&+\|(A_0+V_0)^{-1}(f(u^\varepsilon_*)-f(u^0_*)+V_0(u^\varepsilon_*-u^0_*))\|_{L^2}.\end{aligned}$$

Using (6.1) with $V_0=f'(u^0_*)$, we obtain

$$(A_\varepsilon+V_0)^{-1}-(A_0+V_0)^{-1}=[I-(A_\varepsilon+V_0)^{-1}V_0](A_\varepsilon^{-1}-A_0^{-1})[I-V_0(A_0+V_0)^{-1}].$$

from where it is easy to get that

$$\begin{aligned}\|((A_\varepsilon+V_0)^{-1}-(A_0+V_0)^{-1})\|_{\mathcal{L}(L^2(\Omega),H^1_0(\Omega))}&\le C\|A_\varepsilon^{-1}-A_0^{-1}\|_{\mathcal{L}(L^2(\Omega),H^1_0(\Omega))}\\&\le C\|a_\varepsilon-a_0\|_\infty.\end{aligned}$$

Moreover, if we denote by $z^\varepsilon_*=f(u^\varepsilon_*)-f(u^0_*)+V_0(u^\varepsilon_*-u^0_*)$, using the differentiability of the map $f:H^1_0(\Omega)\to L^2(\Omega)$ proved in Lemma 4.2, we get that for every $\delta>0$ small, there exists $\varepsilon(\delta)>0$ such that $\|z_\varepsilon\|_{L^2(\Omega)}\le\delta\|u^*_\varepsilon-u^*_0\|_{H^1_0(\Omega)}$ for all $0<\varepsilon\le\varepsilon(\delta)$. Hence, for all $0\le\varepsilon\le\varepsilon(\delta)$,

$$\|(A_0+V_0)^{-1}z^\varepsilon_*\|_{H^1_0(\Omega)}\le\delta\|(A_0+V_0)^{-1}\|_{\mathcal{L}(L^2,H^1_0)}\|u^\varepsilon_*-u^0_*\|_{H^1_0(\Omega)}.$$

Choosing $\delta$ small enough so that $\delta\|(A_0+V_0)^{-1}\|_{\mathcal{L}(L^2,H^1_0)}\le1/2$, we get

$$\|u^\varepsilon_*-u^0_*\|_{H^1_0(\Omega)}\le C\|f(u^\varepsilon_*)+V_0u^\varepsilon_*\|_{L^2(\Omega)}\|a_\varepsilon-a_0\|_\infty+\frac{1}{2}\|u^\varepsilon_*-u^0_*\|_{H^1_0(\Omega)}.$$

from where it follows that

$$\|u^\varepsilon_*-u^0_*\|_{H^1_0(\Omega)}\le C\|a_\varepsilon-a_0\|_\infty.\qquad\square$$

Assume that all elements of $\mathcal{E}_0 = \{u_*^1, \dots, u_*^m\}$, are hyperbolic equilibria. Therefore, from the result above, we have that the set of equilibria of (1.3) is also finite and it is given by $\mathcal{E}_\varepsilon = \{u_*^{1,\varepsilon}, \dots, u_*^{m,\varepsilon}\}$ with $0 \le \varepsilon \le \varepsilon_0$, satisfying

$$\|u_*^{i,\varepsilon} - u_*^i\|_{H_0^1(\Omega)} \le C\|a_\varepsilon - a_0\|_{L^\infty(\Omega)}.$$

Writing $V_\varepsilon = f'(u_*^\varepsilon)$ with $u_*^\varepsilon \in \mathcal{E}_\varepsilon$, $\overline{A}_\varepsilon = A_\varepsilon + V_\varepsilon$ for all $0 < \varepsilon \le \varepsilon_0$. If we also denote by $V_\varepsilon$ the operator $H_0^1(\Omega) \ni u \mapsto V_\varepsilon u \in L^2(\Omega)$, that is, the multiplication operator, we have from Theorem 4.2, $V_\varepsilon$ converges to $V_0$ in the uniform operator topology.

Lemma 6.5. *With the definitions above, we have*

$$\|V_\varepsilon A_\varepsilon^{-1} - V_0 A_0^{-1}\|_{\mathcal{L}(L^2(\Omega))} \le C\|a_\varepsilon - a_0\|_\infty$$

*where $C > 0$ is independent of $\varepsilon$.*

Proof. This follows easily using the decomposition

$$V_\varepsilon A_\varepsilon^{-1} - V_0 A_0^{-1} = V_\varepsilon(A_\varepsilon^{-1} - A_0^{-1}) + (V_\varepsilon - V_0)A_0^{-1}$$

and applying Theorem 6.4 and (3.7). □

It is easy to see that the following holds

Proposition 6.6. *The following identity holds*

$$\begin{aligned}\overline{A}_\varepsilon^{-1} - \overline{A}_0^{-1} &= (A_\varepsilon^{-1} - A_0^{-1})(I + V_0 A_0^{-1})^{-1} \\ &\quad - A_\varepsilon^{-1}(I + V_0 A_0^{-1})^{-1}(V_\varepsilon A_\varepsilon^{-1} - V_0 A_0^{-1})(I + V_\varepsilon A_\varepsilon^{-1})^{-1}.\end{aligned}$$

*Furthermore, if $0 \notin \sigma(\overline{A}_0)$, there exists $\varepsilon_0 > 0$ such that $0 \notin \sigma(\overline{A}_\varepsilon)$ for all $\varepsilon \in (0, \varepsilon_0)$. Moreover, we have $\|\overline{A}_\varepsilon^{-1} - \overline{A}_0^{-1}\|_{\mathcal{L}(L^2(\Omega), H_0^1(\Omega))} \le C\|a_\varepsilon - a_0\|_\infty$.*

