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CENTRAL POINTS AND MEASURES, AND DENSE SUBSETS OF COMPACT METRIC SPACES

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Abstract. For every nonempty compact convex subset K of a normed linear space a (unique) point $c_K \in K$, called the generalized Chebyshev center, is distinguished. It is shown that c_K is a common fixed point for the isometry group of the metric space K. With use of the generalized Chebyshev centers, the central measure μ_X of an arbitrary compact metric space X is defined. For a large class of compact metric spaces, including the interval $[0,1]$ and all compact metric groups, another 'central' measure is distinguished, which turns out to coincide with the Lebesgue measure and the Haar one for the interval and a compact metric group, respectively. An idea of distinguishing infinitely many points forming a dense subset of an arbitrary compact metric space is also presented.

1. Introduction

Distinguishing points, subsets or other 'ingredients' related to spaces is important in many parts of mathematics, including algebraic topology (homotopy groups), theory of Lipschitz functions (the base point), theory of locally compact groups (the Haar measure, unique up to a constant factor). In most of algebraic structures the neutral element is a naturally distinguished point. In other areas of mathematics distinguishing appears as a useful tool. For example, the well-known *Chebyshev center* of a nonempty compact convex subset of a *strictly* convex normed linear space (i.e. such a space in which the unit sphere contains

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no segments [6, p. 30]) finds an application in fixed point theory, being a common fixed point for the isometry group of the convex set. The characteristic and important feature of some of the above examples is the uniqueness, in a categorical or weaker sense, of the distinguished ingredients. In such cases this distinguished ingredient may be seen as an integral part of the space (e.g. the Haar measure of a locally compact group or the neutral element of an algebraic structure), while in the others it plays an additional role (e.g. in homotopy groups, spaces of Lipschitz functions). In these cases the distinguishing is just a necessity and it hardly ever finds applications. The foregoing examples show that the situation changes when the distinguished ingredient turns out to be uniquely determined by some natural conditions. (The precise meaning of this in category of metric spaces shall be explained in the next section.)

The aim of this paper is to present a few results dealing with constructive 'applied' distinguishings. In particular, we shall show that every nonempty compact metric space X isometric to a convex subset of a normed linear space (even of a more general class, containing all metric $\mathbb{R}$-trees) contains a unique point c_X (called the *generalized Chebyshev center*) which is in a sense its center. As an application of this, we shall prove that the isometry group of each such space has a common fixed point. This gives a constructive proof of Kakutani's fixed point theorem in a special case. Details are included in Section 3.

In Section 4 we shall apply the results of the previous part to an arbitrary (nonempty) compact metric space X in order to define the *central* (probability Borel) measure μ_X of X by means of the so-called Kantorovich (or Kantorovicz–Rubenstein, cf. [21]) metric induced by the metric of X. In case of a compact metric group G, μ_G turns out to be the Haar measure of G and thus we shall obtain an alternative proof of the Haar measure theorem for compact metrizable groups. However, the problem of whether $\mu_{[0,1]}$ is the one-dimensional Lebesgue measure we leave as open. Section 5 deals with the so-called *quasi-nilpotent* compact metric spaces for which we shall prove another result on distinguishing measures. As a special case we shall obtain the characterizations of the Lebesgue measure on $[0, 1]$ and (again) the Haar measure of a compact metric group. The last, sixth, part is devoted to distinguishing countable dense subsets in arbitrary compact metric spaces, which is related to theory of random metric spaces (see e.g. [19], [20]).

2. Preliminaries

In this paper we deal with categories of metric spaces with additional structures in which every *isomorphism* between spaces is an isometric function between them. For simplicity, let us call each such a category an *iso-category*.

We shall write $K \in \mathcal{K}$ to express that K is a metric space with an additional structure which belongs to an iso-category $\mathcal{K}$.

Let $\mathcal{K}$ be an iso-category. For any two members X and Y of $\mathcal{K}$ let $\mathrm{Iso}_{\mathcal{K}}(X,Y)$ stand for the set of all isomorphisms of X onto Y. We write $\mathrm{Iso}_{\mathcal{K}}(X)$ for $\mathrm{Iso}_{\mathcal{K}}(X,X)$. If no additional structures on metric spaces are needed to describe the category $\mathcal{K}$, we shall write simply $\mathrm{Iso}(X,Y)$ and $\mathrm{Iso}(X)$.

For $X \in \mathcal{K}$ let '$\sim_{\mathcal{K}}$' be the equivalence relation on X given by

$$x \sim_{\mathcal{K}} y \iff \Phi(x) = y \quad \text{for some } \Phi \in \mathrm{Iso}_{\mathcal{K}}(X);$$

let $X^{(1)}$ be the quotient set $X/{\sim_{\mathcal{K}}}$ and $\pi_X^{(1)}\colon X \to X^{(1)}$ the canonical projection. Similarly, for any isomorphism $\Phi \in \mathrm{Iso}_{\mathcal{K}}(Y,Z)$ between spaces $Y,Z \in \mathcal{K}$ let $\Phi^{(1)}\colon Y^{(1)} \to Z^{(1)}$ be the unique function such that $\pi_Z^{(1)} \circ \Phi = \Phi^{(1)} \circ \pi_Y^{(1)}$.

Definition 2.1. Let $\mathcal{K}$ be an iso-category. By a *distinguishing* in $\mathcal{K}$ we mean any assignment $\mathcal{K} \ni X \mapsto C_X \in X^{(1)}$ such that whenever $\Phi \in \mathrm{Iso}_{\mathcal{K}}(K,L)$ with $K, L \in \mathcal{K}$, then $\Phi^{(1)}(C_K) = C_L$.

More natural approach to distinguishing is the following: to each space $X \in \mathcal{K}$ assign a point $c_X \in X$ in such a way that whenever K and L are two isomorphic members of $\mathcal{K}$, there is an isomorphism $\Phi\colon K \to L$ which sends c_K to c_L. However, we are interested in constructive methods of distinguishing (so, without using the axiom of choice) and thus the original Definition 2.1 is more appropriate.

A very special and the most important case of distinguishing appears when the distinguished equivalence class C_K consists of a single point and then we may consider C_K as an element of K. To make this precise, we put

Definition 2.2. By a *strict distinguishing* in an iso-category $\mathcal{K}$ we mean any assignment $\mathcal{K} \ni X \mapsto c_X \in X$ such that

$$\Phi(c_K) = c_L \tag{2.1}$$

for any $K, L \in \mathcal{K}$ and each $\Phi \in \mathrm{Iso}_{\mathcal{K}}(K,L)$.

Strict distinguishings appear very rarely in mathematics, which the following immediate result witnesses to

Proposition 2.3. *If $\mathcal{K} \ni X \mapsto c_X \in X$ is a strict distinguishing in an iso-category $\mathcal{K}$, then for every $K \in \mathcal{K}$, c_K is a common fixed point for the group $\mathrm{Iso}_{\mathcal{K}}(K)$. That is, $\Phi(c_K) = c_K$ for all $\Phi \in \mathrm{Iso}_{\mathcal{K}}(K)$.*

Proof. Just substitute $L = K$ in (2.1). □

Since there are iso-categories $\mathcal{K}$ (even among those of nonempty compact spaces) in which for some spaces $K \in \mathcal{K}$ the group $\mathrm{Iso}_{\mathcal{K}}(K)$ has no common fixed point, a strict distinguishing is not always possible. In the next section we introduce an iso-category for which this is realizable.

3. Weakly convex compact metric spaces

In the literature there are two main approaches to the notion of convexity in metric spaces. The first is related to joining points by line segments, the second relies on emphasizing and focusing on some aspects of the global position of the middle point between two points in convex subsets of normed linear spaces and adapting this to arbitrary metric spaces. Although the key condition is fulfilled by the middle point (in convex sets), usually this condition does not determine the middle point, that is, there are other points which satisfy it. One of such approaches is proposed in the definition below. Other ideas are recalled in Examples 3.2.