## 7. Rate of convergence and attraction of local unstable manifolds

For each $\varepsilon \in [0, \varepsilon_0]$ let $u_*^\varepsilon$ be an equilibrium for (1.3). Assume that there is a constant $C > 0$ such that $\|u_*^\varepsilon - u_*\|_{H_0^1(\Omega)} \le C\|a_\varepsilon - a_0\|_\infty$ for all $\varepsilon \in [0, \varepsilon_0]$ and that $u_*^0 =: u_*$ is hyperbolic. To deal with a neighbourhood of the equilibrium point $u_*^\varepsilon$, we rewrite problem (1.3) as

$$w_t^\varepsilon + \overline{A}_\varepsilon w^\varepsilon = f(w^\varepsilon + u_*^\varepsilon) - f(u_*^\varepsilon) - f'(u_*^\varepsilon)w^\varepsilon, \tag{7.1}$$

where $w^\varepsilon = u^\varepsilon - u_*^\varepsilon$ and $\overline{A}_\varepsilon = A_\varepsilon - f'(u_*^\varepsilon)$. With this, one may prove Lemma 3.4 for $\overline{A}_\varepsilon$ in place of $A_\varepsilon$.

Let $\overline{\gamma}$ be a smooth, closed, simple, rectifiable curve in $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$, oriented counterclockwise and such that the bounded connected component of $\mathbb{C}\backslash\{\overline{\gamma}\}$ (here $\{\overline{\gamma}\}$ denotes the trace of $\overline{\gamma}$) contains $\{z \in \sigma(\overline{A}_0) : \operatorname{Re} z > 0\}$. From part (b) of Proposition 3.3, there is an $\varepsilon_{\overline{\gamma}}$ such that $\{\overline{\gamma}\} \subset \rho(\overline{A}_\varepsilon)$ for all $0 \leq \varepsilon \leq \varepsilon_{\overline{\gamma}}$. Define $\overline{Q}_\varepsilon^+$ by

$$\overline{Q}_\varepsilon^+ = \frac{1}{2\pi i} \int_{\overline{\gamma}} (\lambda - \overline{A}_\varepsilon)^{-1} \, d\lambda$$

for $0 \leq \varepsilon \leq \varepsilon_{\overline{\gamma}}$. The operator $\overline{A}_\varepsilon$ is selfadjoint and there is a $\beta > 0$ and $M \geq 1$ such that, for all $0 \leq \varepsilon \leq \varepsilon_0$,

$$\begin{aligned} \|e^{-\overline{A}_\varepsilon t} \overline{Q}_\varepsilon^+\|_{\mathcal{L}(L^2(\Omega))} &\leq M e^{\beta t}, & t \leq 0, \\ \|e^{-\overline{A}_\varepsilon t} (I - \overline{Q}_\varepsilon^+)\|_{\mathcal{L}(L^2(\Omega), H_0^1(\Omega))} &\leq M t^{-1/2} e^{-\beta t}, & t > 0. \end{aligned}$$

Using the decomposition $H_0^1(\Omega) = \overline{Q}_\varepsilon^+(H_0^1(\Omega)) + (I - \overline{Q}_\varepsilon^+)(H_0^1(\Omega))$, the solution $w^\varepsilon$ of (7.1) can be decomposed as $w^\varepsilon = v^\varepsilon + z^\varepsilon$, with $v^\varepsilon = \overline{Q}_\varepsilon^+ w^\varepsilon$ and $z^\varepsilon = (I - \overline{Q}_\varepsilon^+) w^\varepsilon$. Defining operators $B_\varepsilon := \overline{A}_\varepsilon \overline{Q}_\varepsilon^+$ and $\widetilde{A}_\varepsilon := \overline{A}_\varepsilon (I - \overline{Q}_\varepsilon^+)$, we rewrite equation (7.1) as

$$\begin{cases} v_t^\varepsilon + B_\varepsilon v^\varepsilon = H_\varepsilon(v^\varepsilon, z^\varepsilon), \\ z_t^\varepsilon + \widetilde{A}_\varepsilon z^\varepsilon = G_\varepsilon(v^\varepsilon, z^\varepsilon), \end{cases} \tag{7.2}$$

where

$$\begin{aligned} H_\varepsilon(v^\varepsilon, z^\varepsilon) &:= \overline{Q}_\varepsilon^+ [f(v^\varepsilon + z^\varepsilon + u_*^\varepsilon) - f(u_*^\varepsilon) - f'(u_*^\varepsilon)(v^\varepsilon + z^\varepsilon)], \\ G_\varepsilon(v^\varepsilon, z^\varepsilon) &:= (I - \overline{Q}_\varepsilon^+)[f(v^\varepsilon + z^\varepsilon + u_*^\varepsilon) - f(u_*^\varepsilon) - f'(u_*^\varepsilon)(v^\varepsilon + z^\varepsilon)]. \end{aligned}$$