Definition 3.1. A metric space (X, d) is said to be *weakly convex* if and only if for any two points x and y of X there is a point $z \in X$ such that for each $w \in X$:

(C1) $d(z, w) \leq \max(d(x, w), d(y, w))$,

(C2) $d(x, w) = d(y, w)$ provided $d(z, w) = \max(d(x, w), d(y, w))$.

The set of all points $z \in X$ which satisfy (C1) and (C2) for fixed $x, y \in X$ and all $w \in X$ is denoted by $M(x, y) = M_X(x, y)$.

The class of weakly convex metric spaces includes all known to us convex metric spaces defined by describing the global position of a special point related to two other ones. Examples are given below.

Examples 3.2. (a) Takahashi [18] calls a metric space (X, d) convex if and only if for any $x, y \in X$ and every $\lambda \in (0, 1)$ there is a point $z_\lambda \in X$ such that

$$d(z_\lambda, w) \leq (1 - \lambda)d(x, w) + \lambda d(y, w) \quad \text{for all } w \in X. \tag{3.1}$$

(b) Kijima [8] and Yang and Zhang [22] speak about convexity when (3.1) with $\lambda = 1/2$ is fulfilled.

(c) Kindler [9] says about φ-convexity for any continuous concave, nondecreasing in both variables function $\varphi: \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\varphi(x, y) < \max(x, y) \tag{3.2}$$

whenever $x \neq y$. Namely, a metric space (X, d) is φ-*convex* (or Ψ-convex with respect to φ) if for any two points $x, y \in X$ there is a point $z \in X$ such that $d(w, z) \leq \varphi(d(w, x), d(w, y))$ for all $w \in X$.

Let us show that every 'convex' metric space defined by any of the conditions (a)–(c) is weakly convex. To do this, it suffices to check (c). Indeed, (a) is stronger than (b) and (b) is equivalent to μ-convexity with $\mu(x, y) = \frac{1}{2}x + \frac{1}{2}y$.

Now if φ, (X, d), x, y and z are as in (c), then $z \in M(x, y)$: for any $w \in X$ we have $d(w, z) \leq \varphi(d(w, x), d(w, y)) \leq \max(d(w, x), d(w, y))$ (the last inequality follows from (3.2) and the continuity of φ) which yields (C1); (C2) follows from (3.2). In particular, every convex subset of a normed linear space is weakly convex ($z_\lambda := (1 - \lambda)x + \lambda y$ witnesses (a)). Further examples follow.

(d) All metric $\mathbb{R}$-trees are weakly convex. Recall that a complete metric space T is said to be an *$\mathbb{R}$-tree* if for any two distinct points x and y of T there is a unique homeomorphic copy $\gamma_{x,y}$ of the interval $[0, 1]$ which joins x and y; and $\gamma_{x,y}$ is isometric to a line segment (cf. [10]). Now if T is an $\mathbb{R}$-tree and x and y are arbitrary two distinct points of T (when $x = y$, it is enough to put $z = x$), let e.g. z be the middle point of $\gamma_{x,y}$. For $w \in T$ we distinguish between two cases. When $A := \gamma_{x,y} \cup \gamma_{x,w}$ is homeomorphic to $[0, 1]$, it follows from the definition of an $\mathbb{R}$-tree that A is isometric to a line segment. Thus the assertion (i.e. conditions (C1) and (C2)) follows from the weak convexity of intervals. Hence, we may assume that A is non-homeomorphic to $[0, 1]$. This implies that there is a unique point v such that $v \in \gamma_{x,y} \cap \gamma_{x,w} \cap \gamma_{w,y}$. Then $\gamma_{a,b} = \gamma_{a,v} \cup \gamma_{v,b}$ for distinct $a, b \in \{x, y, w\}$. With no loss on generality we may assume $z \in \gamma_{x,v}$. But then $d(x, w) = d(x, z) + d(z, w)$. Since $d(x, z) > 0$, both (C1) and (C2) are satisfied.

(e) We leave this as an exercise that the set of rationals and $\mathbb{R} \setminus \{0\}$ (with natural metrics) are weakly convex.

The above example (e) shows that there exist totally disconnected as well as disconnected locally compact metric spaces which are weakly convex. In the next result we shall prove that every compact weakly convex space is both connected and locally connected.

A set $A \subset X$ is said to be a *fully convex subset* of a weakly convex space X if and only if $M(a, b) \subset A$ for all $a, b \in A$. It follows from the definition that in a weakly convex metric space X:

(FC1) a fully convex subset A of X is itself a weakly convex metric space as well and $M_A(a, b) \supset M_X(a, b)$ for any $a, b \in A$,

(FC2) a fully convex subset of a fully convex subset of X is a fully convex subset of X,

(FC3) the intersection of a nonempty family of fully convex subsets of X is a fully convex subset of X as well.

Proposition 3.3. *Let (X, d) be a weakly convex metric space.*

(a) *If x and y are distinct points of X, then $\max(d(z, x), d(z, y)) < d(x, y)$ for any $z \in M(x, y)$.*

(b) *Open and closed balls in X are fully convex subsets.*

(c) *X is connected and locally connected provided X is compact.*

Proof. (a) Put $w = x$. Since $d(x,w) \neq d(y,w)$, (C1) and (C2) give $d(z,x) < d(x,y)$. Similarly, $d(z,y) < d(x,y)$.

(b) Let a be the center of a ball $B \subset X$. If $x, y \in B$ and $z \in M(x,y)$, then $d(a,z) \leq \max(d(a,x), d(a,y))$. Consequently, $z \in B$ and we are done.

(c) Assume X is compact. Since closed balls in X are compact and weakly convex (by (b)) as well, it suffices to verify that X is connected. Suppose, for the contrary, that X is disconnected. Let K and L be two nonempty disjoint compact subsets of X such that $X = K \cup L$. Take $x \in K$ and $y \in L$ with

$$d(x,y) = \min\{d(u,v) : u \in K,\ v \in L\} \tag{3.3}$$

and let $z \in M(x,y)$. Say $z \in K$. Then, by (a), $d(z,y) < d(x,y)$ which contradicts (3.3). □

Our aim is to construct a strict distinguishing in the class $\mathcal{WCC}$ of all nonempty weakly convex compact metric spaces (where the category is determined only by metrics). As a corollary, we shall obtain a theorem on common fixed points in weakly convex compact metric spaces.

Now we shall recall the classical atributes of a metric space (see e.g. [3] or [9]). By $\delta(X)$ we denote the *diameter* of a metric space (X,d), that is, $\delta(X) := \sup_{x,y\in X} d(x,y) \in [0,+\infty]$ provided X is nonempty and $\delta(\emptyset) := 0$. For each $x \in X$ let $r_X(x) := \sup_{y\in X} d(x,y)$ and let $r(X) := \inf_{x\in X} r_X(x)$ ($r(\emptyset) := 0$). The number $r(X) \in [0,+\infty]$ is called the *Chebyshev radius* of X. Finally, the *Chebyshev center* of X is the set $C(X) := \{x \in X : r_X(x) = r(X)\}$. If the last set consists of a single point, the unique element of $C(X)$ is also called the *Chebyshev center* of X. The classical result states that $C(K)$ is a singleton provided K is a nonempty compact convex subset of a strictly convex normed linear space. If the assumption of strict convexity of the norm is relaxed, the set $C(K)$ may be infinite. However, $C(X)$ is nonempty for every nonempty compact metric space X.

In order to define the generalized Chebyshev center (as a uniquely determined point of a space), we introduce the following

Definition 3.4. The *n-th Chebyshev center*, $C^n(X)$, of a metric space X is given by the recursive formula: $C^0(X) := X$ and $C^n(X) := C(C^{n-1}(X))$ for $n > 0$. Additionally, let $C^\infty(X) := \bigcap_{n=0}^\infty C^n(X)$.

Our goal is to show that $C^\infty(X)$ consists of a single point provided X is a nonempty weakly convex compact metric space. To show this, we need the next result. It was proved (in a different way) in special cases by Takahashi [18] and Kindler [9].

Lemma 3.5. *If (X,d) is a weakly convex compact metric space having more than one point, then $r(X) < \delta(X)$.*

Proof. Let us first show, by an induction argument, that for every $n \geq 1$ and any $x_1, \dots, x_n \in X$ there is a point $z = z(x_1, \dots, z_n) \in X$ such that for each $w \in X$,

(CC1) $d(z, w) \leq \max(d(x_1, w), \dots, d(x_n, w))$,
(CC2) $d(x_1, w) = \ldots = d(x_n, w)$ provided (CC1) is fulfilled with the equality sign.

For $n = 1$ put $z = x_1$ and for $n = 2$ take any $z \in M(x_1, x_2)$. Now assume $n \geq 3$ and that the above assertion holds true for $n - 1$. Put $z' = z(x_1, \dots, x_{n-1})$ and let $z \in M(z', x_n)$. Then for any $w \in X$,

$$d(z, w) \leq \max(d(z', w), d(x_n, w)) \leq \max(d(x_1, w), \dots, d(x_n, w))$$

(by the definition of z'). And

$$d(z, w) = \max(d(x_1, w), \dots, d(x_n, w)) \implies d(z, w) = \max(d(z', w), d(x_n, w)).$$

So, $d(z', w) = d(x_n, w)$ (since $z \in M(z', x_n)$). We infer from this that $d(z', w) = \max(d(x_1, w), \dots, d(x_{n-1}, w))$ and hence $d(x_1, w) = \ldots = d(x_{n-1}, w) = d(x_n, w)$. This proves (CC1) and (CC2).

Now suppose, for the contrary, that $r(X) = \delta(X)$. By the compactness, there is a maximal finite system $x_1, \dots, x_n$ of elements of X such that $d(x_j, x_k) = \delta(X)$ whenever $j \neq k$. Let $z \in X$ be a point satisfying (CC1) and (CC2) for $x_1, \dots, x_n$. By our assumption, $r_X(z) = \delta(X)$ and hence there is $w \in X$ such that $d(z, w) = \delta(X)$. But then, by (CC2), $d(x_1, w) = \ldots = d(x_n, w) = \delta(X)$ which contradicts the maximality of the system $x_1, \dots, x_n$. □

Proposition 3.6. *For every nonempty weakly convex compact metric space X the set $C^\infty(X)$ consists of a single point.*

Proof. First we shall check that for every weakly convex metric space (Y, ϱ),

(3.4) $$C^1(Y) \text{ is a fully convex subset of } Y.$$

For this, let $a, b \in C^1(Y)$ and $z \in M_Y(a, b)$. We infer from (C1) that $r_Y(z) \leq \max(r_Y(a), r_Y(b)) = r(Y)$. Since always $r(Y) \leq r_Y(z)$, we see that $z \in C^1(Y)$ and (3.4) is proved.

Now we pass to the main proof. By the compactness of X and the closedness of all $C^n(X)$'s, $C^\infty(X)$ is nonempty and compact. What is more, the combination of (3.4) and (FC1)–(FC3) yields that $C^\infty(X)$ is a weakly convex space. Take $z \in C^\infty(X)$. For every natural n, z belongs to $C(C^n(X))$ and hence there is $y_n \in C^n(X)$ for which $d(z, y_n) = r(C^n(X))$. By the definitions of the Chebyshev center and the Chebyshev radius, $r(Y) \geq \delta(C(Y))$ for every metric space Y. We infer from this that $r(C^n(X)) \geq \delta(C^{n+1}(X))$ which implies that

(3.5) $$d(z, y_n) \geq \delta(C^\infty(X)) \quad (n \in \mathbb{N}).$$

Now let $(y_{n_k})_{k=1}^{\infty}$ be a subsequence of $(y_n)_{n=1}^{\infty}$ which converges to some $y \in X$. Then $y \in C^{\infty}(X)$ and hence $d(z,y) = \delta(C^{\infty}(X))$, thanks to (3.5). This shows that $r_{C^{\infty}(X)}(z) = \delta(C^{\infty}(X))$ for every $z \in C^{\infty}(X)$ and thus $r(C^{\infty}(X)) = \delta(C^{\infty}(X))$. Now it suffices to apply Lemma 3.5 to finish the proof. □

Definition 3.7. Let X be a nonempty weakly convex compact metric space. The unique point of $C^{\infty}(X)$ is called the *generalized Chebyshev center* of X and is denoted by c_X.

The reader should notice that if a weakly convex compact metric space X has Chebyshev center (a point), then this point coincides with the generalized Chebyshev center of X (because then $C^1(X) = C^{\infty}(X)$). This justifies the undertaken terminology.

We can now formulate our first result on strict distinguishing.

Theorem 3.8. *Let $\mathcal{WCC}$ be the class of all nonempty weakly convex compact metric spaces. The assignment $\mathcal{WCC} \ni X \mapsto c_X \in X$ is a strict distinguishing.*

Proof. Let $X, Y \in \mathcal{WCC}$ and let $\Phi \in \operatorname{Iso}(X,Y)$. It follows from the definition of the Chebyshev center that $\Phi(C^1(X)) = C^1(Y)$ and hence, by induction, $\Phi(C^n(X)) = C^n(Y)$ for each $n \in \mathbb{N}$. Consequently, $\Phi(C^{\infty}(X)) = C^{\infty}(Y)$ and we are done. □

The above result combined with Proposition 2.3 yields

Corollary 3.9. *Let $X \in \mathcal{WCC}$. For every isometry Φ of X onto X, $\Phi(c_X) = c_X$.*

When X is a convex subset of a normed linear space, Corollary 3.9 is a special case of Kakutani's fixed point theorem on equicontinuous group of affine transformations ([7]; or [17]). The proof presented here is constructive. However, it works only for the specific group — the isometry one.

Remark 3.10. The problem whether every (bijective) isometry between convex subsets of normed linear spaces is affine seems to be still open. (Beside the classical Mazur–Ulam theorem ([13]; or [1, 14.1]), the author knows only one general result [12] in this direction.) If there was a compact convex subset in a normed linear space admitting a non-affine isometry, then Corollary 3.9 would be stronger than Kakutani's fixed point theorem (in this specific case).

4. Central measure

In this section we apply the results of the previous part to distinguish a measure on a compact metric space. To do this, let us fix a nonempty compact metric space (X,d). Denote by $\mathrm{Prob}(X)$ the set of all probabilistic Borel measures on X. Equip $\mathrm{Prob}(X)$ with the metric $\widehat{d}$ given by the formula

$$\widehat{d}(\mu,\nu) = \sup\left\{\left|\int_X f\,d\mu - \int_X f\,d\nu\right| : f \in \mathrm{Contr}(X,\mathbb{R})\right\} \tag{4.1}$$

where $\mu,\nu \in \mathrm{Prob}(X)$ and $\mathrm{Contr}(X,\mathbb{R})$ stands for the family of all d-nonexpansive maps of X into $\mathbb{R}$. The metric $\widehat{d}$ is called the *Kantorovich* (or *Kantorovich–Rubenstein*, cf. [21, Definition 2.3.1]) metric induced by d.