The functions $H_\varepsilon$ and $G_\varepsilon$ are continuously differentiable with $H_\varepsilon(0,0) = 0 = G_\varepsilon(0,0) \in L^2(\Omega)$ and $H_\varepsilon'(0,0) = 0 = G_\varepsilon'(0,0) \in \mathcal{L}(H_0^1(\Omega), L^2(\Omega))$. Hence, given $\rho > 0$, there are $0 < \overline{\varepsilon} = \overline{\varepsilon}_\rho \leq \varepsilon_{\overline{\gamma}}$ and $\delta = \delta_\rho > 0$ such that if $\|v\|_{\overline{Q}_\varepsilon^+ H_0^1(\Omega)} + \|z\|_{H_0^1(\Omega)} < \delta$ and $\varepsilon \leq \overline{\varepsilon}$, then

$$\|H_\varepsilon(v,z)\|_{\overline{Q}_\varepsilon^+(H_0^1(\Omega))} \leq \rho \quad \text{and} \quad \|G_\varepsilon(v,z)\|_{L^2(\Omega)} \leq \rho; \tag{7.3}$$

$$\|H_\varepsilon(v,z) - H_\varepsilon(\overline{v},\overline{z})\|_{\overline{Q}_\varepsilon^+(H_0^1(\Omega))} \leq \rho(\|v - \overline{v}\|_{\overline{Q}_\varepsilon^+(H_0^1(\Omega))} + \|z - \overline{z}\|_{H_0^1(\Omega)}); \tag{7.4}$$

$$\|G_\varepsilon(v,z) - G_\varepsilon(\overline{v},\overline{z})\|_{L^2(\Omega)} \leq \rho(\|v - \overline{v}\|_{\overline{Q}_\varepsilon^+(H_0^1(\Omega))} + \|z - \overline{z}\|_{H_0^1(\Omega)}). \tag{7.5}$$

Theorem 7.1. *Given $D > 0$ and $\Delta > 0$, $\vartheta \in (0,1)$ and $\rho_0 > 0$ be such that:*

$$\begin{gathered} \rho M \beta^{-1/2} \Gamma\Big(\frac{1}{2}\Big) \leq D, \qquad \rho M\, \Gamma\Big(\frac{1}{2}\Big) \frac{M(1+\Delta)}{(2\beta - \rho M(1+\Delta))^{1/2}} \leq \Delta, \\ \rho M \beta^{-1/2} \Gamma\Big(\frac{1}{2}\Big) \bigg[1 + \frac{\rho M(1+\Delta)\beta^{-1/2}}{(2\beta - \rho M(1+\Delta))^{1/2}}\bigg] \leq \vartheta \end{gathered} \tag{7.6}$$

*are satisfied for all $\rho \in (0, \rho_0)$. Assume that $H_\varepsilon$ and $G_\varepsilon$ satisfies (7.3)–(7.5), with $0 < \rho \leq \rho_0$ for all $(v,z) \in \overline{Q}_\varepsilon^+ H_0^1(\Omega) \times (I - \overline{Q}_\varepsilon^+) H_0^1(\Omega)$. Then, there exists $s_\varepsilon^*\colon \overline{Q}_\varepsilon^+ H_0^1(\Omega) \to (I - Q_\varepsilon^+) H_0^1(\Omega)$ such that the unstable manifold of $u_*^\varepsilon$ is given as the graph of the map $s_\varepsilon^*$,*

$$W^u(u_*^\varepsilon) = \big\{(v,z) \in H_0^1(\Omega) : z = s_\varepsilon^*(v),\ v \in \overline{Q}_\varepsilon^+ H_0^1(\Omega)\big\}.$$

*The map $s_\varepsilon^*$ satisfies:*

$$\|s_\varepsilon^*\| := \sup_{v\in\overline{Q}_\varepsilon^+(H_0^1(\Omega))} \|s_\varepsilon^*(v)\|_{H_0^1(\Omega)} \le D,$$
$$\|s_\varepsilon^*(v) - s_\varepsilon^*(\widetilde{v})\|_{H_0^1(\Omega)} \le \Delta\|v-\widetilde{v}\|_{\overline{Q}_\varepsilon^+(H_0^1(\Omega))},$$

*and for $0<\theta<1$ there is a $C_\theta>0$, such that*

$$\|s_\varepsilon^* - s_0^*\| \le C_\theta\|a_\varepsilon - a_0\|_\infty^\theta.$$

*Furthermore, given $0<\gamma<\beta$, there is $0<\rho_1\le\rho_0$ and $C>0$, independent of $\varepsilon$, such that, for any solution $[t_0,\infty)\ni t\mapsto(v^\varepsilon(t),z^\varepsilon(t))\in H_0^1(\Omega)$ of (7.2),*

$$\|z^\varepsilon(t) - s_\varepsilon^*(v^\varepsilon(t))\|_{H_0^1(\Omega)} \leqslant Ce^{-\gamma(t-t_0)}\|z^\varepsilon(t_0) - s_\varepsilon^*(v^\varepsilon(t_0))\|_{H_0^1(\Omega)} \tag{7.7}$$

*for all $t\ge t_0$.*

PROOF. Consider the set

$$\Sigma_\varepsilon = \Big\{ s\colon \overline{Q}_\varepsilon^+(H_0^1(\Omega)) \to (I-Q_\varepsilon^+)(H_0^1(\Omega)) : \\ \|s\|\le D,\ \|s(v)-s(\widetilde{v})\|_{H_0^1(\Omega)} \le \Delta\|v-\widetilde{v}\|_{\overline{Q}_\varepsilon^+ H_0^1(\Omega)}\Big\}.$$

It is not difficult to see that $(\Sigma_\varepsilon, \|\cdot\|)$ is a complete metric space.

Given $s_\varepsilon\in\Sigma_\varepsilon$ and $\eta\in\overline{Q}_\varepsilon(H_0^1(\Omega))$, denote by $v^\varepsilon(t)=\psi(t,\tau,\eta,s_\varepsilon)$ the solution of

$$\begin{cases} v_t^\varepsilon(t) + B_\varepsilon v^\varepsilon(t) = H_\varepsilon(v^\varepsilon(t), s_\varepsilon^*(v^\varepsilon(t))), & t<\tau,\\ v^\varepsilon(\tau)=\eta. \end{cases}$$

Define $\Psi_\varepsilon\colon\Sigma_\varepsilon\to\Sigma_\varepsilon$ by

$$\Psi_\varepsilon(s_\varepsilon)\eta = \int_{-\infty}^{\tau} e^{-\widetilde{A}_\varepsilon(\tau-s)} G_\varepsilon(v^\varepsilon(s), s_\varepsilon(v^\varepsilon(s)))\,ds.$$

Note that, from (7.3) and (7.6), we have $\|\Psi(s_\varepsilon)(\eta)\|_{H_0^1(\Omega)}\le D$.

Now, if $\eta,\ \widetilde{\eta}\in\overline{Q}_\varepsilon(H_0^1(\omega))$, $s_\varepsilon,\widetilde{s}_\varepsilon\in\Sigma$, $v^\varepsilon(t)=\psi(t,\tau,\eta,s_\varepsilon)$ and $\widetilde{v}^\varepsilon(t) = \psi(t,\tau,\eta,\tilde{s}_\varepsilon)$, it is easy to see that

$$\phi(t)\le M\|\eta-\tilde{\eta}\|_{H_0^1(\Omega)} + M\rho(1+\Delta)\int_t^\tau \phi(s)\,ds + M\rho\beta^{-1}\|s_\varepsilon-\widetilde{s}_\varepsilon\|$$

where $\phi(t)=e^{-\beta(t-\tau)}\|v^\varepsilon(t)-\widetilde{v}^\varepsilon(t)\|_{H_0^1(\Omega)}$ and using Gronwall's inequality

$$\|v^\varepsilon(t)-\widetilde{v}^\varepsilon(t)\|_{H_0^1(\Omega)} \le M(\|\eta-\widetilde{\eta}\|_{H_0^1(\Omega)} + \rho\beta^{-1}\|s_\varepsilon-\widetilde{s}_\varepsilon\|)e^{(\beta-M\rho(1+\Delta))(t-\tau)}.$$

From this we obtain that

$$\begin{aligned}\|\Psi(s_\varepsilon)(\eta) - \Psi(\widetilde{s}_\varepsilon)(\widetilde{\eta})\|_{H^1_0(\Omega)} &\leq \rho M\beta^{-1/2}\Gamma\bigg(\frac{1}{2}\bigg)\bigg[1+\frac{\rho M(1+\Delta)\beta^{-1/2}}{(2\beta-\rho M(1+\Delta))^{1/2}}\bigg]\|\!|s_\varepsilon-\widetilde{s}_\varepsilon|\!\| \\ &\quad+\frac{\rho M^2(1+\Delta)}{(2\beta-\rho M(1+\Delta))^{1/2}}\Gamma\bigg(\frac{1}{2}\bigg)\|\eta-\widetilde{\eta}\|_{\overline{Q}_\varepsilon^+ H^1_0(\Omega)}\end{aligned}$$

and

$$\|\Psi(s_\varepsilon)(\eta) - \Psi(\widetilde{s}_\varepsilon)(\widetilde{\eta})\|_{H^1_0(\Omega)} \leq \Delta\|\eta-\widetilde{\eta}\|_{\overline{Q}_\varepsilon^+ H^1_0(\Omega)} + \vartheta\|\!|s_\varepsilon-\widetilde{s}_\varepsilon|\!\|.$$

Hence, $\Psi$ is a contraction. Therefore, there a fixed point $s_\varepsilon^* = \Psi(s_\varepsilon^*)$ in $\Sigma_\varepsilon$.

Now, we prove that $\{(v^\varepsilon, s_\varepsilon^*(v^\varepsilon)) : v \in \overline{Q}_\varepsilon^+ H^1_0(\Omega)\}$ is invariant for (7.2). Let $(v_0^\varepsilon, z_0^\varepsilon) \in W^u(u_*^\varepsilon)$, $z_0^\varepsilon = s_\varepsilon^*(v_0^\varepsilon)$. Denote by $v_*^\varepsilon(t)$ the solutions of the initial value problems

$$\begin{cases} v_t^\varepsilon + B_\varepsilon v^\varepsilon = H_\varepsilon(v^\varepsilon, s_\varepsilon^*(v^\varepsilon)), \\ v^\varepsilon(0) = v_0^\varepsilon. \end{cases}$$

This defines a curve $(v_\varepsilon^*(t), s_\varepsilon^*(v_\varepsilon^*(t))) \in W^u(u_*^\varepsilon)$, $t \in \mathbb{R}$. Also, the only solution of

$$z_t^\varepsilon + \widetilde{A}_\varepsilon z^\varepsilon = G_\varepsilon(v_*^\varepsilon(t), s_\varepsilon^*(v_*^\varepsilon(t)))$$

which remains bounded as $t \to -\infty$ must be

$$z_*^\varepsilon(t) = \int_{-\infty}^t e^{\widetilde{A}_\varepsilon(t-s)} G_\varepsilon(v_*^\varepsilon(s), s_\varepsilon^*(v_*^\varepsilon(s)))\,ds = s_\varepsilon^*(v_*^\varepsilon(t)).$$