The space $(\mathrm{Prob}(X),\widehat{d}\,)$ is compact and $\widehat{d}$ induces on $\mathrm{Prob}(X)$ the topology inherited, thanks to the Riesz characterization theorem, from the weak-* topology of the dual Banach space of $\mathcal{C}(X,\mathbb{R})$. To see that $(\mathrm{Prob}(X),\widehat{d}\,)$ is weakly convex, let us briefly show that it is affinely isometric to a convex subset of a normed space. For this, let $\mathcal{M}(X)$ be the real vector space of all Borel signed (that is, real-valued) measures μ on X with $\mu(X)=0$. For each $\mu \in \mathcal{M}(X)$ put

$$\|\mu\| = \sup\left\{\left|\int_X f\,d\mu\right| : f \in \mathrm{Contr}(X,\mathbb{R})\right\}.$$

The above defined function $\|\cdot\|$ is a norm on $\mathcal{M}(X)$ ($\|\mu\| < \infty$ because $\|\mu\| \leq \delta(X)|\mu|(X)$ where $|\mu|$ is the variation of μ). Now if we fix a point $a \in X$, the formula $\mathrm{Prob}(X) \ni \mu \mapsto \mu - \delta_a \in \mathcal{M}(X)$ well defines an affine isometric embedding (here δ_a is Dirac's measure at a, i.e. $\delta_a(A) = 1$ if $a \in A$ and $\delta_a(A) = 0$ otherwise).

We may now introduce

Definition 4.1. The generalized Chebyshev center of $(\mathrm{Prob}(X),\widehat{d}\,)$ is called the *central measure* of X and it is denoted by μ_X.

Notice that every isometry $\Phi\colon X \to X$ induces an affine isometry $\widehat{\Phi}\colon \mathrm{Prob}(X) \to \mathrm{Prob}(X)$ given by the formula $\widehat{\Phi}(\mu) = \mu \circ \Phi^{-1}$ where $\mu \circ \Phi^{-1}$ denotes the transport of the measure $\mu \in \mathrm{Prob}(X)$ under the transformation Φ (that is, $(\mu \circ \Phi^{-1})(A) = \mu(\Phi^{-1}(A))$). We conclude from Corollary 3.9 that $\widehat{\Phi}(\mu_X) = \mu_X$ for all $\Phi \in \mathrm{Iso}(X)$; that is, μ_X is an invariant measure for the isometry group of X. Again, we have obtained a constructive proof that the isometry group of an arbitrary (nonempty) compact metric space admits an invariant measure.

Now suppose that $\mathrm{Iso}(X)$ acts transitively on X, i.e. for each two points x and y of X there is $\Phi \in \mathrm{Iso}(X)$ with $\Phi(x) = y$. It is known that in that case there is a unique measure invariant under every isometry of X (see e.g. [14, Theorem 2.5]). So, we get

Proposition 4.2. *If the isometry group of X acts transitively on X, μ_X is the unique measure invariant under every isometry of X.*

By a *metric* group we mean a metrizable topological group equipped with a left-invariant metric inducing the topology of the group (there exists one, see e.g. [2]). As a special case of Proposition 4.2 we obtain

Corollary 4.3. *Let G be a compact metric group. The central measure of G is the Haar measure of G.*

Corollary 4.3 provides a new constructive proof of the Haar measure theorem for metrizable compact groups.

Although in compact metric spaces with transitive actions of the isometry groups the central measures may be found thanks to their very specific properties, unfortunately computing the central measure in general is very complicated. For example, we do not know the one of $[0,1]$. The reader interested in this problem may try first to compute the central measure of a three-point space (with non-discrete metric).

Problem 4.4. Is $\mu_{[0,1]}$ the Lebesgue measure?

5. Quasi-nilpotent compact metric spaces

Although we do not know whether the central measure of the unit interval is the Lebesgue measure, we are able to make another distinguishing of measures in a special class of compact metric spaces in such a way that the distinguished measure for the unit interval will be the Lebesgue measure. This will be done in this section.

Recall (see Section 2) that for a metric space (X,d) the set $X^{(1)}$ is the set of all orbits of points of X under the natural action of the isometry group of X. It turns out that $X^{(1)}$ may be topologized by an 'axiomatically' defined metric when (X,d) is compact. Precisely, we denote by $d^{(1)}$ the greatest pseudometric on $X^{(1)}$ which makes the canonical projection $\pi_X^{(1)}\colon (X,d) \to (X^{(1)}, d^{(1)})$ nonexpansive. For an arbitrary metric space (X,d), $d^{(1)}$ may not be a metric. However, we have

Proposition 5.1. *For every compact metric space (X,d), $d^{(1)}$ is a metric on $X^{(1)}$. Moreover, for each $x,y \in X$,*

$$(5.1)\quad d^{(1)}(\pi_X^{(1)}(x), \pi_X^{(1)}(y)) = \sup\{|f(\pi_X^{(1)}(x)) - f(\pi_X^{(1)}(y))|\colon \\ f\colon X^{(1)} \to \mathbb{R},\ f \circ \pi_X^{(1)} \text{ is } d\text{-nonexpansive}\}.$$

Proof. First we shall verify (5.1). If $f\colon X^{(1)} \to \mathbb{R}$ is such that $f \circ \pi_X^{(1)}$ is d-nonexpansive, then the formula $\varrho\colon X^{(1)} \times X^{(1)} \ni (\xi,\eta) \mapsto |f(\xi) - f(\eta)| \in \mathbb{R}_+$ defines a pseudometric on $X^{(1)}$ with respect to which $\pi_X^{(1)}$ is nonexpansive. Thus, $d^{(1)} \geq \varrho$. This shows the inequality "$\geq$" in (5.1). To prove the inverse one, for $x \in X$ put $f_x\colon X^{(1)} \ni \xi \mapsto d^{(1)}(\xi, \pi_X^{(1)}(x)) \in \mathbb{R}$. Clearly, f_x is $d^{(1)}$-nonexpansive and consequently (since $\pi_X^{(1)}$ is nonexpansive as well) $f_x \circ \pi_X^{(1)}$ is d-nonexpansive. Moreover, $|f_x(\pi_X^{(1)}(x)) - f_x(\pi_X^{(1)}(y))| = d^{(1)}(\pi_X^{(1)}(x), \pi_X^{(1)}(y))$ for every $y \in X$. Since x was arbitrary, this finishes the proof of (5.1).

Now we shall show that $d^{(1)}$ is a metric. We shall do this with use of the variation of the Gromov–Hausdorff metric [4], [5] (see also [16]). Namely, let $X_j = X \times \{j\}$ ($j = 0,1,2$) and for distinct $j,k \in \{0,1,2\}$ let $\mathfrak{P}(j,k)$ be the family of all pseudometrics p on $X_j \cup X_k$ such that $p((x,s),(y,s)) = d(x,y)$ for any $x,y \in X$ and $s \in \{j,k\}$. Of course, $\mathfrak{P}(1,2)$ may naturally be identified with $\mathfrak{P}(0,1)$ as well as with $\mathfrak{P}(0,2)$, which shall be used later. For a and b in X let

$$\varrho(a,b) = \inf\{p_H(X_1,X_2) + p((a,1),(b,2)) : p \in \mathfrak{P}(1,2)\}$$

where $p_H(A,B)$ is the Hausdorff distance induced by p. (In other words, $\varrho(a,b)$ is a counterpart of the Gromov-Hausdorff distance for pointed metric spaces (X,a) and (X,b).) Let us check that ϱ is a pseudometric on X such that for all $a,b \in X$,

$$\varrho(a,b) = 0 \iff \exists\, \Phi \in \operatorname{Iso}(X) : \Phi(a) = b.$$