This proves the invariance of the graph of $s_\varepsilon^*$. To prove that the graph of $s_\varepsilon^*$ is the unstable manifold assume the exponential attraction of the graph of $s_\varepsilon^*$ uniformly in $\varepsilon$; that is, if $u^\varepsilon(t) = z^\varepsilon(t) + v^\varepsilon(t)$ is a solution of (7.2) with $v^\varepsilon(t) = \overline{Q}_\varepsilon u^\varepsilon(t)$. If, given $\gamma < \beta$, there exists $\rho_1 > 0$ such that (7.7) holds for any $0 < \rho \leq \rho_1$, it is easy to see that, when $z^\varepsilon(t)$ remains bounded as $t \to -\infty$, it follows that (making $t_0 \to -\infty$ in (7.7)) that $z^\varepsilon(t) = s_\varepsilon^*(v^\varepsilon(t))$ for all $t \in \mathbb{R}$.

The proof of (7.7) can be carried out as inequality (A.8) of [12], using the singular Gronwall's inequality instead of the usual one.

Finally, we obtain the rate of of convergence of $s_\varepsilon^*$ to $s_0^*$; that is, we show that for any $R > 0$ there is a constant $C > 0$ such that

$$\|\!|s_\varepsilon^* - s_0^*|\!\|_R := \sup_{\eta \in B_{\overline{Q}_\varepsilon^+ H^1_0(\Omega)}(0,R)} \|s_\varepsilon^*(\eta) - s_0^*(\eta)\|_{H^1_0(\Omega)} \leq C\|a_\varepsilon - a_0\|^{2\theta}.$$

This follows as in the proof of Proposition 6.1 in [5]. With this, Theorem 7.1 is now proved. $\square$

## 8. Further applications of the analysis in the previous sections

The analysis carried out in the previous sections can also be applied to several other singular perturbation problems with parabolic structure. To demonstrate the applicability of this analysis we consider a semilinear parabolic problems of the form

$$\begin{cases} u^\varepsilon_t - \Delta u^\varepsilon + u^\varepsilon = f(u^\varepsilon), & x \in \Omega_\varepsilon,\ t > 0, \\ \dfrac{\partial u}{\partial n} = 0, & x \in \partial\Omega_\varepsilon, \end{cases} \tag{8.1}$$

where $\Omega_\varepsilon \subset \mathbb{R}^N$, $N \geq 2$, is a bounded smooth domain, $\varepsilon \in [0,1]$ is parameter, $\frac{\partial}{\partial n}$ is the ouside normal derivative and $f\colon \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable function which is bounded and has bounded derivatives up to the second order. The domain $\Omega_\varepsilon$ is a dumbbell type domain consisting of two disconnected domains, that we denote by $\Omega$, joined by a thin channel $R_\varepsilon$, which degenerates to a line segment as the parameter $\varepsilon$ approaches zero. We also denote by $P_0$ and $P_1$ the points where the line segment touches $\Omega$. We show that one may obtain the rate of convergence of local unstable manifolds and of attractors in relation to the Lebesgue measure of the thin channel $R_\varepsilon$. The variation in functional spaces with which we will work will determine our parameter.

The "limiting domain" will consist of the domain $\Omega$ and a line. The limiting equation is

$$\begin{cases} w_t(x,t) - \Delta w(x,t) + w(x,t) = f(w(x,t)), & x \in \Omega,\ t > 0, \\ \dfrac{\partial w}{\partial n}(x,t) = 0, & x \in \partial\Omega, \\ v_t(s,t) - Lv(s,t) + v = f(v(s,t)), & s \in R_0, \\ v(0) = w(P_0),\ v(1) = w(P_1), \end{cases} \tag{8.2}$$

where $w$ is a function defined in $\Omega$ and $v$ is defined the segment $R_0$. Moreover, $L$ is a differential operator which depends on the geometry of the channel $R_\varepsilon$, more exactly, on the way the channel $R_\varepsilon$ collapses to the line segment $R_0$. More specifically $Lu = \frac{1}{g}(gu_x)x$ where $g$ is defined below. Under the above assumptions on $f$, for fixed $\varepsilon \in [0,1]$, the problems (8.1), for $\varepsilon > 0$, and (8.2), for $\varepsilon = 0$, have attractors $\mathcal{A}_\varepsilon$ in $H^1(\Omega_\varepsilon)$, for $\varepsilon > 0$, and in $H^1(\Omega) \times H^1(R_0)$, for $\varepsilon = 0$.

The continuity of attractors for (8.1) has been studied in [3]–[5]. Here we complete the analysis done in these works obtaining the rate of convergence of local unstable manifolds and attractors.

Definition 8.1. A dumbbell domain $\Omega_\varepsilon$ consists of a fixed domain $\Omega$ attached to a thin handle $R_\varepsilon$ that approaches a line segment as the parameter $\varepsilon$ approaches zero; that is, $\Omega_\varepsilon = \Omega \cup R_\varepsilon$. More precisely, let $\Omega \subset \mathbb{R}^N$, with $N \geq 2$, be a fixed bounded and smooth domain such that there is an $l > 0$ for which

$$\begin{aligned}
\Omega \cap \{(s,x') : s^2 + |x'|^2 < l^2\} &= \{(s,x') : s^2 + |x'|^2 < l^2,\ s < 0\},\\
\Omega \cap \{(s,x') : (s-1)^2 + |x'|^2 < l^2\} &= \{(s,x') : (s-1)^2 + |x'|^2 < l^2,\ s > 1\},\\
\Omega \cap \{(s,x') : 0 < s < 1,\ |x'| < l\} &= \emptyset,
\end{aligned}$$

with $\{(0,x') : |x'| < l\} \cup \{(1,x') : |x'| < l\} \subset \partial\Omega$. Here, we are using the notation $\mathbb{R}^N \ni x = (s,x')$, with $s \in \mathbb{R}$, $x' = (x_2,\ldots,x_N) \in \mathbb{R}^{N-1}$.

The channel that we consider will be defined as $R_\varepsilon = \{(s,\varepsilon x') : (s,x') \in R_1\}$ and $R_1$ is a smooth domain given by $R_1 = \{(s,x') : 0 \le s \le 1,\ x' \in \Gamma_1^s\}$ where $\Gamma_1^s = \{x' \in \mathbb{R}^{N-1} : (s,x') \in R^1\}$ is diffeomorphic to the unit ball in $\mathbb{R}^{N-1}$, $0 \le s \le 1$. That is, for each $s \in [0,1]$, there exists a $C^1$ dipheomorphism $L_s : B(0,1) \to \Gamma_1^s$ and $(0,1) \times B(0,1) \ni (s,z) \stackrel{L}{\longmapsto} L(s,z) := (s, L_s(z)) \ni R_1$ is a diffeomorphism.

The function $[0,1] \ni s \stackrel{g}{\longmapsto} g(s) := |\Gamma_1^s|$, where $|\Gamma_1^s|$ denotes the $(N-1)$-dimensional Lebesgue measure of the set $\Gamma_1^s$, is a smooth function defined in $[0,1]$ and there are $d_0, d_1 > 0$ such that $d_0 \le g(s) \le d_1$ for all $s \in [0,1]$. With this, $R_\varepsilon$ collapses to the line segment $R_0 = \{(s,0) : 0 \le s \le 1\}$.

To properly compare functions in $\Omega_\varepsilon$ and $\Omega_0 := \Omega \cup R_0$ we introduce appropriate spaces $U_\varepsilon^p$, $1 < p < \infty$ and $\varepsilon \in [0,1]$ as follows, $U_\varepsilon^p := L^p(\Omega_\varepsilon)$, $\varepsilon \in (0,1]$ with the norm

$$\|\cdot\|_{U_\varepsilon^p} := \|\cdot\|_{L^p(\Omega)} + \varepsilon^{(1-N)/p} \|\cdot\|_{L^p(R_\varepsilon)}$$

and $U_0^p := L^p(\Omega) \oplus L_g^p(0,1)$ with the norm $\|(w,v)\|_{U_0^p} := \|w\|_{L^p(\Omega)} + \|v\|_{L_g^p(0,1)}$, where $L_g^p(0,1)$ is the space $L^p(0,1)$ with the norm

$$\|u\|_{L_g^p(0,1)} := \bigg(\int_0^1 g(s)|u(s)|^p\,ds\bigg)^{1/p}.$$

As the solutions of problem (8.1) are defined in different spaces, we will use the mechanism given in [3] to compare functions defined in "domains" $\Omega_0$ and $\Omega_\varepsilon$. First we need an extension operator

$$\begin{aligned}
&E_\varepsilon : U_0^p \to U_\varepsilon^p\\
&(w,v) \mapsto E_\varepsilon(w,v)(x) := \begin{cases} w(x), & x \in \Omega,\\ v(s), & x = (s,y) \in R_\varepsilon.\end{cases}
\end{aligned} \tag{8.3}$$

With this, we may write the problems (8.1) and (8.2) as semilinear abstract problems of the form (1.3), where $H_0^1(\Omega)$ is replaced by $U_\varepsilon^p$, $0 < \varepsilon \le 1$ and $H_0^1(\Omega)$ replaced by $U_0^p$.

Proposition 8.2. *For $\varepsilon \in (0,1]$, $E_\varepsilon : U_0^p \to U_\varepsilon^p$ is a bounded linear operator and*

$$\|E_\varepsilon(w,v)\|_{U_\varepsilon^p} = \|(w,v)\|_{U_0^p} \quad \text{for all } (w,v) \in U_0^p.$$

We also need the following "*projection*" operator $M_\varepsilon : U^p_\varepsilon \to U^p_0$ defined by $M_\varepsilon(\psi_\varepsilon) = (w_\varepsilon, v_\varepsilon)$ where $w_\varepsilon(x) = \psi_\varepsilon(x)$, $x \in \Omega$ and $v_\varepsilon(s) = T^s_\varepsilon \psi_\varepsilon$, $s \in (0,1)$, where

$$T^s_\varepsilon \psi_\varepsilon(x) = \frac{1}{|\Gamma^s_\varepsilon|} \int_{\Gamma^s_\varepsilon} \psi_\varepsilon(s,y)\,dy, \quad \Gamma^s_\varepsilon = \{y : (s,y) \in R_\varepsilon\}.$$

Proposition 8.3. *For $\varepsilon \in (0,1]$, $M_\varepsilon \in \mathcal{L}(U^p_\varepsilon, U^p_0)$ and $\|M_\varepsilon\|_{\mathcal{L}(U^p_\varepsilon, U^p_0)} \le 1$.*

The proof of the following result follows step by step the proof of Theorem 2.2 using the results proved in [3]–[5].