Observe that ϱ is symmetric, because for every $p \in \mathfrak{P}(1,2)$ also $p^{\#} \in \mathfrak{P}(1,2)$ and $p^{\#}_H(X_1,X_2) = p_H(X_1,X_2)$ where

$$p^{\#}((x,j),(y,k)) = p((x,k),(y,j));$$

and $\varrho(a,\Phi(a)) = 0$ for any $\Phi \in \operatorname{Iso}(X)$ (to convince of that consider a pseudometric p on $X_1 \cup X_2$ given by $p((x,j),(y,k)) = d(\Psi_j(x),\Psi_k(y))$ where $\Psi_1(x) = x$ and $\Psi_2 = \Phi$). To establish the triangle inequality for $a,b,c \in X$, for $\varepsilon > 0$ take pseudometrics $\lambda \in \mathfrak{P}(0,1)$ and $\mu \in \mathfrak{P}(0,2)$ such that $\varrho(a,b) + \varepsilon \geq \lambda_H(X_0,X_1) + \lambda((a,1),(b,0))$ and $\varrho(b,c) + \varepsilon \geq \mu_H(X_0,X_2) + \mu((b,0),(c,2))$. Since λ and μ coincide on the intersection of their domains, that is, on $X_0 \times X_0$, we may extend both these pseudometrics to a pseudometric p on $X_0 \cup X_1 \cup X_2$ by a classical method, namely by putting for $x,y \in X$:

$$p((x,1),(y,2)) = \inf\{\lambda((x,1),(z,0)) + \mu((z,0),(y,2)):\ z \in X\}.$$

Then the restriction of p belongs to $\mathfrak{P}(1,2)$ and therefore

$$\begin{aligned}
\varrho(a,c) &\leq p_H(X_1,X_2) + p((a,1),(c,2)) \\
&\leq p_H(X_1,X_0) + p_H(X_0,X_2) + p((a,1),(b,0)) + p((b,0),(c,2)) \\
&= \lambda_H(X_1,X_0) + \lambda((a,1),(b,0)) + \mu_H(X_0,X_2) + \mu((b,0),(c,2)) \\
&\leq \varrho(a,b) + \varrho(b,c) + 2\varepsilon
\end{aligned}$$

which finishes the proof of the triangle inequality. It remains to prove that if $\varrho(a,b) = 0$, then there is $\Phi \in \operatorname{Iso}(X)$ such that $\Phi(a) = b$. For need of this, we may and do assume that X has more than one point, that is, that $\delta(X) > 0$. Define a metric $\lambda \in \mathfrak{P}(1,2)$ by putting $\lambda((x,1),(y,2)) = \delta(X)$ for every $x, y \in X$ and put $\mathfrak{P} = \{p \in \mathfrak{P}(1,2)\colon\ p \leq \lambda\}$. Additionally, let Λ be a metric on $Z := (X_1 \cup X_2) \times (X_1 \cup X_2)$ given by $\Lambda((\xi_1, \eta_1), (\xi_2, \eta_2)) = \lambda(\xi_1, \xi_2) + \lambda(\eta_1, \eta_2)$ ($\xi_j, \eta_j \in X_1 \cup X_2$). Since for every $p \in \mathfrak{P}(1,2)$, $\min(p, \delta(X)) \in \mathfrak{P}$ and (of course) $\min(p, \delta(X)) \leq p$, we see that

$$\varrho(x, y) = \inf\{p_H(X_1, X_2) + p((x,1),(y,2)) : p \in \mathfrak{P}\}.$$

Observe that $\mathfrak{P} \subset \operatorname{Contr}(Z, [0, \delta(X)])$ (when Z is equipped with the metric Λ) and $\mathfrak{P}$ is closed in the topology of pointwise convergence of $[0, \delta(X)]^Z$. Hence, by the Ascoli theorem, $\mathfrak{P}$ is compact in the topology of uniform convergence. What is more, for every $(\xi, \eta) \in Z$, a function $\mathfrak{P} \ni p \mapsto p(\xi, \eta) \in \mathbb{R}$ is (Λ-)nonexpansive. We infer from this that also functions $\mathfrak{P} \ni p \mapsto \inf_{\xi \in X_j} p(\eta, \xi) \in \mathbb{R}$ and $\mathfrak{P} \ni p \mapsto \sup_{\eta \in X_k}(\inf_{\xi \in X_j} p(\eta, \xi)) \in \mathbb{R}$ are nonexpansive as well. Consequently, $\mathfrak{P} \ni p \mapsto p_H(X_1, X_2) \in \mathbb{R}$ is continuous and it follows from the compactness of (Z, Λ) that in fact

$$\varrho(x, y) = \min\{p_H(X_1, X_2) + p((x,1),(y,2)) : p \in \mathfrak{P}\}.$$

So, if $\varrho(a, b) = 0$, there exists $p \in \mathfrak{P}$ for which

$$p_H(X_1, X_2) = 0 \quad \text{and} \quad p((a,1),(b,2)) = 0. \tag{5.3}$$

Since p makes X_1 and X_2 compact metric spaces, the first relation in (5.3) implies that for every $x \in X$ there is a *unique* point $\Phi(x) \in X$ with $p((x,1),(\Phi(x),2)) = 0$. We have obtained in this way a function $\Phi\colon X \to X$. We conclude from the triangle inequality that $p((\Phi(x),2),(\Phi(y),2)) = p((x,1),(y,1))$ for any $x, y \in X$. Since $p \in \mathfrak{P}(1,2)$ and every isometric map of a compact metric space into itself is onto [11], this implies that $\Phi \in \operatorname{Iso}(X)$. Finally, the uniqueness of $\Phi(a)$ and the second relation in (5.3) yield $\Phi(a) = b$. This proves (5.2).

Now (5.2) implies that ϱ induces a *metric* ϱ^* on $X^{(1)}$ in such a way that $\varrho^*(\pi_X^{(1)}(x), \pi_X^{(1)}(y)) = \varrho(x, y)$ for all $x, y \in X$. Since $\varrho \leq d$ (for $p((x,1),(y,2)) := d(x,y)$ one obtains $p_H(X_1, X_2) = 0$ and consequently $\varrho(x,y) \leq p_H(X_1, X_2) + p((x,j),(y,k)) = d(x,y)$), $\pi_X^{(1)}$ is nonexpansive with respect to the metrics d and ϱ^*, and thus $d^{(1)} \geq \varrho^*$. $\square$

By Proposition 5.1, $(X, d)^{(1)} := (X^{(1)}, d^{(1)})$ is a compact metric space provided (X, d) is so. Thus we may repeat this construction to obtain subsequent spaces $X^{(2)}$, $X^{(3)}$ and so on. Namely, for a compact metric space let $(X^{(0)}, d^{(0)}) = (X, d)$ and $(X^{(n)}, d^{(n)}) = (X^{(n-1)}, d^{(n-1)})^{(1)}$ for $n > 0$. Notice that $\delta(X^{(n)}) \leq \delta(X^{(n-1)})$ and thus the sequence $(\delta(X^{(n)}))_{n=1}^{\infty}$ is convergent. We introduce the following

Definition 5.2. A compact metric space X is said to be *quasi-nilpotent* if and only if

$$\lim_{n \to \infty} \delta(X^{(n)}) = 0.$$

The class of quasi-nilpotent compact metric spaces includes all spaces on which their isometry groups act transitively. One may think that such spaces have to have rich isometry groups. The next example shows that it is not the rule.

Example 5.3. Let (X,d) be the interval $[a,b]$ with the natural metric. Observe that the isometry group of X is very poor — there is only one isometry on X different from the identity map, namely $\psi\colon X \ni x \mapsto a+b-x \in X$. However, X is quasi-nilpotent. To see this, it suffices to show that $X^{(1)}$ is isometric to $[a/2,b/2]$. Since $\mathrm{Iso}(X) = \{\mathrm{id}_X, \psi\}$, the set $X^{(1)}$ (with no metric) and the canonical projection $\pi_X^{(1)}$ may be represented as (respectively)

$$Y = \left[a, \frac{a+b}{2}\right] \quad \text{and} \quad \tau\colon X \ni x \mapsto \frac{a+b}{2} - \left|x - \frac{a+b}{2}\right| \in Y.$$

With use of (5.1) we shall check that the natural metric of Y corresponds under the above identification to $d^{(1)}$. In what follows the term 'nonexpansive' is understood with respect to natural metrics in both X and Y. It is easily seen that τ is nonexpansive. Finally, if $f\colon Y \to \mathbb{R}$ is any function such that $f \circ \tau$ is nonexpansive, then f is nonexpansive as well, because $\tau\big|_Y = \mathrm{id}_Y$. So, the assertion follows from (5.1).

Now we shall distinguish a special measure on a quasi-nilpotent (nonempty) compact metric space, which may also be called central. For a convex subset K of a normed linear space let $\mathrm{Fix}(K)$ be the set of all fixed points under every affine isometry of K onto K. The set K is convex as well (however, it may be empty). Further, let $\mathrm{Fix}^0(K) := K$ and for natural $n > 0$ let $\mathrm{Fix}^n(K) = \mathrm{Fix}(\mathrm{Fix}^{n-1}(K))$. Finally, put $\mathrm{Fix}^\infty(K) = \bigcap_{n=0}^\infty \mathrm{Fix}^n(K)$. Note that $\mathrm{Fix}^\infty(K)$ is convex and if K is compact, $\mathrm{Fix}^\infty(K)$ is nonempty.

Now let X be a nonempty compact metric space and $\mathrm{Prob}(X)$ be equipped with the Kantorovich metric induced by the metric of X.

Let $\Delta(X) = \mathrm{Fix}^\infty(\mathrm{Prob}(X))$. In the sequel we shall prove that $\Delta(X)$ consists of a single measure iff X is quasi-nilpotent. The proof of this fact is based on the next lemmas, some of which are already known.

For a continuous function $f\colon X \to Y$ between compact metrizable spaces X and Y let $f_*\colon \mathrm{Prob}(X) \to \mathrm{Prob}(Y)$ be given by $(f_*(\mu))(B) = \mu(f^{-1}(B))$ where $\mu \in \mathrm{Prob}(X)$ and B is a Borel subset of Y (in other words, $f_*(\mu)$ is the transport of μ under f).

The following result is well known. For reader's convenience, we give its proof.

Lemma 5.4. *For a compact metric space* (X,d)*, the assignment* $\Phi \mapsto \Phi_*$ *establishes a one-to-one correspondence between isometries* $[\Phi]$ *of* (X,d) *and affine isometries* $[\Phi_*]$ *of* $(\mathrm{Prob}(X), \widehat{d}\,)$.

Proof. First we shall check that Φ_* is an affine isometry provided Φ is an isometry. It follows from the definition that Φ_* is affine. Further, since every isometric map of a compact metric space into itself is onto [11], it suffices to check that Φ_* is isometric. But this follows from (4.1) and the following two relations: $\int_X f\, d\nu = \int_X f \circ \Phi\, d\mu$ where $\nu = \Phi_*(\mu)$; and $\mathrm{Contr}(X, \mathbb{R}) = \{f \circ \Phi : f \in \mathrm{Contr}(X, \mathbb{R})\}$.

Now assume that Ψ is an affine isometry of $\mathrm{Prob}(X)$. Then Ψ sends the set of all extreme points of $\mathrm{Prob}(X)$ onto itself. But extreme points of $\mathrm{Prob}(X)$ are precisely Dirac's measures. So, Ψ induces a bijection $\Phi: X \to X$ such that $\Psi(\delta_x) = \delta_{\Phi(x)}$ for any $x \in X$. Further, the relation $\widehat{d}(\delta_x, \delta_y) = d(x, y)$ implies that $\Phi \in \mathrm{Iso}(X)$. Finally, since Φ_* and Ψ are two affine isometries of $\mathrm{Prob}(X)$ which coincide on the set of extreme points, the Kreĭn–Milman theorem gives $\Psi = \Phi_*$. $\square$

The above result implies that $\mathrm{Fix}(\mathrm{Prob}(X))$ coincides with the set of all probabilistic Borel measures invariant under every isometry of X. This observation allows us to formulate the following two results.

Lemma 5.5 (special case of [14, Theorem 2.5]). *For every nonempty compact metric space* (X, d)*, the function*

$$\mathrm{Fix}(\mathrm{Prob}(X)) \ni \mu \mapsto (\pi_X^{(1)})_*(\mu) \in \mathrm{Prob}(X^{(1)})$$

is an affine homeomorphism.

Lemma 5.6 (special case of [15, Proposition 2.5]). *Let* (X, d) *be a compact metric space. For every continuous function* $v: X \to \mathbb{R}$ *the closed convex hull* (*in the topology of uniform convergence*) *of the set* $\{v \circ \Phi\colon\ \Phi \in \mathrm{Iso}(X)\}$ *contains a map* $w: X \to \mathbb{R}$ *such that*

$$w \circ \Phi = w \quad \text{for all } \Phi \in \mathrm{Iso}(X). \tag{5.4}$$

Lemma 5.6 may also be deduced from Kakutani's fixed point theorem (for equicontinuous groups of affine transformations of a compact convex set).

Now we have

Theorem 5.7. *For a nonempty compact metric space* (X, d) *the function*

$$\Psi: \mathrm{Fix}(\mathrm{Prob}(X)) \ni \mu \mapsto (\pi_X^{(1)})_*(\mu) \in \mathrm{Prob}(X^{(1)})$$

is an affine isometry of $\mathrm{Fix}(\mathrm{Prob}(X))$ *onto* $\mathrm{Prob}(X^{(1)})$*. In particular,*

$$\delta(\mathrm{Fix}(\mathrm{Prob}(X))) = \delta(X^{(1)}).$$

Proof. Since $\delta(\mathrm{Prob}(Y)) = \delta(Y)$ for every compact metric space Y, it suffices to prove the first assertion. By Lemma 5.5, Ψ is an affine bijection. So, we only need to check that Ψ is isometric. Fix $\mu_1, \mu_2 \in \mathrm{Fix}(\mathrm{Prob}(X))$ and put $\nu_j = \Psi(\mu_j)$. If $u \in \mathrm{Contr}(X^{(1)}, \mathbb{R})$, then

$$\int_{X^{(1)}} u\, d\nu_j = \int_X u \circ \pi_X^{(1)}\, d\mu_j.$$

This, combined with (4.1), gives $\widehat{d}(\mu_1, \mu_2) \geq \widehat{d^{(1)}}(\nu_1, \nu_2)$.

Conversely, if $v \in \mathrm{Contr}(X, \mathbb{R})$, then (since $\mu_j \in \mathrm{Fix}(\mathrm{Prob}(X))$ and thanks to Lemma 5.4) $\int_X v\, d\mu_j = \int_X v \circ \Phi\, d\mu_j$ for every $\Phi \in \mathrm{Iso}(X)$. Now by Lemma 5.6, the closed convex hull of the set $\{v \circ \Phi\colon\ \Phi \in \mathrm{Iso}(X)\}$ contains a map $w\colon X \to \mathbb{R}$ such that (5.4) is satisfied. This implies that $w \in \mathrm{Contr}(X, \mathbb{R})$ and $\int_X v\, d\mu_j = \int_X w\, d\mu_j$. We infer from (5.4) that there is $w_0\colon X^{(1)} \to \mathbb{R}$ such that $w = w_0 \circ \pi_X^{(1)}$. The last connection and (5.1) yield that $w_0 \in \mathrm{Contr}(X^{(1)}, \mathbb{R})$. So, we finally obtain

$$\begin{aligned}\left| \int_X v\, d\mu_1 - \int_X v\, d\mu_2 \right| &= \left| \int_X w_0 \circ \pi_X^{(1)}\, d\mu_1 - \int_X w_0 \circ \pi_X^{(1)}\, d\mu_2 \right| \\ &= \left| \int_{X^{(1)}} w_0\, d\nu_1 - \int_{X^{(1)}} w_0\, d\nu_2 \right| \leq \widehat{d^{(1)}}(\nu_1, \nu_2),\end{aligned}$$

which finishes the proof. □

Proposition 5.8. *If X is a nonempty compact metric space, then for each natural n, $\delta(\mathrm{Fix}^n(\mathrm{Prob}(X))) = \delta(X^{(n)})$.*