Theorem 8.4. *Let $\{T_\varepsilon(t) : t \ge 0\}$ the solution operator associated to (8.1) and (8.2) and $\mathcal{A}_\varepsilon$ be its global attractor, $\varepsilon \in [0,1]$. Then, there are $\varepsilon_0 > 0$, $L > 0$, $\beta > 0$, $0 < \gamma < 1$ and $C > 0$ such that*

$$\begin{aligned}
\|T_\varepsilon(t)u_\varepsilon - E_\varepsilon T_0(t) M_\varepsilon v_\varepsilon\|_{L^p(\Omega_\varepsilon)} &\le C e^{\beta t} t^{-\gamma} (\|u_\varepsilon - v_\varepsilon\|_{U^p_\varepsilon} + \varepsilon^{\theta N/q}), \quad t > 0,\\
\|T_\varepsilon(t)u_\varepsilon - E_\varepsilon T_0(t) M_\varepsilon v_\varepsilon\|_{U^p_\varepsilon} &\le C e^{\beta t} t^{-\gamma} (\|u_\varepsilon - v_\varepsilon\|_{U^p_\varepsilon} + \varepsilon^{\theta/q}), \quad t > 0.
\end{aligned}$$

*for each $p > N$, $\theta \in (1/2, 2p/(N+2p))$.*

*If all equilibrium points $\mathcal{E}_0 = \{u^{1,0}_*, \ldots, u^{\mathbf{n},0}_*\}$ of (8.2) are hyperbolic (hence there are only a finitely many of them), the semigroup $\{T_\varepsilon(t) : t \ge 0\}$ has a set of exactly $n$ equilibria, $\mathcal{E}_\varepsilon = \{u^{1,\varepsilon}_*, \ldots, u^{\mathbf{n},\varepsilon}_*\}$, all of them hyperbolic, for $p > N$, $\|u^{i,\varepsilon}_* - E_\varepsilon u^i_*\|_{L_q(\Omega_\varepsilon)} \le C\varepsilon^{N/p}$, $1 \le i \le n$,*

$$\mathcal{A}_\varepsilon = \bigcup_{i=1}^{\mathbf{n}} W^u(u^{i,\varepsilon}_*);$$

*and there is a $\rho > 0$ such that, if $W^u_\rho(u^*_\varepsilon) = W^u(u^*_\varepsilon) \cap B^{L^q(\Omega_\varepsilon)}_\rho(u^*_\varepsilon)$ (or $W^u_\rho(u^*_\varepsilon) = W^u(u^*_\varepsilon) \cap B^{U^q_\varepsilon}_\rho(u^*_\varepsilon)$), there is a $C_\theta > 0$ such that*

$$\begin{aligned}
\operatorname{dist}^{L^q(\Omega_\varepsilon)}(W^u_\rho(u^*_\varepsilon), E_\varepsilon W^u_\rho(u^*_0)) + \operatorname{dist}^{L^q(\Omega_\varepsilon)}(W^u_\rho(u^*_0), E_\varepsilon W^u_\rho(u^*_\varepsilon)) &\le C_\theta \varepsilon^{\theta N/q}\\
(\textit{or}\quad \operatorname{dist}^{U^q_\varepsilon}(W^u_\rho(u^*_\varepsilon), E_\varepsilon W^u_\rho(u^*_0)) + \operatorname{dist}^{U^q_\varepsilon}(W^u_\rho(u^*_0), E_\varepsilon W^u_\rho(u^*_\varepsilon)) &\le C_\theta \varepsilon^{\theta/q}).
\end{aligned}$$

*Furthermore, for each $\delta > 0$ suitably small, if $u^\varepsilon \in T_\varepsilon(1, B^{L^q(\Omega_\varepsilon)}(u^\varepsilon_*, \delta))$, there is a $\rho_1 > 0$ and $M \ge 1$, independent of $\varepsilon$, such that*

$$\begin{aligned}
\operatorname{dist}^{L^q(\Omega_\varepsilon)}(z^\varepsilon(t), W^u_\rho(u^*_\varepsilon)) &\le M e^{-\rho_1 t} \operatorname{dist}^{L^q(\Omega_\varepsilon)}(z^\varepsilon(1), W^u_\rho(u^*_\varepsilon)), \quad t \ge 1,\\
\operatorname{dist}^{U^q_\varepsilon}(z^\varepsilon(t), W^u_\rho(u^*_\varepsilon)) &\le M e^{-\rho_1 t} \operatorname{dist}^{U^q_\varepsilon}(z^\varepsilon(t_0), W^u_\rho(u^*_\varepsilon)), \quad t \ge 1,
\end{aligned}$$

*as long as $(v^\varepsilon(t), z^\varepsilon(t))$ of $u^\varepsilon$, remains in $T_\varepsilon(1, B^{L^q(\Omega_\varepsilon)}(u^\varepsilon_*, \delta))$.*

*In addition, from the results in [10], [17],*

(a) *There $\rho > 0$ such that given $B \subset U^p_\varepsilon$ bounded, exists $c = c(B) > 0$ such that*

$$\begin{aligned}
\operatorname{dist}^{L^p(\Omega_\varepsilon)}(T_\varepsilon(t)B, \mathcal{A}_\varepsilon) &\le c e^{-\rho t} \quad \text{for all } t \ge 1,\\
\operatorname{dist}^{U^p_\varepsilon}(T_\varepsilon(t)B, \mathcal{A}_\varepsilon) &\le c e^{-\rho t} \quad \text{for all } t \ge 1;
\end{aligned}$$