Proof. It suffices to show that $K_n := \mathrm{Fix}^n(\mathrm{Prob}(X))$ and $\mathrm{Prob}(X^{(n)})$ are affinely isometric. For $n = 0$ this is immediate. Assume K_{n-1} is affinely isometric to $\mathrm{Prob}(X^{(n-1)})$ for some $n > 0$. Then $K_n = \mathrm{Fix}(K_{n-1})$ is affinely isometric to $\mathrm{Fix}(\mathrm{Prob}(X^{(n-1)}))$ as well. But this set is affinely isometric to $\mathrm{Prob}(X^{(n)})$ (by Theorem 5.7) and we are done. □

Now since $\delta(\Delta(X)) = \lim_{n\to\infty} \delta(\mathrm{Fix}^n(\mathrm{Prob}(X)))$, Proposition 5.8 leads to

Corollary 5.9. *Let X be a nonempty compact metric space. $\Delta(X)$ consists of a single measure iff X is quasi-nilpotent.*

Definition 5.10. Let X be a nonempty quasi-nilpotent compact metric space. The unique member of $\Delta(X)$ is denoted by λ_X and it is called the *central measure of X of a second kind.*

It follows from the definition that whenever X is a nonempty quasi-nilpotent compact metric space, so is $X^{(n)}$ for each $n \in \mathbb{N}$. For $n \geq 1$ let

$$\pi_X^{(n)} = \pi_{X^{(n-1)}}^{(1)} \circ \ldots \circ \pi_X^{(1)}\colon X \to X^{(n)}.$$

Another consequence of Theorem 5.7 is

PROPOSITION 5.11. *Let X be a nonempty compact quasi-nilpotent metric space. Then:*

(a) $(\pi_X^{(n)})_*(\lambda_X) = \lambda_{X^{(n)}}$ *for each* $n \geq 1$,

(b) λ_X *is a unique measure* $\lambda \in \mathrm{Fix}(\mathrm{Prob}(X))$ *such that for all positive integers* n, $(\pi_X^{(n)})_*(\lambda) \in \mathrm{Fix}(\mathrm{Prob}(X^{(n)}))$.

PROOF. To show (a), it is enough to check that $(\pi_X^{(1)})_*(\lambda_X) = \lambda_{X^{(1)}}$. By Theorem 5.7, $(\pi_X^{(1)})_*$ is an affine isometry from $\mathrm{Fix}(\mathrm{Prob}(X))$ onto $\mathrm{Prob}(X^{(1)})$. Therefore, it follows from the induction argument that

$$(\pi_X^{(1)})_*(\mathrm{Fix}^{n+1}(\mathrm{Prob}(X))) = \mathrm{Fix}^n(\mathrm{Prob}(X^{(1)}))$$

for each $n \in \mathbb{N}$. Consequently,

$$(\pi_X^{(1)})_*(\mathrm{Fix}^\infty(\mathrm{Prob}(X))) = \mathrm{Fix}^\infty(\mathrm{Prob}(X^{(1)}))$$

and we are done.

Now we pass to (b). Thanks to (a), it remains to establish the uniqueness of λ. Put $\mu_n = (\pi_X^{(n)})_*(\lambda) \in \mathrm{Fix}(\mathrm{Prob}(X^{(n)}))$ for $n > 0$ and $\mu_0 = \lambda \in \mathrm{Fix}(\mathrm{Prob}(X))$. We then have

$$(\pi_{X^{(n)}}^{(1)})_*(\mu_n) = \mu_{n+1}. \tag{5.5}$$

The proof of (a) shows that

$$(\pi_{X^{(n)}}^{(1)})_*(\mathrm{Fix}^{m+1}(\mathrm{Prob}(X^{(n)}))) = \mathrm{Fix}^m(\mathrm{Prob}(X^{(n+1)}))$$

for each $m \geq 1$. This, combined with (5.5) and induction argument, gives $\lambda \in \mathrm{Fix}^n(\mathrm{Prob}(X))$ (since $\mu_{n-1} \in \mathrm{Fix}(\mathrm{Prob}(X^{(n-1)}))$ and $(\pi_{X^{(j)}}^{(1)})_*$ is one-to-one on $\mathrm{Fix}(\mathrm{Prob}(X^{(j)}))$ for each j) which finishes the proof. $\square$

Since $\lambda_X \in \mathrm{Fix}(\mathrm{Prob}(X))$, the central measure of X of a second kind is invariant under every isometry of X. We conclude from this that $\lambda_X = \mu_X$ provided X is a compact metric space such that $X^{(1)}$ is a singleton (i.e. if the isometry group of X acts transitively on X). We end the section with

PROPOSITION 5.12. *The central measure of* $[0,1]$ *of a second kind coincides with the Lebesgue measure on* $[0,1]$.

PROOF. For $n \geq 0$ let $I_n = [0, 1/2^n]$ be equipped with the natural metric. The argument involved in Example 5.3 shows that $I_n^{(1)}$ and $\pi_{I_n}^{(1)}$ may be represented as (respectively) I_{n+1} and $\tau_n : I_n \ni t \mapsto 1/2^{n+1} - |t - 1/2^{n+1}| \in I_{n+1}$. Further, let m denote the Lebesgue measure on $\mathbb{R}$ and $\mu_n \in \mathrm{Prob}(I_n)$ be given by $\mu_n(A) = 2^n m(A)$. Thanks to Lemma 5.4, $\mu_n \in \mathrm{Fix}(\mathrm{Prob}(I_n))$. So, taking into account point (b) of Proposition 5.11, it suffices to check that $(\tau_n)_*(\mu_n) = \mu_{n+1}$.

Let $\Phi_n : \mathbb{R} \ni x \mapsto 1/2^n - x \in \mathbb{R}$ and note that for a Borel subset A of I_{n+1} we have $\tau_n^{-1}(A) = A \cup \Phi_n^{-1}(A)$ and $A \cap \Phi_n^{-1}(A) \subset \{1/2^{n+1}\}$ which yield

$$\begin{aligned}(\tau_n)_*(\mu_n)(A) &= \mu_n(A) + \mu_n(\Phi_n^{-1}(A)) \\ &= 2^n[m(A) + m(\Phi_n^{-1}(A))] = 2^{n+1}m(A) = \mu_{n+1}(A)\end{aligned}$$

and we are done. □

6. Distinguishing dense subsets

We know that strict distinguishing is impossible in general. However, one may still ask whether it is possible to define a distinguishing in the class $\mathcal{K}$ of all nonempty compact metric spaces. This part is devoted to the solution of this problem. We shall show that there is a sequence of distinguishings $\mathcal{K} \ni K \mapsto C_n(K) \in K^{(1)}$ $(n \geq 1)$ such that for every $K \in \mathcal{K}$ there is a dense in K sequence $\{c_n\}_{n=1}^{\infty} \subset K$ such that $\pi_K^{(1)}(c_n) = C_n(K)$.