(b) *There is a constant $c > 0$ such that*

$$\operatorname{dist}^{L^p(\Omega_\varepsilon)}(\mathcal{A}_\varepsilon, E_\varepsilon\mathcal{A}_0) + \operatorname{dist}^{L^p(\Omega_\varepsilon)}(E_\varepsilon\mathcal{A}_0, \mathcal{A}_\varepsilon) \le c\varepsilon^{\theta\rho N/(p(\rho+L))},$$
$$\operatorname{dist}^{U^p_\varepsilon}(\mathcal{A}_\varepsilon, E_\varepsilon\mathcal{A}_0) + \operatorname{dist}^{U^p_\varepsilon}(E_\varepsilon\mathcal{A}_0, \mathcal{A}_\varepsilon) \le c\varepsilon^{\theta\rho/(p(\rho+L))},$$

*where* $\operatorname{dist}^X(A,B) := \sup_{a\in A}\inf_{b\in B}\|a-b\|_X$ *is the Hausdorff semi-distance between the subsets $A, B$ of the Banach space $X$.*

## 9. Further comments

Other examples where this analysis can be carried out are the localized large diffusion problem considered in [20], [8], [13], the viscous Cahn–Hilliard problem considered in [15] or the domain perturbation problem considered in [2]. Each example requires a careful study of the convergence of resolvents for the associated linear operators and a proper functional analytic setting. This analysis in each particular example is far from trivial but the agenda presented in Sections 1–7 usually applies.

## References

[1] J.M. Arrieta and A.N. Carvalho, *Abstract parabolic problems with critical nonlinearities and applications to Navier–Stokes and heat equations*, Trans. Amer. Math. Soc. **352** (2000), 285–310.

[2] ———, *Spectral convergence and nonlinear dynamics of reaction–diffusion equations under perturbations of the domain*, J. Differential Equations **199** (2004), 143–178.

[3] J.M. Arrieta, A.N. Carvalho and G. Lozada-Cruz, *Dynamics in dumbbell domains* I: *Continuity of the set of equilibria*, J. Differential Equations **231** (2006), 551–597.

[4] ———, *Dynamics in dumbbell domains* II: *The limiting problem*, J. Differential Equations **247** (2009), 174–202.

[5] ———, *Dynamics in dumbbell domains* III: *Continuity of attractors*, J. Differential Equations **247** (2009), 225–259.

[6] J.M. Arrieta, A.N. Carvalho and A. Rodriguez-Bernal, *Parabolic problems with nonlinear boundary conditions and critical nonlinearities*, J. Differential Equations **156** (1999), 376–406.

[7] ———, *Attractors of parabolic problems with nonlinear boundary conditions. Uniform bounds*, Comm. Partial Differential Equations **25** (2000), 1–37.

[8] ———, *Upper semicontinuity for attractors of parabolic problems with localized large diffusion and nonlinear boundary conditions*, J. Differential Equations **168** (2000), 33–59.

[9] J.M. Arrieta and E. Santamaria, *Distance of attractors for Morse–Smale systems* (2011), in preparation.

[10] A.V. Babin and M.I. Vishik, *Attractors of evolution equations*, Stud. Math. Appl. **25** (1992), North-Holland.

[11] ———, *Regular attractors of semigroups and evolution equations*, J. Math. Pures Appl. **62** (1983), 441–491.

[12] S.M. Bruschi, A.N. Carvalho, J.W. Cholewa and T. Dlotko, *Uniform exponential dichotomy and continuity of attractors for singularly perturbed damped wave equations*, J. Dynam. Differential Equations **18** (2006), 767–814.

[13] V.L. Carbone, A.N. Carvalho and K. Schiabel-Silva, *Continuity of attractors for parabolic problems with localized large diffusion*, Nonlinear Anal. **68** (2008), 515–535.

[14] A.N. Carvalho and J.W. Cholewa, *Exponential global attractors for semigroups in metric spaces with applications to differential equations*, Ergodic Theory Dynam. Systems (to appear).

[15] A.N. Carvalho and T. Dlotko, *Dynamics of the viscous Cahn–Hilliard equation*, J. Math. Anal. Appl. **344** (2008), 703–325.

[16] A.N. Carvalho and S. Piskarev, *A general approximation scheme for attractors of abstract parabolic problems*, Numer. Funct. Anal. Optim. **27** (2006), 785–829.

[17] J.K. Hale and G. Raugel, *Lower semicontinuity of attractors of gradient systems and applications*, Ann. Mat. Pura Appl. **154** (1989), 281–326.

[18] D.B. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Lecture Notes in Mathematics, vol. 840, Springer–Verlag, 1981.

[19] T. Kato, *Perturbation Theory of Linear Operators*, Springer–Verlag, 1976.

[20] A. Rodriguez-Bernal, *Localized spatial homogenization and large diffusion*, SIAM J. Math. Anal. **29** (1998), 1361–1380.

*Manuscript received September 4, 2011*

José M. Arrieta
Departamento de Matemática Aplicada
Facultad de Matemáticas
Universidad Complutense de Madrid
28040 Madrid, SPAIN

*E-mail address*: arrieta@mat.ucm.es

Flank D.M. Bezerra
Universidade Federal da Paraíba
Centro de Ciências Exatas e da Natureza
Cidade Universitária-Campus I
58051-900 João Pessoa PB, BRAZIL

*E-mail address*: flank.bezerra@gmail.com

Alexandre N. Carvalho
Instituto de Ciências Matemáticas e de Computação
Universidade de São Paulo-Campus de São Carlos
Caixa Postal 668
13560-970 São Carlos SP, BRAZIL

*E-mail address*: andcarva@icmc.usp.br