Let $\mathbb{N} = \{0, 1, \dots\}$. Fix an infinite compact metric space (X, d). Instead of constructing an 'intrinsic' dense subset of X, we shall construct a metric ϱ_X on $\mathbb{N}$ such that $(\mathbb{N}, \varrho_X)$ is isometric to a dense subset of X. Suppose for some $n > 0$ we have defined the metric ϱ_X on $\{0, \dots, n-1\}$ in such a way that the space $(\{0, \dots, n-1\}, \varrho_X)$ is isometrically embeddable into X (ϱ_X is uniquely defined on $\{0\}$). Put

$$F_{n-1} := \{(x_0, \dots, x_{n-1}) \in X^n : d(x_j, x_k) = \varrho_X(j,k),\ j,k = 0, \dots, n-1\} \tag{6.1}$$

and let $f_{n-1} : F_{n-1} \times X \to \mathbb{R}$ be given by

$$f_{n-1}(x_0, \dots, x_{n-1}; x) = \min(d(x_0, x), \dots, d(x_{n-1}, x)). \tag{6.2}$$

By our assumption, F_{n-1} is nonempty. Next, let

$$A_0^n := \{(x; y) \in F_{n-1} \times X : f_{n-1}(x; y) = \max f_{n-1}(F_{n-1} \times X)\}. \tag{6.3}$$

Now inductively define sets A_j^n for $j = 1, \dots, n$ by

$$\begin{aligned}A_j^n := \big\{&(y_0, \dots, y_{n-1}; y) \in A_{j-1}^n : \\ &d(y_{j-1}, y) = \max\{d(x_{j-1}, x) \mid (x_0, \dots, x_{n-1}; x) \in A_{j-1}^n\}\big\}.\end{aligned} \tag{6.4}$$

It follows from induction and the compactness of the space that

$(1)_n$ $A_n^n \subset A_{n-1}^n \subset \ldots \subset A_0^n$ and $A_n^n \neq \emptyset$,

$(2)_n$ for every $(y_0, \dots, y_{n-1}; y_n), (y_0', \dots, y_{n-1}'; y_n') \in A_k^n$ (with $1 \leq k \leq n$),

$$d(y_p, y_q) = \varrho_X(p, q) \quad (p, q \in \{0, \dots, n-1\})$$

and $d(y_j, y_n) = d(y_j', y_n')$ for $0 \leq j < k$.

Now take any $(x_0, \dots, x_{n-1}; x_n) \in A_n^n$ and put $\varrho_X(j,n) := d(x_j, x_n)$ for $j = 0, \dots, n$. Observe that this definition is independent of the choice of

$$(x_0, \dots, x_{n-1}; x) \in A_n^n,$$

by $(2)_n$. It is also clear that ϱ_X is a metric (not only a pseudometric) on $\{0, \dots, n\}$ (because X is infinite – and hence $\max f_{n-1}(F_{n-1} \times X) > 0$).

In this way we obtain a metric ϱ_X on $\mathbb{N}$ such that $\varrho_X = \varrho_Y$ for every space Y isometric to X. We claim that

Proposition 6.1. *For every infinite compact metric space* (X, d)*, the space* $(\mathbb{N}, \varrho_X)$ *is isometric to a dense subset of* X.

Proof. For each $n \in \mathbb{N}$ let P_n be the set of all sequences $(x_m)_{m=0}^{\infty} \in X^{\mathbb{N}}$ such that the function $(\{0, \dots, n\}, \varrho_X) \ni j \mapsto x_j \in (X, d)$ is isometric. By construction of ϱ_X, P_n is nonempty. It is also clear that P_n is closed in $X^{\mathbb{N}}$ and that $P_n \supset P_{n+1}$. Therefore, by the compactness of X, the intersection $\bigcap_{n=0}^{\infty} P_n$ is nonempty. We infer from this that there is an isometric function Φ of $(\mathbb{N}, \varrho_X)$ into (X, d). We claim that $\Phi(\mathbb{N})$ is dense in X. Suppose, for the contrary, that there is $x \in X$ and $r > 0$ such that

$$d(x, \Phi(n)) \geq r \tag{6.5}$$

for every $n \in \mathbb{N}$. Note that $(\Phi(0), \dots, \Phi(n)) \in A_n^n \subset A_0^n \subset F_{n-1} \times X$ for any $n > 0$, where A_j^n's and F_{n-1}'s are given by (6.4), (6.3) and (6.1). So, (6.5) yields $\max f_{n-1}(F_{n-1} \times X) \geq r$ for f_{n-1}'s given by (6.2). Finally, we conclude from the relation $(\Phi(0), \dots, \Phi(n)) \in A_0^n$ and (6.5) that $f_{n-1}(\Phi(0), \dots, \Phi(n-1); \Phi(n)) \geq r$ which means that $d(\Phi(j), \Phi(k)) \geq r$ for $j < k$. But this contradicts the compactness of X. $\square$

By a *representation* of the metric ϱ_X we mean any isometric function of $(\mathbb{N}, \varrho_X)$ into (X, d), provided X is infinite.

If X is finite and has n elements, we may repeat the above construction to obtain a metric ϱ_X on $\{0, \dots, n-1\}$ which makes this set isometric to X. In that case by a *representation* of ϱ_X we mean any function $\Phi\colon \mathbb{N} \to X$ such that Φ is isometric on $\{0, \dots, n-1\}$ (with respect to the metrics ϱ_X and d) and $\Phi(k) = \Phi(n-1)$ for $k > n-1$.

We may ask how many representations has the metric ϱ_X for an arbitrary space X. The answer to this gives

Proposition 6.2. *Let* X *be a nonempty compact metric space and* $\Phi_0\colon \mathbb{N} \to X$ *be a representation of* ϱ_X*. The function* $\Psi \mapsto \Psi \circ \Phi_0$ *establishes a one-to-one correspondence between isometries* $[\Psi]$ *of* X *and representations* $[\Psi \circ \Phi_0]$ *of* ϱ_X.

Proof. Since the composition of two isometric maps is isometric as well, we see that $\Psi \circ \Phi_0$ is a representation of ϱ_X for each $\Psi \in \mathrm{Iso}(X)$. Conversely, the proof of Proposition 6.1 shows that every representation of ϱ_X has dense image in X. So, the assignment $\Psi \mapsto \Psi \circ \Phi_0$ is one-to-one. Finally, if $\Phi: \mathbb{N} \to X$ is any representation of ϱ_X, then the sets $A := \Phi_0(\mathbb{N})$ and $B := \Phi(\mathbb{N})$ are dense in X and a function $\psi = \Phi \circ \Phi_0^{-1}: A \to B$ is an isometry of A onto B. The completeness of X implies that there is $\Psi \in \mathrm{Iso}(X)$ which extends ψ. But then $\Phi = \Psi \circ \Phi_0$ which finishes the proof. □

Proposition 6.2 says that if the isometry group of a space X is poor, there are only few representations of ϱ_X. In the opposite, if there are many representations, the isometry group of X is rich. Both the situations are interesting.

Now we pass to distinguishing of points. Observe that whatever representation $\Phi: \mathbb{N} \to X$ of ϱ_X we take, the function $\pi_X^{(1)} \circ \Phi$ is the same (thanks to Proposition 6.2). We infer from this that the definition $C_n(X) := \pi_X^{(1)}(\Phi(n))$ where $n \in \mathbb{N}$ and Φ is any representation of ϱ_X is correct. We now have

Proposition 6.3. *For each $n \in \mathbb{N}$, the assignment $\mathcal{K} \ni K \mapsto C_n(K) \in K^{(1)}$ is a distinguishing.*

Proof. Let K and L be two isometric compact nonempty metric spaces. Then $\varrho_K = \varrho_L$. Let $\Phi: \mathbb{N} \to K$ be a representation of ϱ_K and $\Psi \in \mathrm{Iso}(K, L)$. Since then $\Phi' := \Psi \circ \Phi$ is a representation of ϱ_L, we obtain that $C_n(L) = \pi_L^{(1)}(\Phi'(n))$. Note that $\Psi^{(1)}(C_n(K)) = C_n(L)$, since $C_n(K) = \pi_K^{(1)}(\Phi(n))$ and $\Psi^{(1)} \circ \pi_K^{(1)} = \pi_L^{(1)} \circ \Psi$. □

In studying the class of separable complete metric spaces, especially in theory of random metric spaces (cf. [19], [20]), one of methods is to consider the set of all metrics $\mathfrak{D}$ on $\mathbb{N}$ and to make the assignment $\mathfrak{D} \ni d \mapsto$ 'the completion of $(\mathbb{N}, d)$'. In other words, the 'world' of infinite separable complete metric spaces may be identified (by this assignment) with the 'world' of metrics on $\mathbb{N}$. This is quite natural approach, however, there is no one-to-one correspondence between the members of these two worlds. The distinguishing of dense subsets of compact metric spaces constructed in this section may be seen as an example of the 'inverse function' to the above assignment after restricting the considerations to totally bounded metrics on $\mathbb{N}$.

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